# Nearly Starobinsky inflation from modified gravity

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We study inflation induced by (power-low) scalar curvature corrections to General Relativity. The class of inflationary scalar potentials $V(\sigma) \sim \exp[n\sigma]$, $n$ general parameter, is investigated in the Einstein frame, and the corresponding actions in the Jordan frame are derived. We found the conditions for which these potentials are able to reproduce viable inflation according to the last cosmological data and lead to large scalar curvature corrections that emerge only at a mass scale larger than the Planck mass. The cosmological constant may appear or be set equal to zero in the Jordan frame action without changing the behavior of the model during inflation. Moreover, polynomial corrections to General Relativity are analyzed in detail. When de Sitter space-time emerges as an exact solution of the models, it is necessary to use perturbative equations in the Jordan framework to study their dynamics during the inflation. In this case, we demonstrate that the Ricci scalar decreases after a correct amount of inflation, making the models consistent with the observable evolution of the Universe.

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## I. INTRODUCTION

A large number of inflationary cosmology models is based on scalar fields, which play an important role in the particle physics theories. The inflation is produced by a homogeneous scalar field, dubbed inflaton, which under suitable conditions may lead to an early-time accelerated expansion. Following the first proposal of Guth [1] and Sato [2], in the last years many inflationary models based on scalar fields (and inspired by modified gravity theories, string theories, quantum effects in the hot universe, etc.) have been proposed.

Typically the magnitude of the scalar field is very large at the beginning of the inflation and then it rolls down towards a potential minimum where the inflation ends (see Ref. [3] as an example of chaotic inflation). In other models the field can fall in a potential hole, where it starts to oscillate and the reheating processes take place [4–7]. Some more complicated models are based on a phase transition between two scalar fields: they are the so-called hybrid or double inflation models [8,9]. For the introduction to the dynamics of inflation, see Ref. [10] and Refs. [11,12].

Recently, cosmological and astrophysical data [13] seem to confirm the predictions of the Starobinsky inflationary model [14]. Such a model is based on the account of the $R^2$ term as the correction in the Einstein equations. This quadratic correction emerges in the Planck epoch and plays a fundamental role in the high curvature limit, when the early-time acceleration takes place. Such theory is conformally equivalent to a scalar-tensor theory in the Einstein frame, where the inflaton drives the expansion in a quasi–de Sitter space-time and slowly moves to the end of inflation, when the reheating processes [15–17] start. Such an inflationary model has been recently revisited in many works. Among them, in Ref. [18] a superconformal generalization of such a model in superconformal theory has been investigated, and in Refs. [19,20] other applications based on the spontaneous breaking of conformal invariance and on the scale-invariant extensions of the Starobinsky model have been presented. In Ref. [21], a generalization of the Starobinsky model represented by a polynomial correction of the Einstein gravity of the type $c_1R^2 + c_2R^n$ has been studied.

In this paper, we will concentrate on inflation caused by scalar curvature corrections to Einstein gravity; namely, we will consider the so-called $F(R)$ gravity, whose action is in the form of $F(R) = R + f(R)$, with $f(R)$ a function of the Ricci scalar (for recent reviews on modified gravity, see Refs. [22–26] and Ref. [27]). This kind of correction may occur due to quantum effects in the hot universe or may be

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motivated by the ultraviolet completion of the quantum theory of gravity. Our aim is to investigate which kinds of viable inflation can be realized in the contest of $F(R)$ gravity beyond the Starobinsky model, whose dynamics is governed in the Einstein frame by a potential of the type $V(\sigma) \sim (1-\exp[-\sigma])^2$, with $\sigma$ being the inflaton.

The paper is organized as follows. In Secs. II and III, we will revisit the conformal transformations which permit us to pass from the Jordan frame to the Einstein one, and we will recall the dynamics of the viable inflation. In Sec. IV we will study inflation for the general class of scalar potentials of the type $V(\sigma) \sim \exp[n\sigma]$, with $n$ being a general parameter, performing the analysis in the Einstein frame and therefore reconstructing the $F(R)$-gravity theories that correspond to the given potentials. Viable inflation must be consistent with the last Planck data (spectral index, tensor-to-scalar ratio…) and must correspond to Einstein theory corrections that emerge only at mass scales larger than the Planck one, namely at high curvatures. We will see the conditions on $n$ for which slow-roll conditions are satisfied, and we will reconstruct the form of the $F(R)$ models during early-time acceleration and at small curvatures. Some specific examples are presented. In Sec. V, following the recent success of higher-derivative gravity, we will revisit and study in detail the specific class of models $F(R) = R + (R + R_0)^n$. The analysis in the Einstein frame reveals that $n$ must be very close to two in order to realize a viable inflation for large and negative values of the scalar field, but other possibilities are allowed by bounding the field in a different way. In particular, when $n > 2$, the de Sitter solution emerges, but in order to study the exit from inflation is necessary to analyze the theory in the Jordan frame, where perturbations make possible an early time acceleration with a sufficient amount of inflation. A summary and outlook are given in Sec. VI. Technical details and further considerations are presented in the Appendixes.

We shall use units in which $c = \hbar = k_{\mathrm{B}} = 1$, with $c$, $\hbar$, $k_{\mathrm{B},}$ respectively, the speed of light, the Planck constant, and the Boltzmann constant. Moreover, we shall denote by $G_N$ the gravitational constant and by $M_{\mathrm{Pl}} = G_N^{-1/2} = 1.2 \times 10^{19}\,\mathrm{GeV}$ the Planck mass. Finally, we shall set $\kappa^2 \equiv 8\pi G_N$.

## II. CONFORMAL TRANSFORMATIONS

In scalar-tensor theories of gravity, a scalar field coupled to the metric appears in the action. The first scalar-tensor theory was proposed by Brans and Dicke in 1961 [28] in the attempt to incorporate the Mach's principle into the theory of gravity, but today the interest in such theories is related to the possibility of reproducing the primordial acceleration of the inflationary universe.

In principle, a modified gravity theory can be rewritten in scalar-tensor or Einstein frame form. Let us start by considering the general action of $F(R)$-modified gravity,

$$I = \int_{\mathcal{M}} d^4x\sqrt{-g}\left[\frac{F(R)}{2\kappa^2}\right], \tag{1}$$

where $F(R)$ is a function of the Ricci scalar $R$, $g$ is the determinant of the metric tensor $g_{\mu\nu}$, and $\mathcal{M}$ is the space-time manifold. Now we introduce the field $A$ into (1),

$$I_{JF} = \frac{1}{2\kappa^2}\int_{\mathcal{M}} \sqrt{-g}[F_A(A)(R-A) + F(A)]d^4x. \tag{2}$$

Here, $JF$ means Jordan frame, and $F_A(A)$ denotes the derivative of $F(A)$ with respect to $A$. By making the variation with respect to $A$, we immediately obtain $A = R$, such that (2) is equivalent to (1). We define the scalar field $\sigma$ (which in fact encodes the new degree of freedom in the theory, namely the scalaron or inflaton) as

$$\sigma := -\sqrt{\frac{3}{2\kappa^2}}\ln[F_A(A)]. \tag{3}$$

By considering the conformal transformation of the metric,

$$\tilde{g}_{\mu\nu} = \mathrm{e}^{-\sigma} g_{\mu\nu}, \tag{4}$$

we finally get the Einstein frame action, line

$$\begin{aligned} I_{EF} &= \int_{\mathcal{M}} d^4x\sqrt{-\tilde{g}}\left\{\frac{\tilde{R}}{2\kappa^2} - \frac{1}{2}\left(\frac{F_{AA}(A)}{F_A(A)}\right)^2 \tilde{g}^{\mu\nu}\partial_\mu A\partial_\nu A \right. \\ &\quad \left. - \frac{1}{2\kappa^2}\left[\frac{A}{F_A(A)} - \frac{F(A)}{F_A(A)^2}\right]\right\} \\ &= \int_{\mathcal{M}} d^4x\sqrt{-\tilde{g}}\left(\frac{\tilde{R}}{2\kappa^2} - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - V(\sigma)\right), \end{aligned} \tag{5}$$

where $\tilde{R}$ denotes the Ricci scalar evaluated in the conformal metric $\tilde{g}_{\mu\nu}$, and $\tilde{g}$ is the determinant of the conformal metric, namely $\tilde{g} = \mathrm{e}^{-4\sigma} g$. Furthermore, one has

$$\begin{aligned} V(\sigma) \equiv \frac{A}{F'(A)} - \frac{F(A)}{F'(A)^2} = \frac{1}{2\kappa^2}\{&\mathrm{e}^{(\sqrt{2\kappa^2/3})\sigma} R(\mathrm{e}^{-(\sqrt{2\kappa^2/3})\sigma}) \\ &- \mathrm{e}^{2(\sqrt{2\kappa^2/3})\sigma} F[R(\mathrm{e}^{-(\sqrt{2\kappa^2/3})\sigma})]\}, \end{aligned} \tag{6}$$

with $R(\mathrm{e}^{-\sqrt{2\kappa^2/3}\sigma})$ being the solution of Eq. (3) where $A = R$, with $R$ a function of $\mathrm{e}^{-\sqrt{2\kappa^2/3}\sigma}$. In what follows, we will omit the tilde to denote all the quantities evaluated in the Einstein frame.

Note that string-inspired inflationary models also contain canonical and/or tachyon scalar (for recent discussion see Refs. [29, 30]).

## III. DYNAMICS OF INFLATION

In this section, for the sake of completeness, we will recall the well-known facts on inflation. The energy density and pressure of the inflaton $\sigma$ are given by

$$\rho_\sigma = \frac{\dot{\sigma}^2}{2} + V(\sigma), \quad p_\sigma = \frac{\dot{\sigma}^2}{2} - V(\sigma), \tag{7}$$

where the dot is the derivative with respect to the cosmological time. The Friedmann equations in the presence of $\sigma$ read

$$\frac{3H^2}{\kappa^2} = \frac{\dot{\sigma}^2}{2} + V(\sigma), \qquad -\frac{1}{\kappa^2}(2\dot{H} + 3H^2) = \frac{\dot{\sigma}^2}{2} - V(\sigma), \tag{8}$$

and the energy conservation law coincides with the equation of motion for $\sigma$ and reads

$$\ddot{\sigma} + 3H\dot{\sigma} = -V'(\sigma), \tag{9}$$

where the prime denotes the derivative of the potential with respect to $\sigma$. From the Friedmann equations we obtain

$$\dot{H} = -\frac{\kappa^2}{2}(\rho_\sigma + p_\sigma) = -\frac{\kappa^2}{2}\dot{\sigma}^2. \tag{10}$$

On the other hand, the acceleration can be expressed as

$$\frac{\ddot{a}}{a} = H^2 + \dot{H} = H^2(1 - \epsilon), \tag{11}$$

where we have introduced the "slow roll" parameter

$$\epsilon = -\frac{\dot{H}}{H^2}. \tag{12}$$

This parameter may be expressed as a function of the inflaton as

$$\epsilon = \frac{\kappa^2\dot{\sigma}^2}{2H^2}. \tag{13}$$

We also have

$$\frac{\ddot{a}}{a} = \frac{\kappa^2}{3}(V - \dot{\sigma}^2). \tag{14}$$

Thus, the condition to have an acceleration is $\epsilon < 1$ or $\dot{\sigma}^2 < V(\sigma)$. There is another slow roll parameter defined by

$$\eta = -\frac{\ddot{H}}{2H\dot{H}} = \epsilon - \frac{1}{2\epsilon H}\dot{\epsilon}. \tag{15}$$

As a function of the inflaton, one has

$$\eta = -\frac{\ddot{\sigma}}{H\dot{\sigma}}. \tag{16}$$

For the inflation to occur and persist for a convenient amount of time, a quasi–de Sitter space is required; namely, $\dot{H}$ has to be very small, and, as a result, also the two slow roll parameters have to be very small, and one has

$$\dot{\sigma}^2 \ll V(\sigma); \tag{17}$$

namely, the kinetic energy of the field has to be small during the inflation. As a result, the Friedmann equations reduce to

$$\frac{3H^2}{\kappa^2} \simeq V(\sigma), \qquad 3H\dot{\sigma} \simeq -V'(\sigma). \tag{18}$$

It is easy to show that within this slow roll regime, the slow roll parameters may be expressed as a function of the inflaton potential as

$$\epsilon = \frac{1}{2\kappa^2}\left(\frac{V'(\sigma)}{V(\sigma)}\right)^2, \qquad \eta = \frac{1}{\kappa^2}\left(\frac{V'(\sigma)}{V(\sigma)}\right). \tag{19}$$

Inflation ends when $\epsilon$, $|\eta| \sim 1$. A useful quantity that describes the amount of inflation is the $e$-foldings number $N$ defined by

$$N \equiv \ln\frac{a_f}{a_i} = \int_{t_i}^{t_e} H dt \simeq \kappa^2 \int_{\sigma_e}^{\sigma_i} \frac{V(\sigma)}{V'(\sigma)} d\sigma, \tag{20}$$

where the indices $i$, $f$ refer to the quantities at the beginning and the end of inflation, respectively. The required $e$-foldings number for inflation is at least $N \simeq 60$. The amplitude of the primordial scalar power spectrum is

$$\Delta_{\mathcal{R}}^2 = \frac{\kappa^4 V}{24\pi^2\epsilon}, \tag{21}$$

and for slow roll inflation the spectral index $n_s$ and the tensor-to-scalar ratio are given by

$$n_s = 1 - 6\epsilon + 2\eta, \quad r = 16\epsilon. \tag{22}$$

The last Planck data constrain these quantities as

$$n_s = 0.9603 \pm 0.0073, \qquad r < 0.11. \tag{23}$$

## IV. RECONSTRUCTION OF $F(R)$ THEORY FROM THE SCALAR POTENTIAL AND ANALYSIS OF THE INFLATION IN THE EINSTEIN FRAME

In this section, we will study some classes of scalar potential which produce inflation in the Einstein frame. The aim is to generalize the Starobinsky model by considering different behavior of the scalar potential as $V(\sigma) \sim \exp[n\sqrt{2\kappa^2/3}\sigma]$, where $n$ is the parameter on which it depends the dynamics of the inflation (slow roll parameters, spectral indices…). This analysis is motivated by the possibility to reconstruct suitable $F(R)$ corrections to General Relativity in the corresponding Jordan frame. By 'suitable' we mean corrections that vanish at mass scales smaller than the Planck mass $M_{\rm Pl}$ and give rise to corrections only in the high curvature limits, namely during the inflationary period. Every scalar potential will be confronted with cosmological data.

In order to reconstruct the $F(R)$ gravity which corresponds to a given potential, we may start from Eq. (6). By dividing such equation to $\exp[2\sqrt{2\kappa^2/3}]$, and then by taking the derivative with respect to $R$, we get

$$RF_R(R) = -2\kappa^2\sqrt{\frac{3}{2\kappa^2}}\frac{d}{d\sigma}\left(\frac{V(\sigma)}{e^{2(\sqrt{2\kappa^2/3})\sigma}}\right). \quad (24)$$

As a result, giving the explicit form of the potential $V(\sigma)$, thanks to the relation (3), we obtain an equation for $F_R(R)$, and therefore the $F(R)$-gravity model in the Jordan frame. In this process, one introduces the integration constant, which has to be fixed by requiring that Eq. (6) holds true.

### A. $V(\sigma) \sim c_0 + c_1 \exp[\sigma] + c_2 \exp[2\sigma]$: $R + R^2 + \Lambda$ models

Let us start with the following inflationary potential

$$V(\sigma) = [c_0 + c_1 e^{\sqrt{\kappa^2/3}\sigma} + c_2 e^{\sqrt{2\kappa^2/3}\sigma}]. \quad (25)$$

This is the simplest example and is the minimal generalization of the Starobinsky model. Equation (24) gives

$$2c_0 F_R^2 + c_1 F_R - RF_R = 0. \quad (26)$$

Assuming $F_R \neq 0$, one gets

$$F_R(R) = -\frac{c_1}{2c_0} + \frac{R}{2c_0}. \quad (27)$$

Thus, the corresponding Lagrangian of $F(R)$ gravity is given by

$$F(R) = -\frac{c_1}{2c_0}R + \frac{R^2}{4c_0} + \Lambda. \quad (28)$$

Here, $\Lambda$ is a constant of integration, which can be determined by Eq. (6). The result is

$$\Lambda = \frac{c_1^2}{4c_0} - c_2. \quad (29)$$

An important remark is in order. In order to have the correct Einstein-Hilbert term, we must put $-c_1/(2c_0) = 1$. Thus, we have the class of modified quadratic models depending on two constants,

$$F(R) = R + \frac{R^2}{4c_0} + c_0 - c_2. \quad (30)$$

Furthermore, in the specific case $c_2 = 0$, one has

$$F(R) = R + \frac{R^2}{4c_0} + c_0. \quad (31)$$

This is an interesting model, and in the Appendix B, we will study its static spherically symmetric solutions.

The other interesting case is the vanishing of cosmological constant, namely

$$c_2 = c_0. \quad (32)$$

Since we are assuming $c_1/(2c_0) = 1$, it also follows $c_2 = -c_1/2$. As a consequence, we recover the Starobinsky model $F(R) = R + \frac{R^2}{4c_0}$, and related Einstein frame potential

$$V(\sigma) = c_0(1 - e^{\sqrt{2\kappa^2/3}\sigma})^2. \quad (33)$$

The model (30) is the extension of the Starobinsky model to the case with cosmological constant and gives a viable inflation. In what follows, we denote

$$c_0 = \frac{\gamma}{4\kappa^2}. \quad (34)$$

The initial value of the inflaton is large (and negative) and it rolls down toward the potential minimum at $\sigma \to 0^-$, $V(0) = -\gamma/(4\kappa^2) < 0$. During inflation ($\sigma \to -\infty$), Eq. (18) leads to

$$H^2 \simeq \frac{\gamma}{12}, \qquad 3H\dot{\sigma} \simeq \left(\frac{\gamma}{\sqrt{6\kappa^2}}\right)e^{\sqrt{2\kappa^2/3}\sigma}. \quad (35)$$

It means, that a quasi–de Sitter solution can be realized. We must require

$$\gamma \propto M^2, \qquad M \ll M_P, \quad (36)$$

where $M_P$ is the Planck mass. As a consequence, the corrections of Einstein gravity emerge at high curvature and one gets the accelerated expansion with $H \propto \sqrt{\alpha}$. The scalar field behaves as

$$\sigma \simeq -\sqrt{\frac{3}{2\kappa^2}} \ln\left[\frac{1}{3}\sqrt{\frac{2\gamma}{3}}(t_0 - t)\right], \quad (37)$$

where $t_0$ is bounded at the beginning of the inflation. If at this time $|\sigma|$ is very large, the slow roll parameters (19),

$$\epsilon = \frac{4}{3}\frac{1}{(2 - e^{-\sqrt{2\kappa^2/3}\sigma})^2} \simeq 0,$$
$$|\eta| = \frac{4}{3}\frac{1}{|2 - e^{-\sqrt{2\kappa^2/3}\sigma}|} \simeq 0, \quad (38)$$

are very small and the field moves slowly. The inflation ends when such parameters are of the order of unit, namely at $\sigma_e \simeq -0.17\sqrt{3/(2\kappa^2)}$. The $e$-foldings number can be evaluated from (20)and reads ($\sigma_i \gg \sigma_e$),

$$N \simeq \frac{3e^{-\sqrt{2\kappa^2/3}\sigma}}{4}\Bigg|_{\sigma_e}^{\sigma_i} \simeq \frac{1}{4}\sqrt{\frac{2\gamma}{3}}t_0. \quad (39)$$

The inflation ends at $t_e = t_0 - 3\sqrt{3/(2\alpha)}$ exp. For example, in order to obtain $N = 60$, we must require $\sigma_i \simeq -\sqrt{3/(2\kappa^2)}4.38 \simeq 1.07\,M_{pl}$. Moreover, we may express the $e$-foldings numbers as

$$\epsilon \simeq \frac{3}{4N^2}, \qquad |\eta| \simeq \frac{1}{N}. \tag{40}$$

The amplitude of primordial power spectrum (21) is

$$\Delta_{\mathcal{R}}^2 \simeq \frac{\kappa^2 \gamma N^2}{72\pi^2} \propto \frac{\kappa^2 M^2 N^2}{72\pi^2} \ll \frac{\kappa^2 M_P^2 N^2}{72\pi^2}, \tag{41}$$

and the indexes (22) result to be

$$n_s \simeq 1 - \frac{2}{N}, \qquad r \simeq \frac{12}{N^2}. \tag{42}$$

Since we have $n_s > 1 - \sqrt{0.11/3} \simeq 0.809$ when $r < 0.11$, $n_s < 1$, we see that these indices are compatible with (23). For example, for $N = 60$, one has $n_s = 0.967$ and $r = 0.003$. We stress that this behavior is the one of Starobinsky model, with a different minimum of the potential and a cosmological constant in the Jordan frame. The appearance of a cosmological constant at large curvature needs some explanation. It can be originated by some quantum effects or may be supported by a modified gravity term which makes it to vanish at small curvatures (see, for example, Refs. [31, 32]). However, if the cosmological constant is set equal to zero (or, if necessary, is set equal to an other value), the feature of the model in Einstein frame does not change during the inflation: we will see that this double possibility, namely, taking a cosmological constant in the Jordan frame, or taking an additional term $\propto \exp[2\sqrt{2\kappa^2/3}\sigma]$ in the Einstein frame, does not modify the proprieties of the scalar potentials during the early time acceleration.

### B. $V(\sigma) \sim \gamma\, \exp[-n\sigma]$, $n > 0$: $c_0 R^{\frac{n+2}{n+1}}$ models

As a second example, we consider the following potential

$$V(\sigma) = \frac{\alpha}{\kappa^2}(1 - e^{\sqrt{2\kappa^2/3}\sigma}) + \frac{\gamma}{\kappa^2} e^{-n\sqrt{2\kappa^2/3}\sigma}, \tag{43}$$

being $\gamma$, $n > 0$ constants. This potential possesses a minimum in which the scalar field may fall at the end of inflation. Since for large and negative values of the scalar field the potential is not flat, we do not expect a de Sitter universe, but if the slow roll conditions are satisfied, we can obtain an acceleration with a sufficient amount of inflation.

By using our reconstruction, from (24) one derives

$$F_R(R) - \frac{1}{2} + \frac{\gamma}{2\alpha}(2+n) F_R(R)^{1+n} = \frac{R}{4\alpha}. \tag{44}$$

In principle, this equation admits many solutions. At the perturbative level, it is easy to find that, by putting

$$\alpha = -\gamma(n+2), \tag{45}$$

we obtain

$$F_R(R \ll \gamma) \simeq 1 + c_1 R + c_2 R^2 + c_3 R^3 + \cdots, \tag{46}$$

when $|c_1 R|$, $|c_2 R^2|$, $|c_3 R^3| \cdots \ll 1$(see Appendix A). It means that, since $c_1 \propto \gamma^{-1}$, $c_2 \propto \gamma^{-2}$, $c_3 \propto \gamma^{-3}$…, if $\gamma$ satisfies (36), we recover the Einstein's gravity when $R \ll \gamma$ and the theory is an high curvature correction to General Relativity. Moreover, when $R \gg \gamma$, the asymptotic solution of Eq. (44) with (45)is given by

$$F_R(R \gg \gamma) \simeq \left(\frac{1}{4(n+2)}\right)^{\frac{1}{1+n}} \left(\frac{R}{\gamma}\right)^{\frac{1}{n+1}},$$
$$F(R \gg \gamma) \simeq \gamma \left(\frac{n+1}{n+2}\right) \left(\frac{1}{4(n+2)}\right)^{\frac{1}{1+n}} \left(\frac{R}{\gamma}\right)^{\frac{n+2}{n+1}}. \tag{47}$$

Here, one important comment is required. In the Einstein frame, $R_{EF} \sim \gamma$ during inflation, but the corresponding curvature in the Jordan frame is $R_{JF} \simeq e^{-\sqrt{2\kappa^2/3}\sigma} R_{EF}$ (we may neglect the kinetic energy of scalar field in the slow roll approximation), such that $R_{JF} \gg \gamma$ when $\sigma \to -\infty$ and expression (47), which is evaluated in the Jordan frame, effectively is valid for inflation (for $n \to 0$ we recover $F(R) \sim R^2$ in the Jordan frame).

The potential finally reads

$$V(\sigma) = -\frac{\gamma(n+2)}{\kappa^2}(1 - e^{\sqrt{2\kappa^2/3}\sigma}) + \frac{\gamma}{\kappa^2} e^{-n\sqrt{2\kappa^2/3}\sigma}. \tag{48}$$

This potential has a minimum ($V'(\sigma_{\min}) = 0$) at $\sigma_{\min} = -\sqrt{3/(2\kappa^2)}\log[(n+2)/n]/(n+1)$, and one gets

$$V(\sigma_{\min}) = \frac{\gamma}{\kappa^2}\left(n^{\frac{1}{n+1}}(n+2)^{\frac{n}{n+1}} + \left(\frac{n+2}{n}\right)^{\frac{1}{n+1}} - (n+2)\right) > 0, \quad \gamma, n > 0. \tag{49}$$

When $\sigma \to -\infty$ (large curvature), the potential goes to infinity, and when $\sigma \to 0^-$, $V(\sigma) = \gamma/\kappa^2$. Since the slow roll parameters are given by

$$\epsilon = \frac{(n - (n+2)e^{(n+1)\sqrt{2\kappa^2/3}\sigma})^2}{3((n+2)e^{n\sqrt{2\kappa^2/3}\sigma} - (n+2)e^{(n+1)\sqrt{2\kappa^2/3}\sigma} - 1)^2},$$
$$|\eta| = \frac{2}{3} \frac{n^2 + (n+2)e^{(n+1)\sqrt{2\kappa^2/3}\sigma}}{|1 + (n+2)e^{(n+1)\sqrt{2\kappa^2/3}\sigma} - (n+2)e^{n\sqrt{2\kappa^2/3}\sigma}|}, \tag{50}$$

one has $\epsilon(\sigma \to -\infty) \simeq n^2/3$ and $|\eta(\sigma \to -\infty)| \simeq 2n^2/3$, which implies $0 < n \ll 1$. The EOMs (18) in the slow roll limit read

$$H^2 \simeq \frac{\gamma}{3} e^{-n\sqrt{2\kappa^2/3}\sigma}, \quad 3H\dot{\sigma} \simeq \sqrt{\frac{2}{3\kappa^2}}\gamma n e^{-n\sqrt{2\kappa^2/3}\sigma}. \quad (51)$$

The solution for the scalar field is

$$\sigma = \frac{2}{n}\sqrt{\frac{3}{2\kappa^2}} \ln\left[\frac{n^2}{3}\sqrt{\frac{\gamma}{3}}(t_0+t)\right], \quad (52)$$

where $t_0$ is bounded to be very small at the beginning of the inflation, such that the field is negative and its magnitude very large. The solution for the Hubble parameter finally reads

$$H = \frac{3}{n^2(t_0+t)},$$
$$\frac{\ddot{a}}{a} = H^2 + \dot{H} = \frac{3}{n^2(t_0+t)^2}\left(\frac{3}{n^2} - 1\right) > 0, \quad (n < \sqrt{3}) \quad (53)$$

and we have an acceleration as soon as $\epsilon < 1$. Despite to the fact that in this kind of models the acceleration is smaller than in the de Sitter universe, the slow roll parameters can be small enough to justify our slow roll approximations. A direct evaluation of the ratio of kinetic energy of the field and potential leads to $(\dot{\sigma}^2/2)/V(\sigma) = n^2/9$, which is much smaller than one when $n \ll 1$. The inflation ends when the slow roll paramters are on the order of unit, before the minimum of the potential. Note that, by definition, since $V'(\sigma_{\min}) = 0$, one has $\epsilon(\sigma_{\min}) = 0$, which corresponds to the minimum of the slow roll parameter. However, before to this point, since the slow roll parameters behave as

$$\epsilon \simeq \frac{n^2}{3((n+2)e^{n\sqrt{2\kappa^2/3}\sigma} - 1)^2},$$
$$|\eta| \simeq \frac{2}{3}\frac{n^2}{|1-(n+2)e^{n\sqrt{2\kappa^2/3}\sigma}|}, \quad (54)$$

we find that $\epsilon$, $|\eta| \simeq 1$ when $\sigma \simeq -\sqrt{3/(2\kappa^2)} \times \log[(6+3n)/(3-n\sqrt{3})]/n$. Finally, from Eq. (20), we get the $N$-foldings number of inflation,

$$N \simeq -\frac{1}{n}\sqrt{\frac{3\kappa^2}{2}}\sigma\Big|_{\sigma_e}^{\sigma_i} \simeq -\frac{3}{n^2} \ln\left[\frac{n^2}{3}\sqrt{\frac{\gamma}{3}}t_0\right]. \quad (55)$$

When the inflation ends, the field falls in the minimum of the potential and starts to oscillate. The reheating process takes place. The amplitude of primordial power spectrum (21) and the spectral indexes (22) can be written as

$$\Delta^2_{\mathcal{R}} \simeq \frac{\kappa^2\gamma e^{\frac{2}{3}Nn^2}}{8\pi^2 n^2}, \qquad n_s \simeq 1 - \frac{2n^2}{3}, \qquad r \simeq \frac{16n^2}{3}. \quad (56)$$

The corrections to $n_s$ and $r$ are on the order of $\exp[-2n^2N/3] \ll 1$. These indexes are compatible with (23) when

$$n \sim \frac{1}{10}, \frac{2}{10}. \quad (57)$$

These are the typical values of $n$ which make the scalar potential (71) able to reproduce a viable inflationary scenario. The result suggests that only the models close to $R^2$ gravity are able to produce this kind of inflation.

In general, $F_R(R)$ in Eq. (46) may lead to a cosmological constant proportional to $\gamma$ in the $F(R)$ model. However, we can set it equal to zero, adding a suitable term in the potential (71) proportional to $\exp[2\sqrt{2\kappa^2/3}\sigma]$, which changes much more slowly than $\exp[-n\sqrt{2\kappa^2/3}\sigma]$ when $0 < n$, and the dynamics of the inflation is the same of above.

### C. $V(\sigma) \sim 3\gamma/4 - \gamma\exp[\sigma/2]$: $R/2 + c_1R^2 + c_2(R+R_0)^{3/2}$ models

We continue our analysis constructing potentials for de Sitter universe during inflation, but with a different behavior with respect to the Starobinsky one (which decreases as $\exp[-\sigma]$). We propose the potential as

$$V(\sigma) = \frac{\alpha}{\kappa^2} - \frac{\gamma}{\kappa^2} e^{\sqrt{2\kappa^2/3}\sigma/2}, \quad (58)$$

with $\alpha$, $\gamma > 0$, as usual, the constants. It follows that[1]

$$F_R(R) = \frac{9\gamma^2 + 8R\alpha + 3\sqrt{16R\alpha\gamma^2 + 9\gamma^4}}{32\alpha^2}. \quad (59)$$

Since we must require $\alpha$, $\gamma \gg 1$ and at small curvature we want to recover the Einstein gravity ($F_R = 1$), we set $\alpha = 3\gamma/4$, $\gamma > 0$, and we get

$$F_R(R) = \frac{1}{2} + \frac{1}{3\gamma}R + \frac{\sqrt{3}}{6}\sqrt{4R/\gamma + 3}. \quad (60)$$

Thus, from Eq. (6) we obtain

$$F(R) = \frac{R}{2} + \frac{R^2}{6\gamma} + \frac{\sqrt{3}}{36}(4R/\gamma + 3)^{3/2} + \frac{\gamma}{4}. \quad (61)$$

Here, we stress that the conformal transformation gives for the Ricci scalar

$$R = 3e^{-\sqrt{2\kappa^2/3}\sigma}(1 + e^{\sqrt{2\kappa^2/3/2}\sigma/2}),\ 3e^{-\sqrt{2\kappa^2/3}\sigma}$$
$$\times (1 - e^{\sqrt{2\kappa^2/3/2}\sigma/2}), \quad (62)$$

but only the second one leads to our potential (namely, is the one that emerges from our reconstruction). For $R \ll \gamma$,

[1]Here, we exclude the solution with the minus sign in front of the square root which leads to an imaginary value of $\sqrt{F_R}$ when $R$ is real.

the model reads $F(R \ll \gamma) \simeq R + \gamma/2$. If we want to recover the General Relativity action $F(R \ll \gamma) \simeq R$, we must set the cosmological constant, namely the last term in (61), equal to $-\gamma/4$: in this case the scalar potential is

$$V(\sigma) = \frac{3\gamma}{4\kappa^2} - \frac{\gamma}{\kappa^2} e^{\sqrt{2\kappa^2/3}\sigma/2} - \gamma \frac{e^{2\sqrt{2\kappa^2/3}\sigma}}{4\kappa^2}. \qquad (63)$$

Let us analyze the possibility to reproduce inflation from the potential (58), which finally reads

$$V(\sigma) = \left(\frac{3}{4\kappa^2} - \frac{e^{\sqrt{2\kappa^2/3}\sigma/2}}{\kappa^2}\right)\gamma. \qquad (64)$$

The initial value of the inflaton is large (and negative) and it rolls down toward the potential minimum at $\sigma \to 0^-$, $V(0) = -\gamma/(4\kappa^2) < 0$. When $\sigma \to -\infty$ the EOMs in the slow roll limit read

$$H^2 \simeq \frac{\gamma}{4}, \qquad 3H\dot{\sigma} \simeq \left(\frac{\gamma}{\sqrt{6\kappa^2}}\right) e^{\sqrt{2\kappa^2/3}\sigma/2}. \qquad (65)$$

It means that $\gamma$ must satisfy condition (36), such that the inflation takes place at the Plank epoch. The de Sitter expansion can be realized and the field behaves as

$$\sigma \simeq -\sqrt{\frac{6}{\kappa^2}} \ln\left[\frac{\sqrt{\gamma}}{9}(t_0 - t)\right], \qquad (66)$$

where $t_0$ is bounded at the beginning of the inflation. If at this time the magnitude of $\sigma$ is very large, the slow roll parameters (19)

$$\epsilon = \frac{4}{3}\frac{1}{(4 - 3e^{-\sqrt{2\kappa^2/3}\sigma/2})^2} \simeq 0,$$
$$|\eta| = \frac{2}{3}\frac{1}{|4 - 3e^{-\sqrt{2\kappa^2/3}\sigma/2}|} \simeq 0, \qquad (67)$$

are small and the field moves slowly. The inflation ends at $\sigma_e \simeq -0.12\sqrt{3/(2\kappa^2)}$, when the slow roll parameters are of the order of unit. The $e$-foldings number can be evaluated from (20) and reads

$$N \simeq \frac{9e^{-\sqrt{2\kappa^2/3}\sigma/2}}{2}\bigg|_{\sigma_e}^{\sigma_i} \simeq \frac{\sqrt{\gamma}}{2} t_0. \qquad (68)$$

As a consequence, the slow roll parameters can be written as

$$\epsilon \simeq \frac{3}{N^2}, \qquad |\eta| \simeq \frac{1}{N}. \qquad (69)$$

The amplitude of primordial power spectrum (21) and the spectral indexes (22) are given by

$$\Delta_{\mathcal{R}}^2 \simeq \frac{\kappa^2\gamma N^2}{96\pi^2}, \qquad n_s \simeq 1 - \frac{1}{N}, \qquad r \simeq \frac{48}{N^2}. \qquad (70)$$

Since from these formulas we have $n_s > 1 - \sqrt{0.11/48} \simeq 0.9521$ when $r < 0.11$, $n_s < 1$, we see that these expressions are compatible with (23). For example, for $N = 60$, one has $n_s = 0.967$ and $r = 0.013$.

We finish this subsection with some considerations on the potential (63), which corresponds to the model with cosmological constant equal to $-\gamma/4$. Since the term $\exp[2\sqrt{2\kappa^2/3}\sigma]$ changes more slowly than $\exp[\sqrt{2\kappa^2/3}\sigma/2]$, the dynamics of inflation is the same of above. In this case, when the inflaton exits from the slow roll region, it falls in the minimum of the potential located at $V(0) = -\gamma/(8\kappa^2)$.

### D. $V(\sigma) \sim \gamma(2 - n)/2 - \gamma \exp[n\sigma]$, $0 < n < 1$: $c_1R^2 + c_2R^{2-n}$ models

Now, we would like to investigate some general features of the inflationary potential,

$$V(\sigma) = \frac{\alpha}{\kappa^2} - \frac{\gamma}{\kappa^2} e^{n\sqrt{2\kappa^2/3}\sigma}, \qquad (71)$$

where $0 < \alpha$, $\gamma$, and $0 < n < 2$. The above potential is explicitly constructed to give the de Sitter solution in the slow roll limit when $\sigma \to -\infty$. Equation (24) leads to

$$F_R(R) + \frac{\gamma}{2\alpha} F_R(R)^{1-n}(n - 2) = \frac{R}{4\alpha}. \qquad (72)$$

At the perturbative level, it is easy to see that, by choosing

$$\alpha = \frac{\gamma(2 - n)}{2} > 0, \qquad (73)$$

if $\gamma$ satisfies (36), at small curvature one gets

$$F_R(R \ll \gamma) \simeq 1 + c_1R + c_2R^2 + c_3R^3 + \cdots, \qquad (74)$$

with $c_1 \propto \gamma^{-1}$, $c_2 \propto \gamma^{-2}$, $c_3 \propto \gamma^{-3}$…, such that our theory is the high curvature correction to General Relativity (see Appendix A). For example, in the previous subsection we have seen an exact solution for the case $n = 1/2$. Since $1 \ll \gamma$, when $R/\gamma \ll 1$ we can expand $F_R(R)$ in (60) as

$$F_R(R \ll \gamma) \simeq 1 + \frac{2}{3\gamma}R - \frac{R^2}{9\gamma^2} + \cdots, \qquad (75)$$

which returns to be (74) by using the coefficients in Appendix A. On the other side, when $R \gg \gamma$, the asymptotic solution of Eq. (72) with (73)is given by

$$F_R(R \gg \gamma) \simeq \left(\frac{R}{2\gamma(2-n)}\right) + \left(\frac{R}{2\gamma(2-n)}\right)^{1-n},$$
$$F(R \gg \gamma) \simeq \frac{1}{2}\left(\frac{R^2}{2\gamma(2-n)}\right) + \frac{1}{2-n}\left(\frac{1}{2\gamma(2-n)}\right)^{1-n} R^{2-n}. \qquad (76)$$

Also in this case, by taking the exact solution of the previous section in the high curvature limit,

$$F_R(R \gg \gamma) \simeq \frac{R}{3\gamma} + \sqrt{\frac{R}{3\gamma}}, \tag{77}$$

we can verify the consistency of expression (76) for $n = 1/2$.

As an other example, let us consider the case $n = 1/3$. The reconstruction leads to

$$F_R(R) = \frac{1}{3} + \frac{3}{10}\left(\frac{R}{\gamma}\right) + \frac{2}{3\times 5^{1/3}}\left[9\left(\frac{R}{\gamma}\right) + 5\right]\frac{1}{\Delta^{1/3}} + \frac{1}{6\times 5^{2/3}}\Delta^{1/3}, \tag{78}$$

where

$$\Delta = 200 + 243\left(\frac{R}{\gamma}\right)^2 + 540\left(\frac{R}{\gamma}\right) + 27\sqrt{81\left(\frac{R}{\gamma}\right)^4 + 40\left(\frac{R}{\gamma}\right)^3}. \tag{79}$$

At small curvature, it is easy to find

$$F_R(R \ll \gamma) \simeq 1 + \frac{9}{10}\left(\frac{R}{\gamma}\right) + \cdots, \tag{80}$$

and in the high curvature limit this model has the following structure,

$$F_R(R \gg \gamma) \simeq \frac{3}{10}\left(\frac{R}{\gamma}\right) + \left(\frac{3}{10}\right)^{\frac{2}{3}}\left(\frac{R}{\gamma}\right)^{\frac{2}{3}}, \tag{81}$$

which corresponds to (76) with $n = 1/3$. Finally, the model (27) is the limiting case of $n \to 1$.

Let us analyze this class of potentials. When the magnitude of the inflaton is large, the scalar potential (71) with $\alpha = \gamma(2-n)/2$,

$$V(\sigma) = \frac{\gamma(2-n)}{2\kappa^2} - \frac{\gamma}{\kappa^2}\mathrm{e}^{n\sqrt{2\kappa^2/3}\sigma}, \tag{82}$$

behaves as $V(\sigma) \simeq \gamma(2-n)/(2\kappa^2)$, and the EOMs in the slow roll limit read

$$H^2 \simeq \left(\frac{\gamma(2-n)}{6}\right), \qquad 3H\dot{\sigma} \simeq \left(\frac{n\gamma\sqrt{2}}{\sqrt{3\kappa^2}}\right)\mathrm{e}^{n\sqrt{2\kappa^2/3}\sigma}. \tag{83}$$

As a consequence, the field results to be

$$\sigma \simeq -\sqrt{\frac{3}{2\kappa^2}}\frac{1}{n}\ \ln\left[\frac{2\sqrt{2}}{3\sqrt{3}}\frac{n^2\sqrt{\gamma}}{\sqrt{2-n}}(t_0 - t)\right], \tag{84}$$

where $t_0$ is bounded at the beginning of the inflation. When $\sigma \to -\infty$, the slow roll parameters became

$$\epsilon = \frac{4n^2}{3}\frac{1}{(2 + (n-2)\mathrm{e}^{-n\sqrt{2\kappa^2/3}\sigma})^2} \simeq 0, \qquad |\eta| = \frac{4n^2}{3}\frac{1}{|2 + (n-2)\mathrm{e}^{-n\sqrt{2\kappa^2/3}\sigma}|} \simeq 0, \tag{85}$$

and are very small. The inflation ends at $\sigma_e \simeq -(1/n)\sqrt{3/(2\kappa^2)}\log[(\gamma/\alpha)(1 - n/\sqrt{3})]$,

such that the slow roll parameters are of the order of unit and the field reaches the minimum of the potential at $V(0) = -\gamma/(4\kappa^2)$. The $e$-foldings number (20) is given by

$$N \simeq \frac{3(2-n)\mathrm{e}^{-n\sqrt{2\kappa^2/3}\sigma}}{4n^2}\bigg|_{\sigma_e}^{\sigma_i} \simeq \sqrt{\frac{2(2-n)\gamma}{6}}t_0, \tag{86}$$

and

$$\epsilon \simeq \frac{3}{4n^2N^2}, \qquad |\eta| \simeq \frac{1}{N}. \tag{87}$$

As a consequence, the amplitude of primordial power spectrum and the spectral indexes read

$$\Delta^2_{\mathcal{R}} \simeq \frac{\kappa^2\gamma(2-n)n^2N^2}{36\pi^2}, \qquad n_s \simeq 1 - \frac{1}{N}, \qquad r \simeq \frac{12}{n^2N^2}. \tag{88}$$

Since $n_s > 1 - (\sqrt{0.11/12})n \simeq 1 - (0.0957)n$ when $r < 0.11$, $n_s < 1$, one has that

$$0.3386 < n < 1, \tag{89}$$

in order to make the spectral indexes compatible with (22). It means, that the potential (82) can reproduce a viable inflation with at least $n = 1/3$, namely we are considering models in the form $F(R \gg M_{Pl}) \simeq c_1R^2 + c_2R^\zeta$, with $1 < \zeta < 5/3$.

Also in this case, $F_R(R)$ in Eq. (75) may lead to a cosmological constant proportional to $\gamma$ in the $F(R)$ model. However, we can set it equal to zero, acquiring an additional term in the potential (71) proportional to $\exp[2\sqrt{2\kappa^2/3}\sigma]$. This term changes slower than $\exp[n\sqrt{2\kappa^2/3}\sigma]$ when $0 < n < 2$, and the dynamics of the inflation is the same of above. To conclude this section, we add some comments about the scalar potentials containing exponential terms like $\exp[n\sqrt{2\kappa^2/3}\sigma]$ with $n > 1$. Since in this case Eq. (73) could lead to $\alpha < 0$ (when $n > 2$) making the potential unable to reproduce inflation, we may generalize the potential to the following form

$$V(\sigma) = \frac{\alpha}{\kappa^2}(1 - \mathrm{e}^{n\sqrt{2\kappa^2/3}\sigma}) - \frac{\gamma}{\kappa^2}\mathrm{e}^{\sqrt{2\kappa^2/3}\sigma}, \tag{90}$$

where $\alpha$, $\gamma > 0$ and $n > 1$. Now, in order to recover the Einstein gravity at small curvature, we have to put

$$\alpha = \frac{\gamma}{2(n-1)} > 0. \tag{91}$$

In the slow roll limit $\sigma \to -\infty$ the EOMs read

$$H^2 \simeq \frac{\gamma}{6(n-1)}, \qquad 3H\dot{\sigma} \simeq \left(\frac{\sqrt{2}\gamma}{\sqrt{3\kappa^2}}\right) e^{\sqrt{2\kappa^2/3}\sigma}, \tag{92}$$

and the analysis of inflation results to be the same of $R^2$ models (with or without the cosmological constant term). Here, we can give some comments about the results of our investigation. Scalar inflationary potentials that satisfy viable conditions for realistic primordial acceleration in the contest of large scalar curvature corrections to General Relativity can be classified in two classes. In the first one, the scalar potential behaves as $V(\sigma) \sim \exp[n\sqrt{2\kappa^2/3}\sigma]$, $n > 0$, and produces acceleration with a sufficient amount of inflation only for $n$ very close to zero, namely the $F(R)$ model must be close to the Starobinsky one. In the second class, $V(\sigma) \sim \alpha - \gamma \exp[-n\sqrt{2\kappa^2/3}\sigma]$, $n > 0$, and a quasi de Sitter solution emerges during inflation. Cosmological data are satisfied if $n > 1/3$. This kind of scalar potentials is originated from large scalar curvature corrections of the type $F(R) \sim c_1 R^2 + c_2 R^\zeta$, with $1 < \zeta < 5/3$ when $1/3 < n < 1$. Finally, the cases with $1 < n$ show in fact the same behavior of $R^2$ model during inflation.

## V. $R + \alpha(R + R_0)^n$ MODELS

In this section, we will consider power law corrections to Einstein gravity in the form

$$F(R) = R + \alpha(R + R_0)^n + \Lambda, \qquad n > 1, \tag{93}$$

where $\alpha > 0$ is a (dimensional) constant parameter and $R_0$, $\Lambda$ are two (cosmological) constants introduced to generalize the model. This class of models, following the success of quadratic correction, have been often analyzed in literature in order to reproduce the dynamics of inflation and recently, in Ref. [21], the cases of polynomial corrections added to the Starobinsky model have been investigated. In what follows, at first we would like to study the inflation in the Einstein frame given by (93) with $n \neq 2$. We will see that inflation for large magnitude values of the field is realized for $n \lesssim 2$. However, other possibilities are allowed by considering intermediate values of the field. In this case, the de Sitter solution may emerge in the models with $n > 2$, but in order to study the exit from inflation is necessary to analyze the theory in the Jordan frame, where perturbations make possible an early time acceleration with a sufficient amount of $N$-foldings number: that is the aim of the second part of this section.

Let us start from the potential in the scalar field representation of (93),

$$V(\sigma) = \frac{1}{2\kappa^2}\left\{\left(\frac{1}{\alpha n}\right)^{\frac{1}{n-1}} (1 - e^{\sqrt{(2\kappa^2/3)}\sigma})^{\frac{n}{n-1}} e^{\frac{(n-2)}{(n-1)}\sqrt{(2\kappa^2/3)}\sigma}\left[\frac{n-1}{n}\right] + R_0 e^{\sqrt{2\kappa^2/3}\sigma}(e^{\sqrt{2\kappa^2/3}\sigma} - 1) - \Lambda e^{2\sqrt{2\kappa^2/3}\sigma}\right\}. \tag{94}$$

We immediately see that only if $n = 2$, when $\sigma \to -\infty$, one gets $V(\sigma) \sim$ const and obtains the de Sitter solution (for $n = 2$, $\Lambda = R_0 = 0$ we recover (33), $V(\sigma) = (\exp[\sqrt{2\kappa^2/3}\sigma] - 1)^2/(8\alpha\kappa^2)$). On the other hand, even if $1 < n < 2$, the scalar field starts from a maximum of the potential and falls toward the minimum at $\sigma \to 0^-$: it corresponds to the case of $V(\sigma \to -\infty) \sim \exp[-(2-n)/(n-1)\sigma]$, which may be used to reproduce an accelerating expansion if the slow roll limits are satisfied.

The derivatives of the potential read

$$V'(\sigma) = \left[-\frac{1}{2\kappa^2}\left(\frac{1}{\alpha n}\right)^{\frac{1}{n-1}} (1 - e^{\sqrt{(2\kappa^2/3)}\sigma})^{\frac{1}{n-1}} e^{\frac{(n-2)}{(n-1)}\sqrt{(2\kappa^2/3)}\sigma} + 2V(\sigma) + \frac{R_0}{2\kappa^2} e^{\sqrt{2\kappa^2/3}\sigma}\right]\left(\sqrt{\frac{2\kappa^2}{3}}\right), \tag{95}$$

$$V''(\sigma) = \frac{1}{3}\left(\frac{1}{\alpha n}\right)^{\frac{1}{n-1}} [(e^{-\sqrt{(2\kappa^2/3)}\sigma} - 1)^{\frac{2-n}{n-1}} - 3(e^{-\sqrt{(2\kappa^2/3)}\sigma} - 1)^{\frac{1}{n-1}} e^{\sqrt{(2\kappa^2/3)}\sigma}] + 4V(\sigma)\left(\frac{2\kappa^2}{3}\right) + R_0 e^{\sqrt{2\kappa^2/3}\sigma}, \tag{96}$$

and the slow roll parameters (19) are derived as

$$\epsilon = \frac{1}{3}\left[2 - \frac{n}{(n-1)}\frac{1}{y} + f(y)\right]^2, \quad \eta = \frac{2n}{3(n-1)}\left[\frac{1}{y^2} - \frac{3}{y}\right] + \frac{8}{3} + g(y), \tag{97}$$

where we have put

$$y=(1-\mathrm{e}^{\sqrt{(\kappa^2/3)}\sigma}),$$

$$f(y)=\frac{[R_0(\frac{2n-1}{n-1})-\frac{\Lambda(1-y)}{y}(\frac{n}{n-1})]}{\left[(\frac{1}{\alpha n})^{\frac{1}{n-1}}\frac{y^{\frac{n}{n-1}}}{(1-y)^{\frac{1}{n-1}}}(\frac{n-1}{n})+R_0y-\Lambda(1-y)\right]},$$

$$g(y)=2f(y)+\frac{\frac{2}{3y}(\frac{n}{n-1})[\frac{\Lambda(1-y)}{y}-R_0]}{\left[(\frac{1}{\alpha n})^{\frac{1}{n-1}}\frac{y^{\frac{n}{n-1}}}{(1-y)^{\frac{1}{n-1}}}(\frac{n-1}{n})+R_0y-\Lambda(1-y)\right]}. \tag{98}$$

We have $y<1$ for negative values of the field. If $\Lambda=R_0=0$, $f(y)=g(y)=0$, and then for $n=2$ we recover (38). For large and negative values of the field, namely when $y\to 1^-$, one finds

$$\epsilon\simeq\frac{(2-n)^2}{3(n-1)^2},\qquad |\eta|\simeq\frac{4|n-2|}{3(n-1)}, \tag{99}$$

and $n$ has to be

$$\frac{2+\sqrt{3}}{1+\sqrt{3}}\simeq 1.36<n<2. \tag{100}$$

Here, we remember that we are considering only the cases $n<2$ whose EOMs in the slow roll limit read,

$$H^2\simeq\frac{1}{6}\left(\frac{1}{\alpha n}\right)^{\frac{1}{n-1}}\mathrm{e}^{\frac{(n-2)}{(n-1)}\sqrt{(2\kappa^2/3)}\sigma}\left[\frac{n-1}{n}\right],$$

$$3H\dot{\sigma}\simeq\sqrt{\frac{1}{6\kappa^2}}\left(\frac{1}{\alpha n}\right)^{\frac{1}{n-1}}\mathrm{e}^{\frac{(n-2)}{(n-1)}\sqrt{(2\kappa^2/3)}\sigma}\left[\frac{2-n}{n}\right]. \tag{101}$$

The solution for the field is given by

$$\sigma=2\sqrt{\frac{3}{2\kappa^2}}\left[\frac{n-1}{2-n}\right]\ln\left[\sqrt{\frac{2}{3n}}\frac{(2-n)^2}{6(n-1)^{3/2}}\left(\frac{1}{n\alpha}\right)^{\frac{1}{2(n-1)}}(t+t_0)\right]. \tag{102}$$

Since $1<n<2$, $\sigma$ is negative for $t_0>0$ very small bounded at the beginning of inflation. The Hubble parameter reads

$$H=\frac{3(n-1)^2}{(2-n)^2}\frac{1}{(t_0+t)},$$

$$\frac{\ddot{a}}{a}=\frac{3(n-1)^2((n+1)^2-2)}{(n-2)^4}\frac{1}{(t_0+t)^2}>0, \tag{103}$$

and we have an acceleration with decreasing Hubble parameter and curvature

$$R=\frac{3(n-1)^2(3(n-1)^2-(2-n))}{(2-n)^4(t_0+t)}. \tag{104}$$

The $N$-foldings number of inflation is given by

$$N\simeq-\frac{(n-1)}{(2-n)}\sqrt{\frac{3\kappa^2}{2}}\sigma\Big|_{\sigma_e}^{\sigma_i}$$

$$\simeq-3\left(\frac{n-1}{2-n}\right)^2\ln\left[\sqrt{\frac{2}{3n}}\frac{(2-n)^2}{6(n-1)^{3/2}}\left(\frac{1}{n\alpha}\right)^{\frac{1}{2(n-1)}}t_0\right]. \tag{105}$$

When the inflation ends, the field reaches the minimum of the potential. We note that the cosmological constants introduced in the model do not play any role at inflationary stage, and change the behavior of the potential only at the end of the inflation. If $\Lambda=R_0=0$, it is easy to see that $V(0)=0$, otherwise the potential possesses a minimum before $\sigma=0$ (in the same way of Sec. IV B), where the field falls and starts to oscillate.

The amplitude of primordial power spectrum (21) and the spectral indexes (22) are

$$\Delta^2_{\mathcal{R}}\simeq\frac{(n-1)^3\kappa^2\mathrm{e}^{\frac{2}{3}N\frac{(n-2)^2}{(n-1)^2}}}{16\pi^2(2-n)^2n}\left(\frac{1}{\alpha n}\right)^{\frac{1}{n-1}},$$

$$n_s\simeq 1-\frac{8(2-n)}{3(n-1)},\qquad r\simeq\frac{16(2-n)^2}{3(n-1)^2}, \tag{106}$$

where we have taken into account that $n$ is close to two. In order to make these indexes compatible with (23), we must require $n\sim 1.8$, 1.9, such that, as a matter of fact, the quadratic corrections with $n=2$ appear to be the only possibilities of the type (93) able to reproduce a viable inflation in the Einstein frame. However, it is interesting to extend our investigation to the Jordan frame.

### A. Inflation in the Jordan frame

In the first part of this section, we have seen how model (93) can produce inflation in the Einstein frame representation. We have an early time acceleration followed by the end of inflation and the slow roll conditions are satisfied if $n$ is very close to two. For the sake of simplicity, in this subsection, we will put $\Lambda=R_0=0$ in (93). Let us return to the Jordan frame.

The Friedmann equation for a generic $F(R)$ model read (in vacuum)

$$(RF_R(R)-F(R))-6H^2F_R(R)-6H\dot{F}_R(R)=0, \tag{107}$$

such that for (93) one derives

$$\alpha(R)^{n-1}[R(1-n)]-6H^2[1+\alpha n(R)^{n-1}]$$
$$=6H\alpha n(n-1)(R)^{n-2}\dot{R}. \tag{108}$$

In the high curvature limit (it means, for large and negative values of the scalar field $\sigma$ in the Einstein frame) we may consider $\alpha n(R)^{n-1}\gg 1$,

$$H^2 \simeq \frac{(n-1)}{6n}\left[R - \frac{6nH\dot{R}}{R}\right]. \qquad (109)$$

During inflation, the Hubble parameter $H$ evolves slowly and we can consider the analagous of the slow roll parameters that have the same formal dependence of the Einstein frame, namely we require $|\dot{H}|/H^2 \ll 1$, $|\ddot{H}/(H\dot{H})| \ll 1$. Thus, we get

$$\frac{\dot{H}}{H^2} \simeq -\left[\frac{4-2n}{n+4(n-1)^2}\right],$$
$$\frac{\ddot{a}}{a} = H^2 + \dot{H} = 1 - \frac{4-2n}{n+4(n-1)^2}. \qquad (110)$$

It follows that the expansion is accelerated only if

$$n > \frac{5}{4} = 1.25. \qquad (111)$$

This value is close to the one given by the left side of (10). Moreover, for $n > 2$, we see that $\dot{H} > 0$ and the Hubble parameter, and therefore the curvature, grows up with the time and the physics of standard model is not reached. This result is in agreement with the behavior of the model discussed in the Einstein frame: when $n > 2$ and the acceleration emerges in high curvature limit, the field cannot reach a minimum of the potential for small values and we do not exit from inflation. We could arrive to this conclusion also by looking for Eq. (102): when $n > 2$, $t_0$ has to be very large at the beginning of inflation and the scalar field grows up with the time. However, it does not mean that models with $n > 2$ do not produce inflation. In scalar field representation we start at very large and negative values of the field (it means, that in Jordan framework we can ignore the $R$ term in the action). It may be interesting to see what happen for intermediate values of the field. We analyze the problem in the Jordan frame. From Eq. (107) we have the de Sitter condition at $R_0 = 12H_0^2$, $H_0$ constant, namely,

$$2F(R_0) - R_0 F_R(R_0) = 0, \qquad (112)$$

which leads in our case to

$$R_0 = \left(\frac{1}{\alpha(n-2)}\right)^{\frac{1}{n-1}}. \qquad (113)$$

The model with $n = 2$ does not possess an exact de Sitter solution, but if $n > 2$, we can realize it. Here, we remember that $\alpha > 0$. In the Einstein frame this solution corresponds to

$$\sigma_{\max} = -\sqrt{\frac{3}{2\kappa^2}}\log\left[\frac{2n-2}{n-2}\right], \qquad (114)$$

where we have used (3). We note that $\sigma < 0$ only if $n > 2$ (we are assuming $n > 1$); otherwise, if $n < 2$, this expression becomes imaginary and meaningless. This solution corresponds to a maximum of the potential, because of

$$V'(\sigma_{\max}) = 0,$$
$$V''(\sigma_{\max}) = \frac{1}{6}\left(\frac{1}{\alpha n}\right)^{\frac{1}{n-1}}\frac{n-2}{n-1}\left(\frac{n}{n-2}\right)^{\frac{(2-n)}{(n-1)}} > 0. \qquad (115)$$

When $n > 2$, the potential in Einstein frame has a maximum. The scalar field produces the de Sitter solution, but the field does not evolve with the time ($\dot{\sigma} = 0$) and we do not have a natural exit form inflation. However, by making use of the perturbative theory, we may study the stability of the model in the Jordan frame.

By perturbating Eq. (107) as $R \to R + \delta R$, $|\delta R| \ll 1$ around the de Sitter solution, we get

$$(\Box - m^2)\delta R \simeq 0, \qquad m^2 = \frac{1}{3}\left(\frac{F'(R)}{F''(R)} - R\right). \qquad (116)$$

Here, $m^2$ is the effective mass of the scalaron, namely the new degree of freedom introduced by the modified gravity through $F_R(R)$, which is proportional to the opposite of the inflaton, namely $-\sigma$. As a consequence, $m^2$ is proportional to the opposite of the scalar potential of the inflaton and when $m^2 < 0$ the solution is unstable. In our case, we get on the de Sitter solution,

$$m^2 = -(n-2)^{\frac{n-2}{n-1}}\left(\frac{1}{\alpha n}\right)^{\frac{1}{n-1}} < 0, \qquad 2 < n. \qquad (117)$$

It means that a small perturbation can cause the exit from inflation. Now, the question is: in which direction the inflaton moves due to a perturbation and which kind of perturbation we need to have a correct $N$-foldings number? To reproduce a viable comsology, we expect that the inflaton moves toward the minimum of the potential at $V(0) = 0$, such that the small curvature regime can be reached and the cosmology of standard model takes place, and $N$ must be at least $N \sim 60$. In principle, the inflaton can also moves to $\sigma \to -\infty$ (large curvature regime), for which the potential tends also to zero.

In the hot universe scenario, we must take into account also the presence of ultrarelativistic matter or radiation, whose energy density is given by

$$\rho_r = \rho_{r(0)} a(t)^{-4}. \qquad (118)$$

Here, $\rho_{r(0)}$ is a constant bounded at the beginning of inflation. Thus, Eq. (108) reads

$$\frac{3H^2}{\kappa^2 2} = \rho_{MG} + \rho_r,$$
$$\rho_{MG} = \frac{1}{2\kappa^2}[\alpha(R)^{n-1}[R(1-n)] - 6H^2\alpha n(R)^{n-1} - 6H\alpha n(n-1)(R)^{n-2}\dot{R}], \qquad (119)$$

where $\rho_{MG}$ encodes the amount of energy density given by the correction term $R^n$ to Eintein gravity. For simplicity, we introduce the red shift parameter $z = -1 + 1/a(t)$. We also remember that $d/dt = -(z+1)H(z)d/dz$.

Let us define [33,34]

$$y_H(z) \equiv \frac{\rho_{\rm MG}}{M^2/\kappa^2} = \frac{3H^2}{M^2} - \tilde{\chi}(z+1)^4, \tag{120}$$

where $M^2$ is a suitable dimensional constant, such that $[M^2] = [H^2]$ and $\tilde{\chi} = \rho_{r(0)}\kappa^2/M^2$. By taking into account that the Ricci scalar $R(z) = [12H(z)^2 - 6(z+1)H(z)\times dH(z)/dz]$ results in

$$R = M^2\left(4y_H - (z+1)\frac{dy_H(z)}{dz}\right), \tag{121}$$

we derive from (108),

$$\frac{d^2y_H(z)}{dz^2} - \frac{dy_H}{dz}\frac{1}{z+1}\left\{3 - \frac{f_R(R)}{2M^2 f_{RR}(R)[y_H(z)+\tilde{\chi}(z+1)^4]}\right\} + \frac{y_H(z)}{(z+1)^2}\left\{\frac{1-f_R(R)}{M^2 f_{RR}(R)[y_H(z)+\tilde{\chi}(z+1)^4]}\right\} + \frac{2f_R(R)\tilde{\chi}(z+1)^4 + f(R)/M^2}{(z+1)^2 2M^2 f_{RR}(R)[y_H(z)+\tilde{\chi}(z+1)^4]} = 0, \tag{122}$$

where in our case

$$f(R) = \alpha R^n, \qquad f_R(R) = \alpha n R^{n-1}, \qquad f_{RR} = \alpha n(n-1)R^{n-2}. \tag{124}$$

Let us study the perturbations around the de Sitter solution (113), namely

$$y_H(z) \simeq y_0 + y_1(z), \quad y_0 = \frac{1}{4M^2}\left(\frac{1}{\alpha(n-2)}\right)^{\frac{1}{n-1}}, \tag{124}$$

where $|y_1(z)/y_0| \ll 1$. Here, we have put $R_0 = 4M^2y_0$ in the de Sitter universe. By assuming the contribution of ultrarelativistic matter much smaller than $y_0$ at the beginning of inflation, Eq. (122) becomes, at first order in $y_1(z)$,

$$\frac{d^2y_1(z)}{dz^2} - \frac{2}{(z+1)}\frac{dy_1(z)}{dz} + \frac{1}{(z+1)^2}\left(-4 + \frac{4(1+f_R(R))}{Rf_{RR}(R)}\right)y_1(z) \simeq 0, \tag{125}$$

where we have also used condition (112). Thus, the solution for $y_1$ is given by

$$y_1(z) = C_0(z+1)^x, \qquad x = \frac{1}{2}\left(3 - \sqrt{25 - \frac{16(1+f_R(R))}{Rf_{RR}(R)}}\right), \tag{126}$$

$C_0$ being constant. Here, we do not consider the solution with the plus sign in front of the square root of $x$, since it does not cause any instability. On the other hand, by making use of (123), one has on the de Sitter solution (113)

$$x = \frac{1}{2}\left(3 - \sqrt{\frac{25n^2 - 57n + 32}{n(n-1)}}\right) < 0, \quad n > 2, \tag{127}$$

since it is easy to demonstrate that $x < 0$ if $0 < 16(n-2)/n$, that is always true when $2 < n$ and we recover condition (117). As a consequence, the perturbation $y_1(z)$ grows up in expanding universe as

$$y_1(z) = y_1(z_{\rm i})\left[\frac{(z+1)}{(z_{\rm i}+1)}\right]^x. \tag{128}$$

Here, we have considered $C_0 = y_1(z_{\rm i})/(z_{\rm i}+1)^x$, $z_i$ being the redshift at the beginning of inflation where perturbation is bounded. When $y_1(z)$ is on the same order of $y_0$, the inflation ends. A classical perturbation on the (vacuum) de Sitter solution may be given by the ultrarelativistic matter in (120), such that Eq. (128) reads

$$y_1(z) = -\tilde{\chi}(z_i+1)^4\left[\frac{(z+1)}{(z_{\rm i}+1)}\right]^x, \qquad y_1(z_i) = -\tilde{\chi}(z_{\rm i}+1)^4. \tag{129}$$

Thus, the $N$-foldings number during inflation is

$$N = \log\left[\frac{z_{\rm i}+1}{z_{\rm e}+1}\right] \simeq \frac{1}{x}\log\left[\frac{\tilde{\chi}(z_{\rm i}+1)^4}{y_0}\right]. \tag{130}$$

Here, $z_e$ denotes the red shift at the end of inflation and we have considered $y_1(z_{\rm e}) \simeq -y_H$. As small $x$ is, much unstable is inflation. It means, that also a small initial perturbation may give arise to a large $N$-foldings. We can write $\tilde{\chi}$ as

$$\tilde{\chi} = \frac{y_0}{(z_i+1)^4} \times \delta, \qquad \delta \ll 1, \tag{131}$$

such that

$$\delta = {\rm e}^{xN}. \tag{132}$$

In order to obtain an $N$-foldings of 70, for $n = 3$ ($x \simeq -0.393$) we need $\delta \sim 10^{-12}$, for $n = 4$ ($x \simeq -0.562$) we need $\delta \sim 10^{-18}$, for $n = 5$ ($x \simeq -0.656$) we need $\delta \sim 10^{-20}$, for $n = 10$ ($x \simeq -0.835$) it is enough $\delta \sim 10^{-26}$.

Moreover, the behavior of the Ricci scalar is given by

$$R = 4y_0M^2 + M^2\left(4y_1 - (z+1)\frac{dy_1}{dz}\right) \simeq 4y_0M^2 - y_0\delta M^2[4-x]\left(\frac{z+1}{z_i+1}\right)^x. \quad (133)$$

Since $x > 0$, the Ricci scalar decreases with the red shift and the physics of standard model can emerge. In Ref. [35] numerical calculations have been executed in different inflationary models provided by the specific $R^n$ term with $2 < n < 3$, which makes inflation unstable and in the presence of ultrarelativistic matter, which leads to the exit from the inflationary stage toward the physics of standard model.

In conclusion, we can say that large scalar curvature corrections to Einstein gravity of the type $R^n$, $n > 2$, may represent a valid inflationary scenario. In this case, the curvature is bounded at a specific value at the beginning of inflation, and the de Sitter space-time is an exact solution of the model, which results to be highly unstable. From (132) we see how extremely small perturbations in hot universe give the correct $N$-foldings number and the exit from inflation. Finally, it is important to note that positive energy density of perturbations brings the curvature to decrease during the early-time acceleration making the model consistent with the observable evolution of our universe.

## VI. CONCLUSIONS

The attention that recently has been paid to modified theories of gravity is caused by the idea of the unified description of early-time and late-time cosmic acceleration [22]. Moreover, the gravitational action of such a kind of theories may describe quantum effects in hot universe scenario, and the last cosmological data seems to be in favor of quadratic corrections to General Relativity during this phase.

In this paper, we have investigated some features of $F(R)$-modified gravity models for inflationary cosmology, by performing our analysis in the Jordan and in the Einstein framework.

At first, we have studied inflation for the class of scalar potentials of the type $V(\sigma) \sim \exp[n\sigma]$, $n$ being a general parameter, in the Einstein frame. As a matter of fact, for such models it is possible to reconstruct the $F(R)$-gravity theories that correspond to the given potentials. Since viable inflation must be consistent with the last Planck data, the potentials have been carefully analyzed, by finding the conditions on the parameters which make possible the early-time acceleration according with $N$-foldings, spectral index and tensor-to-scalar ratio coming from observations. We have derived the form of the $F(R)$ models at the large and small curvature limits, demonstrating that these models in the Jordan frame correspond to corrections to Einstein gravity which emerge only at mass scales larger than the Planck mass. The investigated potentials can be classified in two classes. In the first one, the scalar potential behaves as $V(\sigma) \sim \exp[n\sqrt{2\kappa^2/3}\sigma]$, $n > 0$, and produces acceleration with a sufficient amount of inflation only for $n$ very close to zero; namely, the $F(R)$ model must be closed to the Starobinsky one. In the second class, $V(\sigma) \sim \alpha - \gamma\exp[-n\sqrt{2\kappa^2/3}\sigma]$, $n > 0$, and the quasi de Sitter solution emerges during viable inflation when $n > 1/3$. This kind of scalar potentials is originated from large curvature corrections of the type $F(R) \sim c_1R^2 + c_2R^\zeta$, with $1 < \zeta < 5/3$ when $1/3 < n < 1$, or from quadratic curvature corrections when $n > 1$. It is interesting to note that $R^2$ term which induces the early-time inflation is also responsible for removal of finite-time future singularity in $F(R)$ gravity unifying the inflation with dark energy [36].

In the second part of the paper, we have studied in detail the specific class of models $F(R) = R + (R + R_0)^n$. The analysis in the Einstein frame reveals that $n$ must be very close to two in order to realize a viable inflation for large and negative values of the scalar field, but other possibilities are allowed by starting from intermediate values of the field. To be specific, when $n > 2$, the de Sitter solution emerges, but in order to study the exit from inflation, since in this case the de Sitter space-time is an exact solution and the field does not move and exit from inflation in a natural way, is necessary to analyze the theory in the Jordan frame, where perturbations make possible an early time acceleration with a sufficient amount of inflation. Moreover, we can explicitly demonstrate that the curvature decreases, making the model consistent with the historical evolution of our universe.

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## APPENDIX A

Let us consider the equation

$$F_R(R) - \frac{1}{2} - \frac{F_R(R)^{1+n}}{2} = -\frac{R}{4\gamma(n+2)}, \quad (A1)$$

with $n, \gamma > 0$ [it corresponds to (44) with (45)]. We want to find solutions of the above equation as a function of $R$. A simple possibility is looking for regular solutions as a power series of $R$, namely setting

$$F_R(R) = 1 + c_1R + c_2R^2 + c_3R^3 + \cdots,$$

where the constants $c_{1,2,3\ldots}$ have to be determined. At first, we are interested in solutions where $|c_{1,2,3\ldots}R^{1,2,3\ldots}| \ll 1$, namely we analyze the limit at small curvature. Thus, plugging this ansatz into the above equation, one has

$$(1+c_1R+c_2R^2+c_3R^3+\cdots) \\ -\frac{1}{2}-\frac{1}{2}(1+c_1R+c_2R^2+c_3R^3+\cdots)^{1+n} \\ =-\frac{R}{4\gamma(n+2)}.$$

Putting

$$X=+c_1R+c_2R^2+c_3R^3+\cdots,$$

and assuming $|X|\ll 1$, the following expansion holds:

$$(1+X)^{1+n}=1+(1+n)X+\frac{(1+n)n}{2}X^2 \\ +\frac{n(n+1)(n-1)}{3!}X^3+\cdots.$$

As a consequence, one arrives at the recursive relations

$$c_1=\frac{1}{2\gamma(n-1)(n+2)},\qquad c_2=-\frac{(1+n)nc_1^2}{2(n-1)}, \\ c_3=-\frac{(1+n)c_1^3}{6},$$

from which it follows,

$$c_1R\propto\frac{R}{\gamma},\qquad c_2R^2\propto\frac{R}{\gamma^2},\qquad c_3R^3\propto\frac{R^3}{\gamma^3}.$$

This approximation is valid when $R\ll\gamma$. On the other side, when $R\gg\gamma$ we can check for the solutions in the form

$$F_R(R)=c_0\left(\frac{R}{\gamma}\right)^{\zeta},\qquad \zeta<1.$$

In such a case, from (A1)we get

$$c_0^{1+n}\left(\frac{R}{\gamma}\right)^{\zeta(1+n)}\simeq\frac{1}{4(n+2)}\left(\frac{R}{\gamma}\right),$$

and as a consequence,

$$c_0=\left(\frac{1}{4(n+2)}\right)^{\frac{1}{1+n}},\qquad \zeta=\frac{1}{1+n}<1.$$

Now let us consider the following equation [it corresponds to (72) with (73)],

$$F_R(R)-F_R(R)^{1-n}=\frac{R}{2\gamma(2-n)},\qquad \text{(A2)}$$

where $\gamma>0$ and $0<n<1$. The solution for $R\ll\gamma$ is given by

$$F_R(R)=1+c_1R+c_2R^2+c_3R^3+\cdots,$$

with

$$c_1=\frac{1}{2\gamma n(2-n)},\quad c_2=-\frac{nc_1^2}{2},\quad c_3=\frac{(1-n)(n+1)c_1^3}{6},\ldots$$

On the other hand, when $R\gg\gamma$ one can expand $F_R(R)$ as

$$F_R(R)=\beta_1\left(\frac{R}{\gamma}\right)^{\alpha_1}+\beta_2\left(\frac{R}{\gamma}\right)^{\alpha_2},\qquad \alpha_1>\alpha_2.$$

Since $0<n<1$ at the first order in $R/\gamma$ one finds

$$\beta_1\left(\frac{R}{\gamma}\right)^{\alpha_1}\simeq\frac{R}{2\gamma(2-n)},$$

and so

$$\alpha_1=1,\qquad \beta_1=\left(\frac{1}{2\gamma(2-n)}\right).$$

Moreover, by using (A2) again we obtain

$$\alpha_2=1-n,\qquad \beta_2=\left(\frac{1}{2\gamma(2-n)}\right)^{1-n}.$$

As a final result, we can conclude that at high curvatures the model under consideration reads

$$F_R(R)=\left(\frac{R}{2\gamma(2-n)}\right)+\left(\frac{R}{2\gamma(2-n)}\right)^{1-n}.$$

## APPENDIX B

As is well known, the Einstein-Hilbert Lagrangian density modified by a quadratic term $R^2$ and by a cosmological constant term has the Schwarzshild–de Sitter solution, and this is the most general spherically symmetric, static solution if the parameters that enter in the modified Lagrangian are arbitrary (unrelated).

Here we shall show that there exists a particular choice of such parameters for which the model has a more general spherically symmetric, static solution, which is formally identical to the one corresponding to the Reissner-Nordström-de Sitter black hole.

To this aim we consider the Lagrangian density

$$F(R)=R+\frac{R^2}{4c_0}+c_0,$$

which has been obtained from the reconstruction process described in Sec. IVB. This Lagrangian depends on one free parameter only.

Now it is easy to verify that the most general spherically symmetric, static solution of field equations in vacuum is given by

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega,$$

$$A(r) = B(r) = 1 - \frac{a}{r} - \frac{c_0}{6} r^2 + \frac{b}{r^2},$$

where $d\Omega$ is the metric on the two sphere, while $a$ and $b$ are arbitrary constants. The physical interpretation of such two constants of integration is not an easy task. The presence of the $1/r^2$ term is quite interesting and it will be discussed elsewhere.


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