PHYSICAL REVIEW D **89,** 021502(R) (2014)

A Young-Laplace law for black hole horizons

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(Received 25 September 2013; published 22 January 2014)

Black hole horizon sections, modeled as marginally outer trapped surfaces (MOTS), possess a notion of stability admitting a spectral characterization. Specifically, the "principal eigenvalue" λ_o of the MOTS-stability operator (an elliptic operator on horizon sections) must be nonnegative. We discuss the expression of λ_o for axisymmetric stationary black hole horizons and show that, remarkably, it presents the form of the Young-Laplace law for soap bubbles in equilibrium, if λ_o is identified with a formal pressure difference between the inner and outer sides of the "bubble." In this view, which endorses the existing fluid analogies for black hole horizons, MOTS-stability is interpreted as a consequence of a pressure increase in the black hole trapped region.

DOI: 10.1103/PhysRevD.89.021502 PACS numbers: 04.70.-s, 04.50.Gh, 98.80.Jk

I. INTRODUCTION

Mechanical fluid analogies have played an important role in building our intuition of black hole (BH) horizon dynamics. The comparison with a rotating liquid drop was early discussed [1], providing an interpretation of the BH surface gravity as the corresponding liquid surface tension. More systematically, the analogy of the BH horizon with a 2-dimensional viscous fluid was developed in [2–4] (and references therein) building the so-called "membrane paradigm," of particular interest in astrophysical BH dynamics. Remarkably, aspects of the latter "membrane perspective" have been recently revisited in higher dimensional settings in the context of the CFT/AdS duality, namely the correspondence between the (bulk) gravitational description of an asymptotically Anti-de Sitter spacetime and the dynamics of an appropriate conformal field theory at its boundary (e.g. [5]). Related analogies of BH horizons as "soap bubbles" can be found in [6,7] and, particularly interesting in our present context, have led to the discussion of the Gregory-Laflamme instability of black strings in terms of the classical fluid Rayleigh-Plateau instability [8].

Here we further support these analogies by interpreting the stability of stationary BHs in terms of the Young-Laplace law for "soap bubbles." This relates the pressure difference at the interface between fluids in equilibrium to the interface shape

$$\Delta p = p_{\mathrm{inn}} - p_{\mathrm{out}} = \gamma(1/R_1 + 1/R_2), \tag{1}$$

where at any interface point Δp is the difference between the inner and outer pressures (p_{inn} and p_{out}), γ is the surface tension and $R_{i=1,2}$ are the principal curvature radii (with normal vector pointing outwards). Specifically, we show that MOTS-stability [9] of stationary BH horizons, characterized by the nonnegativity of the so-called principal eigenvalue λ_o of the MOTS-stability operator $L_{\mathcal{S}}$ (see below), can be discussed in terms of the Young-Laplace law in Eq. (1) if λ_o is identified with a formal pressure difference Δp. This provides a first step in the systematic spectral analysis of the MOTS-stability operator, as well as a suggestive interpretation shift that casts this geometric stability problem on fluid physical grounds.

II. MOTS, STABILITY AND QUASILOCAL HORIZONS

Let us introduce the specific notion of stability discussed here. Let us consider a d–dimensional spacetime $(\mathcal{M}, g_{ab})$ with Levi-Civita connection ∇_a and a closed spacelike $(d-2)$-surface $\mathcal{S}$ (we make $G = c = 1$). Let q_{ab} denote the induced metric on $\mathcal{S}$, and D_a and R its associated Levi-Civita connection and Ricci scalar. We span the normal plane $T^{\perp}\mathcal{S}$ by (future) outgoing ℓ^a and ingoing k^a null vectors, normalized as $\ell^a k_a = -1$. Expansions in the outgoing and ingoing directions are $\theta^{(\ell)} = q^{ab}\nabla_a \ell_b$ and $\theta^{(k)} = q^{ab}\nabla_a k_b$.

The surface $\mathcal{S}$ is called (strictly) outer trapped iff $\theta^{(\ell)} < 0$ and a marginally outer trapped surface (MOTS) iff $\theta^{(\ell)} = 0$. MOTSs possess a natural notion of stability [9]: a MOTS surface $\mathcal{S}$ is said to be (strictly) stable if it admits a deformation along k^a that is outer trapped or, equivalently, a deformation along $-k^a$ that is fully non-trapped. In other words, the MOTS $\mathcal{S}$ is stable if there exists a positive function ψ on $\mathcal{S}$ such that $\delta_{\psi(-k)}\theta^{(\ell)} > 0$, where δ denotes the deformation operator of the surface $\mathcal{S}$ discussed in [9,10]. This notion of stability admits a spectral characterization in terms of the MOTS-stability operator $L_{\mathcal{S}}$ defined on $\mathcal{S}$ as

$$L_{\mathcal{S}}\psi \equiv \delta_{\psi(-k)}\theta^{(\ell)} = \Big[-D^a D_a + 2\Omega^{(\ell)}_a D^a - \Big(\Omega^{(\ell)}_a \Omega^{(\ell)a} - D^a\Omega^{(\ell)}_a - \frac{1}{2}R + G_{ab}\ell^a k^b\Big)\Big]\psi, \tag{2}$$

where $\Omega^{(\ell)}_a = -k^c q^b{}_a \nabla_b \ell_c$ is the connection in $T^{\perp}\mathcal{S}$ [10] and G_{ab} is the Einstein tensor. The eigenvalues are generically complex numbers ($L_{\mathcal{S}}$ is not self-adjoint). However the *principal eigenvalue* λ_o, namely the eigenvalue with smallest real part, can be shown to be real [9]. MOTS-stability of $\mathcal{S}$ is then characterized by the nonnegativity of λ_o [9]

$$\lambda_o \geq 0, \tag{3}$$

with positive principal eigenfunction ϕ_o (i.e. $L_{\mathcal{S}}\phi_o = \lambda_o\phi_o$). We also define an operator $L^*_{\mathcal{S}}$ obtained from $L_{\mathcal{S}}$ by imposing Einstein equations, $G_{ab} + \Lambda g_{ab} = 8\pi T_{ab}$, but dropping the (explicit) presence of the cosmological constant Λ:

$$L^*_{\mathcal{S}}\psi \equiv \Big[-D^a D_a + 2\Omega^{(\ell)}_a D^a - \Big(\Omega^{(\ell)}_a \Omega^{(\ell)a} - D^a \Omega^{(\ell)}_a - \frac{1}{2}R + 8\pi T_{ab}\ell^a k^b\Big)\Big]\psi. \tag{4}$$

Quasilocal models for BHs [11,12] can be constructed by considering marginally trapped tubes (MTT), namely hypersurfaces $\mathcal{H}$ admitting a foliation $\{\mathcal{S}_t\}$ by closed MOTS, i.e. $\mathcal{H} = \cup_{t\in\mathbb{R}}\mathcal{S}_t$. Under the null energy condition and assuming MOTS stability (the *outer condition* in [11]), MTTs are either null or spacelike hypersurfaces [10,11]. The former corresponds to nonexpanding horizons whereas the latter, under the future condition $\theta^{(k)} \leq 0$, are dynamical expanding ones. We focus here on the equilibrium case, where the null horizon $\mathcal{H}$ is generated by the null vector ℓ^a and the intrinsic geometry remains invariant under ℓ^a: $\mathcal{L}_\ell q_{ab} = 0$. Crucially for our discussion, any foliation of $\mathcal{H}$ defines a foliation by MOTS. This freedom will be exploited in Theorem 1 below. We introduce the *surface gravity* $\kappa^{(\ell)}$ as the nonaffinity coefficient of ℓ^a, i.e. $\ell^b\nabla_b\ell^a = \kappa^{(\ell)}\ell^a$, with $\kappa^{(\ell)} = -k^a\ell^b\nabla_b\ell_a$.

We will consider a stronger notion of stationarity than that of nonexpanding horizons by requiring also the extrinsic geometry of the null $\mathcal{H}$ to be invariant under a certain ℓ^a fixed up to a constant rescaling. This defines an *isolated horizon* (IH) [12,13]. More specifically, we require the invariance of the unique connection $\hat{\nabla}_a$ induced on the nonexpanding horizon $\mathcal{H}$ by the ambient one ∇_a: $[\mathcal{L}_\ell, \hat{\nabla}_a] = 0$. This implies the invariance of $\Omega^{(\ell)}_a$ and $\kappa^{(\ell)}$, i.e. $\mathcal{L}_\ell\Omega^{(\ell)}_a = \mathcal{L}_\ell\kappa^{(\ell)} = 0$, and the angular constancy of $\kappa^{(\ell)}$: $D_a\kappa^{(\ell)} = 0$. IHs constitute the model for stationary BH horizons discussed here. This includes in particular Killing horizons, in which ℓ^a can be extended to a symmetry in the spacetime neighborhood of $\mathcal{H}$.

III. λ_o EIGENVALUE FOR AXISYMMETRIC IHS

The sign of the principal eigenvalue λ_o controls MOTS-stability, as expressed in (3). It is therefore of interest to have an explicit expression of λ_o in terms of the geometry of $\mathcal{S}$. Although this is a challenging problem when considered in full generality, the very important case of stationary and axisymmetric BH horizons is addressed by the following result [14]:

Theorem 1: (Reiris [15]). *Given an axisymmetric IH $\mathcal{H}$ with null generator ℓ^a and nonaffinity coefficient $\kappa^{(\ell)}$:*

(i) *There exists an (axisymmetric) foliation $\mathcal{H} = \cup_t\mathcal{S}^o_t$ by MOTSs $\mathcal{S}^o_t$ with constant ingoing expansion $\theta^{(k)}$.*

(ii) *The principal eigenvalue λ_o evaluated on sections $\mathcal{S}^o_t$ is*

$$\lambda_o = -\kappa^{(\ell)}\theta^{(k)}. \tag{5}$$

(iii) *The principal eigenfunction ϕ_o is given by $\phi_o = e^{2\chi}$, with $\Omega^{(\ell)}_a = z_a + D_a\chi$ on $\mathcal{S}^o_t$, where $D^a z_a = 0$.*

The result holds in any dimensions, under the topological condition in [16] of $\mathcal{H}$ being foliated by closed MOTSs. Note that λ_o does not depend on the section of $\mathcal{H}$ [16], though the particular form (5) holds in the preferred foliation $\{\mathcal{S}^o_t\}$ in Theorem 1.

IV. YOUNG-LAPLACE LAW FOR STATIONARY HORIZONS

A. BH surface tension and mean curvature

Let us first rewrite expression (5) the following way

$$\lambda_o/(8\pi) = \kappa^{(\ell)}/(8\pi)(-\theta^{(k)}). \tag{6}$$

The right-hand side presents then a particularly suggestive form when compared with the Young-Laplace law in (1). First, from the first law of BH thermodynamics, namely $\delta M = \kappa^{(\ell)}/(8\pi)\delta A + \Omega\delta J$, the factor $\kappa^{(\ell)}/(8\pi)$ is identified in [1] as an effective BH surface tension

$$\gamma_{\mathrm{BH}} = \kappa^{(\ell)}/(8\pi), \tag{7}$$

using its standard equivalence with an energy surface density. Such thermodynamical identification is consistent with the purely mechanical view provided by the analogy of BH horizons as 2–dimensional viscous fluids in the membrane paradigm [2–4]. In the latter, the understanding of the evolution equations for $\theta^{(\ell)}$ and $\Omega^{(\ell)}_a$ as, respectively, energy and momentum (Damour-Navier-Stokes) balance equations requires the interpretation of $\kappa^{(\ell)}/(8\pi)$ as a pressure of the 2–dimensional fluid, i.e. a mechanical surface tension.

Second, regarding the second factor in (6), let us consider the section $\mathcal{S}^o_t$ provided by point *(i)* in Theorem 1, and let us extend it to a $(d-1)$-dimensional spatial slice Σ_t in the bulk. Such Σ_t can be locally boosted so that the IH null generator ℓ^a and the ingoing null normal to $\mathcal{S}^o_t$ are, respectively, written as $\ell^a = n^a + s^a$ and $k^a = (n^a - s^a)/2$, with n^a the timelike normal to Σ_t and s^a the outgoing spacelike normal to $\mathcal{S}^o_t$ tangent to Σ_t. The mean curvature H of $(\mathcal{S}^o_t, q_{ab})$ into (Σ_t, γ_{ab}), with γ_{ab} induced from the ambient g_{ab}, is written as

$$H = q^{ab}\nabla_a s_b = \tilde{D}_a s^a, \quad (8)$$

with $\tilde{D}_a$ the connection compatible with γ_{ab}. For a 2−surface embedded in an Euclidean 3-space, the form $H = (1/R_1 + 1/R_2)$ in (1) is recovered. Combining $\theta^{(\ell)}$ and $\theta^{(k)}$, we write $H = -\theta^{(k)} + \frac{1}{2}\theta^{(\ell)}$, so that in our MOTS $\theta^{(\ell)} = 0$ case

$$H = -\theta^{(k)}. \quad (9)$$

From (7) and (9) we see that (6) matches the form (1) of the Young-Laplace law, if $\lambda_o/(8\pi)$ is *formally* identified with a pressure difference between the *interior* and the *exterior* of the BH horizon. We justify now such a heuristic identification.

B. The principal eigenvalue λ_o as a pressure

The principal eigenvalue λ_o admits the interpretation of a pressure. First we note that λ_o shares physical nature with the cosmological constant Λ. Indeed, the (explicit) effect of switching on the cosmological constant, as compared with the reference situation in the absence of Λ, is to produce a shift in the eigenvalue λ_o characterising MOTS-stability [17]

$$L_{\mathcal{S}}\phi = \lambda\phi, \quad L^*_{\mathcal{S}}\phi = \lambda^*\phi \Longrightarrow \lambda^*_o = \lambda_o + \Lambda, \quad (10)$$

that follows from (2) and (4) when imposing $G_{ab} + \Lambda g_{ab} = 8\pi T_{ab}$. Therefore, physical dimensions of Λ are shared by λ_o.

Second, the cosmological constant Λ admits the natural interpretation of a pressure, $p_{\rm cosm} = -\Lambda/(8\pi)$, for a perfect fluid stress-energy tensor. Based on these remarks, we propose the interpretation of $\lambda_o/(8\pi)$ as a pressure, specifically a pressure difference between the interior and exterior of the BH horizon

$$\Delta p = p_{\rm inn} - p_{\rm out} \equiv \lambda_o/(8\pi), \quad (11)$$

with $p_{\rm inn}$ and $p_{\rm out}$ the *formal* inner and outer pressures.

C. The MOTS-stability operator $L_{\mathcal{S}}$ as a "pressure operator"

Beyond the interpretation of λ_o in (11), the whole stability operator $L_{\mathcal{S}}$ can be understood as a "pressure operator." To justify this claim, we consider the equation of a MTT. The horizon evolution vector h^a, tangent to $\mathcal{H} = \cup_{t\in\mathbb{R}}\mathcal{S}_t$ and normal to MOTS sections $\mathcal{S}_t$, Lie-drags the section $\mathcal{S}_t$ to $\mathcal{S}_{t+\delta t}$. It can be written as $h^a = \ell^a - Ck^a$, where C is a (dimensionless) function on $\mathcal{H}$ such that the MTT is null, spacelike or timelike for $C = 0$, $C > 0$ or $C < 0$, respectively. The MTT condition $\delta_h\theta^{(\ell)} = 0$ is then expressed in terms of the MOTS-stability operator $L_{\mathcal{S}}$. By using $\delta_h\theta^{(\ell)} = \delta_\ell\theta^{(\ell)} - \delta_{Ck}\theta^{(\ell)}$, the MTT condition is rewritten as $\delta_{C(-k)}\theta^{(\ell)} = -\delta_\ell\theta^{(\ell)}$, so that

$$L_{\mathcal{S}}C = \sigma^{(\ell)}{}_{ab}\sigma^{(\ell)ab} + 8\pi T_{ab}\ell^a\ell^b, \quad (12)$$

where $\sigma^{(\ell)}{}_{ab} = q^c{}_a q^d{}_b \nabla_c \ell_d - 1/(d-2)\theta^{(\ell)}q_{ab}$ is the shear associated with the outgoing null normal and we have made use of the null Raychaudhuri equation. The right-hand side of Eq. (12) fixes the physical dimensions of the stability operator as $[L_{\mathcal{S}}/(8\pi)] = \text{Energy}\cdot\text{Time}^{-1}\cdot\text{Area}^{-1}$, $(G = c = 1)$. Such an interpretation is natural in dynamical scenarios, where the horizon growth is controlled by the presence of matter or gravitational energy fluxes. In purely stationary contexts, as in the spectral problem of (10), physical dimensions of $L_{\mathcal{S}}$ can be recast in a better suited form by simply noting $\text{Energy}\cdot\text{Time}^{-1}\cdot\text{Area}^{-1} \approx \text{Force}\cdot\text{Area}^{-1}$, so that

$$[L_{\mathcal{S}}/(8\pi)] = \text{Pressure}. \quad (13)$$

This provides additional support to the proposed physical interpretation of (the real) $\lambda_o/(8\pi)$ as a pressure. But, in addition, it also suggests a role of the whole spectrum of $L_{\mathcal{S}}$ (including complex eigenvalues) in horizon stability issues.

D. MOTS-STABILITY FROM A BH YOUNG-LAPLACE LAW PERSPECTIVE

We can now revisit MOTS-stability for stationary axisymmetric BHs in the following soap-bubble analogy form:

BH Young-Laplace "law": *For axisymmetric IHs, there exists a foliation in which the identifications*

$$\frac{\kappa^{(\ell)}}{8\pi} \to \gamma_{\rm BH}, \quad -\theta^{(k)} \to H, \quad \frac{\lambda_o}{8\pi} \to \Delta p = p_{\rm inn} - p_{\rm out}, \quad (14)$$

permit one to recast the principal eigenvalue in the form of a Young-Laplace law: $\Delta p = p_{\rm inn} - p_{\rm out} = \gamma_{\rm BH}H$. In this view, MOTS-stability ($\lambda_o \geq 0$) is interpreted as the result of an increase in the pressure of the BH trapped region.

V. PERSPECTIVES FROM A YOUNG-LAPLACE VIEW

Apart from the appeal of casting Theorem 1 in the physical terms of equilibrium bubbles, the main outcome of the Young-Laplace perspective is the identification of λ_o as a pressure. This interpretation extends beyond stationarity and axisymmetry, providing a new twist on MOTS-stability that suggests new avenues and questions motivated by the fluid analogy. The heuristic proposals in the rest of the article illustrate this.

A. BH horizon dynamical time scale

The identification of $\kappa^{(\ell)}/(8\pi)$ as a surface tension, together with the integrated expressions for the BH mass $M = 2\kappa^{(\ell)}/(8\pi)A + 2\Omega J$, led Smarr [1] to consider BH horizon instabilities in analogy with the case of rotating liquid drops.

Although MOTS-stability does not correspond to the notion of dynamical stability, it provides a condition for equilibrium that can be used to estimate the characteristic time scale of dynamical perturbations. This is illustrated for fluids in the Rayleigh-Plateau instability, where the time scale $\tau_{\rm RP}$ of the zero-mode dominating at large times can be determined solely from the equilibrium Young-Laplace law: $\tau_{\rm RP} = \sqrt{4\pi a^3\rho/\gamma}$, with a the radius of the fluid jet, ρ its density and γ the surface tension. In this spirit, our geometrical setting suggests the following proposal for a BH horizon dynamical time scale

$$\tau_{\rm dyn} \equiv \sqrt{1/\Delta p} = \sqrt{8\pi/\lambda_o}. \tag{15}$$

If this time scale corresponds to an instability, or rather to a relaxation process, it must be determined by other methods (e.g. [18]). The first case is illustrated by the Gregory-Laflamme instability of d-dimensional black strings, where $\lambda_o = R_{S^{d-3}(r_{\rm H})}/2 = (d-3)(d-4)/(2r_{\rm H}^2)$, with $r_{\rm H}$ the horizon areal radius, and (15) produces $\tau_{\rm BS} = \sqrt{16\pi/[(d-3)(d-4)]}r_{\rm H}$. For $d = 5$

$$\tau_{\rm BS} = \sqrt{8\pi} r_{\rm H} = \sqrt{8\pi \cdot 4M^2} = \sqrt{M/\gamma_{\rm BH}}, \tag{16}$$

where the last expression stresses the analogy with the Rayleigh-Plateau instability shown in [8], when introducing the effective mass $m_{\rm eff} = 4\pi a^3\rho$ in $\tau_{\rm RP}$ above. Regarding stable scenarios, Reissner-Nordström ($d = 4$) provides a nontrivial example in which (15) leads to a dynamical time scale

$$\tau_{\rm RN} = \frac{\sqrt{4\pi}(M + \sqrt{M^2 - Q^2})^2}{\sqrt{M(M + \sqrt{M^2 - Q^2}) - Q^2}}. \tag{17}$$

The shortest time scale occurs at $Q/M = \sqrt{3}/2$. Interestingly, this number coincides with the value for heat capacity change of sign in Reissner-Nordström [19] (and with Smarr's proposal for the critical J/M^2 for Kerr "rotating instabilities").

B. Full spectral analysis of $L_{\mathcal{S}}$ and BH horizon instabilities

Beyond the role of λ_o in setting a dominating time scale, the full spectrum of $L_{\mathcal{S}}$ may provide a more refined probe into the stability/dynamical properties of the horizon. This is suggested by the presentation of the whole stability operator in (13) as a "pressure operator." Such an approach is particularly rich in the rotating case since higher eigenvalues $\lambda_{n>o}$'s are then generically complex, due to the $2\Omega_a^{(\ell)}D^a$ term, with the imaginary part encoding rotational information [20]. In particular, it is of interest to study a possible imprint of superradiance in the imaginary part of the spectrum. In brief, we propose here the systematic study of the full spectrum of $L_{\mathcal{S}}$ in a line of research that, inspired by the inverse spectral problem for the Laplacian [21,22], can be paraphrased as: *"can one hear the stability of a black hole horizon?"*. Although the exact resolution of the spectral problem is a formidable task in the generic case, semiclassical tools (e.g. [23]) may offer relevant insight into the statistical properties of the spectrum.

C. Inner and outer pressures and the cosmological constant

The Young-Laplace law says nothing about the absolute values of $p_{\rm inn}$ and $p_{\rm out}$ [24]. One can, however, speculate about the implications of the following two possibilities: (i) "Bubble in a room": fix $p_{\rm out}$ to the pressure existing in the absence of the BH, namely the cosmological pressure. Then

$$\begin{aligned} p_{\rm out} &= p_{\rm cosm} = -\Lambda/(8\pi), \\ p_{\rm inn} &= (\lambda_o - \Lambda)/(8\pi). \end{aligned} \tag{18}$$

(ii) "Casimir-like effect": Equations (10) and (11) imply $p_{\rm inn} - p_{\rm out} = -\Lambda/(8\pi) - (-\lambda_o^*)/(8\pi)$, motivating the identification

$$p_{\rm inn} = p_{\rm cosm} = -\Lambda/(8\pi), \qquad p_{\rm out} = -\lambda_o^*/(8\pi). \tag{19}$$

The outer pressure $p_{\rm out} = -\lambda_o^*/(8\pi) = -(\Lambda + \lambda_o)/(8\pi)$ decreases in the formation of a stable BH horizon ($\lambda_o \geq 0$). Equivalently, an effective (bulk) cosmological constant $\Lambda_{\rm eff} \equiv \Lambda + \lambda_o$ increases due to the presence of an inner BH boundary. This provides an ingredient for a physical mechanism correlating the increase of the (effective) cosmological constant to BH cosmological dynamics (note the similarities of such $\Lambda_{\rm eff}$-"enhancing" mechanism with the "neutralization" of Λ through the quantum creation of closed membranes [25]).

D. BH volume

A thermodynamic notion of BH volume has been formulated [26,27] by considering the cosmological constant as an independent intensive variable in the BH first law, so that a volume V is introduced as its corresponding conjugate extensive variable. The present Young-Laplace fluid analogy suggests to "shift" $-\Lambda/(8\pi)$ to $\lambda_o/(8\pi) = \Delta p$ [cf. Eq. (10)], as the appropriate intensive variable to be employed. That is

$$\delta M = T\delta S + \Omega_i \delta J_i + \Phi_\alpha \delta Q_\alpha + V_{\rm BH}\delta(\lambda_o/(8\pi)), \tag{20}$$

where M corresponds to a BH *enthalpy* and $V_{\rm BH}$ is now a volume explicitly associated with the BH. Interestingly, as in [27], such a thermodynamic volume provides the Euclidean $V_{\text{BH}} = 4\pi/3 \cdot r_{\text{H}}^3$ in (3-dimensional) spherical symmetry.

E. BH rest frame

The BH Young-Laplace law holds for a preferred (local) spacetime slicing $\{\Sigma_t\}$. This suggests the proposal:

(i) A "BH rest frame" is introduced as the one in which $H = -\theta^{(k)}$ is constant and the BH Young-Laplace law holds.

(ii) Given a unit vector ξ^a tangent to the preferred 3-slice Σ_t, but transverse (i.e. admitting normal components) to the horizon section $\mathcal{S}_t$, a quasilocal linear momentum along ξ^a is proposed as the dipolar part of the mean curvature H

$$P(\xi) \equiv \frac{1}{8\pi} \int_{\mathcal{S}_t} (\xi^a s_a) H dA. \tag{21}$$

A horizon slicing is fixed by setting the value of $D^a \Omega_a^{(\ell)}$. In [13] a "natural" BH rest-frame was introduced by choosing a vanishing divergence. From point *(iii)* in Theorem 1, the present Young-Laplace proposal amounts to a geometric choice in terms of the principal eigenfunction: $D^a \Omega_a^{(\ell)} = D^a D_a \ln \sqrt{\phi_o}$. Finally, note that $P(\xi)$, devised for measuring a vanishing linear momentum in the BH rest frame, is just the dipolar part of the Brown-York quasilocal energy [28].

ACKNOWLEDGMENTS

I thank M. Reiris for sharing his result on λ_o and M. Mars for key insights. I thank A. Ashtekar, R. Emparan, A. Harte, B. Krishnan, F. Pannarale, I. Rácz, L. Rezzolla, and W. Simon for discussions.

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