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Two-loop perturbative corrections to the thermal effective potential in gluodynamics

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The thermodynamics of pure glue theories can be described in terms of an effective action for the Polyakov loop. This effective action is of the Landau-Ginzburg type, and its variables are the angles parametrizing the loop. In this paper we compute perturbative corrections to this action. Remarkably, two-loop corrections turn out to be proportional to the one-loop action, independent of the eigenvalues of the loop. By a straightforward generalization of the 't Hooft coupling, this surprisingly simple result holds for any of the classical and exceptional groups.

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I. INTRODUCTION

Understanding of the deconfined phase QCD at high temperature T is growing as a result of the heavy ion experiments at RHIC and the LHC and theoretical and numerical lattice work. The latter, in particular, has shown that the conformal anomaly, which is the energy density minus three times the pressure, in the deconfined phase of $SU(N)$ pure gauge theories is approximately proportional to T^2, up to temperatures several times the critical temperature T_c for deconfinement [1]. This demonstrates that $e-3p$ is not dominated by a constant "bag pressure." Even without a detailed understanding of the physical origin of this behavior, previous work has shown that it can be parametrized as a dimension-two constant times a condensate for the *eigenvalues* of the Polyakov loop which "evaporates" at high T [2–4]. At very high temperature then, the behavior of $(e-3p)/T^2$ for $SU(3)$ is described very well by "hard thermal loop" resummed perturbation theory [5].

The goal of the present paper is to compute perturbative quantum corrections to the pressure of that classical condensate at two-loop order. That is, we compute the leading correction due to interactions among gluons in the presence of the condensate. Our main tool is the effective action as a function of the *eigenvalues* of the Polyakov loop. While the two-loop correction to the effective potential does not affect the interaction measure,[1] it does of course modify the pressure and the energy density of the gluon plasma. We hope that our results may be useful for improving the models [2–4,6], which typically employ the one-loop effective potential. Furthermore, our efforts show that the models mentioned above can be improved systematically, at least in regard to the perturbative component, rather than offering mere parametrizations of the lattice results.

We study hot gluodynamics for any number of colors, and make use of the global $Z(N)$ symmetry [7] in that system. However, lattice simulations for groups without a center [8–10] show that deconfinement does not require a global symmetry. Hence, aside from $SU(N)$ we perform our perturbative two-loop calculations also for all other classical gauge groups, including the exceptional group $G(2)$.

For $SU(N)$ gluodynamics, there is an order parameter associated with the global symmetry $Z(N)$, i.e. the Polyakov loop,

$$\mathbf{L}(\vec{x}) = \mathcal{P}\exp\left(i\int_0^{1/T} d\tau A_0(\vec{x},\tau)\right). \tag{1}$$

The global symmetry acts on the loop as a large gauge transformation $\Omega_k(\vec{x},\tau)$. It develops a $Z(N)$ valued

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[1]$\partial g^2(T)/\partial T = \mathcal{O}(g^4)$ is beyond the order considered here.

discontinuity $\exp(ik\frac{2\pi}{N})$ when we move in the periodic Euclidean time direction τ.

The loop $\mathbf{L}(\vec{x})$ transforms like an adjoint field and hence its trace,

$$\frac{1}{N}\,\mathrm{Tr}\mathbf{L}(\vec{x}), \tag{2}$$

picks up a $Z(N)$ phase,

$$\frac{1}{N}\,\mathrm{Tr}\mathbf{L}(\vec{x}) \to \exp\left(ik\frac{2\pi}{N}\right)\frac{1}{N}\,\mathrm{Tr}\mathbf{L}(\vec{x}). \tag{3}$$

The effective action is simply the traditional path integral over the gauge fields subject to a constraint [11]. This constraint is obviously that the integration is done while preserving the value of the Polyakov loop at some fixed value ℓ. Doing so, one generates a probability distribution for the eigenvalues determined by ℓ. We are interested in a loop which is constant in space, so the constraint should be a delta function with the argument

$$\ell - \frac{1}{N}\,\mathrm{Tr}\bar{\mathbf{L}}, \tag{4}$$

involving the spatially averaged loop,

$$\mathrm{Tr}\bar{\mathbf{L}} = \frac{1}{V}\int_V d\vec{x}\,\mathrm{Tr}\mathbf{L}(\vec{x}). \tag{5}$$

Clearly, to fix all independent phases Φ, one has to take as many powers of the loop as there are independent phases. To avoid clutter we do not write these higher powers explicitly; in this simplified notation the effective action becomes

$$\exp(-V\mathcal{V}(\ell)) = \int DA_\mu\,\delta\left(\ell - \frac{1}{N}\,\mathrm{Tr}\bar{\mathbf{L}}\right)\exp\left(-\frac{1}{g^2}S(A)\right). \tag{6}$$

Hence, for $SU(2)$, where there is only one independent phase, this expression fully fixes the phase of the loop. So, there is just one real constraint on the full path integral. We take ℓ to be a trace over a diagonal matrix, without loss of generality (see Sec. II). The loop $\bar{\mathbf{L}}$ is, in general, not diagonal because the fluctuating scalar potential A_0 is arbitrary so long as it satisfies the constraint. As we will see, in perturbation theory the constraint amounts to taking out the $N-1$ zero modes of the fluctuation matrix, for $SU(N)$.

The constraint is nonlinear in the fluctuations, since the Polyakov loop is so. As consequence, at two and higher loop order there is an extra vertex involving the zero mode. It generates diagrams with radiative corrections inserted into the Polyakov loop [12,18]. They are crucial for gauge invariance of the effective potential, and we will use them in this paper.

To compute $\mathcal{V}(\ell)$ at small coupling fluctuations around the background of a constant Polyakov loop, ℓ are integrated over. This leads to the gluon black body radiation contribution plus a $Z(N)$ invariant polynomial of fourth order in the phases Φ of the loop. To be specific, we consider the $SU(2)$ gauge group. The loop has only one independent phase, $2q_1 = -2q_2 = q$. In terms of the variable q, one obtains [13,14]

$$\mathcal{V}_{\mathrm{pert}}(q) = -\frac{\pi^2}{15}T^4 + \frac{4\pi^2}{3}T^4 q^2(1-q)^2. \tag{7}$$

In this expression, q is defined modulo 1 and a $Z(2)$ transformation corresponds to $q \to 1-q$. The minima of this $Z(2)$ invariant polynomial are at 0 and 1, where the loop $\ell = \pm 1$. The motivation of this paper is to establish how radiative corrections affect this potential. An earlier answer to this question [15] presented an elegant but formal proof using the Vafa-Witten trick [16], bypassing issues related to infrared divergences.

The implication of our work is that

i) *in perturbation theory* the eigenvalue distribution $\sim\exp(-V\mathcal{V}_{\mathrm{pert}})$ is not affected by two-loop corrections.[2] In particular, the expectation value of the Polyakov loop calculated at these minima remains $\ell = \pm 1$.

ii) the pressure calculated from the minimum of the potential equals the known perturbative pressure calculated at $q = 0$.

However, we stress that when a nonperturbative contribution is added [3,4], that the two-loop corrections to the perturbative potential do modify the total result. We postpone a detailed fit to lattice results to a future publication.

The roman lowercase letters points above are corroborated by the two loop contribution to the perturbative potential which we compute explicitly in Sec. III. Section II contains a discussion of simple properties of the effective action and the gauge independence of our corrections. In Sec. III we discuss the explicit result at two loops; the simplified expressions for the insertion diagram are given in Sec. IV; the last section contains the conclusions.

II. GENERALITIES OF THE EFFECTIVE POTENTIAL

For the sake of notation and clarity we will mostly work in this section with the $SU(2)$ gauge group. In the first subsection we discuss the relation between various ways of defining the effective action. Its expansion about a constant Polyakov loop background is analyzed next, and finally extended Becchi-Rouet-Stora (BRS) identities are derived which give us a very useful control over the perturbative expansion.

A. Two ways of obtaining the effective action

First we introduce a definition of the effective potential which is manifestly gauge invariant, manageable on the

[2]However, it may change at higher orders.

lattice and, most importantly for this paper, has relatively simple Feynman rules. It is the constrained effective action, defined in a large three volume V in Euclidean space, and periodic in Euclidean time direction with period $1/T$.

The Polyakov loop was defined above as

$$\mathbf{L}(\vec{x}) = \mathcal{P}\exp\left(i\int_0^{1/T} A_0(\vec{x},\tau)d\tau\right). \tag{8}$$

The effective potential is [11]

$$\exp(-V\mathcal{V}(\ell)) \equiv \int DA_\mu \delta\left(\ell - \frac{1}{2}\,\mathrm{Tr}\bar{\mathbf{L}}\right)\exp\left(-\frac{1}{g^2}S(A)\right). \tag{9}$$

Here ℓ is some *a priori* specified number.

The partition function Z equals

$$Z \equiv \int DA_\mu \exp\left(-\frac{1}{g^2}S(A)\right), \tag{10}$$

$$\mathrm{Tr}\bar{\mathbf{L}} \equiv \frac{1}{V}\int_V d\vec{x}\,\mathrm{Tr}\mathbf{L}(\vec{x}). \tag{11}$$

The integration is over fields which are periodic in the Euclidean time direction with period $1/T$. Note that the $SU(2)$ matrix $\mathbf{L}(\vec{x})$ can be diagonalized at every point $\vec{x}$ by a gauge transformation. Hence the space-averaged trace of the loop becomes a spatial average over a cosine, a number not larger than 1. If all the eigenphases of $\mathbf{L}(\vec{x})$ are aligned, the space average, of course, becomes the cosine of the common eigenphase.

Thus ℓ is bound by[3]

$$-1 \le \ell \le 1 \tag{12}$$

and can be parametrized as the trace of a constant SU(2) matrix,

$$\ell(\mathbf{q}) = \frac{1}{2}\,\mathrm{Tr}\exp(2\pi i\mathbf{q}), \tag{13}$$

with

$$\mathbf{q} = \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}, \quad q_1 = -q_2. \tag{14}$$

Now we will show that in the large volume limit the definition (9) is equivalent to the traditional definition of the effective potential where a source term,[4]

$$jV\,\mathrm{Tr}\bar{\mathbf{L}} = j\int_V d\vec{x}\,\mathrm{Tr}\mathbf{L}(\vec{x}), \tag{15}$$

is introduced into the path integral Z,

$$\exp(-VW(j)) = \int DA_\mu \exp\left(-\frac{1}{g^2}S(A) - j\int_V d\vec{x}\,\mathrm{Tr}\mathbf{L}(\vec{x})\right), \tag{16}$$

with

$$\ell \equiv \langle \mathrm{Tr}\bar{\mathbf{L}}\rangle = \frac{\partial W}{\partial j}, \qquad G(\ell) \equiv W(j) - j\ell. \tag{17}$$

The effective action $G(\ell)$ depends on the source j only through ℓ, and it satisfies

$$\frac{\partial G}{\partial \ell} = -j. \tag{18}$$

To compare this definition of the effective action $G(\ell)$ to the one in Eq. (9), we Laplace transform the latter with $\int d\ell \exp(-Vj\ell)$,

$$\int d\ell\, \exp(-Vj\ell)\exp(-V\mathcal{V}(\ell)) = \exp(-VW(j)). \tag{19}$$

Steepest descent of this integral in the large V limit tells us that the effective potential obeys

$$\frac{\partial \mathcal{V}(\ell)}{\partial \ell} = -j, \qquad \mathcal{V}(\ell) + j\ell = W(j), \tag{20}$$

and so

$$G(\ell) = \mathcal{V}(\ell) + f(T). \tag{21}$$

This means that both definitions give the same effective potential, up to a temperature dependent but ℓ-independent function. However, the constrained version is well adapted to lattice calculations and admits a straightforward saddle point expansion around any value of ℓ. This expansion is discussed in the next two sections.

The source term $j\int_V d\vec{x}\,\mathrm{Tr}\mathbf{L}(\vec{x})$ is manifestly gauge invariant. Hence, we expect the effective potential to be the same in whatever gauge we calculate it. This will turn out to be true although the expectation value of the space-averaged loop $\langle \mathrm{Tr}\bar{\mathbf{L}}\rangle$ is gauge dependent (except at $j = 0$). This gauge dependence precisely cancels that of the free- energy contributions as we will expound in Secs. II E and II F.

After Fourier transforming the delta function constraint, we can write the constrained path integral as

[3]Modulo renormalization effects, see e.g. [17].

[4]We absorb the $1/N$ normalization of the trace of $\mathbf{L}$ into the source j.

$$\exp(-V\mathcal{V}(\ell)) \equiv \int DA_\mu d\varepsilon \, \exp\left(-\frac{1}{g^2} S_{\rm con}(A,\varepsilon)\right),$$
$$S_{\rm con}(A,\varepsilon) = i\varepsilon\left(\ell - \frac{1}{2}\,{\rm Tr}\bar{\mathbf{L}}\right) + S(A). \quad (22)$$

We have traded the constraint for an extra field ε in the path integral, and we added a phase to the original gauge action. The new field ε is therefore gauge invariant, like the constraint it generates.

B. The effective potential in perturbation theory

Below we shall give explicitly the Feynman rules for fluctuations around a fixed background ε_c and B_μ. The background is supposed to be a minimum of $S_{\rm con}$. A simple choice of background is

$$\varepsilon_c = 0, \qquad A_\mu = B\delta_{\mu,0}, \quad B \text{ constant in space time.} \quad (23)$$

This is indeed an extremum of $S_{\rm con}$ when minimizing over the fluctuations in

$$A_\mu = B\delta_{\mu,0} + gQ_\mu, \quad \text{and} \quad \varepsilon = \varepsilon_c + g\varepsilon_q. \quad (24)$$

The gauge zero modes have to be tamed by introducing gauge-fixing and ghost terms,

$$S_{\rm gauge} = S(A) + S_{gf} + S_{gh} = S_{\rm free} + gS_{\rm int}. \quad (25)$$

Our choice of gauge fixing is covariant background gauge,

$$S_{gf} = \frac{1}{\xi}\int d\vec{x}\; d\tau\, {\rm Tr}(D_\mu(B)Q_\mu)^2,$$
$$D_\mu(B) = \partial_\mu + i[B\delta_{\mu,0}, \ldots], \quad (26)$$

and the constrained action $S_{\rm con}$ changes accordingly into

$$S_{\rm con} = i\varepsilon\left(\ell - \frac{1}{2}\,{\rm Tr}\bar{\mathbf{L}}\right) + S_{\rm gauge}. \quad (27)$$

Expand $S_{\rm con}(A,\varepsilon)$ in terms of Q and ε_q as

$$S_{\rm con}(B + gQ, \varepsilon_c + g\varepsilon_q) = \sum_{n\geq 0}^{\infty} g^n S^{(n)}_{\rm con}. \quad (28)$$

To avoid clutter in the formulas we shall use the following notation for the expansion in powers of the fluctuation field Q_0:

$$\mathbf{L}(B + gQ_0) = \mathbf{L}(B) + gQ_0 \cdot \mathbf{L}'(B) + g^2 Q_0^2 \cdot \mathbf{L}''(B) + \mathcal{O}(g^3 Q_0^3), \quad (29)$$

and similar for the action $S_{\rm gauge}$. The operator "$\cdot$" means integration over space time and summation over (Lorentz and) color indices.

One mode becomes particularly important in this expansion. It is the zero Matsubara frequency of the zero momentum mode $\bar{Q}_0(\tau) = \int_V d\vec{x} Q_0(\vec{x},\tau)/V$,

$$\bar{\bar{Q}}_0 \equiv \int_0^{1/T} d\tau \bar{Q}_0(\tau). \quad (30)$$

It is the *only* mode that can produce linear terms in the expansion around the space time independent background. All other modes $Q_\mu(\vec{x},\tau)$ are orthogonal to B.

The terms linear in the fluctuations [i.e. in $S^{(1)}_{\rm con}(\mathbf{q})$] are required to vanish. They are

$$i\varepsilon_c {\rm Tr}(\bar{\bar{Q}}_0 \mathbf{L}'(B)) + {\rm Tr}(\bar{\bar{Q}}_0 S'_{\rm guage}(B)) = 0,$$
$$\varepsilon_q\left(\ell - \frac{1}{2}{\rm Tr}\mathbf{L}(B)\right) = 0. \quad (31)$$

From the first condition it follows that $\varepsilon_c = 0$, because $S'_{\rm guage}(B) = 0$. The second fixes B in terms of ℓ,

$$\ell - \frac{1}{2}\,{\rm Tr}\mathbf{L}(B) = 0. \quad (32)$$

Hence, from (13) we have the background B fixed in terms of the phases q_1 and $q_2 = -q_1$ of the Polyakov loop,

$$B = 2\pi\mathbf{q}T. \quad (33)$$

Hence $S^{(0)}_{\rm con}(\mathbf{q}) = 0$.

In what follows we write the zero momentum and zero Matsubara frequency mode projected onto $\mathbf{L}'(\mathbf{q})$ as

$${\rm Tr}(\bar{\bar{Q}}_0 \mathbf{L}'(\mathbf{q})) \equiv \hat{Q}_0. \quad (34)$$

The quadratic term in the expansion of $S_{\rm con}$ is therefore the first nonvanishing term,

$$S^{(2)}_{\rm con} = \int d\vec{x}d\tau({\rm Tr}Q_\mu(\vec{x},\tau)(-D^2(\mathbf{q}))\delta_{\mu\nu} + (1-\xi)D_\mu(\mathbf{q})D_\nu(\mathbf{q})Q_\nu(\vec{x},\tau)) - i\varepsilon_q\hat{Q}_0, \quad (35)$$

where we wrote the explicit form of $S''_{\rm gauge}$.

Thus the expansion of the effective potential (η and ω are the ghost fields) becomes

$$\exp(-V\mathcal{V}) = \int DQ_\mu D\bar{\eta}D\omega d\varepsilon_q \times \exp(-Q^2 \cdot S''_{\rm gauge}(\mathbf{q}) - i\varepsilon_q\hat{Q}_0)(1-R). \quad (36)$$

The last factor equals

$$1-R=1-gQ^3\cdot S'''_{\rm gauge}(\mathbf{q})-g^2Q^4\cdot S''''_{\rm gauge}(\mathbf{q}) + g^2 i\varepsilon_q \mathrm{Tr}(\bar{Q}_0^2\cdot \mathbf{L}''(\mathbf{q}))Q^3\cdot S'''_{\rm gauge}(\mathbf{q})+\cdots. \tag{37}$$

Let us discuss (36) and (37). The terms in the exponent are familiar, except for the last one. This latter term, after integration over ε_q, is restoring the delta function constraint. It tells us *not* to integrate over $\hat{Q}_0$.

C. One-loop determinant

First we neglect all interactions, i.e. the term R in (36), leaving the determinant without the zero mode $\hat{Q}_0$.

It is useful to generalize at this point the discussion from $SU(2)$ to $SU(N)$. For $SU(N)$, the $N-1$ independent eigenvalues are fixed by a product of $N-1$ delta functions; their respective arguments are

$$\delta\left(\ell_n-\frac{1}{N}\,\mathrm{Tr}\overline{(\mathbf{L}(A_0))^n}\right),\quad 1\le n\le N-1. \tag{38}$$

The generalization of the ϵ field to $SU(N)$ follows immediately: it is such that it couples to the Polyakov loop winding n times around the thermal circle. To fix the eigenvalues we need $N-1$ windings $\mathrm{Tr}\mathbf{L}^n$. Hence, there are $N-1$ fields ε_n and they generate the delta function constraints via the following term in the exponent:

$$i\sum_n \varepsilon_n(\ell_n-\mathrm{Tr}\overline{(\mathbf{L}(A_0))^n}) \tag{39}$$

with

$$\mathrm{Tr}\overline{(\mathbf{L}(A_0))^n}\equiv\frac{1}{V}\int_V d\vec{x}\,\mathrm{Tr}(\mathbf{L}(A_0))^n. \tag{40}$$

After expanding around the saddle point, the analogue of $\hat{Q}_0$ in Eq. (34) is labeled by the number of windings,

$$\hat{Q}_0^n\equiv\mathrm{Tr}(\bar{\bar{Q}}_0(\mathbf{L}^n(\mathbf{q}))')\equiv\sum_d \bar{\bar{Q}}_0^d\; t_d^n(\mathbf{q}). \tag{41}$$

We introduced the matrix $t_d^n(\mathbf{q})\equiv\mathrm{Tr}(\lambda_d(\mathbf{L}^n(\mathbf{q}))')$ for later use. It connects the winding basis labeled by n to the diagonal Cartan basis labeled by d.

By analogy to $SU(2)$, the ℓ_n can be written as

$$\ell_n=\frac{1}{N}\,\mathrm{Tr}\,\exp(2\pi i n\mathbf{q}). \tag{42}$$

The matrix $\mathbf{q}$ is taken to be diagonal with its N eigenvalues q_j obeying $\sum_j q_j=0$.

The diagonalization of $S''_{\rm gauge}$ is well known [13,14] and simply employs the plane wave basis $Q_\mu(p_0,\vec{p})$. The color basis is the well-known Cartan basis, spelled out in Sec. III for all classical groups. In this section we just need one salient property of this basis.

It is divided into diagonal elements H_d and off-diagonal elements E_α, eigenmatrices of the H_d,

$$[H_d,E_\alpha]=\alpha_d E_\alpha. \tag{43}$$

The coefficients α_d are the components of an r-dimensional vector, the root $\vec{\alpha}$, where r is the rank of the group.

For the fundamental representation of $SU(N)$, this is as follows. The fluctuation variables are labeled by $d=1,2,\ldots,N-1$ corresponding to the $N-1$ diagonal matrices. The off-diagonal fluctuation variables correspond to the $N(N-1)$ off-diagonal matrices λ^{ij}, where the indices $i,j=1,2,\ldots,N$ with $i\neq j$. The off-diagonal matrices are a direct generalization of the off-diagonal Pauli matrices for $SU(2)$,

$$(\lambda^{ij})_{kl}=\frac{1}{\sqrt{2}}\delta_{ik}\delta_{jl}. \tag{44}$$

As a consequence the Matsubara frequency in $D_0(\mathbf{q})$, Eq. (26), acting on the off-diagonal mode $Q_\mu^{ij}(p_0,\ldots)$ is shifted by $q_i-q_j\equiv q_{ij}$ but remains unchanged if it acts on a diagonal mode Q_μ^d,

$$\begin{aligned} D_0(\mathbf{q})Q_\mu^{ij}(p_0,\ldots)&=i(p_0+2\pi q_{ij})Q_\mu^{ij}(p_0,\ldots),\\ D_0(\mathbf{q})Q_\mu^{d}(p_0,\ldots)&=ip_0Q_\mu^{d}(p_0,\ldots). \end{aligned} \tag{45}$$

Hence the shifted four-momenta are either $p^{ij}=(p_0+q_{ij},\vec{p})$, or $p^d=(p_0,\vec{p})$, with $p_0=2\pi T n_0$ where n_0 is an integer. We will use this notation throughout the paper. These rules generalize to any other classical group (see Sec. III).

After these preliminaries we can easily compute the one loop determinant; in dimensional regularization we obtain the well-knowm result [13,14],

$$\Gamma_f=-p_{SB}+\frac{2\pi^2}{3}T^4\sum_{i\neq j}B_4(q_{ij}). \tag{46}$$

The Bernoulli polynomial B_4 is given in Appendix A.

D. Interactions without the ϵ fields

These interactions are fully contained in the interaction terms in the first line of (37). They give the usual free-energy diagrams Γ_f as in Fig. 2 with propagators and vertices determined by $\ell=\cos(2\pi q)$. Only the zero momentum mode $\hat{Q}_0$ is not integrated over.

The Feynman rules in the presence of the color diagonal background $\mathbf{q}$ are simple. They have been discussed in the previous section and amount to replacing the momenta as in Eq. (45) and below.

With these rules it is straightforward to obtain the contribution Γ_f due to the free- energy diagrams in Fig. 2 to one and two loop order in Feynman gauge $\xi = 1$,

$$\Gamma_f = -p_{SB} + \sum_a \hat{B}_4(q_a) + g^2 \sum_{a,b,c} |f^{a,b,c}|^2 \hat{B}_2(q_b)\hat{B}_2(q_c). \tag{47}$$

The first two terms correspond to the one loop result (46), the last term is the two loop correction. The indices a, b, c run through the diagonal indices d and the off-diagonal indices ij. So, $q_d = 0$ and $q_{ij} = q_i - q_j$. The Bernoulli polynomials are simple and defined in Appendix A.

E. The insertion diagram due to the constraint

We now consider the interactions involving the fluctuations ε_q in the second line of Eq. (36). They play an essential role at two and more loops [18]. They originate in terms $S^{(3)}_{\rm con}$, $(S^{(3)}_{\rm con})^2$ etc. Among other contributions the latter gives the term we wrote explicitly,

$$g^2 i\varepsilon_q \mathrm{Tr}(\bar{Q}_0^2 \cdot \mathbf{L}''(\mathbf{q}))(Q^3 \cdot S'''_{\rm gauge}(\mathbf{q})). \tag{48}$$

To avoid inessential complications we first discuss $SU(2)$.

The first factor in this expression is the Polyakov loop expanded to second order in the fluctuation Q_0. To two-loop order, this is the only term that contributes at $\mathcal{O}(V)$ to $\mathcal{V}$. No other term does because of the absence of infrared divergences.[5]

To perform the integration over ε_q, we replace $i\varepsilon_q$ by $\frac{\partial}{\partial \hat{Q}_0}$ acting on the last term in (35). By inspection, the only contraction of $\mathcal{O}(V)$ is

$$\langle \bar{Q}_0^2 \cdot \mathrm{Tr}\mathbf{L}''(\mathbf{q})\rangle \left\langle \frac{\partial}{\partial \hat{Q}_0} Q^3 \cdot S'''_{\rm gauge}(\mathbf{q}) \right\rangle. \tag{49}$$

The first contraction is the expectation value of the Polyakov loop through one gluon exchange. The second contraction is the one point function at zero momentum. The one point function is nonzero in thermal physics because Lorentz invariance is reduced to rotational invariance, so Q_0 is a scalar. It is shown in Fig. 1.

This term would vanish if $\mathbf{L}''(\mathbf{q}) = 0$, i.e. if the constraint were linear. But it is not or else it would not be gauge invariant. Indeed, in addition to the usual free-energy terms, this contribution renders the total result independent of the gauge choice. This will become clear from the BRS analysis below.

In the $SU(2)$ case the spatial averages and the Polyakov loop average simplify to

[5]At three-loop order there are, however, linear infrared divergences.

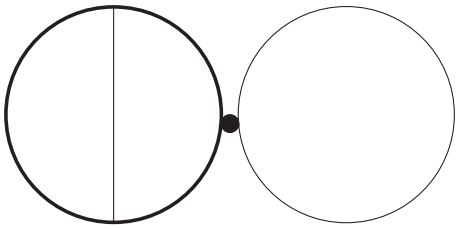

FIG. 1. The insertion diagram, Eq. (49). The two blobs correspond to the two contractions. The Polyakov loop is the fat circle, with the gluon going across. The dot is where $\hat{Q}_0$ acts to create the one point function shown by the thin circle.

$$\langle \mathrm{Tr}\bar{Q}_0^2 \cdot \mathbf{L}''(\mathbf{q})\rangle = (3-\xi)\hat{B}_1(q_{12})\sin(\pi q_{12}),$$
$$\hat{Q}_0 = \mathrm{Tr}(\bar{\bar{Q}}_0\sigma_3)\sin(\pi q_{12}), \tag{50}$$

where we used Eq. (34). Also, Eq. (49) becomes

$$\langle \mathrm{Tr}\bar{Q}_0^2 \cdot \mathbf{L}''(\mathbf{q})\rangle \left\langle \frac{\partial}{\partial \hat{Q}_0} Q^3 \cdot S'''_{\rm gauge}(\mathbf{q}) \right\rangle = 4(3-\xi)\hat{B}_1(q_{12})\hat{B}_3(q_{12}). \tag{51}$$

Note that all reference to the unitary nature of the loop in the constraint has dropped out. The derivative of the loop, $\sin(\pi q_{12})$ drops out of the insertion of the radiative correction for the Polyakov loop into the one loop effective action. For groups larger than $SU(2)$ this is true as well, though much less trivial (see Appendix B).

In the $SU(N)$ case the contribution (48) becomes a sum over $N-1$ winding contributions,

$$\Gamma_i = \sum_n \langle \bar{Q}_0^2 \cdot \mathrm{Tr}L^n(\mathbf{q})''\rangle \left\langle \frac{\partial}{\partial \hat{Q}_0^n} Q^3 \cdot S'''_{\rm gauge}(\mathbf{q}) \right\rangle. \tag{52}$$

In Appendix B we prove the crucial identity

$$\langle \mathrm{Tr}(\bar{Q}_0^2 \cdot (\mathbf{L}^n)'')\rangle = \langle \mathrm{Tr}\bar{Q}_0^2 \cdot \mathbf{L}''(\mathbf{q})\rangle_d \cdot t_d^n(\mathbf{q}). \tag{53}$$

This identity relates the one-loop expectation values of multiply winding loops to that for single winding, using the matrix defined in (41). In conjunction with (41) it eliminates the summation over windings n in (52) and reduces it to a summation over diagonal indices d.

$$\Gamma_i^{(2)} = (3-\xi)g^2 \sum_{d,b,c} f^{d,b,-b} f^{d,c,-c} \hat{B}_1(q_b)\hat{B}_3(q_c). \tag{54}$$

We used for this result that the VEV for single winding is the $\mathcal{O}(g^2)$ correction to the Polyakov loop $\mathrm{Tr}\mathbf{L}(\mathbf{q})$ in the background $\mathbf{q}$,

$$\langle \mathrm{Tr}\bar{Q}_0^2 \cdot \mathbf{L}''(\mathbf{q})\rangle_d = (3-\xi)\sum_{d,ij} f^{d,ij,ji}\hat{B}_1(q_{ij}). \tag{55}$$

Note that this expectation value refers to the traced loop without the normalization factor $1/N$. Further, that it is gauge choice dependent and proportional to the Bernoulli function $\hat{B}_1(q_{ij})$. This function is linear, periodic mod 1, antisymmetric and vanishes at $q_{ij} = 1/2$. Hence it is a sawtooth function, with nonzero values at $q_{ij} = 0, 1$. As usual, d refers to the diagonal index while b and c run through off-diagonal indices only. In Sec. IV we simplify the summation over the index d.

As $\hat{B}_3$ vanishes linearly for small argument it follows that $\Gamma_i^{(2)}$ vanishes, to this order, in the limit of zero background.

F. BRS identities and the gauge independence of the effective potential

We need to understand *why* the ξ dependence in the diagram for the free-energy contribution Γ_f cancels against that in the insertion diagram Γ_i. There is a simple way to see this, using the BRS identities in the presence of a thermal background [18].

All we have to do is to take the free-energy contribution for an arbitrary value of ξ and to note that the ξ dependence is due exclusively to the gluon propagators. It does not appear anywhere else in the free-energy diagrams.

Varying the gluon propagators in the three diagrams (a1), (a2) and (a3) shown in Fig. 2 multiplies each by 3, 2 and 1, respectively. Combining these factors with the combinatorial factors in the figure turns the result into the one loop gluon self-energy $\Pi_{\mu,\nu}^{(1)a,b}$ [see (b1)–(b3) in Fig. 2] folded into the gauge part of the propagator,[6]

$$\frac{\partial \Gamma_f^{(2)}}{\partial \xi} = \not{\sum}_p \Pi_{\mu,\nu}^{(1)a,b} \frac{p_\mu^a p_\nu^b}{(p^a)^4}. \tag{56}$$

We use the shorthand notation

$$T\sum_{n_0} \int \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}} \equiv \not{\sum}_p. \tag{57}$$

If $a = b = d$, with d a diagonal index, the BRS identity tells us, as in the case without background, that the one-loop self-energy is transverse,

$$\Pi_{\mu,\nu}^{(1)d,d} p_\mu^d p_\nu^d = 0. \tag{58}$$

In case $a = (ij)$, $b = (ji)$ is off-diagonal, the BRS identity relates the two-point function to the one-point function Γ_d,

$$\Gamma_d^{(1)} = \left\langle \frac{\partial S_{\rm int}}{\partial Q_0^d} \right\rangle. \tag{59}$$

Only the scalar, color diagonal one-point function is nonvanishing in thermal field theory. We obtain

[6]This identity is a special case of an identity valid for any field theory.

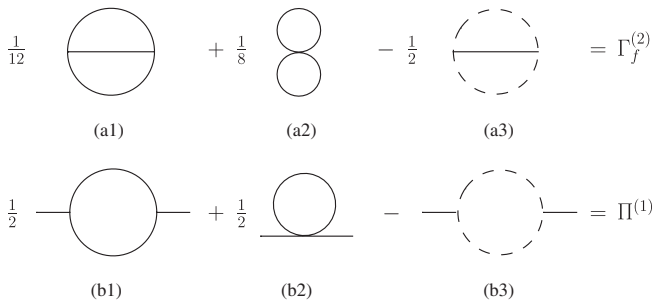

FIG. 2. The two-loop free-energy contributions $\Gamma_f^{(2)}$ to the effective potential are shown in (a1), (a2) and (a3). The one-loop self-energy is shown in (b1)–(b3).

$$\Pi_{\mu,\nu}^{(1)ij,ji} p_\mu^{ij} p_\nu^{ji} = f^{d,ij,ji} p_0^{ij} \Gamma_d^{(1)}(\mathbf{q}),$$
$$\Gamma_d^{(1)}(q) = \sum_{k,l} f^{d,kl,lk} \hat{B}_3(q_{kl}). \tag{60}$$

Unless explicitly shown there is no summation over color indices in (58) and (59). The second equality relates the one point function to the background field derivative of the free energy. The final result for the gauge variation follows from Eq. (56) and the BRS identities (58) and (60),

$$\frac{\partial \Gamma_f^{(2)}}{\partial \xi} = g^2 \sum_{ijkl} \not{\sum}_p \frac{p_0^{ij}}{(p^{ij})^4} \sum_d f^{d,ij,ji} f^{d,kl,lk} \hat{B}_3(q_{kl}). \tag{61}$$

Inspection of the first factor in this expression shows (see Appendix A on Bernoulli functions) that

$$\not{\sum}_p \frac{p_0^{ij}}{(p^{ij})^4} = \hat{B}_1(q_{ij}). \tag{62}$$

Hence, the gauge dependence cancels precisely with the gauge variation of the insertion diagram (54).

III. TWO-LOOP CORRECTION: EXPLICIT RESULT

Now that we have seen the cancellation of the gauge artifacts in the two contributions to the effective action, we evaluate them for various groups. Because the free-energy contribution Γ_f, Eq. (46), is so simple in $\xi = 1$ gauge, we calculate Eq. (54) for $\xi = 1$ as well. It shows that the insertion diagram not only guarantees gauge parameter independence, but also a surprisingly simple outcome for the effective action.

We should mention also that although the specific forms of Eqs. (46) and (54) are based on our discussion of $SU(N)$, they in fact apply to all the groups that we are going to consider. For later use, we rewrite the effective potential up to two-loop order with $\xi = 1$ as

$$\Gamma^{(1)} = -\frac{\pi^2 T^4 d(A)}{45} + \sum_a \hat{B}_4(q_a), \tag{63}$$

$$\Gamma_f^{(2)} = g^2 \sum_{a,b,c} |f^{a,b,c}|^2 \hat{B}_2(q_b)\hat{B}_2(q_c), \tag{64}$$

$$\Gamma_i^{(2)} = 2g^2 \sum_{d,b,c} f^{d,b,-b} f^{d,c,-c} \hat{B}_1(q_b)\hat{B}_3(q_c). \tag{65}$$

Here, the one-loop effective action depends on the dimension of the adjoint representation of the group which is denoted as $d(A)$. It equals N^2-1 for $SU(N)$, $2N^2-N$ for $SO(2N)$, 14 for $G(2)$, while for both $Sp(N)$ and $SO(2N+1)$ we have $d(A) = 2N^2+N$. The index a runs through the off-diagonal indices. In $\Gamma_f^{(2)}$, the indices a, b and c run over both diagonal and off-diagonal indices. In $\Gamma_i^{(2)}$, each structure constant contains the diagonal indices d, while b and c denote off-diagonal indices. If b is a typical off-diagonal index, the index $-b$ is defined as follows: let E^b being some off-diagonal generator, then $E^{-b} \equiv (E^b)^\dagger$. The definition of these indices will become more clear in the following.

Our calculation will show that the two-loop effective potential is simply a multiplicative and background *independent* renormalization of the one-loop result. In terms of the quadratic Casimir invariant $C_2(A)$ in the adjoint representation,

$$\frac{\Gamma^{(2)}}{\Gamma^{(1)}} = -\frac{5g^2 C_2(A)}{16\pi^2}, \tag{66}$$

where the Casimir invariant is given by

$$C_2(A)\delta_{ce} = f^{a,b,c} f^{a,b,e}. \tag{67}$$

From this definition it follows in particular that for two diagonal indices $c = d$ and $e = d'$,

$$C_2(A) d(r) = f^{a,b,d} f^{a,b,d}, \tag{68}$$

where $d(r)$ is the rank of the group, i.e. the dimension of the Cartan space. We have that $C_2(A) = N-1$ for $SO(2N)$, $N-\frac{1}{2}$ for $SO(2N+1)$, $N+1$ for $Sp(2N)$, 2 for $G(2)$, and finally $C_2(A) = N$ for the $SU(N)$ groups. For the $SU(N)$ groups the result (66) was in fact known since long for straight paths[7] from the origin, $\mathbf{q} = 0$, to the degenerate $Z(N)$ minima [19]. These paths run along the edges of the $SU(N)$ Weyl chamber and a combinatorial proof of Eq. (66) exists [20]. We do not (yet) know how the combinatorics works out inside the Weyl chamber.

Below we gather the tools to produce explicit expressions for the two loop insertion $\Gamma_i^{(2)}$ and the free energy $\Gamma_f^{(2)}$. Due to the increasing number of independent variables of the background field and the complication of the indices of the structure constants, we developed a MATHEMATICA program [21] for all classical groups to evaluate explicitly the above two contributions, Eq. (64) and Eq. (65). However, we have not succeeded in finding a general proof of Eq. (66) which does not require explicit evaluation by brute force.

A. Generalities on the classical Lie algebras

We start with the commutation relations in the Cartan basis for any semi-simple Lie algebra,

$$[\vec{H}, E_\alpha] = \vec{\alpha} E_\alpha \tag{69}$$

$$[E_\alpha, E_{-\alpha}] = \vec{\alpha}\cdot\vec{H} \tag{70}$$

$$[E_\alpha, E_\beta] = f^{\alpha,\beta,-\alpha-\beta} E_{\alpha+\beta}, \quad \text{if } \alpha+\beta \text{ is a root; if not, it vanishes.} \tag{71}$$

We define the structure constants from the generators in the fundamental representation of the group, with the generators normalized as

$$\mathrm{Tr}(E_\alpha E_{-\alpha}) = \mathrm{Tr}(H_d^2) = 1/2. \tag{72}$$

The components of $\vec{H}$ are the orthonormal matrices spanning the Cartan subalgebra and they are the diagonal generators in the Cartan basis. The orthonormal E_α, labelled by the roots α, are vectors in Cartan space. They are the off-diagonal generators.

The roots themselves are labeled by an off-diagonal index. For a typical off-diagonal index, say a, we have

$$[H^d, E_a] = f^{d,a,-a} E_a. \tag{73}$$

Hence, the dth component of a root (labeled by an off-diagonal index a) is the structure constant $f^{d,a,-a}$. Besides the structure constants involving a diagonal component, we have another kind of structure constants $f^{\alpha,\beta,-\alpha-\beta}$ which connect off-diagonal generators. With our normalization, the absolute values of $f^{\alpha,\beta,-\alpha-\beta}$ are all equal to $\frac{1}{\sqrt{2}}$ for $SU(N)$. For all the other classical groups they are $\frac{1}{2}$. For the exceptional group $G(2)$, they are given in Sec. IIIE. In the following, we shall discuss the commutation relations of the generators and the corresponding structure constants for each group separately.

[7]However, in general the minimum of the potential does not exactly follow a straight path as a function of temperature [4].

B. Calculation for $SU(N)$

Starting from Eqs. (64) and (65), we are able to calculate the two-loop perturbative correction to the effective potential. First of all, we need to know the structure constants. They can be obtained from the commutation relations of the generators in Cartan basis.

For $SU(N)$, there are $N(N-1)$ off-diagonal generators $E^{ij} \equiv \lambda^{ij}$ with $i, j = 1, \ldots, N$ and $i \neq j$. The explicit forms are given in Eq. (44). In addition, we have $N-1$ traceless diagonal generators $H^d \equiv \lambda^d$ with $d = 1, \ldots, N-1$,

$$\lambda^d = \frac{1}{\sqrt{2d(d+1)}} \operatorname{diag}(1, 1, \ldots, -d, 0, 0, \ldots, 0). \quad (74)$$

The commutators between diagonal generators are obviously zero. The nonvanishing commutators we need are[8]

$$[H^d, E^{ij}] \equiv f^{d,ij,lk} E^{kl} = (\lambda^d_{ii} - \lambda^d_{jj}) E^{ij}, \quad (75)$$

$$[E^{ij}, E^{kl}] \equiv f^{ij,kl,ts} E^{st} = \frac{1}{\sqrt{2}} (\delta_{jk} E^{il} - \delta_{il} E^{kj}). \quad (76)$$

Here, λ^d_{ii} is the ith diagonal component of λ^d. From Eq. (75) we find that the roots $\vec{\alpha}^{ij} = (\vec{\lambda}_{ii} - \vec{\lambda}_{jj})$. As mentioned before, the dth component of $\vec{\alpha}^{ij}$ is the structure constant $f^{d,ij,ji}$.

We can define a diagonal matrix,

$$\Lambda^{ij} \equiv \frac{\vec{\lambda} \cdot \vec{\alpha}^{ij}}{(\vec{\alpha}^{ij})^2}, \quad (77)$$

and it is easily to find the following commutator:

$$[\Lambda^{ij}, E^{ij}] = E^{ij}. \quad (78)$$

Using the explicit form of E^{ij}, we get $\Lambda^{ij} = \frac{1}{2} \operatorname{diag}(0, 0, \ldots, 1, 0, \ldots, 0, -1, 0, \ldots, 0)$, i.e. the ith component is 1, the jth component is -1 and all others are zero.

Taking the square of Eq. (77) and then the trace on both sides, the roots satisfy

$$(\vec{\alpha}^{ij})^2 = 1. \quad (79)$$

In other words, the roots can be written in terms of an orthonormal basis $\{\vec{e}_i\}$ spanning an N-dimensional space,

$$\vec{\alpha}^{ij} = \frac{1}{\sqrt{2}} (\vec{e}_i - \vec{e}_j). \quad (80)$$

[8]For $SU(N)$, if a typical off-diagonal index b is denoted by $b = ij$, then we have $-b = ji$.

Using Eqs. (68) and (79), we have $C_2(A) = N$ for $SU(N)$. Notice that there are $N^2 - N$ off-diagonal indices and that the rank of $SU(N)$ is $N-1$.

Using the total antisymmetry of the structure constants, all the nonvanishing structure constants without diagonal index can be read off from Eq. (76). It is obvious that the absolute values of these structure constants are $1/\sqrt{2}$.

Since the explicit form of the generators is known, the calculations of the structure constants is straightforward but rather tedious as N becomes large. In fact, we can rewrite Eqs. (75) and (76) to obtain the following expressions for the structure constants:

$$f^{d,ij,kl} = 2 \operatorname{Tr}(E^{kl} \cdot [H^d, E^{ij}]),$$
$$f^{ij,kl,st} = 2 \operatorname{Tr}(E^{st} \cdot [E^{ij}, E^{kl}]). \quad (81)$$

These expressions permit a straightforward computation of the structure constants using MATHEMATICA.

When using Eqs. (63) to (65) to compute the effective potential for $SU(N)$, one needs to observe that for a diagonal index $q_a = 0$. However, for an off-diagonal index ij one has $q_a = q_i - q_j$. We have $N-1$ independent q_i, for $i = 1, 2, \ldots, N-1$. Thus, the background field can be parametrized as $Q = \operatorname{diag}(q_1, q_2, \ldots, q_N)$ with the single constraint $q_N = -q_1 - q_2 - \ldots - q_{N-1}$. The above discussion can be understood by using the following commutator:

$$[Q, E^{ij}] = (q_i - q_j) E^{ij}. \quad (82)$$

It is obvious that Q commutes with H^d which leads to $q_d = 0$ as stated.

In general, there is no restriction to the possible values of q_i with $i = 1, 2, \ldots, N-1$. Therefore, modulo functions appear in the Bernoulli polynomials which makes the calculation more involved. However, without loss of generality, we can perform the calculation with a set of q'_i such that the absolute values of the arguments of the Bernoulli polynomials are less than 1, i.e., $-1 < q'_i - q'_j < 1$. It is easy to show that it is always possible to find such a set of q'_i. For example, when considering $SU(3)$, we have $Q = \operatorname{diag}(q_1, q_2, q_3)$ with $q_3 = -q_1 - q_2$. If we define $q'_i = q_i - n_i$ where n_i is an integer, we can achieve $0 \leq q'_i < 1$ by appropriate choice of n_i. Since the Bernoulli polynomials are periodic modulo 1, one can use q'_i instead of q_i to calculate the effective potential and the result is the same. The advantage of using q'_i is to avoid these modulo functions (see Appendix A) which can not be handled easily by MATHEMATICA. We mention that in terms of q'_i the background field is not necessarily traceless. In fact, we have $q'_3 = -q'_1 - q'_2 + n$ with $n = 0$, 1, 2. With the set of q'_i, the Bernoulli polynomials are given by Eq. (A6) with sign functions only. By permutation of the matrix elements of the background field, we can assume

that $q'_1 \geq q'_2 \geq \cdots \geq q'_N$. With this assumption, the sign of $q'_i - q'_j$ becomes definitive which can further simplify the calculation by ignoring the sign functions in the Bernoulli polynomials. The details can be found in Appendix A.

Based on Eq. (81) and the above discussion, we have been able to compute the two-loop perturbative correction to the effective potential for $SU(N)$ for any given but arbitrary N; we have performed this calculation explicitly up to $N = 5$ with MATHEMATICA [21] and verified the relation (66). For example, for $SU(2)$ we find

$$\begin{aligned}\Gamma_f^{(2)} &= \frac{g^2T^4}{24}[1 + 2q'_1(q'_1 - 1) \\ &\quad + 2q'_2(1 + q'_2 - 2q'_1)][1 + 6q'_1(q'_1 - 1) \\ &\quad + 6q'_2(1 + q'_2 - 2q'_1)], \\ \Gamma_i^{(2)} &= \frac{g^2T^4}{3}(1 + 2q'_2 - 2q'_1)^2(q'_1 - q'_2)(1 + q'_2 - q'_1).\end{aligned} \tag{83}$$

The effective potential at one-loop order is given by

$$\Gamma^{(1)} = -\frac{\pi^2T^4}{15} + \frac{4T^4\pi^2}{3}(q'_2 - q'_1)^2(1 + q'_2 - q'_1)^2, \tag{84}$$

so that

$$\frac{\Gamma_f^{(2)} + \Gamma_i^{(2)}}{\Gamma^{(1)}} = -\frac{5g^2}{8\pi^2}, \quad \text{for } N = 2. \tag{85}$$

Note that we write these equations in terms of q'_i to remind the readers that these variables should satisfy $0 \leq q'_i < 1$ and $q'_1 \geq q'_2 \geq \cdots \geq q'_N$. For example, if the background field is given as $Q = \mathrm{diag}(-\frac{5}{3}, \frac{5}{3})$, to get the correct effective potential from Eqs. (83) and (84) one should set $q'_1 = \frac{2}{3}$ and $q'_2 = \frac{1}{3}$. This procedure can be easily generalized to higher N.

C. Calculation for $SO(2N)$ and $SO(2N+1)$

For these groups we use a variant of the notation from Georgi's book [22]. The generators M^{ab} in the fundamental representation have matrix elements,

$$(M^{ab})_{xy} = -\frac{i}{2}(\delta_{ax}\delta_{by} - \delta_{ay}\delta_{bx}). \tag{86}$$

Obviously there is antisymmetry under exchange of the labels a and b, i.e. $M^{ab} = -M^{ba}$.

Furthermore, we can define the off-diagonal generators in the Cartan basis. For both groups, there are $N(2N-2)$ off-diagonal generators $E^{\eta i.\eta' j}$ with $i, j = 1, \ldots, N$ and $i > j$. Here, we define the indices i with an associated sign η. Similarly, j is defined with η'. The signs η or η' are independently ± 1. The explicit form of the generators is

$$E^{\eta i.\eta' j} = \frac{1}{2}(M^{2i-1,2j-1} + i\eta M^{2i,2j-1} + i\eta' M^{2i-1,2j} - \eta\eta' M^{2i,2j}). \tag{87}$$

For $SO(2N+1)$ there are $2N$ additional off-diagonal generators

$$E^{\eta i} = \frac{1}{\sqrt{2}}(M^{2i-1,2N+1} + i\eta M^{2i,2N+1}). \tag{88}$$

For either of the groups the N-dimensional Cartan subalgebra is spanned by mutually commuting and orthogonal generators H^d, with

$$H^d = M^{2d-1,2d}, \quad \text{with} \quad d = 1, 2, \ldots N. \tag{89}$$

So far, we have defined all the generators in the Cartan basis; the structure constants can be obtained from the commutation relation[9]

$$[H^d, E^{\eta j}] \equiv f^{d,\eta j,-\eta' k}E^{\eta' k} = \frac{\eta}{2}\delta_{dj}E^{\eta j} \tag{90}$$

$$\begin{aligned}[H^d, E^{\eta j.\eta' k}] &\equiv f^{d,\eta j.\eta' k,-\rho l.-\rho' m}E^{\rho l.\rho' m} \\ &= \frac{1}{2}(\eta\delta_{dj} + \eta'\delta_{dk})E^{\eta j.\eta' k}\end{aligned} \tag{91}$$

$$\begin{aligned}[E^{\eta i.\eta' j}, E^{\rho k}] &\equiv f^{\eta i.\eta' j,\rho k,-\sigma l}E^{\sigma l} \\ &= \frac{i}{4}(\delta_{ki}(1 - \rho\eta)E^{\eta' j} - \delta_{kj}(1 - \rho\eta')E^{\eta i})\end{aligned} \tag{92}$$

$$\begin{aligned}[E^{\eta i.\eta' j}, E^{\rho k.\rho' l}] &\equiv f^{\eta i.\eta' j,\rho k.\rho' l,-\sigma t.-\sigma' n}E^{\sigma t.\sigma' n} \\ &= \frac{i}{4}(\delta_{ki}(1 - \rho\eta)E^{\eta' j.\rho' l} - \delta_{kj}(1 - \rho\eta')E^{\eta i.\rho' l} \\ &\quad - \delta_{lj}(1 - \rho'\eta')E^{\rho k.\eta i} + \delta_{il}(1 - \eta\rho')E^{\rho k.\eta' j}).\end{aligned} \tag{93}$$

From Eqs. (90) and (91), the roots can be expressed as

$$\vec{\alpha}^{\eta i} = \frac{\eta}{2}\vec{e}_i, \qquad \vec{\alpha}^{\eta i.\eta' j} = \frac{1}{2}(\eta\vec{e}_i + \eta'\vec{e}_j). \tag{94}$$

There are $N(2N-2)$ off-diagonal generators associated with the long roots and $2N$ off-diagonal generators associated with the short roots. For both $SO(2N)$ and

[9]For $SO(2N)$ and $SO(2N+1)$, if the typical off-diagonal index b is denoted as $b = \eta i.\eta' j$ then $-b = -\eta i. - \eta' j$; if $b = \eta i$, then $-b = -\eta i$. In Eq. (93), with our notation, $E^{\rho k.\eta i}$ should be understood as $-E^{\eta i.\rho k}$ if $i > k$. Similarly for $E^{\eta' j.\rho' l}$.

$SO(2N+1)$, $d(r)=N$. Using Eq. (68) we can easily get $C_2(A)=N-\frac{1}{2}$ for $SO(2N+1)$ and $C_2(A)=N-1$ for $SO(2N)$.

Like for $SU(N)$, in order to perform the calculation with MATHEMATICA we express the structure functions as

$$\begin{aligned} f^{d,\eta j,\eta' k} &= 2\,\mathrm{Tr}(E^{\eta' k}\cdot[H^d,E^{\eta j}]) \\ f^{d,\eta j.\eta' k,\rho l.\rho' m} &= 2\,\mathrm{Tr}(E^{\rho l.\rho' m}\cdot[H^d,E^{\eta j.\eta' k}]) \\ f^{\eta i.\eta' j,\rho k,\sigma l} &= 2\,\mathrm{Tr}(E^{\sigma l}\cdot[E^{\eta i.\eta' j},E^{\rho k}]) \\ f^{\eta i.\eta' j,\eta k.\eta' l,\sigma t.\sigma' n} &= 2\,\mathrm{Tr}(E^{\sigma t.\sigma' n}\cdot[E^{\eta i.\eta' j},E^{\rho k.\rho' l}]). \end{aligned} \tag{95}$$

If an index a is an off-diagonal index, we have two different cases: if $a=\eta i$, then $q_a=\eta q_i$; if $a=\eta i.\eta' j$, then $q_a=\eta q_i+\eta' q_j$. Here, we have N independent q_i for $i=1,2,\ldots,N$. Thus, the background field can be parametrized as $Q=\sum_{d=1}^{N}2q_dH^d$. The above discussion can be understood by using the two commutators from Eqs. (90) and (91).

In order to perform the computation with our MATHEMATICA program we again require an appropriate choice of q_i. For $SO(2N+1)$ and $SO(2N)$, we can always start the calculation by using a set of q_i which satisfy $-\frac{1}{2}<q_i\le\frac{1}{2}$. As a result, the arguments of the Bernoulli polynomials are restricted to the interval $-1\le x\le 1$, and we can use Eq. (A6) which does not involve modulo functions. Without loss of generality, we can also assume $q_1\ge q_2\ge\cdots\ge q_N\ge 0$ by a suitable permutation of the matrix elements of Q. At this point we are able to use our program to compute the effective potential for these two groups. For example, with $N=2$, the results for $SO(5)$ are

$$\begin{aligned} \Gamma_f^{(2)} &= \frac{g^2T^4}{8}\left[\frac{5}{6}+(3q_1^2-3q_1)(2-3q_1+3q_1^2)\right. \\ &\quad -(8q_1^2-8q_1+2)q_2 \\ &\quad \left.+(11-36q_1+36q_1^2)q_2^2-6q_2^3+9q_2^4\right], \\ \Gamma_i^{(2)} &= \frac{g^2T^4}{4}\left[3(1-2q_1)^2(1-q_1)q_1+4\left(q_1^2-q_1+\frac{1}{4}\right)q_2\right. \\ &\quad \left.+(48q_1-13-48q_1^2)q_2^2+8q_2^3-12q_2^4\right]. \end{aligned} \tag{96}$$

The effective potential at one-loop order reads

$$\begin{aligned} \Gamma^{(1)} &= -\frac{2\pi^2T^4}{9} \\ &\quad +\frac{4T^4\pi^2}{3}[3(q_1-1)^2q_1^2-2q_2^3+3q_2^4 \\ &\quad +3(q_2-2q_1q_2)^2]. \end{aligned} \tag{97}$$

It is straightforward to show that

$$\frac{\Gamma_f^{(2)}+\Gamma_i^{(2)}}{\Gamma^{(1)}}=-\frac{15g^2}{32\pi^2},\quad \text{for } N=2. \tag{98}$$

We can easily get analogous results for $SO(2N)$ simply by ignoring the off-diagonal generator $E^{\eta i}$. We have verified Eq. (66) for $SO(2N+1)$ and $SO(2N)$ up to $N=5$.

D. Calculation for $Sp(2N)$

In this section we discuss the symplectic groups $Sp(2N)$. They are the pseudoreal part of $SU(2N)$ constructed by defining the charge conjugation matrix,

$$I_{2N}=i\sigma_2\otimes\mathbf{1}_N, \tag{99}$$

and requiring the special unitary matrix U to obey

$$I_{2N}UI_{2N}^{\dagger}=U^*, \tag{100}$$

where σ_i are the Pauli matrices with $i=1,2,3$, and $\mathbf{1}_N$ is the N-dimensional unit matrix.

Writing $U=\exp(i\mathcal{G})$, the symplectic generator is of the form

$$\mathcal{G}=\begin{pmatrix} A & B \\ B^* & -A^* \end{pmatrix}. \tag{101}$$

Here, A is a Hermitian matrix with $A=A^{\dagger}$, and $B=B^t$ is complex. For $N=1$, this form indeed reduces to the generator of $SU(2)$. The Hermitian matrix A is not traceless, but $\mathcal{G}$ is. We therefore have N^2 real degrees of freedom from A, and $N(N+1)$ degrees of freedom from the symmetric complex matrix B. In total, we have $N(2N+1)$. The Cartan space is N dimensional.

The diagonal generators of $Sp(2N)$ is

$$H^d=\frac{1}{\sqrt{2}}\sigma_3\otimes\lambda^d,\quad d=1,\ldots,N. \tag{102}$$

Here, the $N-1$ matrices λ^d are the same as for $SU(N)$, and we need the additional $\lambda^N=\frac{1}{\sqrt{2N}}\mathbf{1}_N$.

The corresponding off-diagonal generators E^{ij} are

$$E^{ij}=\frac{1}{\sqrt{2}}\begin{pmatrix} \lambda^{ij} & 0 \\ 0 & -\lambda^{ji} \end{pmatrix},\quad i,j=1,\ldots,N,\quad \text{and}\quad i\ne j. \tag{103}$$

The E^{ij} produce the roots of $SU(N)$, up to a factor of $\frac{1}{\sqrt{2}}$,

$$[H^d,E^{ij}]=\frac{1}{\sqrt{2}}(\lambda_{ii}^d-\lambda_{jj}^d)E^{ij}. \tag{104}$$

In addition, we have additional $N(N+1)$ off-diagonal generators of the complex symmetric matrix B which are denoted $E^{\eta ij}$; the first index η is a sign index. They are defined by

$$E^{\eta ij} = \left[\frac{1}{\sqrt{2}} + \delta_{ij}\left(\frac{1}{2} - \frac{1}{\sqrt{2}}\right)\right]\sigma^{\eta} \otimes (\lambda^{ij} + \lambda^{ji}),$$
$$i, j = 1, \ldots, N, \quad \text{and} \quad i \geq j, \tag{105}$$

where $\sigma^{\eta} = \frac{1}{2}(\sigma_1 + i\eta\sigma_2)$. Here, the index i can be equal to j which defines $2N$ long roots. The generators $E^{\eta ij}$ produce a new type of roots. For $i > j$, we have

$$[H^d, E^{\eta ij}] = \frac{\eta}{\sqrt{2}}(\lambda^d_{ii} + \lambda^d_{jj})E^{\eta ij}. \tag{106}$$

For $i = j$, we have

$$[H^d, E^{\eta ii}] = \eta\sqrt{2}\lambda^d_{ii}E^{\eta ii}. \tag{107}$$

Like for $SU(N)$, we can write the roots for $Sp(2N)$ in terms of the orthonormal basis $\{\vec{e}_i\}$ introduced in Sec. III B,

$$\vec{\alpha}^{\eta ij} = \frac{\eta}{2}(\vec{e}_i + \vec{e}_j), \quad 1 \leq j < i \leq N,$$
$$\vec{\alpha}^{ij} = \frac{1}{2}(\vec{e}_i - \vec{e}_j), \quad 1 \leq i \leq N, \quad 1 \leq j \leq N \quad \text{and} \quad i \neq j,$$
$$\vec{\alpha}^{\eta i} = \eta\vec{e}_i, \quad 1 \leq i \leq N. \tag{108}$$

Here, the first roots are associated with the generators $E^{\eta ij}$ when $i > j$ and the second roots are associated with the generators E^{ij}. These two kinds of roots have length $\frac{1}{\sqrt{2}}$ and they are the short roots. There are $2N(N-1)$ of those. The roots $\vec{\alpha}^{\eta i}$ (which can be also written as $\vec{\alpha}^{\eta ii}$) come from $E^{\eta ii}$. The $2N$ roots $\vec{\alpha}^{\eta i}$ have length 1 and are the long roots.

Inversion of the roots is called duality and delivers the roots of $SO(2N+1)$. The roots of $Sp(2N)$ are those of $SO(2N+1)$, with the long roots being the short ones, and the short roots being the long ones. With our normalization of the generators there is an overall factor of $1/2$.

From this discussion we can easily get $C_2(A) = N + 1$ for $Sp(2N)$. For $N = 1$, we find $C_2(A) = 2$ which is the same as for $SU(2)$, as expected.

On the other hand, the commutation relations between the off-diagonal generators are

$$[E^{ij}, E^{kl}] = \frac{1}{2}(\delta_{jk}E^{il} - \delta_{il}E^{kj}),$$
$$[E^{+ij}, E^{-kl}] = \frac{1}{2}(\delta_{jk}E^{il} + \delta_{il}E^{jk} + \delta_{jl}E^{ik} + \delta_{ik}E^{jl}),$$
$$[E^{\eta ij}, E^{\eta kl}] = 0. \tag{109}$$

The first line is the commutation relation of $SU(N)$, with the structure constant $1/2$ instead of $1/\sqrt{2}$. The other lines all reflect the symmetry in the indices of the $E^{\eta kl}$.

For $Sp(2N)$, we have four different types of structure constants due to the nonvanishing commutators, namely, $f^{d,\eta ij,\eta' kl}$, $f^{d,ij,kl}$, $f^{tn,ij,kl}$ and $f^{tn,\eta ij,\eta' kl}$. The structure constants can be obtained from the nonvanishing commutators as discussed above.[10] As before, these structure constants can be obtained also by the trace calculation. With the definition of the structure constants, it is very straightforward to write down the equations corresponding to Eqs. (81)and (95).

The background field of $Sp(2N)$ can be parametrized as $Q = \sigma_3 \otimes Q'$ where $Q' = \mathrm{diag}(q_1, q_2, \ldots, q_N)$ is an $N \times N$ diagonal matrix. Using the commutators

$$[Q, E^{ij}] = (q_i - q_j)E^{ij}, \qquad [Q, E^{\eta ij}] = \eta(q_i + q_j)E^{\eta ij}, \tag{110}$$

we found that the arguments of the Bernoulli functions are the following: $q_d = 0$, $q_{ij} = q_i - q_j$ and $q_{\eta ij} = \eta(q_i + q_j)$.

For the N independent variables q_i of the background field, with the same assumption on their values as $SO(2N+1)$ and $SO(2N)$, we can compute the effective potential for $Sp(2N)$ with our program. For example, with $N = 2$, the results for $Sp(4)$ are given by

$$\Gamma^{(2)}_f = \frac{5g^2T^4}{24}\left[\frac{27}{2}q_2^4 - 10q_2^3 + q_2(2q_1 - 2q_1^2 - 1) + q_2^2\left(\frac{9}{2} - 7q_1 + 9q_1^2\right) + \frac{q_1}{2}(17q_1 - 34q_1^2 + 27q_1^3 - 4)\right],$$
$$\Gamma^{(2)}_i = g^2T^4[30q_2^3 - 36q_2^4 - 2(1 - 3q_1)^2q_1(2q_1 - 1) + q_2(1 - 2q_1 + 2q_1^2) - 2q_2^2(6 - 11q_1 + 12q_1^2)]. \tag{111}$$

The effective one-loop potential is

$$\Gamma^{(1)} = -\frac{2\pi^2T^4}{9} + \frac{8T^4\pi^2}{3}[9q_2^4 - 8q_2^3 + q_2^2(3 - 6q_1 + 6q_1^2) + q_1^2(3 - 10q_1 + 9q_1^2)]. \tag{112}$$

Therefore,

[10] For $Sp(2N)$, if the typical off-diagonal index b is denoted as $b = ij$, then $-b = ji$ which is the same as for $SU(N)$; if $b = \eta ij$, then $-b = -\eta ij$.

$$\frac{\Gamma_f^{(2)}+\Gamma_i^{(2)}}{\Gamma^{(1)}}=-\frac{15g^2}{16\pi^2},\quad \text{for } N=2. \tag{113}$$

For $Sp(2N)$, we verify Eq. (66) explicitly up to $N=5$.

E. Calculation for $G(2)$

$G(2)$ is a subgroup of $SO(7)$. It leaves the structure constants of the octonians invariant, and this is the way it is traditionally defined. However, the algebra of $G(2)$ is related in a straightforward way, by simple projections, to that of $SO(7)$ as shown in Fig. 3. In what follows the indices i,j run from 1 to 3 and $i>j$. The relation between the two groups is quite simple: six of the twelve long roots $\vec{\alpha}^{\eta i.\eta' j}$ are in the plane $q_1+q_2+q_3=0$ as shown in Fig. 4). They are the roots associated with the generators $E^{+i.-j}$ and $E^{-i.+j}$. The other six that are not in that plane are projected onto that plane. They are the projections of the six short roots of $SO(7)$ which are associated with the generators $E^{\eta i}$.

Because of the projection, the short roots of $G(2)$ are $\frac{1}{\sqrt{3}}$ in units of the long roots. Recall that the short roots of $SO(7)$ were $\frac{1}{\sqrt{2}}$ in units of the long roots. The projection respects the commutation relations of $SO(7)$, except for the scale factor we just mentioned. For example, in $SO(7)$, we have

$$[E^{\eta i},E^{\eta j}]=\frac{i}{2}E^{\eta i.\eta j}. \tag{114}$$

The generator on the right-hand side projects onto $-\eta\epsilon_{ijk}E^{-\eta k}$. As a result, for $G(2)$, the above commutator reads

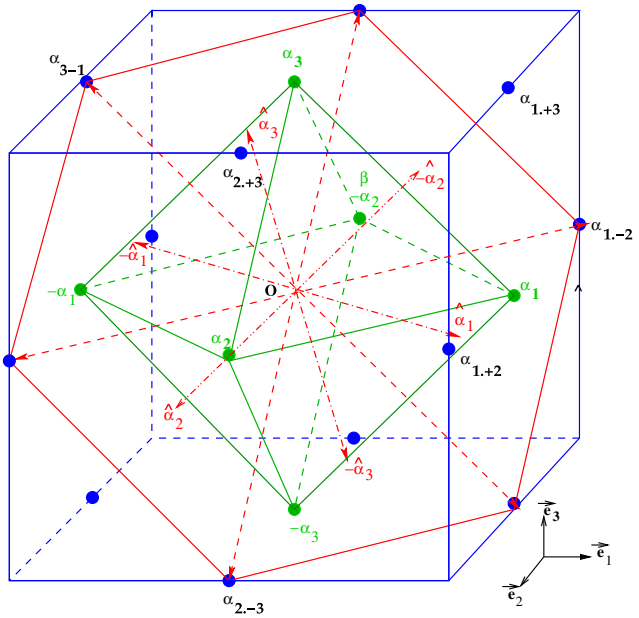

FIG. 3 (color online). The three-dimensional root space of $SO(7)$, with the plane where the six roots $\vec{\alpha}^{\eta i.\eta' j}$ lie. This plane is the root space of $G(2)$, on which the six short roots $\vec{\alpha}^{\eta i}$ of $SO(7)$ are projected. The six projections $\vec{\alpha}^{\eta i}$ are of length $\frac{1}{\sqrt{3}}$ in units of the length of the six roots $\vec{\alpha}^{\eta i.\eta' j}$.

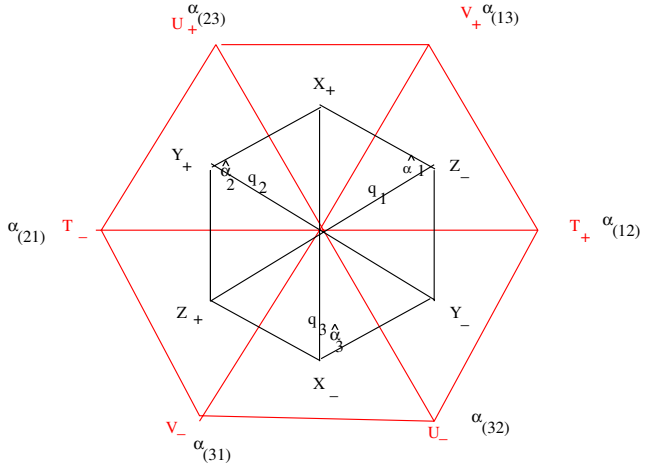

FIG. 4 (color online). The root space of $G(2)$, which is the $q_1+q_2+q_3=0$ plane in Fig. 3, with the same notation for the roots. The roots are related to the matrices $T^{\pm}$ etc. in Ref. [23].

$$[E^{\eta i},E^{\eta j}]=-\frac{i\eta}{\sqrt{3}}\epsilon_{ijk}E^{-\eta k}. \tag{115}$$

The three commuting generators of the $SO(7)$ Cartan algebra reduce to two for $G(2)$, because of the constraint $q_1+q_2+q_3=0$. For $G(2)$, we define the two Cartan generators

$$H^1=\frac{1}{\sqrt{2}}(M^{12}-M^{34}), \tag{116}$$

$$H^2=\frac{1}{\sqrt{6}}(M^{12}+M^{34}-2M^{56}). \tag{117}$$

The prefactor ensures that $\mathrm{Tr}(H^d)^2=\frac{1}{2}$. Together with the other twelve off-diagonal generators $E^{+i.-j}$, $E^{-i.+j}$ and $E^{\eta i}$, we have the explicit form of all the 14 generators for $G(2)$. Except for the one given by Eq. (115), all other commutation relations can be obtained from the corresponding equations of $SO(7)$, i.e. Eqs. (92) and (93). In addition, the commutation relations involving the diagonal generators are

$$\begin{aligned}
[H^1,E^{\eta i}]&=\frac{\eta}{2\sqrt{2}}(\delta_{1i}-\delta_{2i})E^{\eta i},\\
[H^2,E^{\eta i}]&=\frac{\eta}{2\sqrt{6}}(\delta_{1i}+\delta_{2i}-2\delta_{3i})E^{\eta i},\\
[H^1,E^{\eta i.\eta' j}]&=\frac{1}{2\sqrt{2}}(\eta(\delta_{1i}-\delta_{2i})+\eta'(\delta_{1j}-\delta_{2j}))E^{\eta i.\eta' j},\\
[H^2,E^{\eta i.\eta' j}]&=\frac{1}{2\sqrt{6}}(\eta(\delta_{1i}+\delta_{2i}-2\delta_{3i})\\
&\quad+\eta'(\delta_{1j}+\delta_{2j}-2\delta_{3j}))E^{\eta i.\eta' j}.
\end{aligned} \tag{118}$$

The above commutators state that the square of the roots $\vec{\alpha}^{\eta i,\eta' j}$, which are associated with the 6 generators $E^{\eta i,\eta' j}$, equal $1/2$ and the square of the roots $\vec{\alpha}^{\eta i}$ associated with the 6 generators $E^{\eta i}$ equal $1/6$. Since the rank of $G(2)$ is 2, we get $C_2(A) = 2$.

We can obtain all the structure constants that are needed to compute the effective potential through

$$\begin{aligned} f^{d,\eta i,\rho j} &= 2\,\mathrm{Tr}(E^{\rho j}\cdot[H^d, E^{\eta j}]) \\ f^{d,\eta j.-\eta k,\rho l.-\rho m} &= 2\,\mathrm{Tr}(E^{\rho l.-\rho m}\cdot[H^d, E^{\eta j.-\eta k}]), \\ f^{\eta i.-\eta j,\rho k,-\rho l} &= 2\,\mathrm{Tr}(E^{-\rho l}\cdot[E^{\eta i.-\eta j}, E^{\rho k}]), \\ f^{\eta i.-\eta j,\rho k.-\rho l,\sigma t.-\sigma n} &= 2\,\mathrm{Tr}(E^{\sigma t.-\sigma n}\cdot[E^{\eta i.-\eta j}, E^{\rho k.-\rho l}]). \end{aligned} \quad (119)$$

In addition, we have a special one from Eq. (115) which is $f^{\eta i,\eta j,\rho k} = -\frac{i\eta}{\sqrt{3}}\epsilon_{ijk}\delta_{\eta\rho}$. We employ these expressions to compute the structure constants in our MATHEMATICA program.

The background field of $G(2)$ can be parametrized in the same way as $SO(7)$, with an additional constraint that $q_3 = -q_1 - q_2$. Furthermore, for the possible values of q_i, we use the same assumptions as for $SU(3)$. As a result, the constraint becomes $q_3 = -q_1 - q_2 + n$ with $n = 0, 1, 2$. Notice that for $G(2)$, the argument of the Bernoulli functions can be 0, ηq_i and $\eta(q_i - q_j)$. Unlike $SO(7)$, there is no $q_i + q_j$ in the Bernoulli functions and our assumption on the values of q_i enables us to avoid the modulo function and also make the sign function definitive.

The resulting effective potential for $G(2)$ reads

$$\begin{aligned} \Gamma_f^{(2)} = \frac{g^2T^4}{3}\Big[&\frac{7}{12} + q_3 - q_2 + 4q_1^4 + 4q_3^2(1 + q_3 + q_3^2) - q_3(1 + q_3)(1 + 6q_3)q_2 \\ &+ (4 + 5q_3 + 10q_3^2)q_2^2 - 2(1 + 3q_3)q_2^3 + 4q_2^4 - 2q_1^3(4 + 3q_3 + 3q_2) + q_1^2(7 + 8q_3 + 10q_3^2 + 5q_2 + 10q_2^2) \\ &- q_1(3 + 4q_3 + 10q_3^2 + 6q_3^3 + q_2 + 7q_2^2 + 6q_2^3)\Big], \\ \Gamma_i^{(2)} = \frac{g^2T^4}{9}[&-32q_1^4 - q_3(3 + 8q_3)(2 + 3q_3 + 4q_3^2) + q_3(25 + 57q_3 + 52q_3^2)q_2 - (25 + 69q_3 + 72q_3^2)q_2^2 \\ &+ 4(3 + 13q_3)q_2^3 - 32q_2^4 + q_1^3(60 + 52q_3 + 52q_2) - q_1^2(34 + 78q_3 + 72q_3^2 + 69q_2 + 72q_2^2) \\ &+ q_1(6 + 34q_3 + 66q_3^2 + 52q_3^3 + 25q_2 + 57q_2^2 + 52q_2^3)]. \end{aligned} \quad (120)$$

Comparing to the one-loop result

$$\begin{aligned} \Gamma^{(1)} = -\frac{14\pi^2T^4}{45} + \frac{4T^4\pi^2}{3}&[(q_1 - 1)^2q_1^2 + (q_1 - 1 - q_3)^2(q_1 - q_3)^2 + (q_3 - 1)^2q_3^2 \\ &+ (q_1 - 1 - q_2)^2(q_1 - q_2)^2 + (q_3 - q_2)^2(1 + q_3 - q_2)^2 + (q_2 - 1)^2q_2^2], \end{aligned} \quad (121)$$

we see that

$$\frac{\Gamma_f^{(2)} + \Gamma_i^{(2)}}{\Gamma^{(1)}} = -\frac{5g^2}{8\pi^2}. \quad (122)$$

However, unlike for $SU(N)$, to obtain this result we must explicitly use that $q_3 = -q_1 - q_2 + 1$ or $q_3 = -q_1 - q_2 + 2$ or $q_1 = q_2 = q_3 = 0$.

IV. A SIMPLIFIED FORM FOR THE INSERTION

The insertion diagram involves sums over diagonal indices d which can be performed quite easily as they correspond to inner products between the corresponding roots. We use the relation between the roots and the unit vectors mentioned in the previous sections to reduce the inner products to sums of Kronecker δ's. In addition, the antisymmetry of the Bernoulli polynomials $\hat{B}_1$ and $\hat{B}_3$ is needed. For example, $\hat{B}_1(\eta q_i + \eta' q_j) = \eta'\hat{B}_1(\eta\eta' q_i + q_j)$, etc. Those are then applied to the expression for $\Gamma_i^{(2)}$ from Eq. (65).

For $SU(N)$ this is quite simple. Using

$$\vec{\alpha}^{ij}\hat{B}_n(q_i - q_j) = \sqrt{2}\vec{e}_i\hat{B}_n(q_i - q_j), \quad (123)$$

we get

$$\Gamma_i^{(2)}(SU(N)) = 4g^2\sum_{ijl}\hat{B}_1(q_i - q_j)\hat{B}_3(q_i - q_l). \quad (124)$$

Here, the only constraint on the indices is that $i \neq j$ and $i \neq l$. In Eq. (123), $\hat{B}_n$ always refers to $\hat{B}_1$ or $\hat{B}_3$ and this applies throughout this section.

For the orthogonal groups the long and short roots satisfy the following relations[11]

$$\sum_{\eta\eta'}\vec{\alpha}^{\eta i,\eta' j}\hat{B}_n(\eta q_i+\eta' q_j)=\vec{e}_i(\hat{B}_n(q_i+q_j)+\hat{B}_n(q_i-q_j)), \tag{125}$$

$$\sum_{\eta}\vec{\alpha}^{\eta i}\hat{B}_n(\eta q_i)=\vec{e}_i\hat{B}_n(q_i). \tag{126}$$

As a result, the insertion diagram for $SO(2N)$ is reduced to

$$\begin{aligned}\Gamma_i^{(2)}(SO(2N))=2g^2\sum_{i,j,l}&(\hat{B}_1(q_i+q_j)\\&+\hat{B}_1(q_i-q_j))(\hat{B}_3(q_i+q_l)\\&+\hat{B}_3(q_i-q_l)),\end{aligned} \tag{127}$$

with $i\neq j$ and $i\neq l$. For $SO(2N+1)$, the result is

$$\begin{aligned}\Gamma_i^{(2)}(SO(2N+1))=\Gamma_i^{(2)}(SO(2N))&+2g^2\sum_{i,j}[(\hat{B}_1(q_i+q_j)\\&+\hat{B}_1(q_i-q_j))\hat{B}_3(q_i)+(\hat{B}_3(q_i+q_j)\\&+\hat{B}_3(q_i-q_j))\hat{B}_1(q_i)]\\&+2g^2\sum_i\hat{B}_1(q_i)\hat{B}_3(q_i).\end{aligned} \tag{128}$$

and the same constraints $i\neq j$ and $i\neq l$ apply.

Finally, for $Sp(2N)$, we have

$$\sum_{\eta}\vec{\alpha}^{\eta ij}\hat{B}_n(\eta(q_i+q_j))=\vec{e}_i(1+\delta_{ij})\hat{B}_n(q_i+q_j), \tag{129}$$

$$\vec{\alpha}^{ij}\hat{B}_n(q_i-q_j)=\vec{e}_i\hat{B}_n(q_i-q_j). \tag{130}$$

In Eq. (129), the case where $i=j$ is included. In Eq. (130), $i\neq j$ applies. The simplified insertion diagrams read

$$\begin{aligned}\Gamma_i^{(2)}(Sp(2N))=2g^2\sum_i\Bigg(&\sum_j(\hat{B}_1(q_i+q_j)+\hat{B}_1(q_i-q_j))\\&+2\hat{B}_1(2q_i)\Bigg)\Bigg(\sum_l(\hat{B}_3(q_i+q_l)\\&+\hat{B}_3(q_i-q_l))+2\hat{B}_3(2q_i)\Bigg),\end{aligned} \tag{131}$$

where $i\neq j$ and $i\neq l$ apply.

[11]On the left-hand side of Eq. (125), $j>i$ is forbidden according to our notations. However, on the right-hand side of this equation, $j>i$ is permitted. The same is true for Eq. (129).

For $G(2)$, we can also work out the structure constants to get

$$\begin{aligned}\Gamma_i^{(2)}(G(2))=2g^2\sum_{ijl}&\hat{B}_1(q_i-q_j)(\hat{B}_3(q_l-q_j)\\&-\hat{B}_3(q_l-q_i))+\frac{4g^2}{3}\sum_i\hat{B}_1(q_i)\hat{B}_3(q_i)\\&+2g^2\sum_{ij}\hat{B}_1(q_i-q_j)(2\hat{B}_3(q_i-q_j)\\&+\hat{B}_3(q_i)-\hat{B}_3(q_j))\\&+\frac{2g^2}{3}\sum_{il}\hat{B}_1(q_i)(3\hat{B}_3(q_i-q_l)-\hat{B}_3(q_l)),\end{aligned} \tag{132}$$

where $i>j$, $j\neq l$ and $i\neq l$.

Using the explicit expressions for $\Gamma_i^{(2)}$ given in this section, we can compute for larger N more efficiently. At the same time, we can easily prove that the result for $\Gamma_i^{(2)}$ is independent of the value of $\hat{B}_1(n_0)$ for integer n_0. We point out that for the last term in $G(2)$, to show the independence on $\hat{B}_1(n_0)$, we need to use the condition $q_1+q_2+q_3=0$, 1, 2. Notice that although the same condition appears for $SU(N)$, it is not actually needed to show independence of $\hat{B}_1(n_0)$.

For the free energy $\Gamma_f^{(2)}$, according to Eq. (64), there is an obvious simplification if one of the indices in the structure constant is diagonal. In this case, $\sum_d|f^{d,b,c}|^2$ is just the square of the roots' length which is already known. (Here, b and c denote off-diagonal indices.) For instance, for $SU(N)$ such a term becomes

$$g^2\sum_{ij}(2\hat{B}_2(0)\hat{B}_2(q_i-q_j)+(\hat{B}_2(q_i-q_j))^2). \tag{133}$$

It is straightforward to get these contributions for other groups and we don't list the rest here.

For the case where $f^{a,b,c}$ has no diagonal index, there is no obvious simplification. However, the values of these structure constants can be simply read off from the commutators given above.

V. CONCLUSIONS

The main result of this paper is that the two-loop renormalization of the effective potential is very simple: the two-loop potential is proportional to that at one loop, Eq. (66). There is nothing in the way we perform the computation that suggests such simplicity. For $SU(N)$ groups it has long been known [20,24] that this proportionality holds along the edges of the Weyl chamber.

Hence, at this order in perturbation theory the minima of the perturbative action stay put. How this works out to three-loop order is something that remains to be worked out.

The two-loop effective potential found here could now be supplemented by a model for nonperturbative physics, e.g. along the lines of Refs. [2–4], in an attempt to understand the *eigenvalue* distribution of the Polyakov loop in the gauge theories mentioned above.

ACKNOWLEDGMENTS

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APPENDIX A: BERNOULLI POLYNOMIALS

We define the Bernoulli polynomials,

$$\hat{B}_{d-2k}(x) = T\sum_{n_0}\int \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}}\frac{1}{(p^{ij})^{2k}}, \tag{A1}$$

$$\hat{B}_{d-2k+1}(x) = T\sum_{n_0}\int \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}}\frac{p_0^{ij}}{(p^{ij})^{2k}}, \tag{A2}$$

$$\hat{B}_{d}(x) = T\sum_{n_0}\int \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}}(\log(p^{ij})^2 - \log\ p^2). \tag{A3}$$

In these equations, $p_0^{ij} = 2\pi T(n_0 + x^{ij})$ with n_0 an integer. Below, the indices i and j associated with x are omitted for simplicity of notation. Also, $(p^{ij})^2 = (p_0^{ij})^2 + \vec{p}^2$ and $p^2 = (2\pi n_0 T)^2 + \vec{p}^2$.

In $d = 4$ dimensions and for $k = 0$, 1, we have the following four Bernoulli polynomials:

$$\hat{B}_4(x) = \frac{2}{3}\pi^2 T^4 B_4(x), \qquad \hat{B}_3(x) = \frac{2}{3}\pi T^3 B_3(x),$$
$$\hat{B}_2(x) = \frac{1}{2}T^2 B_2(x), \qquad \hat{B}_1(x) = -\frac{T}{4\pi}B_1(x), \tag{A4}$$

with

$$B_4(x) = x^2(1-x)^2, \qquad B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x,$$
$$B_2(x) = x^2 - x + \frac{1}{6}, \qquad B_1(x) = x - \frac{1}{2}. \tag{A5}$$

The above expressions are defined on the interval $0 \le x \le 1$ and they are periodic functions of x, with period 1. For arbitrary values of x, the argument of the above Bernoulli polynomials should be understood as $x - [x]$ with $[x]$ the largest integer less than or equal to x, which is nothing but the modulo function.

If $-1 \le x \le 1$ we can drop the modulo functions and the Bernoulli polynomials reduce to

$$B_4(x) = x^2(1-\epsilon(x)x)^2,$$
$$B_3(x) = x^3 - \frac{3}{2}\epsilon(x)x^2 + \frac{1}{2}x,$$
$$B_2(x) = x^2 - \epsilon(x)x + \frac{1}{6},$$
$$B_1(x) = x - \frac{1}{2}\epsilon(x), \tag{A6}$$

where $\epsilon(x)$ is the sign function.

In fact the Bernoulli polynomials $B_1(x)$ and $B_3(x)$ are odd functions of x, while $B_2(x)$ and $B_4(x)$ are even functions of x, so we can always make the arguments of Bernoulli polynomials positive(or be zero) and ignore the sign functions which can save a lot of computing time. However, we point out that $B_1(x)$ has discontinuities at integer x. For example, the value of $B_1(0)$ depends on the way one approaches zero, from above or from below. If the result of the effective potential depends on $B_1(0)$, we have to know how one approaches zero in order to use the correct values of $B_1(0)$. In this case, the sign function in $B_1(x)$ is very important and can not be dropped even $x \ge 0$. Fortunately, we can prove that the contributions related to $B_1(n_0)$ vanish without specifying the value of $B_1(n_0)$. Therefore, the effective potential does not depend on $B_1(n_0)$ and we can simply drop the sign functions when $x \ge 0$. The proof is straightforward when using the total antisymmetry of the structure constants. Alternatively, one can also prove it by using the simplified expressions of the insertion diagram given in Sec. IV.

APPENDIX B: THE IDENTITY EQUATION (53)

This identity relates the one-gluon correction of the multiply winding Polyakov loop to the one-gluon correction of the Polyakov loop with single winding,

$$\langle \mathrm{Tr}(\overline{Q_0^2}\cdot(\mathbf{L}^n)'')\rangle = \langle \bar{Q}_0^2\cdot\mathbf{L}''\rangle^d t_d^n(\Phi). \tag{B1}$$

Here, $D^{ii} = \sum_j \lambda^{ij}\lambda^{ji}$ and the definition of $\Delta_{(r)}$ is[12]

$$\Delta_{(r)}(\Phi_{ij}) = \not{\sum}_p \frac{1}{(p_0^{ij})^r}\Delta_{00}(p^{ij}), \tag{B2}$$

with

[12]The dependence on the argument Φ_{ij} is through p^{ij}. We have $(p^{ij})^2 = (p_0^{ij})^2 + \vec{p}^2$ and $p_0^{ij} = 2\pi T n_0 + T\Phi_{ij}$.

$$\Delta_{00}(p^{ij}) = \frac{\delta_{00} - (1-\xi)\frac{(p_0^{ij})^2}{(p^{ij})^2}}{(p^{ij})^2}. \tag{B3}$$

The multiply winding loop $\mathrm{Tr}(\mathbf{L}(A_0)^n)$ can be written as a time-ordered product from time $\tau = 0$ to $\tau = \frac{n}{T}$,

$$\mathrm{Tr}(\mathbf{L}(A_0)^n) = \mathrm{Tr}\,\mathcal{P}\,\exp\left(i\int_0^{n/T} d\tau A_0(\vec{x},\tau)\right). \tag{B4}$$

There is a caveat: the field A_0 is periodic modulo $\frac{1}{T}$, *not* $\frac{n}{T}$. This smaller periodicity is guaranteed by the Matsubara frequencies being integer multiples of $2\pi T$. The propagator follows from the action and has the small periodicity $1/T$.

Diagonal gluons do not feel the background field; upon integration over the emission and absorption times they are odd in the Matsubara frequency and so do not contribute. We only have to consider the contractions of off-diagonal $\langle Q_0^{ij}(\tau_2)Q_0^{ji}(\tau_1)\rangle$. These propagators are gauge field propagators in ξ gauge,

$$\langle Q_0^{ij}(\tau_2)Q_0^{ji}(\tau_1)\rangle = \exp(ip_0(\tau_2-\tau_1))\Delta_{00}(p^{ij}). \tag{B5}$$

Note the shift of the Matsubara frequencies in the propagator $\Delta_{00}(p^{ij})$. This follows from the diagonalization in color space of the bilinear part of the action. The propagators in (B5) are still periodic modulo $1/T$.

Thus, the calculation of the one-loop average of $\mathrm{Tr}(\mathbf{L}(A_0)^n)$ boils down to the one- gluon exchange correction in

$$\overline{\mathrm{Tr}\,\mathcal{P}\,\exp\left(i\int_0^{n/T} A_0(\vec{x},\tau)\right)}. \tag{B6}$$

For convenience in what follows we write Φ instead of $\mathbf{q} = \Phi/2\pi$ in the arguments of $\mathbf{L}$ and the exponents. The calculation of the average is now quickly achieved,

$$\begin{aligned}\left\langle \overline{\mathrm{Tr}\,\mathcal{P}\,\exp\left(i\int_0^{n/T} A_0(\vec{x},\tau)\right)}\right\rangle \\ = -g^2\int_0^{n/T}\int_0^{\tau_1} d\tau_1 d\tau_2 \langle \mathrm{Tr}\,\exp(i\Phi T\tau_2)Q_0(\tau_2) \\ \times \exp(i\Phi T(\tau_1-\tau_2))Q_0(\tau_1)\exp(i\Phi(n-T\tau_1))\rangle.\end{aligned} \tag{B7}$$

We now use the propagator (B5) for $\langle Q_0^{ij}(\tau_2)Q_0^{ji}(\tau_1)\rangle$ and the identities

$$\exp(i\Phi T\tau)\lambda_{ij}\,\exp(-i\Phi T\tau) = \exp(i\Phi_{ij}T\tau)\lambda_{ij}, \tag{B8}$$

to shift the Matsubara frequencies from p_0 to $p_0 + \Phi_{ij}T = p_0^{ij}$. The result is that we can drop the heavy quark propagators but shift the frequency p_0 in the propagator in Eq. (B5) to $p_0^{ij} = p_0 + \Phi_{ij}T$.

The time-ordered integrals give two terms,

$$\begin{aligned}\int_0^{n/T}\int_0^{\tau_1} d\tau_1 d\tau_2\,\exp(ip_0^{ij}(\tau_2-\tau_1)) \\ = \frac{(1-\exp(-in\Phi_{ij}))}{(p_0^{ij})^2} - \frac{n}{iTp_0^{ij}}.\end{aligned} \tag{B9}$$

We do the same for the mode with $\langle Q_0^{ji}(\tau_2)Q_0^{ij}(\tau_1)\rangle$. As expected it gives Eq. (B9) with $i\leftrightarrow j$. If we sum over $-p_0$ instead of p_0, we see that the denominator of the first resp. second term are even resp. odd under interchange of i and j. Substituting into Eq. (B7) we get the combination (remember that $D^{ii} = \lambda^{ij}\lambda^{ji}$)

$$\begin{aligned}&\langle \mathrm{Tr}(\bar{Q}_0^2\cdot(\mathbf{L}^n)'')\rangle \\ &= -\frac{g^2}{2}\sum_{ij}\mathrm{Tr}\left[\frac{n}{iT}\,\exp(in\Phi)(D_{ii}-D_{jj})\Delta_{(1)}(\Phi_{ij})\right. \\ &\quad + \exp(in\Phi)\Delta_{(2)}(\Phi_{ij})(D^{ii}(1-\exp(-inT\Phi_{ij})) \\ &\quad \left. + D^{jj}(1-\exp(inT\Phi_{ij})))\right].\end{aligned} \tag{B10}$$

The second term, proportional to $\Delta_{(2)}$ drops out after taking the trace. The reason is that $\mathrm{Tr}(\exp(in\Phi)D^{ii}) = \frac{1}{2}\exp(in\Phi_i)$. Clearly, the untraced loop contains unphysical results like the divergent $\Delta_{(2)}$ but the trace projects them out.

The latter argument is not only valid for $SU(N)$ but also for the other classic groups (by using the roots $\vec{e}_i$).

Remarkably, the matrix $t_d^n(\Phi)$ factors out and we obtain Eq. (53),

$$\langle \mathrm{Tr}(\bar{Q}_0^2\cdot(\mathbf{L}^n)'')\rangle = \langle \bar{Q}_0^2\cdot\mathbf{L}''\rangle_d t_d^n(\Phi). \tag{B11}$$

This the desired factorization, and the first factor $\langle \bar{Q}_0^2\cdot\mathbf{L}''\rangle_d$ is the projection on λ_d of the one-gluon corrected Polyakov loop.

Recall that the insertion diagram in Fig. 1 involves only summation over the looping index n. The derivative acting on the gauge field vertices is, according to (41),

$$\hat{Q}_0^n \equiv \sum_d t_d^n(\mathbf{q})\bar{\bar{Q}}_0^d. \tag{B12}$$

Hence, the summation over these loop indices n drops out, because of the factorization we just obtained,

$$\Gamma_i = \sum_{n=1}^{N-1} \langle \mathrm{Tr}(\bar{Q}_0^2 \cdot (\mathbf{L}^n)'') \rangle \left\langle \frac{\partial S_{\mathrm{int}}}{\partial \hat{Q}_0^n} \right\rangle = \sum_{d=1}^{N-1} \langle \bar{Q}_0^2 \cdot \mathbf{L}'' \rangle_d \left\langle \frac{\partial S_{\mathrm{int}}}{\partial \bar{Q}_0^d} \right\rangle. \tag{B13}$$

$$\langle \bar{Q}_0^2 \cdot \mathbf{L}'' \rangle_d = g^2 \frac{1}{2T} \sum_{ij} f^{d,ij,ji} \Delta_{(1)}(\Phi^{ij}). \tag{B14}$$

This is the result for all covariant background gauges and all classical groups. We leave it to the reader to isolate the part proportional to the gauge parameter ξ. This follows immediately from the expression for $\Delta_{(1)}$ in Eq. (B2).

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