Quantitative studies of the homogeneous Bethe-Salpeter equation in Minkowski space

Tobias Frederico,[1] Giovanni Salmè,[2] and Michele Viviani[3]

[1]*Departamento de Física, Instituto Tecnológico de Aeronáutica, DCTA, 12.228-900 São José dos Campos, São Paulo, Brazil*
[2]*Istituto Nazionale di Fisica Nucleare, Sezione di Roma, Piazzale Moro 2, I-00185 Roma, Italy*
[3]*Istituto Nazionale di Fisica Nucleare, Sezione di Pisa, Largo Pontecorvo 3, 56100 Pisa, Italy*

The Bethe-Salpeter equation for a bound system, composed by two massive scalars exchanging a massive scalar, is quantitatively investigated in ladder approximation, within the Nakanishi integral representation approach. For the S-wave case, numerical solutions with a form inspired by the Nakanishi integral representation are calculated. The needed Nakanishi weight functions are evaluated by solving two different eigenequations, obtained directly from the Bethe-Salpeter equation applying the light-front projection technique. A remarkable agreement, in particular, for the eigenvalues, is achieved, numerically confirming that the Nakanishi uniqueness theorem for the weight functions demonstrated in the context of the perturbative analysis of the multileg transition amplitudes and playing a basic role in suggesting one of the two adopted eigenequations can be extended to a nonperturbative realm. The detailed, quantitative studies are completed by presenting both probabilities and light-front momentum distributions for the valence component of the bound state.

I. INTRODUCTION

Solving the Bethe-Salpeter equation (BSE)[1] in Minkowski space, even for scalar theories, is still a challenging problem, and not too many numerical investigations able to address the issue can be found in the literature. Seemingly, one of the most effective tools for facing such a task with high numerical accuracy (see, e.g., Refs. [2–8] for an illustration of actual calculations) is represented by the so-called perturbation theory integral representation (PTIR) of the multileg transition amplitudes proposed by Nakanishi in the 1960s (see, e.g., Refs. [9–12]). Such an approach originates from the parametric formula for Feynman integrals and leads to a spectral representation of any multileg transition amplitude expressed through an infinite series of Feynman diagrams. Then, a transition amplitude can be written as a suitable folding of a nonsingular weight function, the so-called Nakanishi weight function, divided by a denominator containing the analytic structure of the amplitude. In particular, the PTIR of the three-leg amplitude, i.e., the PTIR vertex function (related to the BS amplitude through the inverse of the constituent propagators), plays a basic role in the quest of physically motivated, actual solutions of the BSE in Minkowski space. This is shown by the nice results in Refs. [2–8], where a wide range of (i) systems (bosonic and fermionic), (ii) approximated kernels (ladder and cross ladder), and (iii) constituent propagators (free and dressed) have been explored. Loosely speaking, applying PTIR in order to solve the BSE can be seen as a generalization of the approach proposed by Wick [13] and Cutkosky [14]

for obtaining explicit solutions of the scalar-scalar BSE but with a massless-scalar exchange.

Among the attractive features of PTIR, one has (i) the dependence of the nonsingular weight functions upon real variables, whose number is related to the number of independent invariants of the problem, and (ii) the explicit analytic structure of the transition amplitude that allows one to perform analytic integrations when requested (indeed, this will be the case). These properties have been exploited in order to obtain equations for the Nakanishi weight function starting from the BSE. In particular (cf. Refs. [2–8]), one can single out two different equations for determining the Nakanishi weight function, but both of them share the first step: One assumes a form like the PTIR vertex for the BS amplitude and puts such an expression in the BSE. Then, one can proceed by invoking the Nakanishi *uniqueness theorem* for the weight functions [12] and obtain the eigenequation for the Nakanishi weight function adopted in Refs. [2–5], where the ladder approximation of the BS kernel has been assumed and some elaborations, like either free or dressed constituent propagators, have been proposed. Indeed, the uniqueness theorem was demonstrated within the perturbative analysis of the transition amplitudes; therefore, one could ask if and to what extent this can be applied in a nonperturbative framework: this is the first question we have addressed in the present paper. Moreover, in Refs. [2–5], the truncated kernel was elaborated by performing the needed analytical integrations using the standard Minkowski variables. Differently, in Refs. [6–8], an explicitly covariant light-front (LF) approach [15], with the set of LF variables $k^\pm = k^0 \pm k^3$

and $\mathbf{k}_\perp \equiv \{k^1, k^2\}$ was adopted for determining another integral equation for the Nakanishi weight function. In particular, through an exact relation based on a suitable analytic integration of the Nakanishi representation of the BS amplitude, one can extract the so-called *valence component* of the interacting-system state (see also Refs. [16,17]), i.e., the first contribution to the Fock expansion of the state. Applying the same integration on both sides of the BSE, one gets an integral equation for the weight function that has on the lhs the valence component and on the rhs the Nakanishi function combined with a proper kernel (obtained from the BS 4D kernel). It should be pointed out that such an integral equation is a generalized eigenequation, and, therefore, substantially different from the one of Refs. [2–5] that is a true eigenequation once the ladder approximation is adopted.

A formal investigation for establishing a direct bridge between the above-described approaches also extending the analysis from the homogeneous (bound states) to inhomogeneous (scattering states) ladder BSE was performed in Ref. [18]. To accomplish this, a nonexplicitly covariant LF framework was adopted together with a LF projection technique (see Refs. [19–23] for details). In particular, the integration over the LF variable k^- was exploited for obtaining the valence component from the BS amplitude, arriving at the same generalized eigenequation of Ref. [6] for the Nakanishi weight function but deduced within the explicitly covariant LF framework (see Ref. [15]). Moreover, by a suitable elaboration of the kernel present in the generalized eigenequation, one becomes ready for applying the Nakanishi uniqueness theorem [12]. It should be pointed out that all the formal developments in Ref. [18] benefit from the well-known virtue of the LF variables to make the analytical integrations simpler (see Ref. [24] for an elementary introduction to the issue).

The aim of the present work is the numerical investigation of the above-mentioned eigenequations (i.e., with and without the application of the uniqueness theorem) in order to evaluate the Nakanishi weight functions of BS amplitudes, solutions of a ladder BSE for an S-wave bound system composed by two massive scalars interacting through the exchange of a massive scalar. In particular, for both eigenequations, we have calculated (i) eigenvalues and eigenfunctions corresponding to binding energies and masses of the exchanged scalar of Refs. [3,6] and (ii) valence probabilities and LF distributions in both the longitudinal-momentum fraction ξ and the transverse momentum $|\mathbf{k}_\perp|$ that notably can be evaluated once the Nakanishi weight functions are determined. The comparison between the numerical results obtained from the two eigenequations allows us to check to which extent the uniqueness theorem of the Nakanishi weight function is valid and to assess the reliability of quantities like valence probabilities and the LF distribution that are quite relevant

in the phenomenological studies of interacting relativistic systems.

The paper is organized as follows. In Sec. II, the general formalism of the BSE onto the null plane is introduced, as well as the valence component of the BS amplitude. In Sec. III, the kernel of the ladder BSE is recast in a form suitable for applying the uniqueness theorem by Nakanishi. In Sec. IV, the LF momentum distributions are defined. In Sec. V, the numerical results are discussed. Finally, in Sec. VI, the conclusions are drawn.

II. THE HOMOGENEOUS BETHE-SALPETER EQUATION ONTO THE NULL PLANE

In this section, the general formalism adopted for obtaining the eigenequations for the Nakanishi weight functions within the LF framework of Ref. [18] is quickly reviewed in order to have the full matter under control and proceed in the following sections to the numerical analysis. Moreover, it is illustrated (Appendix A contains the details) a shorter way to deduce the eigenequation based on the uniqueness theorem from the one based on the LF valence wave. It should be pointed out that the BSE we have considered does not contain either self-energy or vertex corrections, but it is worth mentioning that one could rely upon a Dyson-Schwinger framework for dressing the constituent propagators (see, e.g., Ref. [4]).

Let us start by recalling that the BS amplitude for a bound state fulfills the following BSE:

$$\Phi_b(k, p) = G_0^{(12)}(k, p) \int \frac{d^4 k'}{(2\pi)^4} i\mathcal{K}(k, k', p)\Phi_b(k', p), \quad (1)$$

where $p = p_1 + p_2$ is the total momentum of the interacting system with total mass $p^2 = M^2$, $k = (p_1 - p_2)/2$ is the relative momentum, and $i\mathcal{K}$ the interaction kernel that contains all the irreducible diagrams. As mentioned above, the self-energy is disregarded, and, therefore, one has to consider the free propagator of the two constituents $G_0^{(12)}$ given by

$$G_0^{(12)}(k, p) = G_0^{(1)} G_0^{(2)}$$
$$= \frac{i}{(\frac{p}{2} + k)^2 - m^2 + i\epsilon} \frac{i}{(\frac{p}{2} - k)^2 - m^2 + i\epsilon}. \quad (2)$$

In what follows, we look for S-wave solutions of Eq. (1) that can be written as the PTIR vertex function; i.e., a proper folding of a nonsingular weight function that depends upon real variables, and a factor that contains the analytic structure (see, also, Refs. [2–4,6–8,18]), namely,

$$\Phi_b(k,p) = i \int_{-1}^{1} dz' \int_{0}^{\infty} d\gamma' \frac{g_b(\gamma',z';\kappa^2)}{[\gamma' + \kappa^2 - k^2 - p \cdot kz' - i\epsilon]^{2+n}},$$
$$(3)$$

where $g_b(\gamma', z'; \kappa^2)$ is the Nakanishi weight function, and κ^2 is defined by

$$\kappa^2 = m^2 - \frac{M^2}{4}, \qquad (4)$$

with m the constituent mass. Notice that by definition, one has $\kappa^2 > 0$ for bound states, while $\kappa^2 < 0$ for scattering states. The power n in the denominator of Eq. (3) can be any value $n \geq 1$. The minimal value $n = 1$ ensures the convergence of the 4D integral, and in what follows, we adopt this choice, as in Refs. [3,6]. Increasing the value of n should produce a solution for $g_b(\gamma', z'; \kappa^2)$ more and more soft [3,25]. The factor of 2 in the exponent of the denominator of Eq. (3) comes from the fact we are dealing with the BS amplitude and not with the vertex function.

It is worth noting that the dependence upon z' of $g_b(\gamma', z'; \kappa^2)$ is even as expected by the symmetry property of the BS amplitude for the two-scalar system. As a matter of fact, when the exchange between the two particles is performed, the scalar product $k \cdot p$ in the denominator in Eq. (3) changes sign. In order to recover the expected symmetry of the BS amplitude, the Nakanishi weight function must be even in z'. Moreover, as pointed out in Refs. [3,9–11], z-odd $g_b(\gamma', z'; \kappa^2)$ functions correspond to odd-parity BS amplitudes with respect to the change $k^0 \rightarrow -k^0$ (recall that in the rest frame $p \cdot k = Mk^0$). It turns out (see Refs. [9–11] for more details) that such BS amplitudes have negative norm.

As is well known (see, e.g., Refs. [18,26] for the non-explicitly covariant LF approach and Refs. [6,15] for the explicitly covariant case), one can obtain the valence component of the interacting state from the corresponding BS amplitude through the suitable analytic integration, namely, the integration over k^-. Once the expansion of the interacting state on a Fock basis is introduced, the

valence component corresponds to the contribution with the minimal number of constituents, that in the present case amounts to two scalars. For the sake of clarity, it is useful to briefly recall the above-mentioned procedure within the LF framework adopted in our previous work [18], since in the following sections the valence probability and the LF distributions will be discussed and numerically evaluated.

In the Fock space, one can introduce the completeness given by

$$\sum_{n \geq 2} \int [d^3 \tilde{q}_i] |n; \tilde{q}_i\rangle\langle \tilde{q}_i; n| = \mathcal{I}, \qquad (5)$$

with $\mathcal{I}$ the identity in the Fock space, $\tilde{q}_i \equiv \{q_i^+, \mathbf{q}_{i\perp}\}$ the LF three-momenta, and

$$|n; \tilde{q}_i\rangle = (2\pi)^{3n/2} \frac{1}{\sqrt{n!}} \sqrt{2q_1^+} \ldots \ldots \sqrt{2q_n^+} a_{\tilde{q}_1}^\dagger \ldots \ldots a_{\tilde{q}_n}^\dagger |0\rangle. \quad (6)$$

The normalization for the single-particle free state is $\langle \tilde{q}' | \tilde{q} \rangle = 2q^+(2\pi)^3 \delta^3(\tilde{q}' - \tilde{q})$ that leads to the standard LF phase space, viz.,

$$\int [d^3 \tilde{q}_i] = \prod_{i=1}^{n} \int \frac{d^3 \tilde{q}_i}{2q_i^+ (2\pi)^3}. \qquad (7)$$

The free Fock states in Eq. (6) have the following ortho-normalization:

$$\langle \tilde{q}_i'; n' | n; \tilde{q}_i \rangle = \frac{1}{n!} \delta_{n,n'} \sum_{[j_1 \ldots j_n]_{\text{perm}}} \prod_{i=1}^{n} 2q_i^+ (2\pi)^3 \delta^3(\tilde{q}_i' - \tilde{q}_{j_i}), \quad (8)$$

where the sum has to be performed over all the $n!$ permutations of $1 \ldots \ldots n$, as shortly indicated by $[j_1 \ldots j_n]_{\text{perm}}$. Then, the interacting state can be expanded as follows (see, e.g., Ref. [26]):

$$|\tilde{p}; \Psi_{\text{int}}\rangle = 2(2\pi)^3 \sum_{n \geq 2} \int [d\xi_i][d^2 \mathbf{k}_{i\perp}] \psi_{n/p}(\xi_i, \mathbf{k}_{i\perp}) |n; \xi_i p^+, \mathbf{k}_{i\perp} + \xi_i \mathbf{p}_\perp\rangle, \qquad (9)$$

where (i) $|n; \xi_i p^+, \mathbf{k}_{i\perp} + \xi_i \mathbf{p}_\perp\rangle$ is the Fock state with n particles, and the variables $\tilde{q}_i$ have been expressed in terms of the intrinsic variables $\{\xi_i, \mathbf{k}_{i\perp}\}$, as follows: $q_i^+ = \xi_i p^+$ and $\mathbf{q}_{i\perp} = \mathbf{k}_{i\perp} + \xi_i \mathbf{p}_\perp$; (ii) $\psi_{n/p}(\xi_i, \mathbf{k}_{i\perp})$ are the so-called LF wave functions that allow one to describe the intrinsic dynamics and are related to the overlap $\langle n; \xi_i p^+, \mathbf{k}_{i\perp} + \xi_i \mathbf{p}_\perp | \tilde{p}; \Psi_{\text{int}}\rangle$, as discussed below. Notice that the global motion and the intrinsic structure have been kept separate in $|\tilde{p}; \Psi_{\text{int}}\rangle$ given the kinematical nature of the

LF boosts. Finally, let us remember that the interacting system is on-mass-shell, i.e., $p^- = (M^2 + |\mathbf{p}_\perp|^2)/p^+$, and the set of intrinsic variables $\{\xi_i, \mathbf{k}_{i\perp}\}$ satisfy the following relations:

$$\sum_{i=1}^{n} \xi_i = 1, \qquad \sum_{i=1}^{n} \mathbf{k}_{i\perp} = 0. \qquad (10)$$

The phase-space factors in Eq. (9) are given by

$$\int [d\xi_i] \equiv \prod_{i=1}^{n} \int \frac{d\xi_i}{2(2\pi)\xi_i} \delta\left(1 - \sum_{j=1}^{n} \xi_j\right) = p^+ \prod_{i=1}^{n} \int \frac{dq_i^+}{2q_i^+(2\pi)} \delta\left(p^+ - \sum_{j=1}^{n} q_j^+\right),$$

$$\int [d^2\mathbf{k}_{i\perp}] \equiv \prod_{i=1}^{n} \int \frac{d^2\mathbf{k}_{i\perp}}{(2\pi)^2} \delta^2\left(\sum_{j=1}^{n} \mathbf{k}_{j\perp}\right). \tag{11}$$

Since the intrinsic motion is kinematically separated from the global one within the LF framework, the overlap $\langle n; \xi_i p^+, \mathbf{k}_{i\perp} + \xi_i \mathbf{p}_\perp | \tilde{p}; ; \Psi_{\text{int}}\rangle$ can be trivially factorized into the product of a momentum-conserving delta function and the intrinsic LF wave function as follows:

$$\langle n; \xi_i p^+, \mathbf{k}_{i\perp} + \xi_i \mathbf{p}_\perp | \tilde{p}; \Psi_{\text{int}}\rangle = 2p^+(2\pi)^3 \delta^3\left(\tilde{p} - \sum_{i=1}^{n} \tilde{q}_i\right) \psi_{n/p}(\xi_i, \mathbf{k}_{i\perp})$$

$$= 2(2\pi)^3 \delta\left(1 - \sum_{i=1}^{n} \xi_i\right) \delta^{(2)}\left(\sum_{i=1}^{n} \mathbf{k}_{i\perp}\right) \psi_{n/p}(\xi_i, \mathbf{k}_{i\perp}). \tag{12}$$

From Eq. (9) and reminding that the CM plane waves have the standard normalization that can be factorized out, one can obtain the normalization of the intrinsic state and in turn (i) the overall normalization of the LF wave functions and (ii) the probability of each Fock component. As a matter of fact, one can write

$$\langle \tilde{p}'; \Psi_{\text{int}} | \tilde{p}; \Psi_{\text{int}}\rangle = 2p^+(2\pi)^3 \delta^3(\tilde{p}' - \tilde{p}) \langle \Psi_{\text{int}} | \Psi_{\text{int}}\rangle$$

$$= [2p^+(2\pi)^3]^2 \sum_{n\geq 2} \int [d^3\tilde{q}_i] \delta^3\left(\sum_{i=1}^{n} \tilde{q}_i - \tilde{p}\right) \psi_{n/p}(\xi_i, \mathbf{k}_{i\perp}) \delta^3\left(\sum_{i=1}^{n} \tilde{q}_i - \tilde{p}'\right) \psi_{n/p'}(\xi_i, \mathbf{k}_{i\perp})$$

$$= 2p^+(2\pi)^3 \delta^3(\tilde{p}' - \tilde{p}) 2(2\pi)^3 \sum_{n\geq 2} \int [d\xi_i][d^2\mathbf{k}_{i\perp}] |\psi_{n/p}(\xi_i, \mathbf{k}_{i\perp})|^2. \tag{13}$$

Then, the LF wave functions $\psi_{n/p}(\xi_i, \mathbf{k}_{i\perp})$ are normalized according to

$$\langle \Psi_{\text{int}} | \Psi_{\text{int}}\rangle = 2(2\pi)^3 \sum_{n\geq 2} \int [d\xi_i][d^2\mathbf{k}_{i\perp}] |\psi_{n/p}(\xi_i, \mathbf{k}_{i\perp})|^2 = 1. \tag{14}$$

This equation clearly shows the physical content associated to the LF wave functions: $|\psi_{n/p}(\xi_i, \mathbf{k}_{i\perp})|^2$ yields the probability distributions to find n constituents with intrinsic coordinates $\{\xi_i, \mathbf{k}_{i\perp}\}$ inside the interacting-system state. In view of this, the basic role played by LF wave functions in extracting the probability content hidden inside the BS amplitude should be pointed out. Notice that a factor $2(2\pi)^3$ is missing in the corresponding equation [i.e., Eq. (15)] of Ref. [18].

In particular, the probability to find the valence component in the bound state (see Sec. IV) is given by

$$N_2 = 2(2\pi)^3 \int \frac{d\xi_1}{2(2\pi)\xi_1} \int \frac{d\xi_2}{2(2\pi)\xi_2} \delta(1 - \xi_1 - \xi_2) \times \int \frac{d^2\mathbf{k}_{1\perp}}{(2\pi)^2} \int \frac{d^2\mathbf{k}_{2\perp}}{(2\pi)^2} \delta^2(\mathbf{k}_{1\perp} + \mathbf{k}_{2\perp}) |\psi_{n=2/p}(\xi_1, \mathbf{k}_{1\perp})|^2$$

$$= \frac{1}{(2\pi)^3} \int \frac{d\xi}{2\xi(1 - \xi)} \int d^2\mathbf{k}_\perp |\psi_{n=2/p}(\xi, \mathbf{k}_\perp)|^2, \tag{15}$$

where the notation has been simplified, putting $\xi = \xi_1$ and $\mathbf{k}_\perp = \mathbf{k}_{1\perp}$. In general, the probability N_n of the nth Fock component can be evaluated through the corresponding LF wave function.

In general, the valence wave function can be obtained by integrating the BS amplitude $\Phi_b(k, p)$ over k^- (see Ref. [18] for details). Once we assume the expression for the BS amplitude suggested by the PTIR approach [12], we get (see, also, Ref. [6])

$$\psi_{n=2/p}(\xi, k_\perp) = \frac{p^+}{\sqrt{2}}\xi(1-\xi)\int \frac{dk^-}{2\pi}\Phi_b(k,p) = \frac{1}{\sqrt{2}}\xi(1-\xi)\int_0^\infty d\gamma' \frac{g_b(\gamma', 1-2\xi; \kappa^2)}{[\gamma' + k_\perp^2 + \kappa^2 + (2\xi-1)^2\frac{M^2}{4} - i\epsilon]^2},\qquad(16)$$

where the integration over k^- leads to fix the value of the variable z' in Eq. (3) to $1-2\xi$. The factor $1/\sqrt{2}$ comes from the normalization of the Fock state with $n=2$, given the statistics property.

From Eq. (16) and the physically motivated request that the density in the transverse variable $\mathbf{b}_\perp$, conjugated to $\mathbf{k}_\perp$, be finite for $|\mathbf{b}_\perp| = 0$, one can deduce that $g_b(\gamma', 1-2\xi; \kappa^2)$ must vanish for $\gamma' \to \infty$. As a matter of fact, one has

$$\tilde{\psi}_{n=2/p}(\xi, b_\perp) = \int \frac{d\mathbf{k}_\perp}{(2\pi)^2} e^{i\mathbf{k}_\perp \cdot \mathbf{b}_\perp}\psi_{n=2/p}(\xi, k_\perp)\qquad(17)$$

and

$$\tilde{\psi}_{n=2/p}(\xi, b_\perp = 0) = \pi\frac{1}{\sqrt{2}}\xi(1-\xi)\int_0^\infty \frac{dk_\perp^2}{(2\pi)^2}\int_0^\infty d\gamma' \frac{g_b(\gamma', 1-2\xi; \kappa^2)}{[\gamma' + k_\perp^2 + \kappa^2 + (2\xi-1)^2\frac{M^2}{4} - i\epsilon]^2}$$

$$= \frac{1}{4\pi\sqrt{2}}\xi(1-\xi)\int_0^\infty d\gamma' \frac{g_b(\gamma', 1-2\xi; \kappa^2)}{[\gamma' + \kappa^2 + (2\xi-1)^2\frac{M^2}{4} - i\epsilon]}.\qquad(18)$$

If the transverse density at the origin, i.e., $|\tilde{\psi}_{n=2/p}(\xi, b_\perp = 0)|^2$, is finite, one can immediately realize the needed falloff of $g_b(\gamma', 1-2\xi; \kappa^2)$. Notice that the denominator is always positive for a bound state. By introducing the variables (γ, z), as in Ref. [6],

$$\gamma = k_\perp^2 \qquad 1 \geq z = 1-2\xi \geq -1,\qquad(19)$$

one can rewrite the valence wave function as follows:

$$\psi_{n=2/p}(z, \gamma) = \frac{(1-z^2)}{4\sqrt{2}}$$
$$\times \int_0^\infty d\gamma' \frac{g_b(\gamma', z; \kappa^2)}{[\gamma' + \gamma + z^2 m^2 + (1-z^2)\kappa^2 - i\epsilon]^2}.\qquad(20)$$

The announced integral equation for the Nakanishi weight function $g_b(\gamma, z; \kappa^2)$ is obtained by inserting Eq. (3) in both sides of the LF-projected BS equation (see, also, Refs. [19–23]), viz.,

$$\int \frac{dk^-}{2\pi}\Phi_b(k,p) = \int \frac{dk^-}{2\pi}G_0^{(12)}(k,p)$$
$$\times \int \frac{d^4k'}{(2\pi)^4}i\mathcal{K}(k,k',p)\Phi_b(k',p).\quad(21)$$

Then, one gets [18] (see Ref. [6] for the corresponding elaboration within the explicitly covariant LF framework)

$$\int_0^\infty d\gamma' \frac{g_b(\gamma', z; \kappa^2)}{[\gamma' + \gamma + z^2 m^2 + (1-z^2)\kappa^2 - i\epsilon]^2}$$
$$= \int_0^\infty d\gamma' \int_{-1}^1 dz' V_b^{\mathrm{LF}}(\gamma, z; \gamma', z')g_b(\gamma', z'; \kappa^2),\quad(22)$$

where the new kernel V_b^{LF}, that we call the Nakanishi kernel for the sake of brevity, is related to the BS kernel $i\mathcal{K}$ in Eq. (1), as follows:

$$V_b^{\mathrm{LF}}(\gamma, z; \gamma', z') = ip^+ \int_{-\infty}^\infty \frac{dk^-}{2\pi}G_0^{(12)}(k,p)$$
$$\times \int \frac{d^4k'}{(2\pi)^4}\frac{i\mathcal{K}(k,k',p)}{[k'^2 + p\cdot k' z' - \gamma' - \kappa^2 + i\epsilon]^3}.\qquad(23)$$

Still starting from the LF-projected BSE (21), a different equation for $g_b(\gamma, z; \kappa^2)$ can be obtained, if one takes into account (i) the uniqueness of the Nakanishi weight function, as ensured by a theorem in Ref. [12] and (ii) the PTIR expressions for both the BS amplitude and the BS kernel, i.e., a four-leg transition amplitude (see, e.g., Refs. [3,18] for the actual PTIR of the off-shell T matrix). Then, in place of Eq. (22), one could write the following eigenequation (see Refs. [2–5,18,27,28] for the ladder case):

$$g_b(\gamma, z; \kappa^2) = \int_0^\infty d\gamma' \int_{-1}^1 dz' \mathcal{V}_b(\gamma, z; \gamma', z'; \kappa^2)g_b(\gamma', z'; \kappa^2).\qquad(24)$$

Within the PTIR framework, it is very important to notice that both Eqs. (22) and (24) are equivalent to the initial BSE (1) if the uniqueness theorem holds. Once the weight function $g_b(\gamma', z; \kappa^2)$ is known, then one can fully reconstruct in Minkowski space [see Eq. (3)] the BS amplitude *that belongs to the class of physically acceptable solutions* (with positive norm and suitable for an investigation within an S-matrix framework). Moreover, it is not surprising that

through the information stored in the valence component, one can map the full BS amplitude, since the whole and rich content of the BS amplitude can be transferred to the LF kernel, i.e., the kernel projected onto the null plane. This result is quite general and holds both in perturbative and nonperturbative regimes, and, even more, for both bound and scattering states [19–23].

III. LF NAKANISHI KERNEL IN LADDER APPROXIMATION

At the present stage, our numerical investigation is restricted to the ladder approximation of the BSE, where the BS kernel is given by

$$i\mathcal{K}^{(Ld)}(k, k') = \frac{i(-ig)^2}{(k - k')^2 - \mu^2 + i\epsilon}, \tag{25}$$

with μ the mass of the exchanged scalar. Explicit expressions for both ladder and cross-ladder approximations of

$V_b^{\text{LF}}(\gamma, z; \gamma', z')$ obtained within the covariant LF framework can be found in Refs. [6,7].

In Ref. [18], where a nonexplicitly covariant LF framework was chosen, the scattering case was analyzed in great detail, and the ladder approximation of the Nakanishi kernel in the S-wave bound state [see Eq. (22)] $V_b^{(Ld)}$ was obtained through a proper limit of the scattering kernel. In what follows, a more direct and simple way to obtain $V_b^{(Ld)}$ is presented (see Appendix A for more details), eventually achieving an expression suitable for exploiting the uniqueness theorem of the Nakanishi weight function [12]. As to the numerical calculations, the results evaluated with our LF approach and the ones shown in Refs. [3,6] are compared in Sec. V.

In a reference frame, where $\mathbf{p}_\perp = 0$ and $p^\pm = M$, the ladder approximation of $V_b^{\text{LF}}(\gamma, z; \gamma', z')$ to be inserted in the integral equation (22) is written as follows (see Appendix A for details):

$$
\begin{aligned}
V_b^{(Ld)}(\gamma, z; \gamma', z') = {}&-g^2 p^+ \int \frac{d^4 k''}{(2\pi)^4} \frac{1}{[k''^2 + p \cdot k'' z' - \gamma' - \kappa^2 + i\epsilon]^3} \\
&\times \int_{-\infty}^{\infty} \frac{dk^-}{2\pi} \frac{1}{[(\frac{p}{2} + k)^2 - m^2 + i\epsilon]} \frac{1}{[(\frac{p}{2} - k)^2 - m^2 + i\epsilon]} \frac{1}{(k - k'')^2 - \mu^2 + i\epsilon} \\
= {}&-\frac{g^2}{2(4\pi)^2} \int_{-\infty}^{\infty} d\gamma'' \frac{\theta(\gamma'')}{[\gamma + \gamma'' + z^2 m^2 + \kappa^2(1 - z^2) - i\epsilon]^2} \\
&\times \left[\frac{(1+z)}{(1+\zeta')} \theta(\zeta' - z) h'(\gamma'', z; \gamma', \zeta', \mu^2) + \frac{(1-z)}{(1-\zeta')} \theta(z - \zeta') h'(\gamma'', -z; \gamma', -\zeta', \mu^2) \right],
\end{aligned} \tag{26}
$$

where

$$
\begin{aligned}
h'(\gamma'', z; \gamma', \zeta', \mu^2) = {}&\theta\left[\gamma'' \frac{(1+\zeta')}{(1+z)} - \gamma' - \mu^2 - 2\mu\sqrt{\zeta'^2 \frac{M^2}{4} + \kappa^2 + \gamma'} \right] \\
&\times \left[-\frac{\mathcal{B}_b(z, \zeta', \gamma', \gamma'', \mu^2)}{\mathcal{A}_b(\zeta', \gamma', \kappa^2) \Delta(z, \zeta', \gamma', \gamma', \kappa^2, \mu^2)} \frac{1}{\gamma''} + \frac{(1+\zeta')}{(1+z)} \int_{y_-}^{y_+} dy \frac{y^2}{[y^2 \mathcal{A}_b(\zeta', \gamma', \kappa^2) + y(\mu^2 + \gamma') + \mu^2]^2} \right] \\
&- \frac{(1+\zeta')}{(1+z)} \int_0^{\infty} dy \frac{y^2}{[y^2 \mathcal{A}_b(\zeta', \gamma', \kappa^2) + y(\mu^2 + \gamma') + \mu^2]^2},
\end{aligned} \tag{27}
$$

with

$$
\begin{aligned}
\mathcal{A}_b(\zeta', \gamma', \kappa^2) &= \zeta'^2 \frac{M^2}{4} + \kappa^2 + \gamma' \geq 0 \\
\mathcal{B}_b(z, \zeta', \gamma', \gamma', \mu^2) &= \mu^2 + \gamma' - \gamma'' \frac{(1+\zeta')}{(1+z)} \leq 0 \\
\Delta^2(z, \zeta', \gamma', \gamma'', \kappa^2, \mu^2) &= \mathcal{B}_b^2(z, \zeta', \gamma', \gamma'', \mu^2) - 4\mu^2 \mathcal{A}_b(\zeta', \gamma', \kappa^2) \geq 0 \\
y_\pm &= \frac{1}{2\mathcal{A}_b(\zeta', \gamma', \kappa^2)} [-\mathcal{B}_b(z, \zeta', \gamma', \gamma'', \mu^2) \pm \Delta(z, \zeta', \gamma', \gamma'', \kappa^2, \mu^2)].
\end{aligned} \tag{28}
$$

It is relevant for what follows that for $z \to (-1)$, (i) the theta function does not yield a constraint anymore, and (ii) the function $\mathcal{B}_b \to -\infty$, and (iii) the two integrals on y cancel each other. Then, one gets

$$\frac{(1+z)}{(1+\zeta')} h'(\gamma'', z; \gamma', \zeta', \mu^2) \to \frac{(1+z)}{(1+\zeta')} \frac{1}{(\gamma'' \mathcal{A}_b(\zeta', \gamma', \kappa^2))}$$

$$\to 0.$$

An analogous result can be obtained for $z \to 1$ when the term containing $h'(\gamma'', -z; \gamma', \zeta', \mu^2)$ is considered.

Notably, in Eq. (26) the denominator $1/[\gamma + \gamma'' + z^2m^2 + \kappa^2(1-z^2) - i\epsilon]^2$ has been factored out, making

possible the application of the uniqueness theorem to the ladder approximation of Eq. (22) that reads

$$\int_0^\infty d\gamma' \frac{g_b^{(Ld)}(\gamma', z; \kappa^2)}{[\gamma' + \gamma + z^2m^2 + (1-z^2)\kappa^2 - i\epsilon]^2}$$

$$= \int_0^\infty d\gamma' \int_{-1}^1 dz' \, V_b^{(Ld)}(\gamma, z; \gamma', z') g_b^{(Ld)}(\gamma', z'; \kappa^2). \tag{29}$$

As a matter of fact, one can rewrite Eq. (29) as follows:

$$\int_0^\infty d\gamma' \frac{g_b^{(Ld)}(\gamma', z; \kappa^2)}{[\gamma' + \gamma + z^2m^2 + (1-z^2)\kappa^2 - i\epsilon]^2} = \frac{g^2}{2(4\pi)^2} \int_0^\infty d\gamma' \int_{-1}^1 d\zeta' g_b^{(Ld)}(\gamma', \zeta'; \kappa^2)$$

$$\times \int_0^\infty d\gamma'' \frac{1}{[\gamma + \gamma'' + z^2m^2 + \kappa^2(1-z^2) - i\epsilon]^2}$$

$$\times \left[\frac{(1+z)}{(1+\zeta')} \theta(\zeta' - z) h'(\gamma'', z; \gamma', \zeta', \mu^2) + \frac{(1-z)}{(1-\zeta')} \theta(z - \zeta') h'(\gamma'', -z; \gamma', -\zeta', \mu^2) \right], \tag{30}$$

and after changing the name of the integration variable in the lhs, one gets

$$\int_0^\infty d\gamma'' \frac{1}{[\gamma'' + \gamma + z^2m^2 + (1-z^2)\kappa^2 - i\epsilon]^2} \left[g_b^{(Ld)}(\gamma'', z; \kappa^2) - \frac{g^2}{2(4\pi)^2} \int_0^\infty d\gamma' \int_{-1}^1 d\zeta' g_b^{(Ld)}(\gamma', \zeta'; \kappa^2) \right.$$

$$\times \left. \left(\frac{(1+z)}{(1+\zeta')} \theta(\zeta' - z) h'(\gamma'', z; \gamma', \zeta', \mu^2) + \frac{(1-z)}{(1-\zeta')} \theta(z - \zeta') h'(\gamma'', -z; \gamma', -\zeta', \mu^2) \right) \right]$$

$$= 0. \tag{31}$$

If the uniqueness theorem holds, then the ladder approximation of Eq. (24) reads

$$g_b^{(Ld)}(\gamma'', z; \kappa^2) = \int_0^\infty d\gamma' \int_{-1}^1 d\zeta' \mathcal{V}_b^{(Ld)}(\gamma'', z; \gamma', \zeta'; \kappa^2)$$

$$= \frac{\alpha m^2}{2\pi} \int_0^\infty d\gamma' \int_{-1}^1 d\zeta' g_b^{(Ld)}(\gamma', \zeta'; \kappa^2)$$

$$\times \left[\frac{(1+z)}{(1+\zeta')} \theta(\zeta' - z) h'(\gamma'', z; \gamma', \zeta', \mu^2) + \frac{(1-z)}{(1-\zeta')} \theta(z - \zeta') h'(\gamma'', -z; \gamma', -\zeta', \mu^2) \right], \tag{32}$$

where

$$\alpha = \frac{1}{16\pi} \left(\frac{g}{m} \right)^2$$

is an adimensional quantity, since in our model Lagrangian, $\mathcal{L}_{\text{int}} = g\Phi_a^\dagger \Phi_a \phi_b$, the coupling constant g has the dimension of a mass (as it must be for a ϕ^3 theory). It is worth noting that the kernel between the square brackets is symmetric with respect to the transformation $\{z, \zeta'\} \to \{-z, -\zeta'\}$. Moreover, $g_b(\gamma, z = \pm 1; \kappa^2) = 0$, given the presence of the theta functions and the vanishing value of

$$\frac{(1 \pm z)}{(1 \pm \zeta')} h'(\gamma'', \pm z; \gamma', \pm \zeta', \mu^2)$$

for $z \to \mp 1$, as discussed below Eq. (28) (see the second reference in [9–11] for a discussion of the Wick-Cutkosky model).

In Ref. [3], where a covariant framework was adopted for performing the needed analytic integrations, the uniqueness theorem for the Nakanishi weight function [12] was applied directly to the BSE, obtaining an eigenequation like Eq. (32) and a kernel quite involved. However, the kernel shown in Appendix C of Ref. [3] is more general than the one shown in Eqs. (32) and (26), since a renormalized (at one loop) propagator and a sum of exchanged scalars have been considered. Fortunately, the presented numerical results were evaluated in ladder approximation, as per the calculations shown in Ref. [6] where the ladder approximation of Eq. (23) was adopted. This motivated our investigation to only the ladder approximation for the time being. Indeed, the actual evaluation of the ladder kernel shown in Eq. (26) allows one to appreciate the well-known attractive feature of the LF framework to make less cumbersome the analytic integration, since the complexity of the calculation profitably distributes among two variables: k^+ and k^- (see, e.g., Ref. [24] for a simple discussion of the box diagram). Finally, it is important to emphasize that Eq. (32) is an eigenequation, with eigenvalue $1/\alpha$, once the mass M of the interacting system is assigned. Such a simple structure is a direct consequence of the linear dependence upon α of the kernel $i\mathcal{K}$, in ladder approximation. Differently, Eq. (29) is a generalized eigenequation (cf. Ref. [6]).

As to the γ dependence, we have already noted that for physical reasons [see Eqs. (17) and (18)], $g_b(z, \gamma; \kappa^2)$ must decrease for large values of γ. Moreover, one can check in ladder approximation that such a property is valid, since a constant $g_b^{(Ld)}(z, \gamma; \kappa^2)$ for $\gamma \to \infty$ leads to a different behavior for the lhs and rhs of Eq. (32). This can be seen by taking into account that in Eq. (27) the difference between the second term and the third one becomes vanishing for large γ'', and one remains with a $1/\gamma''$ falloff on the rhs, namely, the first term in Eq. (32), in contrast with the assumed constant behavior of $g_b^{(Ld)}(z, \gamma''; \kappa^2)$.

IV. LF MOMENTUM DISTRIBUTIONS

It is attractive to perform numerical comparisons that in perspective could be useful for an experimental investigation of actual interacting systems. In view of this, it is very interesting to consider (see the next section for the actual calculations) the LF distributions of the valence component [cf. Eqs. (16) and (20)]. As shown below, those distributions can be evaluated through $g_b(z, \gamma; \kappa^2)$. Moreover, the normalization of the valence component, once the BS amplitude itself is properly normalized (see Appendix B for a short review of this issue and Refs. [9–11,29] for details), yields the probability to find the valence contribution in the Fock expansion of the interacting two-scalar state (see, e.g., Refs. [26,30]), viz.,

$$P_{\text{val}} = \frac{1}{(2\pi)^3} \int_0^1 \frac{d\xi}{2\xi(1-\xi)} \int d\mathbf{k}_\perp \psi_{n=2/p}^2(\xi, k_\perp)$$

$$= \frac{1}{(16\pi)^2} \int_{-1}^1 dz(1-z^2) \int_0^\infty d\gamma \left[\int_0^\infty d\gamma' \frac{g_b(\gamma', z; \kappa^2)}{[\gamma' + \gamma + z^2 m^2 + (1-z^2)\kappa^2]^2} \right]^2, \quad (33)$$

where Eq. (16) has been inserted in the last step, $\xi = (1-z)/2$ and $d\mathbf{k}_\perp = d\phi d\gamma/2$. It should be reminded that $P_{\text{val}} \equiv N_2$, which is given in Eq. (15).

As is well known, one can define the probability distribution to find a constituent with LF longitudinal fraction $\xi = p_i^+/P^+$ in the valence state as follows:

$$\phi(\xi) = \frac{1}{(2\pi)^3} \frac{1}{2\xi(1-\xi)} \int d\mathbf{k}_\perp \psi_{n=2/p}^2(\xi, k_\perp)$$

$$= 2 \frac{(1-z^2)}{(16\pi)^2} \int_0^\infty d\gamma \left[\int_0^\infty d\gamma' \frac{g_b(\gamma', z; \kappa^2)}{[\gamma' + \gamma + z^2 m^2 + (1-z^2)\kappa^2]^2} \right]^2, \quad (34)$$

with the obvious normalization $\int_0^1 d\xi \phi(\xi) = P_{\text{val}}$. Furthermore, one can consider the probability distribution in $\gamma = |\mathbf{k}_\perp|^2$, i.e.,

$$\mathcal{P}(\gamma) = \frac{1}{2(2\pi)^3} \int_0^1 \frac{d\xi}{2\xi(1-\xi)} \int_0^{2\pi} d\varphi \, \psi_{n=2/p}^2(\xi, k_\perp)$$

$$= \frac{1}{(16\pi)^2} \int_{-1}^1 dz(1-z^2) \left[\int_0^\infty d\gamma' \frac{g_b(\gamma', z; \kappa^2)}{[\gamma' + \gamma + z^2 m^2 + (1-z^2)\kappa^2]^2} \right]^2, \quad (35)$$

with the normalization $\int_0^\infty d\gamma \mathcal{P}(\gamma) = P_{\text{val}}$.

Two final remarks are in order. First, let us remind that for $\mu \to 0$ and $n = 2$ in Eq. (3), the Nakanishi amplitude factorizes as $g_b(\gamma', z; \kappa^2) \to \delta(\gamma')f(z; \kappa^2)$ (see, e.g., Ref. [30]), and, therefore, in the Wick-Cutkosky model, one gets

$$\psi_{n=2/p}^{WiC}(\xi, k_\perp) \propto \frac{f(z; \kappa^2)}{[\gamma + z^2 m^2 + (1 - z^2)\kappa^2]^2}. \quad (36)$$

Second, we would emphasize that the valence wave function behaves as expected (see Ref. [26]) for large values of $k_\perp^2 = \gamma$ once we choose $n = 2$. As a matter of fact, the Nakanishi weight function drops out for increasing γ', and one has for $\gamma \to \infty$,

$$\psi_{n=2/p}(\xi, k_\perp) = \frac{(1 - z^2)}{4\sqrt{2}} \int_0^\infty d\gamma' \frac{g_b(\gamma', z; \kappa^2)}{[\gamma' + \gamma + z^2 m^2 + (1 - z^2)\kappa^2]^2}$$
$$\to \frac{C(z)}{\gamma^2}, \quad (37)$$

with a γ tail independent of the mass of the exchanged scalar.

In the next section, the numerical results of the LF distributions obtained in ladder approximation are presented. We can anticipate that such LF distributions evaluated by using the solutions of Eqs. (29)) and (32) for a given mass of the exchanged meson and binding energy overlap, though the numerical Nakanishi weight functions $g_b^{(Ld)}(\gamma', z; \kappa^2)$ show few-percent differences for low values of γ, as discussed in what follows.

V. NUMERICAL COMPARISONS

In order to implement the quantitative studies of the Nakanishi weight function for the S-wave BS amplitude of a two-scalar system with a massive scalar exchange, we have adopted a proper basis. This basis allows us to expand the nonsingular weight function by taking into account the features of $g_b^{(Ld)}(\gamma, z; \kappa^2)$ discussed in Secs. II and III, namely, (i) the symmetry with respect to z, (ii) the constraint $g_b^{(Ld)}(\gamma, z = \pm 1; \kappa^2) = 0$, and (iii) the falloff in γ. In particular, Gegenbauer polynomials with proper indexes have been chosen for describing the z dependence, while the Laguerre polynomials have been adopted for the γ dependence. In short, we have expanded the Nakanishi weight function as follows:

$$g_b^{(Ld)}(\gamma, z; \kappa^2) = \sum_{\ell=0}^{N_z} \sum_{j=0}^{N_g} A_{\ell j} G_\ell(z) \mathcal{L}_j(\gamma), \quad (38)$$

where (i) the functions $G_\ell(z)$ are given in terms of even Gegenbauer polynomials $C_{2\ell}^{(5/2)}(z)$ by

$$G_\ell(z) = 4(1 - z^2)\Gamma(5/2)\sqrt{\frac{(2\ell + 5/2)(2\ell)!}{\pi\Gamma(2\ell + 5)}} C_{2\ell}^{(5/2)}(z), \quad (39)$$

and (ii) the functions $\mathcal{L}_j(\gamma)$ are expressed in terms of the Laguerre polynomials $L_j(a\gamma)$ by

$$\mathcal{L}_j(\gamma) = \sqrt{a}\, L_j(a\gamma)e^{-a\gamma/2}. \quad (40)$$

The following orthonormality conditions are fulfilled:

$$\int_{-1}^1 dz\, G_\ell(z)G_n(z) = \delta_{\ell n},$$
$$\int_0^\infty d\gamma \mathcal{L}_j(\gamma)\mathcal{L}_\ell(\gamma) = a \int_0^\infty d\gamma e^{-a\gamma} L_j(a\gamma)L_\ell(a\gamma) = \delta_{j\ell}. \quad (41)$$

In order to speed up the convergence, in the actual calculations the parameter $a = 6.0$ has been adopted, and the variable γ has been rescaled according to $\gamma \to 2\gamma/a_0$ with $a_0 = 12$. It is worth noting that the two parameters a and a_0 control, loosely speaking, the range of relevance of the Laguerre polynomials and the structure of the kernel, respectively. Finally, the integration over the variable z has been performed by using a Gauss-Legendre quadrature rule, while the Gauss-Laguerre quadrature has been adopted for the variable γ.

A. Eigenvalues and eigenvectors

We have first solved Eq. (29), i.e., the one proposed in Ref. [6], but using our basis instead of the spline basis adopted there. With the spline basis for both z and γ, some instabilities appear, and in Ref. [6] a small parameter was introduced to achieve stable results (see also below). Our basis allows us to overcome such a problem, since it contains the above-mentioned general features of $g_b^{(Ld)}(\gamma, z = \pm 1; \kappa^2)$. This first step was necessary to gain confidence in our basis through the comparison with the results in Ref. [6] (see what follows). As a second step, we evaluated the eigenvalues and eigenvectors of Eq. (33), which was deduced by invoking the uniqueness theorem. As for this equation, it should be pointed out that a completely different numerical method was chosen in Ref. [3]. In particular, it was applied an iterative procedure, suggested by the structure of the ladder kernel obtained in [3].

In the following tables, a detailed comparison between our results and the ones obtained in Refs. [3,6] is presented. Let us remind that in Ref. [3], though the proposed ladder kernel contains dressed propagators and a sum of exchanged meson, the numerical evaluations were performed without such extras, and, therefore, their results can be directly compared to ours and the ones in Ref. [6], with only the caveat of a different definition of the coupling constant α. As already pointed out in Refs. [3,6], the kernel contains a highly nonlinear

TABLE I. Values of $\alpha = g^2/(16\pi m^2)$ obtained by solving the eigenequations (29) and (32) (i.e., the eigenequation with the application of the uniqueness theorem). Results correspond to $\mu/m = 0.15$, 0.50 varying the binding energies, B/m. The second column contains the results obtained in Ref. [6] by using the spline basis and Eq. (29); the third column shows our results obtained from Eq. (29) by using our basis [Eqs. (38)–(40)] with $N_z = 18$, $N_g = 32$, and $a = 6$ in Eq. (40); the fourth column contains our results obtained from the eigenequation (32) and our basis.

	$\mu/m = 0.15$		
B/m	α [6]	α Eq. (29)	α Eq. (32)
0.01	0.5716	0.5716	0.5716[a]
0.10	1.437	1.437	1.437
0.20	2.100	2.099	2.099
0.50	3.611	3.610	3.611
1.00	5.315	5.313	5.314

	$\mu/m = 0.50$		
B/m	α [6]	α Eq. (29)	α Eq. (32)
0.01	1.440	1.440	1.440
0.10	2.498	2.498	2.498
0.20	3.251	3.251	3.251
0.50	4.901	4.901	4.901
1.00	6.712	6.711	6.711

[a]For $\mu/m = 0.15$ and $B/m = 0.01$, the stability of the coupling constant ($\alpha < 1$) is reached for $N_g \geq 46$.

dependence upon the mass M of the interacting system, but a linear dependence upon the coupling constant α, given the adopted ladder approximation. Therefore, it is customary (i) first to choose a value for the binding energy in the interval

$$0 \leq \frac{B}{m} = 2 - \frac{M}{m} \leq 2,$$

TABLE II. Values of $\alpha = g^2/(16\pi m^2)$ obtained by solving the eigenequations (32) (i.e., with the application of the uniqueness theorem) and (29). Results correspond to $\mu/m = 0.50$ varying the binding energies, B/m. The second column shows the values obtained in Ref. [3], where the uniqueness theorem was exploited and an iterative method was adopted; the third column corresponds to the solution of Eq. (32) by using our basis [cf. Eqs. (38)–(40)]; the fourth column contains our results from Eq. (29).

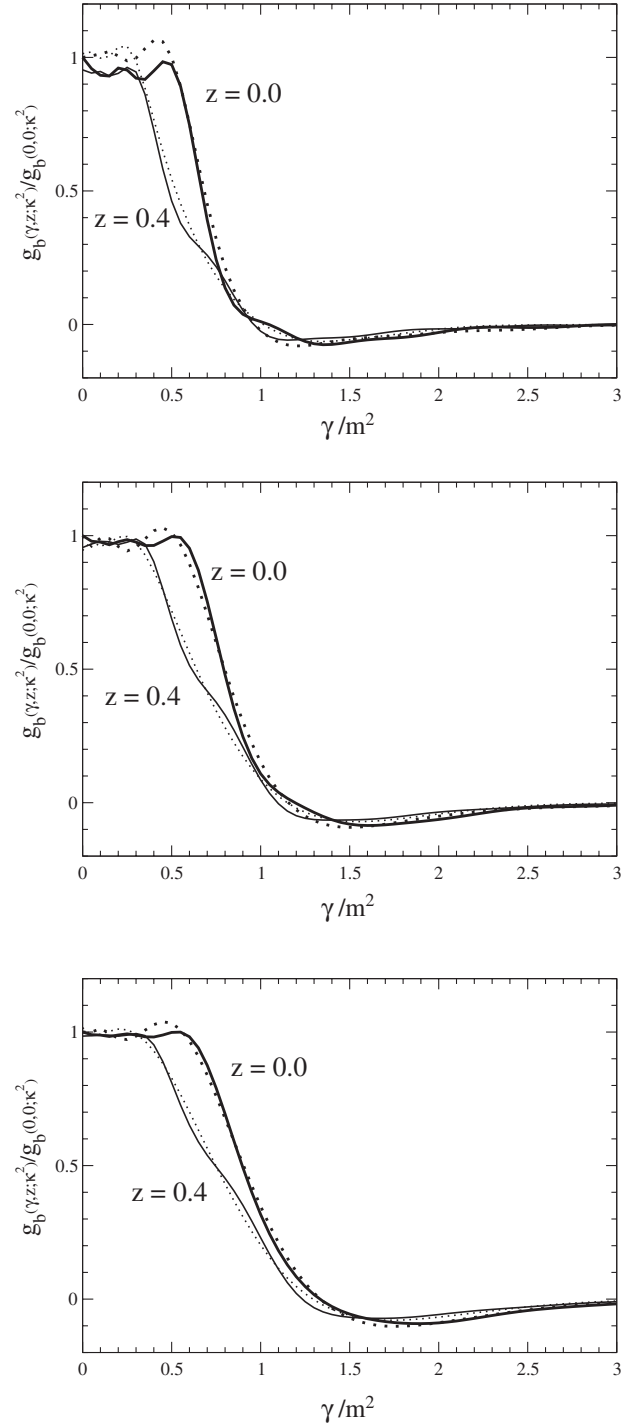|	$\mu/m = 0.50$		
B/m	α[3]	α Eq. (32)	α Eq. (29)
0.002	1.211	1.216	1.216
0.02	1.624	1.623	1.623
0.20	3.252	3.251	3.251
0.40	4.416	4.415	4.416
0.80	6.096	6.094	6.094
1.20	7.206	7.204	7.204
1.60	7.850	7.849	7.849
2.00	8.062	8.061	8.061

FIG. 1. The Nakanishi weight function $g_b^{(Ld)}(\gamma, z; \kappa^2)$ for $\mu/m = 0.5$ and $B/m = 0.2$, 0.5, 1.0 (from the top) vs γ/m^2 and two values of z. Thick lines refer to $z = 0$ and thin lines to $z = 0.4$, as indicated by the inset. Solid lines: results from Eq. (32). Dotted lines: results from Eq. (29).

and (ii) then to look for the minimal value of the coupling constant that allows such a binding energy. A comment on the range of the usually chosen interval is in order. As is well known (see Ref. [31]), all the ϕ^3 models do not show any
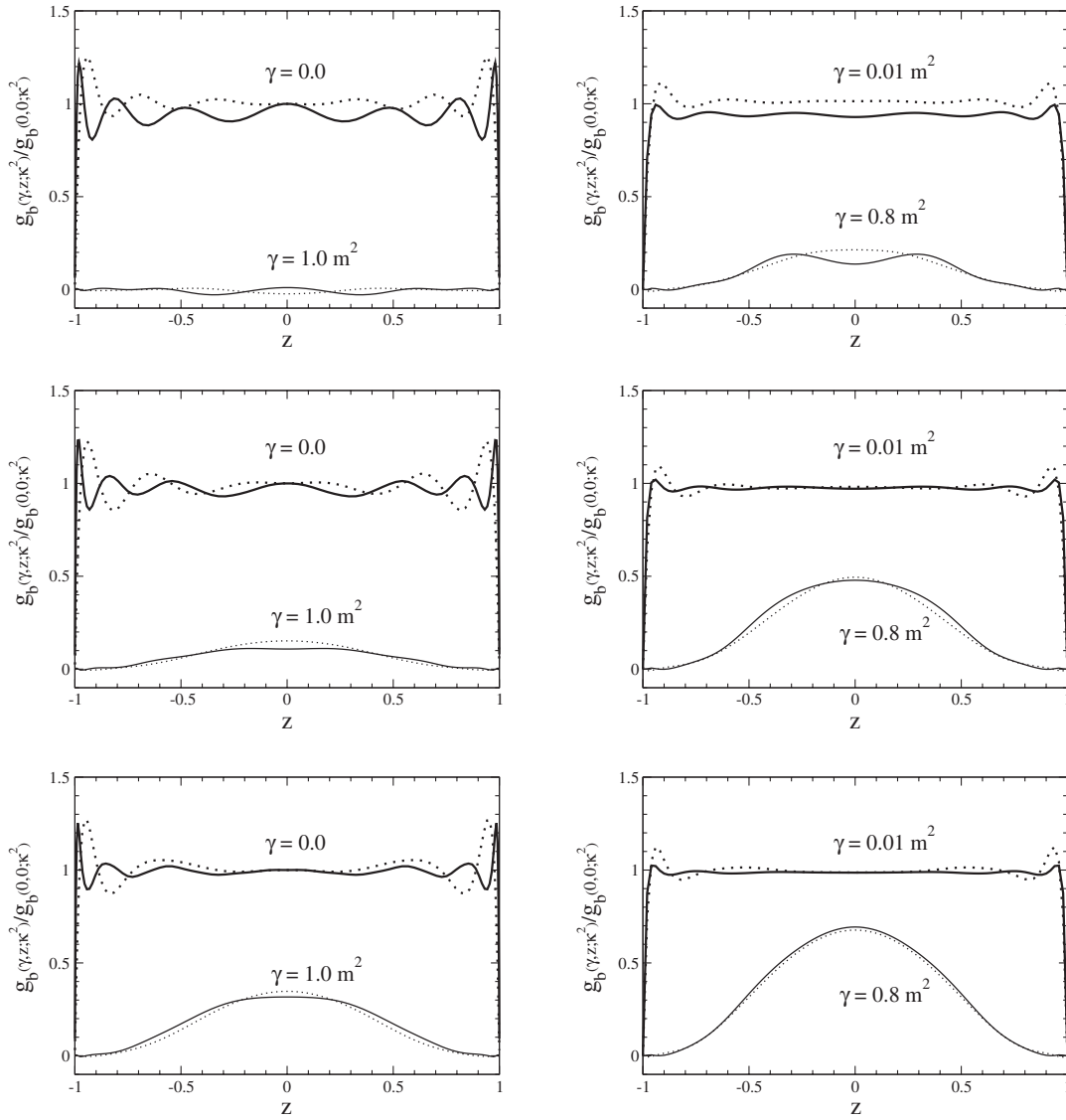
FIG. 2. The Nakanishi weight function $g_b^{(Ld)}(\gamma, z; \kappa^2)$ for $\mu/m = 0.5$ and $B/m = 0.2, 0.5, 1.0$ (from the top) vs z and four values of γ/m^2. Thick lines refer to $\gamma = 0$ and $\gamma = 0.01m^2$, while thin lines to $\gamma = 0.8m^2$ and $\gamma = 1m^2$, as indicated by the inset. Solid lines: results from Eq. (32). Dotted lines: results from Eq. (29).

ground state; nonetheless, they are widely adopted for illustrative purposes and for gaining insights into the effectiveness of theoretical tools. Here, we also adhere to this general attitude (see Ref. [32] for some details on how and to what extent it is possible to reconcile the general features of the ϕ^3 models and the actual calculations). After introducing a basis, it should be noticed that in the case of Eq. (29), one has a generalized eigenvalue problem (cf. Ref. [6]) that in a symbolic form reads

$$\frac{1}{\alpha} B(M) g^{(Ld)} = A^{(Ld)}(M) g(Ld), \qquad (42)$$

while for Eq. (32), one has a genuine eigenvalue problem, viz.,

$$\frac{1}{\alpha} g = D^{(Ld)}(M) g(Ld). \qquad (43)$$

The possibility to reduce the first problem to the second one relies on the existence of the inverse of the integral operator $B(M)$ and the numerical feasibility of such inversion with enough accuracy. In particular, in Ref. [6], where the spline basis was adopted, a small parameter was added to the matrix $B(M)$ in order to achieve a good stability. We have investigated if adopting our basis, which includes the expected falloff of the weight function for large values of γ, one has to similarly introduce a small parameter. Fortunately, with our basis, the small quantity to be added to the diagonal terms of $A^{(Ld)}(M)$ is $\epsilon = 10^{-9}$ [the largest number of Gaussian

TABLE III. Values of P_{val} Eq. (33) evaluated by using the weight function $g_b^{(Ld)}(\gamma, z; \kappa^2)$ corresponding to Eq. (32) (i.e., with the application of the uniqueness theorem) are shown for three values of μ/m and varying the binding energy, B/m. Notice that for $B/m = 0.001$, the values $N_z = 16$, $N_g = 48$, and $a = 12$ have been adopted in Eqs. (38) and (40) for obtaining a better convergence.

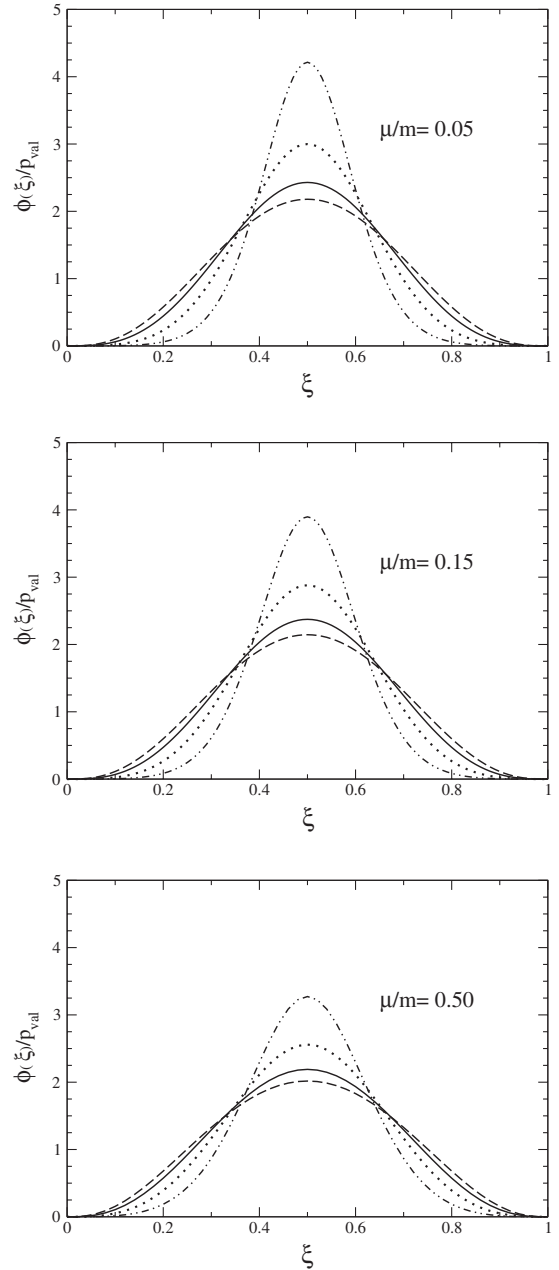| $\mu/m = 0.05$		
B/m	α	P_{val}
0.001	0.1685	0.94
0.01	0.3521	0.89
0.10	1.191	0.75
0.20	1.850	0.72
0.50	3.358	0.68
1.00	5.056	0.66
2.00	6.336	0.65
$\mu/m = 0.15$		
B/m	α	P_{val}
0.001	0.3667	0.97
0.01	0.5716	0.94
0.10	1.437	0.80
0.20	2.099	0.75
0.50	3.611	0.70
1.00	5.314	0.67
2.00	6.598	0.66
$\mu/m = 0.50$		
B/m	α	P_{val}
0.001	1.167	0.98
0.01	1.440	0.96
0.10	2.498	0.87
0.20	3.251	0.83
0.50	4.900	0.77
1.00	6.711	0.74
2.00	8.061	0.72

FIG. 3. The longitudinal LF distribution $\phi(\xi)$ for the valence component Eq. (34) vs the longitudinal-momentum fraction ξ for $\mu/m = 0.05$, 0.15, 0.50. Dash-double-dotted line: $B/m = 0.20$. Dotted line: $B/m = 0.50$. Solid line: $B/m = 1.0$. Dashed line: $B/m = 2.0$. Recall that $\int_0^1 d\xi \phi(\xi) = P_{val}$ (cf. Table III).

points was 80 for each variable in $g^{(Ld)}$]. As for Eq. (32), one has been able to get rid of the numerical inversion of the matrix, since, *de facto*, it has been mathematically performed. Finally, it is important noticing that for both equations, the involved matrices are real but not symmetric, and, therefore, pairs of complex eigenvalues can appear.

In order to achieve a very good convergence for both eigenvalues and eigenvectors [in particular, for Eq. (32)], the numerical studies with the basis in Eqs. (38)–(40) has been extended up to $N_z = 18$ and $N_g = 32$, for all the values of B/m, except for $B/m = 0.01$ where we extend N_g up to 48. Indeed, for $B/m \geq 0.1$ a nice stability of the eigenvalues can be reached already for $N_z = 8$ and $N_g > 24$. In general, the stability of the eigenvalues settles well before that the convergence of the eigenvectors.

In Table I, the results for the coupling constant α corresponding to Eqs. (29) and (32) for $\mu/m = 0.15$, 0.50 and a set of binding energies B/m are shown. In particular, in the second column, the results obtained in Ref. [6] by using the spline basis are reported, while our results corresponding to both Eqs. (29) and (32) are presented in the third column and the fourth one, respectively. It is important to note that for $B/m = 0.01$ and $\mu/m = 0.15$, the stability of the eigenvalue obtained through Eq. (32) is reached with $N_g \geq 46$, when $a = 6$ is chosen, while $N_g = 28$ is enough when $a = 12$
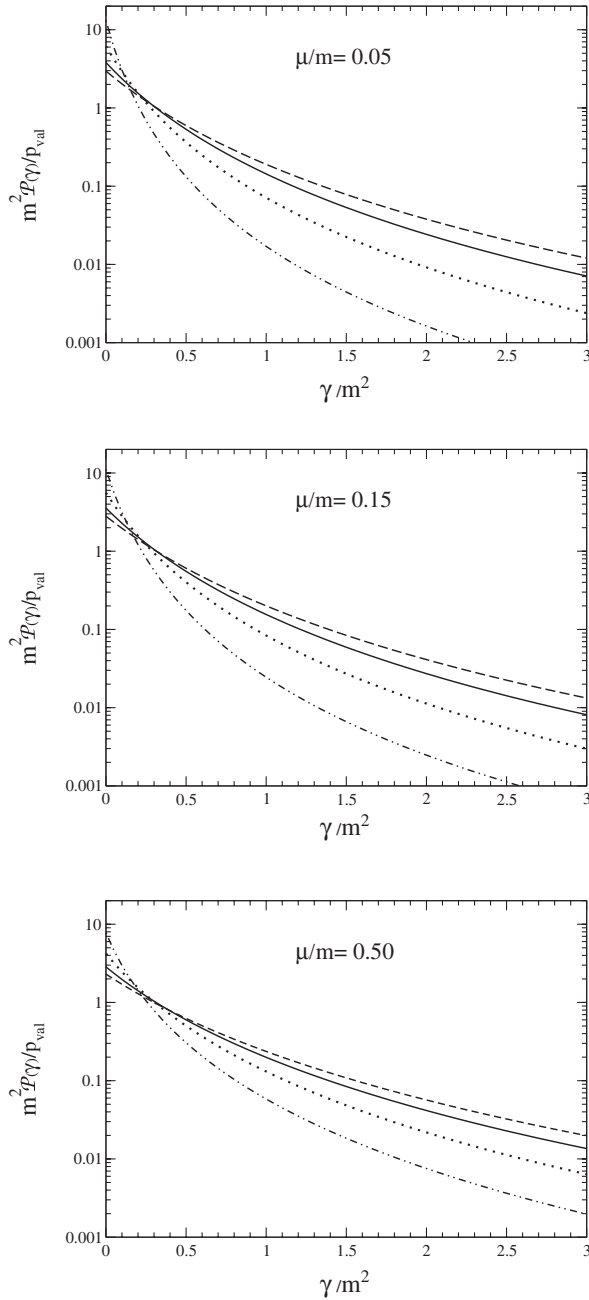
FIG. 4. The transverse LF distribution $\mathcal{P}(\gamma)$ for the valence component Eq. (35) vs the adimensional variable γ/m^2, for $\mu/m = 0.05$, 0.15, 0.50. Dash-double-dotted line: $B/m = 0.20$. Dotted line: $B/m = 0.50$. Solid line: $B/m = 1.0$. Dashed line: $B/m = 2.0$. Recall that $\gamma = k_\perp^2$ and $\int_0^\infty d\gamma \mathcal{P}(\gamma) = P_{\text{val}}$ (cf. Table III).

is adopted (with this value for a, the convergence of the eigenvectors is not satisfactory for $N_g = 28$).

Table II shows the comparison with the results from Ref. [3]. It should be pointed out that in Ref. [3], only the value $\mu/m = 0.50$ was considered, and the coupling constant contained an extra factor π with respect to the definition adopted in the present paper and in Ref. [6]. It is

important to remind that the eigenvalues shown in Ref. [3] compared very favorably with the ones obtained in Ref. [33], where the BS equation in ladder approximation was solved in Euclidean space. Moreover, one can find in Refs. [34,35] more evaluations both within the LF Hamiltonian dynamics and in Euclidean space, which appear in nice agreement with our calculations.

Finally, it should be pointed that all the digits of our results presented in the tables are stable, and the numerical uncertainties affect only the digit beyond the ones shown, at the level of a few units.

In Figs. 1 and 2, the comparison between the weight functions obtained from Eqs. (29)) and (31) is shown for $\mu/m = 0.50$ and $B/m = 0.2$, 0.5, 1.0. Few-percent differences appear for small values of γ and are bigger for small values of the binding energy. In this case, the characteristic momentum associated with the weak-binding energy is much smaller than the mass scale of the system, and, therefore, to appropriately describe the Nakanishi weight function, one should use a larger basis which accurately spans both the small and large momentum regions. This demands more numerical efforts that can be postponed, since our present aim is to validate the Nakanishi approach over the largest range of dynamical regimes, which can be covered by the basis we have chosen (see, e.g., Table I and $B/m = 0.01$ and $\mu/m = 0.15$). Notably, the above-mentioned differences do not have any sizable effect on the LF distributions (see the next subsection).

As a further check, we evaluated the solution of Eq. (29) corresponding to $\mu/m = 0.50$ and $B/m = 1.0$, by introducing a small parameter as in Ref. [6]. In particular, we adopted $\epsilon = 10^{-4}$ for comparing with the weight function presented in Figs. 2 and 3 in Ref. [6], and we obtained the same results. It is worth noting that also by adopting the small parameter $\epsilon = 10^{-4}$, we did not find any sizable effects on the LF distributions.

B. Valence probability and LF distributions

After determining the expansion coefficients of the Nakanishi weight function, as given in Eq. (38), and imposing the normalization condition on the BS amplitude, Eq. (B14)), one can calculate the valence component of the interacting system, Eq. (16). Then, very interesting (in particular, from the phenomenological point of view) quantities can be evaluated. First of all, the valence probability Eq. (33) can be obtained. The results are shown in Table III for $\mu/m = 0.05$, $\mu/m = 0.15$, and $\mu/m = 0.5$. Several values of B/m have been chosen for covering the interval $0 < B/m \leq 2$. It should be recalled that the asymptotic value $P_{\text{val}} = 1$ reached for $B/m \to 0$ is more and more closely approached for smaller and smaller values of B/m (or equivalently smaller values of α) when μ/m decreases. Since in Table III there are also the results for

$\mu/m = 0.05$, a by-product of these calculations is the following interesting remark. For $B/m = 2$ and decreasing μ/m, the values of α show a decreasing behavior toward $\alpha = 2\pi$, namely, the value of α obtained in the Wick-Cutkosky case, i.e., $\mu/m = 0$ (cf. Ref. [30]). Correspondingly, the valence probability approaches the Wick-Cutkosky value $P_{\text{val}} \sim 0.64$ [30].

In Figs. 3 and 4, the valence LF distributions given in Eqs. (34) and (35) are shown for $\mu/m = 0.05, 0.15, 0.5$, and $B/m = 0.2, 0.5, 1.0, 2.0$. The curves correspond to the eigenvectors of Eq. (32), since the ones obtained from Eq. (29) completely overlap with the previous ones, though the weight functions have differences at low values of γ. It should be pointed out that the valence wave function, the main ingredient for calculating the LF distributions, is obtained from the weight function by applying the integral operator symbolically indicated by $B(M)$ in Eq. (42). This eliminates the above-mentioned instabilities that are possibly produced by the inversion of $B(M)$.

VI. CONCLUSION

We have quantitatively investigated the ladder Bethe-Salpeter equation in Minkowski space within the perturbation theory integral representation of the multileg transition amplitudes proposed by Nakanishi in the 1960s [9,12]. The formal analysis leading to the determination of the Nakanishi weight function takes a great benefit from the light-front framework, as shown in Ref. [6] for the bound states and in Ref. [18] for the scattering states. In particular, if one exploits both (i) the equation obtained from BSE for the valence component of the Fock expansion of the interacting-system state and (ii) the Nakanishi theorem [12] on the uniqueness of the nonsingular weight function related to the vertex function in PTIR, one can obtain

Eq. (32). This equation and Eq. (29) allow the numerical evaluation of the weight function corresponding to a given value of the binding energy of the interacting system and the exchanged-boson mass. We have shown that the eigenvalues and the eigenvectors obtained by solving Eqs. (32) and (29) can be substantially taken as the same (the eigenvectors differ at the level of few-percent for $\gamma \to 0$). In particular, if one considers only the phenomenological observables, i.e., eigenvalues and LF distributions, the outcomes of both equations can be even taken as equal. This gives us great confidence in the validity of the uniqueness theorem also in a nonperturbative regime.

An important feature of our analysis is represented by the basis we have chosen for expanding the weight function. Such a basis is able to include all the general properties of the weight function, allowing a good control on the instabilities, that, in principle, could plague the numerical solutions of the above equations.

In perspective, the numerical analysis we have performed appears very encouraging, and it makes compelling the next steps represented by the evaluation of observables related to the scattering states, like the scattering length and the inclusion of the crossed-box diagrams, as already done in Ref. [7], but exploiting the Nakanishi uniqueness theorem.

APPENDIX A: A NEW FORM FOR THE LF KERNEL IN LADDER APPROXIMATION

This appendix contains the details for obtaining the expression of the kernel in Eq. (26), $V_b^{(Ld)}$, which is suitable for applying the uniqueness theorem for the Nakanishi weight function.

In a reference frame where $\mathbf{p}_\perp = 0$ and $p^\pm = M$, the kernel in ladder approximation is (cf. Ref. [18])

$$
\begin{aligned}
V_b^{(Ld)}(\gamma, z; \gamma', z') &= -g^2 p^+ \int \frac{d^4}{(2\pi)^4} \frac{1}{[k''^2 + p \cdot k''z' - \gamma' - \kappa^2 + i\epsilon]^3} \\
&\times \int_{-\infty}^{\infty} \frac{dk^-}{2\pi} \frac{1}{[(\frac{p}{2}+k)^2 - m^2 + i\epsilon]} \frac{1}{[(\frac{p}{2}-k)^2 - m^2 + i\epsilon]} \frac{1}{(k-k'')^2 - \mu^2 + i\epsilon} \\
&= \frac{g^2}{2(4\pi)^2} \frac{1}{[\gamma + (1-z^2)\kappa^2 + z^2 m^2 - i\epsilon]} \int_0^1 dv\, v^2 \mathcal{F}(v, \gamma, z; \gamma', \zeta'),
\end{aligned}
\tag{A1}
$$

where

$$
\begin{aligned}
\mathcal{F}(v, \gamma, z; \gamma', \zeta, \zeta') &= \frac{(1+z)^2}{X^2(v, \zeta, \zeta')} \frac{\theta(\zeta' - z)}{[\gamma + z^2 m^2 + \kappa^2(1 - z^2) + \Gamma(v, z, \zeta', \gamma') - i\epsilon]^2} \\
&+ \frac{(1-z)^2}{X^2(v, \zeta, -\zeta')} \frac{\theta(z - \zeta')}{[\gamma + z^2 m^2 + \kappa^2(1 - z^2) + \Gamma(v, -z, -\zeta', \gamma') - i\epsilon]^2},
\end{aligned}
\tag{A2}
$$

with

$$X(v, \zeta, \zeta') = v(1-v)(1+\zeta')$$

$$\Gamma(v, z, \zeta', \gamma') = \frac{(1+z)}{(1+\zeta')} \left\{ \frac{v}{(1-v)} \left[\zeta'^2 \frac{M^2}{4} + \kappa^2(1+\zeta^2) + \gamma' \right] + \frac{\mu^2}{v} + \gamma' \right\}. \tag{A3}$$

The previous expression coincides with the one in Ref. [6].

For combining the denominators in the last line of Eq. (A1) and in Eq. (A2) the standard Feynman trick can be used, viz.,

$$\frac{1}{BA^2} = \lim_{\lambda \to 0^+} \frac{1}{\lambda} \left[\frac{1}{BA} - \frac{1}{B(A+\lambda)} \right] = \lim_{\lambda \to 0^+} \frac{1}{\lambda} \left\{ \int_0^1 d\xi \frac{1}{[B - \xi(B-A)]^2} - \int_0^1 d\xi \frac{1}{[B - \xi(B-A) + \xi\lambda]^2} \right\}, \tag{A4}$$

with

$$A = \gamma + z^2 m^2 + \kappa^2(1-z^2) + \Gamma(v, \pm z, \pm\zeta', \gamma') - i\epsilon,$$

$$B = \gamma + z^2 m^2 + \kappa^2(1-z^2) - i\epsilon, \tag{A5}$$

obtaining the following expression

$$V_b^{(Ld)}(\gamma, z; \gamma', \zeta') = -\frac{g^2}{2(4\pi)^2} \left[\frac{(1+z)}{(1+\zeta')} \theta(\zeta' - z)\mathcal{H}'(\gamma, z; \gamma', \zeta', \mu^2) + \frac{(1-z)}{(1-\zeta')} \theta(z - \zeta')\mathcal{H}'(\gamma, -z; \gamma', -\zeta', \mu^2) \right], \tag{A6}$$

where

$$\mathcal{H}'(\gamma, z; \gamma', \zeta'\mu^2) = \lim_{\lambda \to 0^+} \frac{1}{\lambda} [\mathcal{H}(\gamma, z; \gamma', \zeta', \mu^2, \lambda) - \mathcal{H}(\gamma, z; \gamma', \zeta', \mu^2, 0)], \tag{A7}$$

with

$$\mathcal{H}(\gamma, z; \gamma', \zeta', \mu^2, \lambda) = \frac{(1+z)}{(1+\zeta')} \int_0^1 \frac{dv}{(1-v)^2} \int_0^1 d\xi \int_{-\infty}^\infty d\gamma'' \frac{\delta[\gamma'' - \xi\Gamma(v, z, \zeta', \gamma') - \xi\lambda]}{[\gamma + \gamma'' + z^2 m^2 + \kappa^2(1-z^2) - i\epsilon]^2}. \tag{A8}$$

The positivity of γ' [cf. Eq. (22)] entails the positivity of $\Gamma(v, z, \zeta', \gamma')$ and, eventually, of γ''. Given the linear dependence upon ξ in the delta function, one can productively perform first the integration on ξ, obtaining

$$\mathcal{H}(\gamma, z; \gamma', \zeta', \mu^2, \lambda) = \int_{-\infty}^\infty d\gamma'' \frac{\theta(\gamma'')h(\gamma'', z; \gamma', \zeta', \mu^2, \lambda)}{[\gamma + \gamma'' + z^2 m^2 + \kappa^2(1-z^2) - i\epsilon]^2}, \tag{A9}$$

where

$$h(\gamma'', z; \gamma', \zeta', \mu^2, \lambda) = \frac{(1+z)}{(1+\zeta')} \int_0^1 \frac{dv}{(1-v)^2} \int_0^1 d\xi \delta[\gamma'' - \xi\Gamma(v, z, \zeta', \gamma') - \xi\lambda]$$

$$= \frac{(1+z)}{(1+\zeta')} \int_0^1 \frac{dv}{(1-v)^2} \frac{\theta(\Gamma(v, z, \zeta', \gamma') - \gamma'' + \lambda)}{\Gamma(v, z, \zeta', \gamma') + \lambda}, \tag{A10}$$

since ξ must belong to the interval [0,1]. The derivative of $h(\gamma'', z; \gamma', \zeta', \mu^2, \lambda)$ implied by Eq. (A7) is given by

$$h'(\gamma'', z; \gamma', \zeta', \mu^2) = \frac{(1+z)}{(1+\zeta')} \frac{1}{\gamma''} \int_0^1 \frac{dv}{(1-v)^2} \delta(\Gamma(v, z, \zeta', \gamma') - \gamma'') - \frac{(1+z)}{(1+\zeta')} \int_0^1 \frac{dv}{(1-v)^2} \frac{\theta(\Gamma(v, z, \zeta', \gamma') - \gamma'')}{[\Gamma(v, z, \zeta', \gamma')]^2}$$

$$= \frac{(1+z)}{(1+\zeta')} \frac{1}{\gamma''} \int_0^\infty dy \, \delta(\Gamma(y, z, \zeta', \gamma') - \gamma'') - \frac{(1+z)}{(1+\zeta')} \int_0^\infty dy \frac{\theta(\Gamma(y, z, \zeta', \gamma') - \gamma'')}{[\Gamma(y, z, \zeta', \gamma')]^2}$$

$$= \frac{1}{\gamma''} \int_0^\infty dy \, \delta\left[\frac{1}{y}(y^2 \mathcal{A}_b(\zeta', \gamma', \kappa^2) + y \mathcal{B}_b(z, \zeta', \gamma', \gamma'', \mu^2) + \mu^2)\right]$$

$$- \frac{(1+\zeta')}{(1+z)} \int_0^\infty dy \, y^2 \frac{\theta[y^2 \mathcal{A}_b(\zeta', \gamma', \kappa^2) + y \mathcal{B}_b(z, \zeta', \gamma', \gamma'', \mu^2) + \mu^2]}{[y^2 \mathcal{A}_b(\zeta', \gamma', \kappa^2) + y(\mu^2 + \gamma') + \mu^2]^2}, \tag{A11}$$

where the transformation $v \to y/(1+y)$ has been performed. In Eq. (A11), the function $\Gamma(y, z, \zeta', \gamma')$ is given by

$$\Gamma(y, z, \zeta', \gamma') = \frac{(1+z)}{(1+\zeta')} \left\{ y\left[\zeta'^2 \frac{M^2}{4} + \kappa^2 + \gamma'\right] + \frac{1+y}{y}\mu^2 + \gamma' \right\} = \frac{(1+z)}{(1+\zeta')} \frac{1}{y}\{y^2 \mathcal{A}_b(\zeta', \gamma', \kappa^2) + y(\mu^2 + \gamma') + \mu^2\} \tag{A12}$$

and

$$\mathcal{A}_b(\zeta', \gamma', \kappa^2) = \zeta'^2 \frac{M^2}{4} + \kappa^2 + \gamma' \geq 0$$

$$\mathcal{B}_b(z, \zeta', \gamma', \gamma'', \mu^2) = \mu^2 + \gamma' - \gamma'' \frac{(1+\zeta')}{(1+z)}. \tag{A13}$$

The two contributions to $h'(\gamma'', z; \gamma', \zeta', \mu^2)$ will be discussed separately in what follows. The first term is

$$I_1 = \frac{1}{\gamma''} \int_0^\infty dy \, \delta\left[\frac{1}{y}(y^2 \mathcal{A}_b(\zeta', \gamma', \kappa^2) + y \mathcal{B}_b(z, \zeta', \gamma', \gamma'', \mu^2) + \mu^2)\right]$$

$$= \frac{1}{\gamma''} \int_0^\infty dy \left[\theta(y_+)y_+ \frac{\delta(y-y_+)}{\mathcal{A}_b(\zeta', \gamma', \kappa^2)|y_+ - y_-|} + \theta(y_-)y_- \frac{\delta(y-y_-)}{\mathcal{A}_b(\zeta', \gamma', \kappa^2)|y_+ - y_-|}\right] \theta(\Delta^2(z, \zeta', \gamma', \gamma'', \kappa^2, \mu^2)), \tag{A14}$$

where y_i are the two solutions of

$$y^2 \mathcal{A}_b(\zeta', \gamma', \kappa^2) + y \mathcal{B}_b(z, \zeta', \gamma', \gamma'', \mu^2) + \mu^2 = 0, \tag{A15}$$

namely,

$$y_\pm = \frac{1}{2\mathcal{A}_b(\zeta', \gamma', \kappa^2)}[-\mathcal{B}_b(z, \zeta', \gamma', \gamma'', \mu^2) \pm \Delta(z, \zeta', \gamma', \gamma'', \kappa^2, \mu^2)], \tag{A16}$$

with

$$\Delta^2(z, \zeta', \gamma', \gamma'', \kappa^2, \mu^2) = \mathcal{B}_b^2(z, \zeta', \gamma', \gamma'', \mu^2) - 4\mu^2 \mathcal{A}_b(\zeta', \gamma', \kappa^2) \geq 0. \tag{A17}$$

Notice that

$$y_+ y_- = \frac{\mu^2}{\mathcal{A}_b(\zeta', \gamma', \kappa^2)}. \tag{A18}$$

This means that the two solutions have the same sign. Only for positive solutions, I_1 is not vanishing.

From the requested positivity of the two solutions, one deduces that

$$0 \geq \mathcal{B}_b(z, \zeta', \gamma', \gamma'', \mu^2) = \mu^2 + \gamma' - \gamma'' \frac{(1+\zeta')}{(1+z)}. \tag{A19}$$

Therefore, the two constraints $\theta(y_+)$ and $\theta(y_-)$, once $y_\pm$ exist, are simultaneously fulfilled if

$$\theta\left[\gamma''\frac{(1+\zeta')}{(1+z)}-\gamma'-\mu^2\right]. \tag{A20}$$

In conclusion, Eq. (A14) becomes

$$I_1 = \frac{1}{\gamma''}\theta\left[\gamma''\frac{(1+\zeta')}{(1+z)}-\gamma'-\mu^2\right]\frac{y_+ + y_-}{\mathcal{A}_b(\zeta',\gamma',\kappa^2)|y_+ - y_-|}\theta(\Delta^2(z,\zeta',\gamma',\gamma'',\kappa^2,\mu^2))$$

$$= -\frac{\mathcal{B}_b(z,\zeta',\gamma',\gamma'',\mu^2)}{\mathcal{A}_b(\zeta',\gamma',\kappa^2)\Delta(z,\zeta',\gamma',\gamma'',\kappa^2,\mu^2)}\frac{1}{\gamma''}\theta\left[\gamma''\frac{(1+\zeta')}{(1+z)}-\gamma'-\mu^2\right]\theta(\Delta^2(z,\zeta',\gamma',\gamma'',\kappa^2,\mu^2)). \tag{A21}$$

The second term in Eq. (A11), i.e.,

$$I_2 = -\frac{(1+\zeta')}{(1+z)}\int_0^\infty dy\, y^2 \frac{\theta[y^2\mathcal{A}_b(\zeta',\gamma',\kappa^2)+y\mathcal{B}_b(z,\zeta',\gamma',\gamma'',\mu^2)+\mu^2]}{[y^2\mathcal{A}_b(\zeta',\gamma',\kappa^2)+y(\mu^2+\gamma')+\mu^2]^2} \tag{A22}$$

can be analyzed as follows. One has to discuss two cases: (i) if $\mathcal{B}_b(z,\zeta',\gamma',\gamma'',\mu^2) \geq 0$, the argument of the theta function is positive for any y; (ii) if $\mathcal{B}_b(z,\zeta',\gamma',\gamma'',\mu^2) < 0$, one has to check the sign of $\Delta^2(z,\zeta',\gamma',\gamma'',\kappa^2,\mu^2)$. In conclusion, one can single out the following three contributions:

$$I_2^{(a)} = -\frac{(1+\zeta')}{(1+z)}\theta(\mathcal{B}_b(z,\zeta',\gamma',\gamma'',\mu^2))\int_0^\infty dy\frac{y^2}{[y^2\mathcal{A}_b(\zeta',\gamma',\kappa^2)+y(\mu^2+\gamma')+\mu^2]^2},$$

$$I_2^{(b)} = -\frac{(1+\zeta')}{(1+z)}\theta(-\mathcal{B}_b(z,\zeta',\gamma',\gamma'',\mu^2))\theta(\Delta^2(z,\zeta',\gamma',\gamma'',\kappa^2,\mu^2))\int_0^\infty dy\frac{y^2[\theta(y-y_+)+\theta(y_- - y)]}{[y^2\mathcal{A}_b(\zeta',\gamma',\kappa^2)+y(\mu^2+\gamma')+\mu^2]^2}$$

$$= -\frac{(1+\zeta')}{(1+z)}\theta(-\mathcal{B}_b(z,\zeta',\gamma',\gamma'',\mu^2))\theta(\Delta^2(z,\zeta',\gamma',\gamma'',\kappa^2,\mu^2))\int_0^\infty dy\frac{y^2[1-\theta(y_+ - y)+\theta(y_- - y)]}{[y^2\mathcal{A}_b(\zeta',\gamma',\kappa^2)+y(\mu^2+\gamma')+\mu^2]^2},$$

$$I_2^{(c)} = -\frac{(1+\zeta')}{(1+z)}\theta(-\mathcal{B}_b(z,\zeta',\gamma',\gamma'',\mu^2))\theta(-\Delta^2(z,\zeta',\gamma',\gamma'',\kappa^2,\mu^2))\int_0^\infty dy\frac{y^2}{[y^2\mathcal{A}_b(\zeta',\gamma',\kappa^2)+y(\mu^2+\gamma')+\mu^2]^2}. \tag{A23}$$

Therefore,

$$I_2 = I_2^{(a)}+I_2^{(b)}+I_2^{(c)} = -\frac{(1+\zeta')}{(1+z)}\int_0^\infty dy\frac{y^2}{[y^2\mathcal{A}_b(\zeta',\gamma',\kappa^2)+y(\mu^2+\gamma')+\mu^2]^2}$$

$$+\frac{(1+\zeta')}{(1+z)}\theta\left(\gamma''\frac{(1+\zeta')}{(1+z)}-\gamma'-\mu^2\right)\theta(\Delta^2(z,\zeta',\gamma',\gamma'',\kappa^2,\mu^2))\int_{y_-}^{y_+} dy\frac{y^2}{[y^2\mathcal{A}_b(\zeta',\gamma',\kappa^2)+y(\mu^2+\gamma')+\mu^2]^2}. \tag{A24}$$

Collecting the above results, Eq. (A11) can be cast in the following form:

$$h'(\gamma'',z;\gamma',\zeta',\mu^2) = \theta\left[\gamma''\frac{(1+\zeta')}{(1+z)}-\gamma'-\mu^2\right]\theta(\Delta^2(z,\zeta',\gamma',\gamma'',\kappa^2,\mu^2))\left\{-\frac{\mathcal{B}_b(z,\zeta',\gamma',\gamma'',\mu^2)}{\gamma''\mathcal{A}_b(\zeta',\gamma',\kappa^2)\Delta(z,\zeta',\gamma',\gamma'',\kappa^2,\mu^2)}\right.$$

$$\left.+\frac{(1+\zeta')}{(1+z)}\int_{y_-}^{y_+} dy\frac{y^2}{[y^2\mathcal{A}_b(\zeta',\gamma',\kappa^2)+y(\mu^2+\gamma')+\mu^2]^2}\right\}$$

$$-\frac{(1+\zeta')}{(1+z)}\int_0^\infty dy\frac{y^2}{[y^2\mathcal{A}_b(\zeta',\gamma',\kappa^2)+y(\mu^2+\gamma')+\mu^2]^2}. \tag{A25}$$

The two theta functions can be simplified taking the profit of their interplay.

The explicit form for Δ^2 is

$$
\begin{aligned}
\Delta^2(\pm z, \pm \zeta', \gamma', \gamma'', \kappa^2, \mu^2) &= \left[\gamma'' \frac{(1 \pm \zeta')}{(1 \pm z)} - \gamma' - \mu^2\right]^2 - 4\mu^2\left(\zeta'^2 \frac{M^2}{4} + \kappa^2 + \gamma'\right) \\
&= \left[\gamma'' \frac{(1 \pm \zeta')}{(1 \pm z)} - \gamma' - \mu^2 - 2\mu\sqrt{\zeta'^2 \frac{M^2}{4} + \kappa^2 + \gamma'}\right] \\
&\quad \times \left[\gamma'' \frac{(1 \pm \zeta')}{(1 \pm z)} - \gamma' - \mu^2 + 2\mu\sqrt{\zeta'^2 \frac{M^2}{4} + \kappa^2 + \gamma'}\right].
\end{aligned}
\tag{A26}
$$

In order to have $\Delta^2(\pm z, \pm \zeta', \gamma', \gamma'', \kappa^2, \mu^2) \geq 0$, given the presence of the first theta function in Eq. (A25), it is enough that

$$
\gamma'' \frac{(1 \pm \zeta')}{(1 \pm z)} - \gamma' - \mu^2 \geq 2\mu\sqrt{\zeta'^2 \frac{M^2}{4} + \kappa^2 + \gamma'} \geq 0.
\tag{A27}
$$

Summarizing the above discussion, one can write the kernel as follows:

$$
\begin{aligned}
h'(\gamma'', z; \gamma', \zeta', \mu^2) &= \theta\left[\gamma'' \frac{(1 + \zeta')}{(1 + z)} - \gamma' - \mu^2 - 2\mu\sqrt{\zeta'^2 \frac{M^2}{4} + \kappa^2 + \gamma'}\right] \\
&\quad \times \left[-\frac{\mathcal{B}_b(z, \zeta', \gamma', \gamma'', \mu^2)}{\mathcal{A}_b(\zeta', \gamma', \kappa^2)\Delta(z, \zeta', \gamma', \gamma'', \kappa^2, \mu^2)} \frac{1}{\gamma''} + \frac{(1 + \zeta')}{(1 + z)} \int_{y_-}^{y_+} dy \frac{y^2}{[y^2 \mathcal{A}_b(\zeta', \gamma', \kappa^2) + y(\mu^2 + \gamma') + \mu^2]^2}\right] \\
&\quad - \frac{(1 + \zeta')}{(1 + z)} \int_0^\infty dy \frac{y^2}{[y^2 \mathcal{A}_b(\zeta', \gamma', \kappa^2) + y(\mu^2 + \gamma') + \mu^2]^2}.
\end{aligned}
\tag{A28}
$$

APPENDIX B: THE NORMALIZATION OF THE BS AMPLITUDE

In this appendix, the normalization of the BS amplitude in ladder approximation for a two-scalar system in S wave is presented. The reader can find more details in Refs. [9,11,29,36].

In general, but disregarding self-energy contributions, the BS amplitude is normalized through the following constraint:

$$
\int \frac{d^4 k}{(2\pi)^4} \int \frac{d^4 k'}{(2\pi)^4} \bar{\Phi}_b(k', p) \left\{\frac{\partial}{\partial p^\mu}[G_0^{-1}(12)(k, p)(2\pi)^4 \delta^4(k - k') - i\mathcal{K}(k, k', p)]\right\}\bigg|_{p^2 = M^2} \Phi_b(k, p) = i2p_\mu,
\tag{B1}
$$

with

$$
G_0(12)(k, p) = G_0\left(\frac{p}{2} + k\right) G_0\left(\frac{p}{2} - k\right) = \frac{i}{(\frac{p}{2} + k)^2 - m^2 + i\epsilon} \frac{i}{(\frac{p}{2} - k)^2 - m^2 + i\epsilon}.
\tag{B2}
$$

In ladder approximation, fortunately, the kernel $i\mathcal{K}(k, k', p)$ becomes independent of p, viz.,

$$
i\mathcal{K}^{(Ld)}(k, k', p) = \frac{i(-i)^2}{(k - k')^2 - \mu^2 + i\epsilon}.
\tag{B3}
$$

Therefore, the ladder BS amplitude is normalized through

$$
(-i) \int \frac{d^4 k}{(2\pi)^4} \bar{\Phi}_b^{(Ld)}(k, p) \left[\left(\frac{p_\mu}{2} + k_\mu\right) G_0^{-1}\left(\frac{p}{2} - k\right) + G_0^{-1}\left(\frac{p}{2} + k\right)\left(\frac{p_\mu}{2} - k_\mu\right)\right] \Phi_b^{(Ld)}(k, p) = i2p_\mu.
\tag{B4}
$$

Since $\Phi_b^{(Ld)}(k, p)$ is symmetric under the exchange $1 \to 2$, and recalling that k^μ changes sign under such a transformation, one can rewrite Eq. (B4) as follows:

$$-\int \frac{d^4k}{(2\pi)^4} \bar{\Phi}_b^{(Ld)}(k, p)(M^2 + 2k \cdot p)\left[\left(\frac{p}{2} - k\right)^2 - m^2\right]\Phi_b^{(Ld)}(k, p)$$

$$= \int \frac{d^4k}{(2\pi)^4} \bar{\Phi}_b^{(Ld)}(k, p)[M^2(\kappa^2 - k^2) + 2(k \cdot p)^2]\Phi_b^{(Ld)}(k, p) = i2M^2, \tag{B5}$$

where $\kappa^2 = m^2 - M^2/4$ and the odd contributions in k^μ have been eliminated given the symmetry of the BS amplitude. By introducing the expression of $\Phi_b^{(Ld)}(k, p)$ in terms of the Nakanishi weight function, Eq. (3), with $n = 1$ (as explained in the main text), one gets

$$\int \frac{d^4k}{(2\pi)^4}[M^2(\kappa^2 - k^2) + 2(k \cdot p)^2] \int_{-1}^1 dz' \int_0^\infty d\gamma' \frac{g_b^{(Ld)}(\gamma', z'; \kappa^2)}{[k^2 + p \cdot kz' - \gamma' - \kappa^2 + i\epsilon']^3} \int_{-1}^1 dz \int_0^\infty d\gamma \frac{g_b^{(Ld)}(\gamma, z; \kappa^2)}{[k^2 + p \cdot kz - \gamma - \kappa^2 + i\epsilon]^3}$$

$$= \int_{-1}^1 dz' \int_0^\infty d\gamma' g_b^{(Ld)}(\gamma', z'; \kappa^2) \int_{-1}^1 dz \int_0^\infty d\gamma g_b^{(Ld)}(\gamma, z; \kappa^2) \mathcal{F}(\gamma', z', \gamma, z) = i2M^2. \tag{B6}$$

It is worth noting that (i) the S-wave weight function is real, and (ii) the boundary condition $i\epsilon$ has to be chosen in $\bar{\Phi}_b^{(Ld)}(k, p)$ for ensuring the correct propagation in time (see, e.g., Refs. [29,36]).

In order to evaluate $\mathcal{F}(\gamma', z', \gamma, z)$, let us apply the Feynman trick as follows (cf. Refs. [30,37]):

$$\frac{1}{[k^2 + p \cdot kz' - \gamma' - \kappa^2 + i\epsilon']^3} \frac{1}{[k^2 + p \cdot kz - \gamma - \kappa^2 + i\epsilon]^3}$$

$$= \int_0^1 dv \frac{30v^2(1 - v)^2}{[v(k^2 + p \cdot kz' - \gamma' - \kappa^2 + i\epsilon') + (1 - v)(k^2 + p \cdot kz - \gamma - \kappa^2 + i\epsilon)]^6} =$$

$$= \int_0^1 dv \frac{30v^2(1 - v)^2}{[k^2 - \kappa^2 + p \cdot k(vz' + (1 - v)z) - \gamma'v - \gamma(1 - v) + i\eta]^6} =$$

$$= \int_0^1 dv \frac{30v^2(1 - v)^2}{[q^2 - \kappa^2 - \frac{p^2}{4}\lambda^2 - \gamma'v - \gamma(1 - v) + i\eta]^6}, \tag{B7}$$

reduces to

$$\mathcal{F}(\gamma', z', \gamma, z) = \int \frac{d^4k}{(2\pi)^4} \frac{[M^2(\kappa^2 - k^2) + 2(k \cdot p)^2]}{[k^2 + p \cdot kz' - \gamma' - \kappa^2 + i\epsilon']^3} \frac{1}{[k^2 + p \cdot kz - \gamma - \kappa^2 + i\epsilon]^3}$$

$$= \int \frac{d^4q}{(2\pi)^4} \int_0^1 dv \frac{[M^2(\kappa^2 - q^2 + \frac{M^2}{4}\lambda^2) + 2(p \cdot q)^2 - M^2\lambda(p \cdot q)]30v^2(1 - v)^2}{[q^2 - \kappa^2 - \frac{M^2}{4}\lambda^2 - \gamma'v - \gamma(1 - v) + i\eta]^6}$$

$$= \int \frac{d^4q}{(2\pi)^4} \int_0^1 dv \frac{[M^2(\kappa^2 - q^2 + \frac{M^2}{4}\lambda^2) + 2(p \cdot q)^2]30v^2(1 - v)^2}{[q^2 - \kappa^2 - \frac{M^2}{4}\lambda^2 - \gamma'v - \gamma(1 - v) + i\eta]^6}, \tag{B8}$$

where the term $(p \cdot q)$ yields a vanishing contribution after integrating over d^4q.

Then, by choosing a reference frame where $p^\mu \equiv \{M, \mathbf{0}\}$ for the sake of simplicity, and performing a Wick rotation, given the positions of the poles, the integration on d^4q can be evaluated in a Euclidean 4D space, i.e., $d^4q \to id^4q_E$. One obtains the following result (cf. Refs. [30,37]):

$$\mathcal{F}(\gamma',z',\gamma,z) = \int \frac{d^4q}{(2\pi)^4} \int_0^1 dv \frac{[M^2(\kappa^2-q^2+\frac{M^2}{4}\lambda^2)+2(p\cdot q)^2]30v^2(1-v)^2}{[q^2-\kappa^2-\frac{M^2}{4}\lambda^2-\gamma'v-\gamma(1-v)+i\eta]^6}$$

$$= i \int \frac{d^4q_E}{(2\pi)^4} \int_0^1 dv \frac{[M^2(\kappa^2+q_E^2+\frac{M^2}{4}\lambda^2)-2M^2(q_E^0)^2]30v^2(1-v)^2}{[q_E^2+\kappa^2+\frac{M^2}{4}\lambda^2+\gamma'v+\gamma(1-v)-i\eta]^6}$$

$$= \frac{iM^2}{(2\pi)^4} \int_0^1 dv\, 30v^2(1-v)^2 \int d\rho\rho^3 \int_0^{2\pi} d\varphi \int_{-1}^1 d\cos(\theta_1) \int_0^\pi \sin^2(\theta_2)d\theta_2 \frac{[\kappa^2+\rho^2+\frac{M^2}{4}\lambda^2-2\rho^2\cos^2(\theta_2)]}{[\rho^2+\kappa^2+\frac{M^2}{4}\lambda^2+\gamma'v+\gamma(1-v)-i\eta]^6},$$

$$\text{(B9)}$$

with

$$d^4q_E \to \rho^3 d\rho d\phi \, \sin(\theta_1)d\theta_1 \sin^2(\theta_2)d\theta_2.$$

Finally, by using

$$\int_0^{2\pi} d\phi \int_{-1}^1 d\cos(\theta_1) \int_0^\pi \sin^2(\theta_2)d\theta_2 \int_0^\infty d\rho \frac{\rho^3}{(\rho^2+A)^6} = 2\pi^2 \frac{1}{2} \int_0^\infty dy \frac{y}{(y+A)^6} = \frac{\pi^2}{20A^4} \qquad \text{(B10)}$$

and

$$\int_0^{2\pi} d\phi \int_{-1}^1 d\cos(\theta_1) \int_0^\pi d\theta_2 \sin^2(\theta_2)[1-2\cos^2(\theta_2)] \int_0^\infty d\rho \frac{\rho^5}{(\rho^2+A)^6} = \pi^2 \frac{1}{2} \int_0^\infty dy \frac{y^2}{(y+A)^6} = \frac{\pi^2}{60A^3}, \qquad \text{(B11)}$$

one gets

$$\mathcal{F}(\gamma',z',\gamma,z) = \frac{iM^2\pi^2}{2(2\pi)^4} \int_0^1 dv\, v^2(1-v)^2 \frac{[3(\kappa^2+\frac{M^2}{4}\lambda^2)+(\kappa^2+\frac{M^2}{4}\lambda^2+\gamma'v+\gamma(1-v))]}{[\kappa^2+\frac{M^2}{4}\lambda^2+\gamma'v+\gamma(1-v)-i\eta]^4}. \qquad \text{(B12)}$$

Recollecting the above results, the normalization condition Eq. (B6) reads

$$\int_{-1}^1 dz' \int_0^\infty d\gamma' g_b^{(Ld)}(\gamma',z';\kappa^2) \int_{-1}^1 dz \int_0^\infty d\gamma\, g_b^{(Ld)}(\gamma,z;\kappa^2)\mathcal{F}(\gamma',z',\gamma,z)$$

$$= \frac{iM^2}{2(4\pi)^2} \int_{-1}^1 dz' \int_0^\infty d\gamma' g_b^{(Ld)}(\gamma',z';\kappa^2) \int_{-1}^1 dz \int_0^\infty d\gamma\, g_b^{(Ld)}(\gamma,z;\kappa^2) \int_0^1 dv v^2(1-v)^2 \frac{[4(\kappa^2+\frac{M^2}{4}\lambda^2)+\gamma'v+\gamma(1-v)]}{[\kappa^2+\frac{M^2}{4}\lambda^2+\gamma'v+\gamma(1-v)-i\eta]^4}$$

$$= i2M^2. \qquad \text{(B13)}$$

In conclusion, from Eq. (B13) one obtains the following normalization of the Nakanishi weight function for the S-wave bound state of a two-scalar system:

$$\frac{1}{(8\pi)^2} \int_{-1}^1 dz' \int_0^\infty d\gamma' g_b^{(Ld)}(\gamma',z';\kappa^2) \int_{-1}^1 dz \int_0^\infty d\gamma\, g_b^{(Ld)}(\gamma,z;\kappa^2) \int_0^1 dv v^2(1-v)^2 \frac{[4(\kappa^2+\frac{M^2}{4}\lambda^2)+\gamma'v+\gamma(1-v)]}{[\kappa^2+\frac{M^2}{4}\lambda^2+\gamma'v+\gamma(1-v)-i\eta]^4} = 1.$$

$$\text{(B14)}$$

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