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Conformal scaling and the size of *m*-hadrons

Luigi Del Debbio* and Roman Zwicky†

School of Physics and Astronomy, University of Edinburgh, Edinburgh EH9 3JZ, Scotland

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The scaling laws in an IR theory are dictated by the critical exponents of relevant operators. We have investigated these scaling laws at leading order in two previous papers. In this work we investigate further consequences of the scaling laws, trying to identify potential signatures that could be studied by lattice simulations. From the first derivative of the form factor we derive the behavior of the mean charge radius of the hadronic states in the theory. We obtain $\langle r_H^2 \rangle \sim m^{-2/(1+\gamma_m^*)}$ which is consistent with $\langle r_H^2 \rangle \sim 1/M_H^2$. The mean charge radius can be used as an alternative observable to assess the size of the physical states, and hence finite size effects, in numerical simulations. Furthermore, we discuss the behavior of specific field correlators in coordinate space for the case of conformal, scale-invariant, and confining theories making use of selection rules in scaling dimensions and spin. We compute the scaling corrections to correlations functions by linearizing the renormalization group equations. We find that these corrections are potentially large close to the edge of the conformal window. As an application we compute the scaling correction to the formula $M_H \sim m^{1/(1+\gamma_m^*)}$ directly through its associated correlator as well as through the trace anomaly. The two computations are shown to be equivalent through a generalization of the Feynman-Hellmann theorem for the fermion mass and the gauge coupling.

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I. INTRODUCTION

Gauge theories with an infrared fixed point (IRFP) are studied currently for building models of strongly interacting electroweak symmetry breaking [1–4]. At large distances the couplings flow toward their fixed point values, and the theory becomes scale-invariant. Theories with an IRFP are said to lie within the *conformal window*, see e.g. Refs. [5,6] for analytical results in the perturbative regime.

In the absence of supersymmetry, it is difficult to identify a fixed point in the nonperturbative regime of the theory. Lattice simulations provide a first principle tool to investigate the low-energy dynamics of asymptotically free gauge theories. Breaking scale invariance explicitly, e.g. by introducing a fermion mass term in the action and studying the scaling of field correlators as the breaking parameter tends to zero, has become a common way to characterize IRFPs in lattice studies e.g. [7]. A theoretical understanding of the scaling laws is a necessary tool for these analyses, and a number of useful (hyperscaling) relations have already been investigated in our previous work [8,9]. Working out these scaling relations is an interesting theoretical problem, independently of its application to the analysis of lattice data. For a recent discussion of lattice results, we refer the reader to the comprehensive review that appeared in Ref. [10].

Extending our previous work on mass-deformed conformal gauge theories (mCGT) [8,9], we discuss here the application of the scaling laws to a number of interesting physical cases, namely the scaling of the hadron size, the scaling corrections, and the determination of selection rules for field correlators.

The fact that hadrons emerge in a mCGT is a nontrivial empirical fact. At least at weak coupling this can be understood as a consequence of the fermions decoupling below the mass m, so that the low-energy dynamics should be described by a pure Yang-Mills effective theory, which is believed to be of confining nature [11]. In practical lattice simulations confinement is identified through a nonvanishing expectation value for the Polyakov loop, and it is characterized by the spectrum of the bound states that determine the correlators of gauge invariant interpolating fields as in QCD. In such a theory all hadronic parameters are controlled to leading order by the coupling m, which breaks explicitly scale invariance, and whose scaling exponent characterizes the long-distance dynamics. This is clearly at odds with the behavior observed in QCD, where chiral symmetry breaking requires the Goldstone bosons to be massless in the chiral limit, while the rest of the spectrum has a finite mass, which is dictated by some typical hadronic scale. We shall refer to the bound states of a mCGT as m-hadrons in what follows. For a mCGT the properties of these m-hadrons are very different from the ones commonly encountered in QCD-like theories. Being able to characterise the size of m-hadrons, and to compute the scaling of the size with the fermion mass is crucial in order to understand finite-size effects (FSE) in the results of numerical simulations. When the volume of the lattice is not large enough to accommodate the m-hadrons, FSE distort the spectrum, and may well obscure the scaling behavior that one is trying to identify. This is an important source of systematic errors in lattice studies, and has been a major concern in the interpretation of the most recent (and

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precise) studies of the spectrum of mCGT, see e.g. Refs. [12,13] for recent discussions.

There has been a recent renewed interest in the existence of theories that are scale invariant without being symmetric under the full conformal group. For recent work on scale-invariant (SFT) versus conformal field theories (CFT) see e.g. Refs. [14–18], and references therein for earlier work on the subject. It emerges from our analysis that the scaling laws for field correlators in a neighborhood of a fixed point provide a criterion to distinguish SFTs from CFTs. As a consequence, we discuss the possibility of identifying the existence of a fixed point describing a CFT by looking at the scaling behavior of the correlators when the theory is deformed by a mass term.

This paper is organized as follows. In Sec. II we rederive briefly the scaling laws, emphasizing the features that will be useful in the rest of our study. In Sec. III we apply the scaling relations to form factors of conserved currents, and deduce a scaling law for the radius of the charge distribution inside the (pseudo)scalar meson. In Sec. IV we use the scaling laws to formulate a criterion that allows us to distinguish a scale-invariant theory from a conformal-invariant one as well confining theories. Finally in Sec. V, we investigate the subleading corrections to the scaling laws for generic correlation functions. The corrections are explicitly calculated for the hadronic mass in two ways and their equivalence is shown using a Feynman-Hellmann type relation for the gauge coupling. The relation of the charge radius to the derivative of the form factor is summarized in Appendix A 2 for the reader's convenience.

II. CONFORMAL SCALING

Let us concentrate here on a theory with only one relevant perturbation at the IRFP, whose coupling we denote by m, and let us introduce an UV cutoff Λ; O_1 and O_2 are two local operators. The generic two-point correlator, evaluated on two arbitrary physical states $\varphi_{a,b}$, in the regulated bare theory:

$$C(x, m, \Lambda) = \langle \varphi_a | O_1(x) O_2(0) | \varphi_b \rangle \tag{1}$$

depends on the distance x, the coupling m and the scale Λ. In the expression above we rescale the dimensionful coupling m by some reference scale m_0, so that the correlator depends on the dimensionless coupling $\hat{m} \equiv m/m_0$. We denote the scaling dimension of the coupling m by $y_m \equiv d_m + \gamma_m$ where d_m and γ_m are the engineering and anomalous dimension, respectively (and clearly $d_m = 1$). We shall adopt the same conventions as in Refs. [8,9], denoting by d_{O_i} and γ_{O_i} the classical and anomalous dimensions of O_i and therefore the scaling dimension of the operator O reads: $\Delta_{O_i} \equiv d_{O_i} + \gamma_{O_i}$. For the sake of clarity, anticipating Sec. V, we shall denote by γ^* (and thus Δ^*) any anomalous dimension at the fixed point in order to distinguish it from the one away from the fixed point.

In computing the leading scaling we perform, as usual, a renormalization group (RG) transformation $\Lambda \to \Lambda/b$:

$$\begin{aligned} C(x, \hat{m}, \Lambda) &= b^{-(\gamma^*_{O_1} + \gamma^*_{O_2})} C(x, b^{y^*_m} \hat{m}, \Lambda/b), \\ y^*_m &= 1 + \gamma^*_m \end{aligned} \tag{2}$$

followed by a rescaling of all mass scales by a factor of b, on the right-hand side of (2):

$$C(x, b^{y^*_m} \hat{m}, \Lambda/b) = b^{-(d_{O_1} + d_{O_2} + d_{\varphi_a} + d_{\varphi_b})} C(x/b, b^{y^*_m} \hat{m}, \Lambda). \tag{3}$$

A crucial observation is that the physical states are free of anomalous scaling [9]. Combining Eqs. (2) and (3) we get

$$C(x, \hat{m}, \Lambda) = b^{-(\Delta^*_{O_1} + \Delta^*_{O_2})} C(x/b, b^{y^*_m} \hat{m}, \Lambda). \tag{4}$$

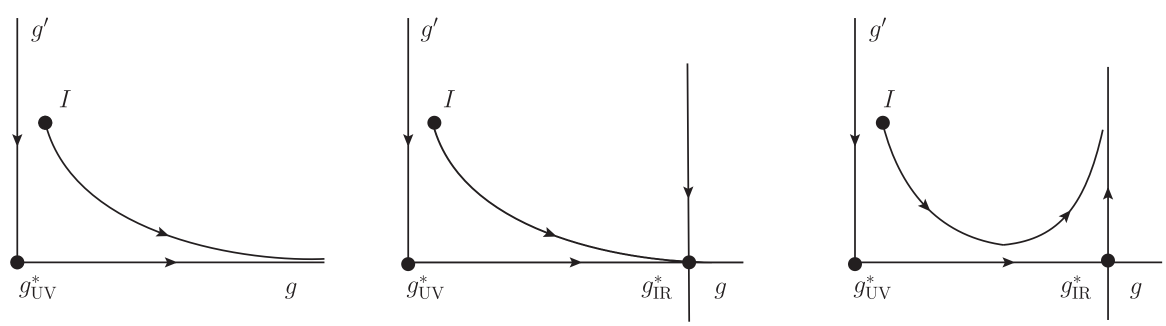

FIG. 1. Overview of behavior of relevant and irrelevant directions at UV and IR fixed points (FPs). The couplings, say $\mathcal{L}^{\rm eff} \sim g_i O_i$ with $y_i = 4 - \Delta_i$, fall into relevant ($y_{g_i} > 0$), irrelevant ($y_{g_i} < 0$) and marginal ($y_{g_i} = 0$) classes at the FPs. In all cases there is a trivial ($g^*_{UV} = 0$) ultraviolet fixed point UVFP and the y-axis corresponds to its critical surface. (Left) The couplings g' and g are irrelevant and relevant at the UVFP. An example of which is QCD with g being the gauge coupling and g' the quark mass. (Middle) nontrivial IR fixed point ($g^*_{IR} \neq 0$). The direction g is relevant and irrelevant at UV and IRFP, respectively, whereas g' is irrelevant at both fixed points. An example is IR-conformal gauge theory (with $m = 0$) and g' is four quark operator provided $y_{g'} < 4$ is really the case. (Right) The same as before but g' is relevant at IR fixed point. Examples are, assuming $y_m > 0$, mCGT where g' is the mass m. Another example is IR-conformal gauge theory with $m = 0$ where g' is four quark operator where this time $y_{g'} > 0$.

We can exploit the arbitrariness of b and choose it such that $b = \sqrt{x^2} m_0$. This then implies that

$$C(x, \hat{m}, \Lambda) = (\hat{x}^2)^{-\alpha} (m_0)^{d_{O_1}+d_{O_2}+d_{\varphi_a}+d_{\varphi_b}} F(\hat{x}^{y_m^*}\hat{m}, \hat{\Lambda}) \quad (5)$$

with $\alpha \equiv (\Delta^*_{O_1} + \Delta^*_{O_2} + d_{\varphi_a} + d_{\varphi_b})/2$, $\hat{\Lambda} = \Lambda/m_0$ and F a dimensionless function. We will use this particular form of the scaling law to derive some physical consequences in the following sections. The application to mCGTs can be inferred indirectly from the caption of Fig. 1. Discussion of finite size effects to Eq. (5) can be found in Appendix B 1.

A. Comment on additional relevant directions in mCGT

In derivations like the one shown in the previous section it was assumed that there is only one relevant operator driving the system away from the IRFP. Current lattice results seem to suggest that this is indeed the case for the theories that have been investigated so far. Nevertheless it might be the case that four quark operators

$$\mathcal{L}^{\mathrm{eff}} = \frac{c_{\bar{q}q\bar{q}q}}{\Lambda^2_{ETC}} \bar{q}q\bar{q}q \quad (6)$$

that do appear for example in extended technicolor (TC) models, become relevant, i.e. $\Delta_{\bar{q}q\bar{q}q} < 4$. In this case a situation like the one shown in Fig. 1 (right) will apply: in the very far IR this operator will grow and drive the system away from the fixed point; both the mass of the fermions and this additional coupling need to be tuned for the system to be on the critical surface. Academically one could hope to hit a trajectory that goes directly in the UVFP for which $c_{\bar{q}q\bar{q}q}|_{UVFP} = 0$, and then flow out of the UVFP along the renormalized trajectory flowing into the IRFP. This would be the equivalent of finding a perfect action for the IRFP. In practice, e.g. when setting the bare parameters in a simulation at finite lattice spacing, it is impossible to tune the system exactly to this point. The simple plaquette action does contain higher dimensional couplings by construction, and an infinite amount of tuning is needed to find a perfect action. Thus summa summarum the study of the scaling dimension of higher dimensional operators within mCGT will remain an important topic in practice.

III. SIZE OF m-HADRONS FROM FORM FACTORS

In this section we characterize the size of hadronic states in mCGTs by studying the radius of their charge distribution. The radius of the charge distribution is defined from the derivative of the form factor of the state; the latter is defined in turn from the matrix element of the conserved vector current between hadronic states. Scaling laws for the derivatives of the form factor can be deduced from the scaling laws we have obtained for the matrix elements in our previous paper [9].

In the following let us consider a matrix element where a scalar particle H probes a conserved vector current. On the grounds of Lorentz covariance the matrix element may be parametrized as follows [1]

$$\langle H(p_1)|V_\mu|H(p_2)\rangle = (p_1 + p_2)_\mu f_+^H(q^2), \quad J^{\mathrm{PC}}(H) = 0^{\mathrm{PC}}, \quad (7)$$

where $q \equiv p_1 - p_2$ is the momentum transfer to the current. Note that the structure $(p_1 - p_2)_\mu f_-^H(q^2)$ vanishes by virtue of current conservation: $\partial \cdot V = 0$. The function $f_+^H(q^2)$ is known as a form factor: its value at zero momentum corresponds to the charge of H under the current V_μ, and its derivative corresponds to the square of the charge distribution cf. Appendix A 2. For instance for the pion form factor in QCD,

$$f_+^{\pi_\pm}(0) = \pm 1, \quad \langle r^2_{\pi_\pm}\rangle = 6\frac{d}{dq^2} f_+^{\pi_\pm}(q^2)\Big|_{q^2=0}. \quad (8)$$

We wish to emphasize that (8) is not related to the pion's special role in QCD as should be clear from the notes in Appendix A 2. We shall later on contrast the behavior of the pion charge radius in QCD with the charge radius of a generic m-hadron. In order to determine the scaling exponents, following the notation in [8,9], we define

$$f^H_{+,n} \equiv \frac{d^n}{d(q^2)^n} f_+^H(q^2)\Big|_{q^2=0} \sim m^{\eta_{f_n}}, \quad (9)$$

and shall assume that the derivatives exist. Our main interest is to establish the behavior of the size of the m-hadrons as a function of the relevant perturbation m. We will proceed in two steps: (i) we derive the relative difference $\eta_{f_{n+1}} - \eta_{f_n}$, and (ii) we determine η_{f_0}.

(i) The mass dependence of the form factor, $f(q^2) \equiv f_+^H(q^2)$ for shorthand, is summarized in a scaling law akin to Eq. (5):

$$f(q^2) = \tilde{f}(\hat{q}^2/\hat{m}^{2/y_m^*}) = \tilde{f}(0) + \tilde{f}'(0)\left(\frac{\hat{q}^2}{\hat{m}^{2/y_m^*}}\right) + \frac{1}{2}\tilde{f}'(0)\left(\frac{\hat{q}^2}{\hat{m}^{2/y_m^*}}\right)^2 + \cdots, \quad (10)$$

where the dots stand for higher terms in the Taylor expansion. Note there is no dependence on the RG scale as the current is conserved. From Eq. (10) it is immediate to deduce

$$\eta_{f_{n+1}} - \eta_{f_n} = -2/y_m^* \quad (11)$$

[1] The current V_μ, which we do not specify any further at this point, might be in the flavor singlet or adjoint representation. The main point is that H couples to it. Subtle cases in the real hadronic world are $f_+^{\pi_0} = 0$ by virtue of C-covariance; yet $f_+^{K_0} \neq 0$ as K_0 is not C-eigenstate.

(ii) Second, we shall show $\eta_{f_0} = 0$. It follows directly from our master formula [9]:

$$\langle \varphi_2 | O(0) | \varphi_1 \rangle \sim (\hat{m})^{(\Delta^*_O + d_{\varphi_1} + d_{\varphi_2})/y^*_m} \quad (12)$$

where $\varphi_{1,2}$ are physical states. We note that $\Delta_{V_\mu} = 3$ (since V_μ is a conserved current) and that $d_{\varphi_1} = d_{\varphi_2} = -1$ which implies that $f^H_{+,1}(0)(p_1 + p_2)_\mu \sim m^{1/y^*_m}$. Since the energy momentum vector is free from anomalous scaling it counts like the engineering dimension in the formula in the nominator of the exponent in (12) and therefore $\tilde{f}(0) = f(0) \sim \mathcal{O}(1)$ (i.e. $\eta_{f_0} = 0$). Another way to arrive at the same result is to notice that $\tilde{f}(0)$ is equal to the charge and since the latter cannot scale with external parameters like the mass this implies that $\tilde{f}(0)$ is independent of the mass and thus $\eta_{f_0} = 0$.

Putting the two results together we get:

$$\eta_{f_n} = \frac{-2n}{y^*_m} \equiv \frac{-2n}{1+\gamma^*_m}, \quad (13)$$

and for the mean charge radius squared (8) we obtain:

$$\langle r^2_H \rangle = 6 \frac{d}{dq^2} f^H_+(q^2)\Big|_{q^2=0} \sim m^{\eta_{f_1}} = m^{-2/y^*_m} \sim \frac{1}{M^2_H}, \quad (14)$$

where M_H denotes the mass of the hadron H and we have used the general result $M_H \sim m^{1/y^*_m}$ derived for the entire hadronic spectrum in Ref. [9]. Thus, in summary, the size of the m-hadrons is inversely proportional to the hadronic mass. Whereas this result does not seem surprising, it is of importance for controlling FSE on the lattice. Whereas the scaling law gives information on the relative size of hadrons for different values of m, it does not determine its absolute size $\langle r^2_H \rangle = K_{r^2_H} M^{-2}_H$. The determination of $K_{r^2_H} \sim \mathcal{O}(1)$ could then be pursued by a measurement of the slope of the form factor (7) through Eq. (14). Using twisted boundary conditions could help in improving the momentum resolution, and hence in resolving better the slope of the form factor. The discussion of finite size effects in the context of the form factor can be found in Appendix B 2.

It would seem that the arguments of the form factor of a scalar coupled to a conserved current (7) ought to generalize to higher spin hadrons. The application to the analogue of the proton electromagnetic form factor should be rather straightforward. In general a more detailed analysis would necessitate the consideration of the corresponding polarization tensors. Suppose two higher spin hadrons couple to an operator O that is not necessarily related to a physical charge. Even though $\eta_{f_0}(O) \neq 0$ in general, we anticipate that the extension of the overlap with the operator O is determined by (11) based (10) which in turn follows from generic scaling arguments.

Let us briefly open a parenthesis here. Since $M_H \approx K_{M_H} m^{1/y^*_m} \Lambda_{\mathrm{ETC}}^{1-1/y^*_m}$ with $m \ll \Lambda_{\mathrm{ETC}}$ (c.f. Fig. 2 for an explanation of Λ_{ETC}), $K_{M_H} = \mathcal{O}(1)$, one concludes that for

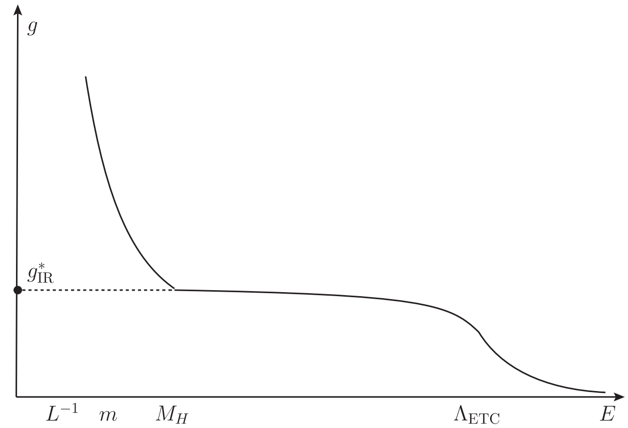

FIG. 2. Sketch of the RG flow for an IR-conformal gauge theory. At high energies the theory is asymptotically free, and at lower energies it reaches a fixed point g^*_{IR}. The mass parameter m, or equivalently the mass scale $M_H > m$ (of the hadronic bound states), drives the theory away from the FP. This is the picture that heuristic computations of the quark condensate suggest e.g. [8,19]. Note the inverse of the lattice box size has to be significantly smaller than M_H in order for FSE effects to be under control. More precisely as long as $LM_H \gg 1$, FSE are of the order $\exp(-M_H L)$. If this condition is not met the effects are powerlike with known exponents e.g. [13]. We should point out that we have not attempted to indicate the effect on the coupling of the actual value of L on the curve on the graph.

$$y^*_m \equiv 1 + \gamma^*_m > 1 \Rightarrow m < M_H, \quad (15)$$

at least for sufficiently small m to overcome the unknown $\mathcal{O}(1)$-coefficient discussed above. Other than that the hierarchy is controlled by the positivity of γ^*_m which is of course dependent on the actual gauge theory. Furthermore whereas the unitarity bound implies $\gamma_m \leq 2$ no lower bound exists other than the fact that for $\gamma_m < -1$ the operator becomes irrelevant which goes against our working assumption as well as all results, known to the authors, in the literature.

It is interesting to contrast the behavior of the mean charge radius of the (pseudo)scalar meson in mCGT to the one obtained for the Goldstone boson in QCD. More precisely, since in both cases the masses vanish in the limit $m \to 0$ it is clear from a heuristic viewpoint that, for a state with sharp momentum, the particle cannot be localized and therefore one expects the charge radius to diverge. The functional behavior of the divergence is not clear a priori. In a theory where chiral symmetry is spontaneously broken, the dynamics of the light Goldstone bosons is described by chiral perturbation theory. The mean charge radius can be computed in perturbation theory, and it is found to diverge logarithmically with the pion mass [20]. This difference suggests that the existence of a conformal fixed point could be characterized by studying the scaling of $\langle r^2_H \rangle$ for the pseudoscalar meson. We wish to reemphasize [8] that the scaling laws for the mass parameter imply that there is no remnant of the pion as a pseudo-Goldstone boson in a mCGT.

IV. EXPLOITING SELECTION RULES OF CFT-CORRELATORS

We discuss in this section how to exploit selection rules for two-point vacuum correlators originating from scaling dimensions and spin of the quasiprimary (to be commented on further below) operators. In subsection IV A we contrast these aspects from the viewpoint of distinguishing CFTs from SFTs (cf. [21] for lecture notes on this topic), while in subsection IV B we focus on differentiating conformal from confining behavior.

A. CFT vs SFT

Consider first a scale invariant theory, and specifically (quasi)primary fields $\mathcal{O}_{1,2}$ and $\mathcal{O}_3^\mu$ with respective scaling dimension $\Delta^*_{\mathcal{O}_1} = \Delta^*_{\mathcal{O}_3} \neq \Delta^*_{\mathcal{O}_2}$. In the absence of symmetry breaking, the short distance correlator obeys the following selection rules:

(1) Scaling dimension [22][2]

$$C(x) = \langle 0|\mathcal{O}_1(x)\mathcal{O}_2(0)|0\rangle \sim \begin{cases} (x^2)^{-\alpha} & \text{SFT} \\ 0 & \text{CFT} \end{cases}, \quad (16)$$

with $\alpha \equiv (\Delta^*_{\mathcal{O}_1} + \Delta^*_{\mathcal{O}_2})/2$.

(2) Spin:

$$C^\mu(x) = \langle 0|\mathcal{O}_1(x)\mathcal{O}_3^\mu(0)|0\rangle \sim \begin{cases} x^\mu (x^2)^{-(\alpha+1/2)} & \text{SFT} \\ 0 & \text{CFT} \end{cases}, \quad (17)$$

with $\alpha \equiv (\Delta^*_{\mathcal{O}_1} + \Delta^*_{\mathcal{O}_3})/2$. Eq. (17) follows from the investigations in [23].

The equations above state that, in order to have a nonvanishing correlator, the scaling dimension as well as the spin structure of the two operators in question have to be identical [23]. Let us add that it is the local nature of the special conformal transformations which is responsible for the selection rules quoted above. These transformations are precisely the difference between the symmetries of a CFT and a SFT.

Using Eq. (5) we get:

$$F(t,y) \overset{t\to 0}{\to} \begin{cases} \text{constant} & \text{SFT} \\ 0 & \text{CFT} \end{cases}, \quad (18)$$

for $y = \Lambda/m_0 = \Lambda\sqrt{x^2}/b$ such that the system is suitably close to the fixed point. More precisely for fixed x^2 and Λ, y (or b) has to be such that the system is close to the fixed point. In general we expect the constant to be finite with the possible caveat that the correlator, which is generally not a physical observable, is affected by IR divergences.

This criterion is unfortunately of limited use for standard gauge theories. In recent years the understanding has emerged [16] that limit cycles are the only possibility for four-dimensional unitary quantum field theories to be scale but not conformal invariant. On the other hand limit cycles have only been found in theories with flavor dependent couplings, a.k.a. Yukawa terms [16]. These couplings are absent in the gauge theories currently studied on the lattice, and therefore it would seem that IR-conformal theories are indeed IR-conformal and not just IR-scale-invariant. Let us add to this end that, currently, the only logical possibility for scale invariant theories to exist is if the theories can evade the strong version of the a-theorem at the nonperturbative level [17], as the latter has been shown to be valid in perturbation theory some time ago [25].

B. CFT (IR-conformal) vs confining theory

In the previous paragraph we discussed a possible recipe for discerning theories that have only one (flavor independent) coupling, and that are CFTs and not SFTs. The selection rules can also be useful in distinguishing CFTs (IR-conformal) from confining theories. In the following we shall assume that in a gauge theory without IR fixed points, chiral symmetry breaking and confinement occur together.

For that purpose, let us analyze the dimension, and the spin selection rules for a number of example operators. We consider the case of (i) quasiprimary operators from the viewpoint of the CFT candidate theory, (ii) whose correlation function does not vanish by virtue of non-CFT selection rules such as parity symmetry for example.

(i) Scaling dimension:[3]

$$\mathcal{O}_1 = \frac{1}{g^2}G^2, \qquad \Delta^*_{G^2} = 4,$$
$$\mathcal{O}_2 = \bar{q}q, \qquad \Delta^*_{\bar{q}q} = 3 - \gamma^*_m. \quad (19)$$

We note that the correlator (16) with (19) vanishes to all order in perturbation theory in the massless limit as the gauge theory Lagrangian is even under $q \to \gamma_5 q$

[2]For more elaborate forms under open flavor and Lorentz indices of the SFT correlators we refer the reader to Refs. [16,24].

[3]The actual implementation on the lattice might still be nonstraightforward as a gluon field strength tensor is known to mix with $m\bar{q}q$. Yet in the limit this mixing would disappear. A more delicate issue is the mixing with the identity, which corresponds to the disconnected part of the correlator. Whereas in dimensional regularization the mixing occurs with $m^4\mathbf{1}$ only, which is of no problem for the same reason as above, in lattice cutoff regularization terms of the form $m^2\Lambda^2\mathbf{1}$ and $\Lambda^4\mathbf{1}$ (with $\Lambda = 1/a$ with a being the lattice spacing) are expected to occur. Whereas both have hitherto prohibited a clean extraction of the gluon condensate for instance, it is the latter which seems to pose a problem for the case discussed above.

and $m \to -m$ whereas the correlator is odd (since $\bar{q}q \to -\bar{q}q$ and $G^2 \to G^2$). Thus the correlator probes the nonperturbatve regime or more precisely chiral symmetry breaking through $\langle \bar{q}q \rangle \neq 0$.

(ii) Spin:

$$\mathcal{O}_1 = P_5^a = \bar{q} i\gamma_5 t^a q, \quad \Delta^*_{P_5^a} = 3 - \gamma_m^*,$$
$$\mathcal{O}_3^\mu = A^{a\mu} = \bar{q} i\gamma^\mu \gamma_5 t^a q, \qquad \Delta^*_{A^{a\mu}} = 3, \tag{20}$$

where t^a is a $SU(N_f)$ representation matrix acting on flavor space. For the same reason as above the correlator (17) with (20) vanishes to all orders in perturbation theory in the massless limit. It is, however, nonvanishing in the theory with chiral symmetry breaking since the pion couples to both currents:

$$\langle 0|A_\mu^a(0)|\pi^b\rangle = \delta^{ab} i f_\pi p_\mu, \quad \langle 0|P^a(0)|\pi^b\rangle = \delta^{ab} g_\pi,$$
$$g_\pi = \frac{f_\pi m_\pi^2}{2m}, \tag{21}$$

where it is noted in particular that g_π is finite and nonvanishing in the chiral limit for $m_\pi^2 \sim m$ in a chirally broken phase. Conversely f_π is only nonvanishing if $m_\pi \sim m$ at least which is the case for the Goldstone bosons only. It seems worthwhile to elaborate a bit further on this point. The correlation function (17) assumes the following form in the chirally broken phase,

$$C^\mu(x) = \langle 0|\mathcal{O}_1(x)\mathcal{O}_3^\mu(0)|0\rangle$$
$$= \underbrace{\int \frac{d^4p}{(2\pi)^4} \frac{i p_\mu f_\pi g_\pi}{p^2 + m_\pi^2} e^{ip\cdot x}}_{\equiv C_0^\mu(x)} + \mathcal{O}(m), \tag{22}$$

with all other contributions to the spectrum vanishing in the chiral limit[4]. More precisely

$$C_0^\mu(x) = f_\pi g_\pi \partial_\mu \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot x}}{p^2 + m_\pi^2}$$
$$= f_\pi g_\pi \partial_\mu \frac{m_\pi K_1(m_\pi x)}{(2\pi)^2 x}$$
$$= f_\pi g_\pi \partial_\mu \left(\frac{1}{x^2} + \mathcal{O}(m_\pi \ \ln x) \right)$$
$$= \langle \bar{q}q \rangle \frac{2x^\mu}{x^4} + \mathcal{O}(m), \tag{23}$$

where $x = \sqrt{x^2}$, for odd powers of x and in the last equality we have made use of the Gell-Mann Oakes Renner relation $f_\pi^2 m_\pi^2 = -2m\langle \bar{q}q \rangle$.

[4]Multiparticle pion states also come with zero invariant mass but at the same time have zero phase space and therefore vanish in the limit $m \to 0$.

Let us briefly comment on the scaling dimensions of the operators quoted in Eqs. (19, 20) which were already exploited in our previous work [8,9]. The scaling dimension of the gluon field strength tensor is four as it appears in the trace anomaly which is related to the physical mass. The scaling dimension of quark condensate times the mass is four, $\Delta^*_{\bar{q}q} + (1 + \gamma_m^*) = 4$, for the same reason and therefore $\Delta^*_{\bar{q}q} = 3 - \gamma_m^*$. The scaling dimension of $A^{a\mu}$ is three because it is a partially conserved current affected only by explicit breaking. The scaling dimension of P_5^a can be obtained from Ward identities as presented in Appendix B.1 of Ref. [8]. It would seem worthwhile to point out that the operators quoted in Eqs. (19,20) are of the (quasi)primary type as the non-primary operators derive from the latter through derivatives.

Finally, in essence we get, as a replacement of Eq. (18) for the case at hand,

$$F(t,y) \overset{t\to 0}{\to} \begin{cases} \neq 0 & \text{confining} \\ 0 & \text{CFT} \end{cases}, \tag{24}$$

for y such that the system is suitably close to the fixed point as previously discussed. We note that $F(t,y)$ is known explicitly (23) for the second example considered.

C. Comments on finite volume effects in lattice simulations

Equations (3), and (18) show that the behavior of $F(t,y) \to 0$, as $t \to 0$, provides the possibility to distinguish between a CFT and an SFT or a confining theory. In practice one would keep x fixed and study the behavior of the correlator for $m \to 0$. In lattice simulations one would need to work for given values of m with sufficiently large volumes, $L \gg M_H{}^{-1}$, as the limits $m \to 0$ and $V \equiv L^4 \to \infty$ are known not to commute. Further comments can be inferred from Fig. 2 where the relative scales are sketched against a typical behavior of a (gauge) coupling. In regard to this figure we would like to draw the reader's attention to the fact that the actual value of the coupling is scheme dependent, whereas the question of whether there is a fixed point or not is scheme independent as it shows up in physical measurable quantities in terms of scaling laws.

V. FIRST ORDER CORRECTION TO THE FIXED POINT

We have already discussed the scaling of field correlators as a function of the mass m for $g = g^*$. When the coupling is not tuned to its critical value, scaling corrections appear. In this section we compute these corrections at first order in the $\delta g \equiv g - g^*$. In Sec. V A we introduce the notation and discuss the linearized RG equations. In Sec. V B we compute the scaling corrections to field correlators of local operators. In Sec. V C we apply these results and compute the scaling corrections to the hadronic masses first by using the trace anomaly, and then by analyzing the mass correlator. Furthermore we show that the two expressions for the scaling corrections are equal by using an extension of the Feynman-Hellmann theorem [26].

A. Linearization around the IR fixed-point

We assume that the bare couplings at the cutoff scale g and $\hat{m}$, which correspond to the point I in Fig. 1 (left) with the identification $(g, g') = (g, \hat{m})$, are chosen such that the system is on a trajectory that is close to the fixed point. We are going to linearize the RG flow equations in the deviations from the fixed point, that is to say in the variable $\delta g \equiv g - g^*$, where we use the notation $g^* = g^*_{\rm IR}$ throughout this section. We shall comment on the aspects of this expansion at the end of Sec. V B 1. For the beta function, the mass anomalous dimension γ_m, and the anomalous dimension matrix $\gamma_{ij} \equiv (\gamma_O)_{ij}$[5] of a generic set of operators $\{O_i\}$ that mix under the RG-flow, we may linearize the system around the IRFP as follows:

$$\beta = \beta_1 \delta g + \mathcal{O}(\delta g^2), \quad \delta g \equiv g - g^*, \quad \gamma_m = \gamma_m^* + \gamma_m^{(1)} \delta g + \mathcal{O}(\delta g^2),$$
$$\gamma_{ij} = \gamma_{ij}^* + \gamma_{ij}^{(1)} \delta g + \mathcal{O}(\delta g^2), \quad (\gamma_{ij} \equiv (\gamma_O)_{ij}). \qquad (25)$$

We have verified that in a mass-independent scheme β_1 is universal (scheme independent) whereas $\gamma^{(1)}_{m/ij}$ are not. We remind the reader that the anomalous dimensions associated with gauge invariant operators (such as γ_m^*) are universal. When working with renormalized quantities we shall choose notation accordingly. The behavior of the beta function as a function of the coupling is illustrated in Fig. 3; β_1 corresponds to the slope where the curve crosses the IR fixed point. We note that for the beta function described in Fig. 3, the coefficient β_1 is positive as there are no further zeros between $g = 0$ and $g = g^*$. The beta function equation is easily integrated to that order,

$$\beta(g) = \Lambda \frac{d}{d\Lambda}(\delta g) = \beta_1 \delta g + \mathcal{O}(\delta g^2) \Rightarrow \delta g(\Lambda)$$
$$= \delta g(\Lambda_0) \left(\frac{\Lambda}{\Lambda_0} \right)^{\beta_1}, \qquad (26)$$

where Λ is a UV cutoff as will become clear in the next subsection.

B. Scaling corrections to correlators

We shall use the language of Wilsonian renormalization group for which the theory is defined at some fixed UV cutoff $\Lambda_{\rm UV} \equiv \Lambda$[6]. Let us consider a correlation function $O_i(g, m, \Lambda)$, as a function of the bare parameters g, m, and the UV cutoff Λ. We shall denote by Z_{ij} the matrix that describes the mixing of O_i under renormalization. O_i satisfies an RG equation (also known as 't Hooft-Weinberg or Callan-Symanzik equations), e.g. Ref. [27],

$$\left(\Lambda \frac{\partial}{\partial \Lambda} \delta_{ij} + \beta(g) \frac{\partial}{\partial g} \delta_{ij} - \gamma_m m \frac{\partial}{\partial m} \delta_{ij} - \gamma_{ij} \right) O_j(g, m, \Lambda)$$
$$= 0, \qquad (27)$$

where summation over j is implied and

$$\beta(g) = \Lambda \frac{d}{d\Lambda} g, \quad \gamma_m = -\Lambda \frac{d}{d\Lambda} \ln m, \quad \gamma_{ij} = \Lambda \frac{d}{d\Lambda} \ln Z_{ij}. \qquad (28)$$

We now wish to reformulate the theory using a different UV cutoff $\Lambda'_{UV} \equiv \Lambda'$

$$\frac{\Lambda}{\Lambda'} = b, \qquad (29)$$

where the parameter b has the interpretation of a blocking factor, and $b > 1$ if high-energy modes are to be integrated out. The formal solution to Eq. (27) is given by:

$$O_i(g, m, \Lambda) = Z_{ij}^{-1}(b) O_j(g(b), m(b), \Lambda/b), \qquad (30)$$

where

$$\frac{d}{d \ln b} \ln Z_{ij}(b) = -\gamma_{ij}(g(b)), \quad Z(1) = 1,$$
$$\frac{d}{d \ln b} g(b) = -\beta(g(b)), \quad g(1) = g,$$
$$\frac{d}{d \times \ln b} \ln m(b) = \gamma_m(g(b)), \quad m(1) = m. \qquad (31)$$

We assume here that we are working in a mass-independent scheme, and therefore the beta function and the anomalous dimensions only depend on the gauge coupling g. The solution (30, 31) is known by the name of the method of characteristics, see e.g. Ref. [27]. Assuming the fixed point is in the linear regime (26), the three equations above can be solved to order $\mathcal{O}(\delta g)$:

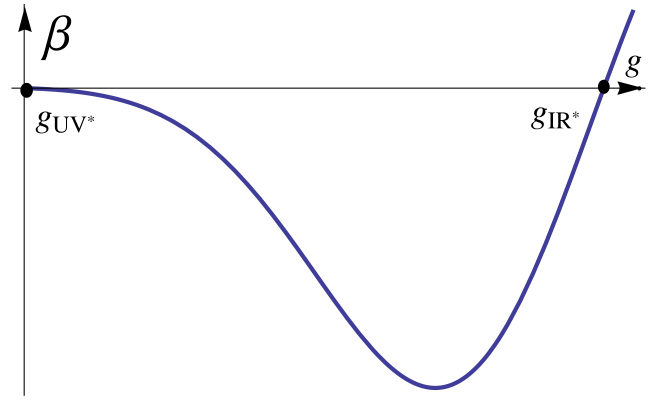

FIG. 3 (color online). Sketch of the β function in terms of the coupling for a system exhibiting asymptotic freedom $g^*_{\rm UV} = 0$ and a nontrivial IR fixed point at some value $g^*_{\rm IR} > 0$. In Sec. V a system in the vicinity of the IR fixed point is considered as indicated in the figure.

[5]In statistical mechanics the anomalous dimensions of operators are often denoted by the symbol η rather than γ in order to distinguish it from anomalous dimensions of parameters such as the mass for instance.

[6]In the context of a lattice field theory the lattice spacing a is related to the UV cutoff as $a = \Lambda_{\rm UV}^{-1}$.

$$g(b) = g^* + \delta g(b) = g^* + \delta g b^{-\beta_1},$$
$$m(b) = m b^{\gamma_m^*} \exp\left[-\frac{\gamma_m^{(1)}}{\beta_1}\delta g f(b)\right],$$
$$Z_{ij}(b) = \exp\left[\gamma^* \ln b - \frac{\gamma^{(1)}}{\beta_1}\delta g f(b)\right]_{ij}, \quad (32)$$

where we have introduced the notation

$$f(b) \equiv b^{-\beta_1} - 1, \quad (33)$$

which parametrizes the distance from the initial point in blocking space. Equation (30) may be written using the relation (32) as:

$$\begin{aligned} O_i(g,m,\Lambda) &= Z_{ij}(b)^{-1} O_j(g(b), m(b), \Lambda/b) \\ &= \exp\left[-\gamma^* \ln b + \frac{\gamma^{(1)}}{\beta_1}\delta g f(b)\right]_{ij} O_j\left(g(b), m b^{\gamma_m^*} \exp\left[-\frac{\gamma_m^{(1)}}{\beta_1}\delta g f(b)\right], \Lambda/b\right) \\ &= \exp\left[-\Delta^* \ln b + \frac{\gamma^{(1)}}{\beta_1}\delta g f(b)\right]_{ij} O_j\left(g(b), m b^{-(1+\gamma_m^*)} \exp\left[-\frac{\gamma_m^{(1)}}{\beta_1}\delta g f(b)\right], \Lambda\right), \end{aligned}$$

where in the last equality we have rescaled all dimensionful quantities by a factor b. The matrix $\Delta_{ij} = d_i \delta_{ij} + \gamma_{ij}$, where d_i is the classical dimension of O_i, yields the scaling dimensions of the operators. In order to get the δg corrections we need to expand in that variable. In our opinion this is best done from the expression in the second line of the equation above. The last step can be done after the expansion for each individual term. Furthermore, in order to avoid path ordering in coupling space, we shall assume that γ_{ij}^* is diagonal. The corrections are parametrized as follows,

$$O_i(g,m,\Lambda) = b^{-\gamma_{ii}^*}([O_i]^* + \delta g O_i^{(1)} + \mathcal{O}(\delta g^2)), \quad (34)$$

where

$$O_i^{(1)} = \left(\frac{\gamma_{ii}^{(1)}}{\beta_1}[O_i]^* f(b) - \frac{\gamma_m^{(1)}}{\beta_1} m^* [O_i]^*_{,m} f(b) + [O_i]^*_{,g} b^{-\beta_1}\right) \quad (35)$$

and

$$\begin{aligned} [O_i]^* &= O_i(g(b), m(b), \Lambda/b)|_{\delta g=0}, \\ [O_i]^*_{,m} &= \frac{\partial}{\partial m(b)} O_i(g(b), m(b), \Lambda/b)|_{\delta g=0}, \\ [O_i]^*_{,g} &= \frac{\partial}{\partial g(b)} O_i(g(b), m(b), \Lambda/b)|_{\delta g=0}, \\ m^* &= m(b)|_{\delta g=0}, \quad (g^* = g(b)|_{\delta g=0}). \end{aligned} \quad (36)$$

We wish to emphasize that when g^* is tuned to the fixed point coupling, $m^* = m b^{\gamma_m^*}$ corresponds to the leading scaling of the mass at the fixed point.

The scaling corrections as a function of m can be made explicit by rescaling all dimensionful quantities by the appropriate power of b in the last step in Eq. (34), and then using the arbitrariness of $b > 1$ to impose:

$$m b^{-(1+\gamma_m^*)} = m_0 \Rightarrow b^{-1} = \hat{m}^{1/(1+\gamma_m^*)}, \quad \hat{m} \equiv \frac{m}{m_0}. \quad (37)$$

As a result, we obtain a scaling formula that includes the scaling corrections at first order in δg:

$$O_i(g,m,\Lambda) = \hat{m}^{\frac{\Delta_{ii}}{1+\gamma_m^*}}[O_i]^*\left(1 + \delta g\left(A + B\hat{m}^{\frac{\beta_1}{1+\gamma_m^*}}\right)\right) + \mathcal{O}(\delta g^2), \quad (38)$$

with

$$\begin{aligned} A &= \left\{-\frac{\gamma_{ij}^{(1)}}{\beta_1} + \frac{\gamma_m^{(1)}}{\beta_1} m^* \frac{[O_i]^*_{,m}}{[O_i]^*}\right\}_b \\ B &= \left\{+\frac{\gamma_{ij}^{(1)}}{\beta_1} - \frac{\gamma_m^{(1)}}{\beta_1} m^* \frac{[O_i]^*_{,m}}{[O_i]^*} + \frac{[O_i]^*_{,g}}{[O_i]^*}\right\}_b, \end{aligned} \quad (39)$$

where the curly brackets with a b superscript indicate that all physical units are to be scaled by b, e.g. $\{[Q_i^*]\}_b \overset{(36)}{=} \{O_i(g(b),\, m(b),\, \Lambda/b)|_{\delta g=0}\}_b \;\rightarrow\; b^{d_O} O_i(g(b),$ $\times bm(b),\, \Lambda)|_{\delta g=0}$. It is interesting to note that the scaling corrections above simplify when $\beta_1 \to 0$:

$$\left(A + B\hat{m}^{\frac{\beta_1}{1+\gamma_m^*}}\right) \to \frac{[O_i]^*_{,g}}{[O_i]^*} + O(\beta_1). \quad (40)$$

This situation is expected to be realized at the lower edge of the conformal window in the Banks-Zaks limit.

1. *Discussion of scaling corrections*

The expression (34) yields the corrections to scaling for small fermion mass $\hat{m}$, while the irrelevant coupling g is at a distance δg from the fixed point. Clearly when δg vanishes, so do the scaling violations. We note that for fixed initial value g, $\delta g \equiv g - g^*$ is proportional to the value of the IR fixed point coupling g^*, as can be inferred from Fig. 1. The linear approximation discussed here therefore becomes less reliable if the IR fixed point is at strong coupling coupling, unless g is tuned to reduce the size of δg. Note that for large δg the linear corrections tend to grow. This can be compensated by going to smaller initial masses $\hat{m}$ since the first order (relative) scaling corrections are determined by the combination $B\delta g \hat{m}^{\beta_1/(1+\gamma_m^*)}$.

We would like to add an important point concerning the size of the corrections at the lower edge of the conformal window. For a strong coupling fixed point, one would expect large values of $\gamma_{ij}^{(1)}$, as well as $\gamma_m^{(1)}$, whereas the value of β_1 is expected to be small as the fixed point is to be lost which in turn is consistent with g^* being large. Moreover, unless the bare coupling g is fine-tuned, one can expect to have rather large values of δg, driven by our ignorance in guessing the exact location of the fixed point. Thus in summary the precoefficient $B\delta g$ should be expected to be large at the lower edge of the conformal window. On the other hand, the exponent $\beta_1/(1+\gamma_m^*)$ is then small and leads to a suppression. In the previous statements large and small are meant relative to the region away from the lower edge of the conformal window. Which of the two counteracting effects dominates is unclear a priori but the argument suggests that it is important to go to small masses $\hat{m}$ at the lower edge of the conformal window to suppress potentially large scaling corrections. This is of practical importance as many of the lattice simulations have been performed precisely at the lower edge of the conformal window in search of a theory of walking technicolor.

The signs of A and B, in Eq. (38), are determined by the dynamics. Since in general we cannot make statements about the derivative of the operators the sign of A and B are thus not known a priori. This is somewhat different for the hadronic masses, that is to say for the operators Q and G, which is what we are going to exploit in the next section.

C. Scaling corrections to the mass formula

We shall first introduce some notation and justify the formulas needed for the comparison of the two derivations of the scaling corrections to the hadronic mass in subsection V C 2.

1. Preliminary formulas

The following notation,

$$\langle X \rangle_{E_H} \equiv \langle H(E,\vec{p})|X|H(E,\vec{p})\rangle_c, \tag{41}$$

shall prove convenient throughout this section. The subscript c denotes the connected part of the matrix element, while $|H(E,\vec{p})\rangle$ is a physical state with definite spatial momentum and energy and X is a (local) operator. Above we have explicitly indicated the energy dependence of the hadronic state H which we occasionally suppress in the remaining part of this work. Note that the disconnected part of the correlator is related to the vacuum energy, that is to say the cosmological constant. As usual the Lorentz invariant state normalization is given by:

$$\langle H(E',\vec{p'})|H(E,\vec{p})\rangle = 2E(\vec{p})(2\pi)^3\delta^{(3)}(\vec{p}-\vec{p'}). \tag{42}$$

The expectation value of the energy momentum tensor $T_{\mu\nu}$ in a single-particle state is:

$$\langle T_{\mu\nu}\rangle_{E_H} = 2p_\mu p_\nu, \tag{43}$$

where $p_0 = E$. In order to keep a compact notation we are going to extend the notation (41) for two specific matrix elements to:

$$Q_{E_H} \equiv N_f m \langle \bar{q}q\rangle_{E_H}, \qquad G_{E_H} \equiv \left\langle \frac{1}{g^2}G^2 \right\rangle_{E_H}. \tag{44}$$

We remind the reader that the notation (41) refers to the connected part of the matrix element only.

For the discussion in this section it is convenient to use renormalized quantities. Accordingly we denote the renormalized couplings by $\bar{g}$ and $\bar{m}$, and the matrix elements of the renormalized operators $\bar{G}_{E_H}$ and $\bar{Q}_{E_H}$, respectively. The renormalized coupling $\bar{g}$ is defined as

$$g = Z_g(g)\bar{g}. \tag{45}$$

En passant we note that physical quantities such as the energy momentum tensor and thus the hadronic mass do not renormalize (i.e. $T_{\mu\nu} = \bar{T}_{\mu\nu}$). In the neighborhood of the IRFP, the renormalization constant is expanded similar to (30) as

$$Z_g(g) = Z_g^* + Z_g^{(1)}\delta g + \mathcal{O}(\delta g^2), \tag{46}$$

which implies

$$\delta g = (Z_g^* + gZ_g^{(1)})\delta\bar{g} + \mathcal{O}(\delta g^2), \qquad g\frac{\partial}{\partial g} = \kappa\bar{g}\frac{\partial}{\partial\bar{g}},$$
$$\kappa = \left(1 - \frac{Z_g^{(1)}}{Z_g^*}\right) + \mathcal{O}(\delta g^2). \tag{47}$$

The trace anomaly can be written in terms of the renormalized quantities as [28]

$$2M_H^2 = \left(\frac{\bar{\beta}}{2\bar{g}}\right)\bar{G}_{M_H} + (1+\bar{\gamma}_m)\bar{Q}_{M_H}, \tag{48}$$

and is a RG invariant. More precisely since $\bar{Q}_{E_H}$ is a RG invariant, $\bar{\beta}/(2\bar{g})\bar{G}_{E_H} + \bar{\gamma}_m\bar{Q}_{E_H}$ inherits this property by virtue of Eq. (48). This entails that $G_{E_H} \neq \bar{G}_{E_H}$. In identifying the two computation the following relations are of importance:

$$\bar{m}\frac{\partial}{\partial \bar{m}}E_H^2 = \bar{Q}_{E_H}, \qquad \bar{g}\frac{\partial}{\partial \bar{g}}E_H^2 = -\frac{1}{2}\bar{G}_{E_H}. \tag{49}$$

The first relation is a straightforward application of the Feynman-Hellmann theorem and is widely used, as for instance in our previous work [9]. The second relation is akin to a Feynman-Hellmann relation but has been derived recently [26] through a RGE, the trace anomaly (48) as well as the first relation.

2. Two pathways to mass-scaling corrections

Let us now compute the corrections to scaling in two different ways by using results from the previous section: the corrections are obtained up to order $\delta\bar{g} \equiv (\bar{g}-\bar{g}^*)$ and the symbol δ on other quantities denotes the linear variation in the $\delta\bar{g}$ variable. Recall that $\beta_1 = \bar{\beta}_1$ (at least in a mass-independent scheme) and $\gamma_m^{(1)} \neq \bar{\gamma}_m^{(1)}$ in general and we shall therefore use notation accordingly.

(1) First we compute $\delta(2M_H^2)$ directly from the RG scaling formulas (38) for renormalized quantities, combined with the relation (49):

$$\delta(2M_H^2) = \delta\bar{g}\left([2M_H^2]^*_{,\bar{g}}b^{-\beta_1} - \frac{\bar{\gamma}_m^{(1)}}{\beta_1}\bar{m}^*[2M_H^2]^*_{,\bar{m}}f(b)\right) + \mathcal{O}(\delta\bar{g}^2) \overset{(49)}{=} \delta\bar{g}\left(-\frac{1}{\bar{g}^*}[\bar{G}_{M_H}]^*b^{-\beta_1} - 2\frac{\bar{\gamma}_m^{(1)}}{\beta_1}[\bar{Q}_{M_H}]^*f(b)\right) + \mathcal{O}(\delta\bar{g}^2). \tag{50}$$

(2) Second we compute $\delta(2M_H^2)$ through the trace anomaly (48):

$$\delta(2M_H^2) = \delta\bar{g}b^{-\beta_1}\left(\frac{\beta_1}{2\bar{g}^*}[\bar{G}_{M_H}]^* + \bar{\gamma}_m^{(1)}[\bar{Q}_{M_H}]^*\right) + (1+\gamma_m^*)\delta\bar{Q}_{M_H}, \tag{51}$$

which necessitates the computation of $\delta\bar{Q}_{M_H}$. The latter is given by Eq. (38):

$$\delta\bar{Q}_{M_H} = \delta\bar{g}\left([\bar{Q}_{M_H}]^*_{,\bar{g}}b^{-\beta_1} - \frac{\bar{\gamma}_m^{(1)}}{\beta_1}\bar{m}^*[\bar{Q}_{M_H}]^*_{,\bar{m}}f(b)\right). \tag{52}$$

The expressions (50), and (52) yield the scaling corrections as a function of δg and the mass m. These expressions all have the same scaling exponents, yet it is not clear from these formulas that the corresponding prefactors are equal. To compare the prefactors we ought to compute $[\bar{Q}_{M_H}]^*_{,\bar{g}}$ and $[\bar{Q}_{M_H}]^*_{,\bar{m}}$ to leading order in $\delta\bar{g}$. The latter is simply given by the leading order scaling (12)

$$\bar{m}^*[\bar{Q}_{M_H}]^*_{,\bar{m}} = \frac{2}{1+\gamma_m^*}[\bar{Q}_{M_H}]^* + \mathcal{O}(\delta\bar{g}), \tag{53}$$

up to corrections which are beyond the aimed accuracy. The computation of $[\bar{Q}_{M_H}]^*_{,\bar{g}}$ is slightly more involved; it is obtained by differentiating $2M_H^2$ with respect to $\bar{g}$ using (49):

$$\frac{\partial}{\partial\bar{g}}(2M_H^2) = -\frac{1}{\bar{g}}\bar{G}_{M_H} = -\frac{1}{\bar{g}^*}[\bar{G}_{M_H}]^* + \mathcal{O}(\delta g^2) \tag{54}$$

as well as the right-hand side (RHS) of Eq. (48),

$$\begin{aligned}\frac{\partial}{\partial\bar{g}}(2M_H^2) &= \left(\frac{\beta}{2\bar{g}}\right)'\bar{G}_{M_H} + \left(\frac{\beta}{2\bar{g}}\right)\bar{G}'_{M_H} + \gamma'_m\bar{Q}_{M_H} + (1+\gamma_m^*)\bar{Q}'_{M_H} + O(\delta\bar{g}) \\ &= \frac{\beta_1}{2\bar{g}^*}[\bar{G}_{M_H}]^* + \bar{\gamma}_m^{(1)}[\bar{Q}_{M_H}]^* + (1+\gamma_m^*)[\bar{Q}_{M_H}]^*_{,\bar{g}} + O(\delta\bar{g}),\end{aligned} \tag{55}$$

where $\square'$ denotes differentiation with respect to $\bar{g}$. We have dropped the term $\sim\beta\bar{G}'_{M_H}$ from passing from the first to the second line since it is of relative order $\mathcal{O}(\delta\bar{g})$. By equating Eqs. (54) and (55) we may solve for $[\bar{Q}_{M_H}]^*_{,\bar{g}}$ and insert it into (52) and finally into (51) to obtain:

$$\begin{aligned}\delta(2M_H^2) &= \delta\bar{g}b^{-\beta_1}\left(\frac{\beta_1}{2\bar{g}^*}[\bar{G}_{M_H}]^* + \bar{\gamma}_m^{(1)}[\bar{Q}_{M_H}]^*\right) + \delta\bar{g}\left(-2\frac{\bar{\gamma}_m^{(1)}}{\beta_1}[\bar{Q}_{M_H}]^*f(b) \underbrace{-\frac{1}{\bar{g}^*}[\bar{G}_{M_H}]^*b^{-\beta_1} - b^{-\beta_1}\left(\frac{\beta_1}{2\bar{g}^*}[\bar{G}_{M_H}]^* + \bar{\gamma}_m^{(1)}[\bar{Q}_{M_H}]^*\right)}_{[\bar{Q}_{M_H}]^*_{,\bar{g}}}\right) \\ &= \delta\bar{g}\left(-\frac{1}{\bar{g}^*}[\bar{G}_{M_H}]^*b^{-\beta_1} - 2\frac{\bar{\gamma}_m^{(1)}}{\beta_1}[\bar{Q}_{M_H}]^*f(b)\right),\end{aligned} \tag{56}$$

which equals Eq. (50), as expected. We note that the second line in Eq. (56) is equal to $(1+\gamma_m^*)\delta\bar{Q}_{M_H}$ at leading order.

An interesting question is whether we can say something about the sign of the correction in Eq. (50). That is to say we would like to know whether l_{M_H} in $M_H^2 = k_{M_H} + l_{M_H}\delta\bar{g}$ is positive or negative. We should add that $\delta g < 0$ as can be inferred from Fig. 3. As we are interested in the long-distance dynamics of the theory that is defined at the UV Gaussian fixed point, the coupling lies in the interval $[0, g^*]$.

As previously stated $\beta_1 > 0$ in Eq. (30) by virtue of no zero crossings of the β-function between the UV and IRFP. If the anomalous dimension increases monotonically from the UVFP $\gamma^*_{m,UV} = 0$ to $\gamma^*_{m,IR} = \gamma^*_m$ then $\bar{\gamma}_m^{(1)} > 0$, which is not compelling but to be expected. Furthermore, $[Q_{M_H}]^* > 0$ since $M_H^2 = (1+\gamma_m^*)[Q_{M_H}]^* + \mathcal{O}(\delta g)$ and $(1+\gamma_m^*) > 0$ as we have assumed m to be a relevant direction and $f(b) < 0$ for $b > 1$. Finally we see that everything depends on the sign of G_{M_H} for which we cannot make a definite assertion. It is well known that naive positivity of operators, effective in quantum mechanics, is not necessarily maintained in quantum field theory. In the case at hand there is the additional complication that only the connected part of the matrix element is required. That is to say even if the total matrix element were positive the connected part might still be negative. It seems worthwhile to point out that in QCD for $m \to 0$ and $\beta < 0$ Eq. (48) implies that $G_{M_H} < 0$ indeed. Summa summarum we cannot say anything definite about the sign of l_{M_H} as the sign of G_{M_H} seems uncertain.

VI. CONCLUSIONS

We have explored the consequences of conformal scaling in a number of interesting cases. One of our main findings is the scaling of the radius of the m-hadrons as a function of the fermion mass. Our results show that the typical size of the m-hadrons, defined from the average charge density, is a linear function of the inverse mass of the hadron (14). Characterizing the size of m-hadrons is very important in order to understand how to tame FSE in numerical studies, and hence obtain reliable results from Monte Carlo simulations. It is worthwhile to emphasise that the dependence of the mean charge radius on the mass of the m-hadrons is radically different from the logarithmic scaling obtained in chiral perturbation theory for the Goldstone boson in a chirally broken theory [20]. The latter provides yet another way to asses the difference between a conformal and a confining phase.

By exploiting selection rules for scaling dimensions and spin we propose to use coordinate space correlation function, deformed by a mass term, to distinguish CFTs from SFTs as well as confining theories.

We investigated the scaling corrections to correlation functions by linearizing the RGE in the variable $\delta g = g - g^*$ which is the distance of the initial coupling from the, presumably unknown, fixed point value. In essence this corresponds to the scaling corrections due to the IR-irrelevant coupling g. The generic result is given in Eq. (38) and (39). In subsection V B 1 we note in particular that scaling corrections can be expected to be large at the lower edge of the conformal window. This can be counteracted by going to smaller masses. We computed the scaling corrections to the hadron mass explicitly, once directly through its associated correlation function and second through the trace anomaly. The results are given in Eqs. (50), (51) and their equivalence is made manifest in Eq. (56). The latter was established by using the Feynman-Hellmann relation for the mass and an analogous relation for the gauge coupling (49). The derivation of the latter is given in a separate paper [26].

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APPENDIX A: CHARGE AND CHARGE RADIUS OF PION FORM FACTOR

In this appendix we shall give a derivation of the charge radius in terms of the form factor as stated in Eq. (8) as the derivation of the latter has become sparse in modern textbooks. We shall work in Minkowski-space in this section with metric signature $(+,-,-,-)$. Starting from the zeroth component of (7)

$$\langle H(p_1)|V_0(y)|H(p_2)\rangle = (E_{p_1} + E_{p_2}) f_+^H(q^2) e^{i(p_1-p_2)\cdot y}, \tag{A1}$$

where $E_p = \sqrt{\vec{p}^2 + M_H^2}$ and $q \equiv p_1 - p_2$ as usual. Note for on-shell states the equality of the 3-vectors $\vec{p_1} = \vec{p_2}$ then implies the vanishing of the 4-vector $q = 0$. We define the $D-1 = 3$-dimensional Fourier transform of the form factor

$$f^H(q^2) = \int \frac{d^3x}{(2\pi)^3} \hat{f}^H(\vec{x}, q_0^2) e^{i\vec{x}\cdot\vec{q}}, \tag{A2}$$

for latter convenience. The scalar product with arrow vector denotes the 3-dimensional scalar product.

1. Charge

The charge of the state H is obtained by integrating the charge density over the space

$$\int d^3x \langle H(p_1)|V_0(x)|H(p_1)\rangle = 2E_p Q_H((2\pi)^3\delta^{(3)}(0)),$$
$$\stackrel{(A.1)}{=} 2E_p f_+^H(0)\left(\int_V d^3x\right), \tag{A3}$$

remembering the normalization $\langle H(p_1)|H(p_2)\rangle = 2E_{p_1}(2\pi)^3\delta^{(3)}(\vec{p}_1 - \vec{p})_2)$, setting $\vec{p_1} = \vec{p_2}$ on the first line and using the definition on the second line. To the more mathematically inclined reader this equation might look better if $\vec{p}_1 = \vec{p}_2$ is not assumed before identifying $\int_V d^3x = (2\pi)^3\delta^{(3)}(0)$. The latter identification leads to the first result of this appendix:

$$\Rightarrow f_H(0) = Q_H, \tag{A4}$$

and suggests that

$$\hat{f}_H(\vec{x}, 0)/(2\pi)^3 = \rho(\vec{x}) \tag{A5}$$

is the charge density which we shall use below.

2. Charge radius

We shall define the 3-dimensional Laplace operator Δ acting on Fourier space as follows:

$$\Delta\circ\mathcal{F}(q) = \sum_{a=1}^{3} i\frac{d}{dq_a} i\frac{d}{dq_a}\mathcal{F}(q)|_{q=0}. \tag{A6}$$

We let it act on the form factor directly and through its Fourier transform:

$$\begin{aligned}\Delta\circ f^H(q^2) &\overset{(A.6)}{=} \left(6\frac{d}{dq^2}f^H(q^2) + 4\vec{q}^2\frac{d^2}{d(q^2)^2}f^H(q^2)\right)|_{q=0} = 6\frac{d}{dq^2}f^H(q^2)|_{q=0} \\ &\overset{(A.2)}{=} \int\frac{d^3x}{(2\pi)^3}\vec{x}^2\hat{f}^H(\vec{x}, q_0^2)e^{i\vec{x}\cdot\vec{q}}|_{q=0} = \int\frac{d^3x}{(2\pi)^3}\vec{x}^2\hat{f}^H(\vec{x}, 0).\end{aligned} \tag{A7}$$

This leads, using (A5), to the second result of this appendix:

$$\Rightarrow \langle r_H^2\rangle = \int d^3x\vec{x}^2\rho_H(\vec{x}) \overset{(A.7)}{=} 6\frac{d}{dq^2}f^H(q^2)|_{q=0}. \tag{A8}$$

Thus we have now justified the results quoted in Eq. (8) through (A4) and (A8).

APPENDIX B: FINITE SIZE EFFECTS

The aim of this appendix is to present an extension of the presentation in the main text to include finite size effects.

1. Generic two point function

Finite-size effects to Eq. (5) can be easily incorporated. Writing explicitly the dependence of the correlators on the size L of the physical volume, the RG equation becomes:

$$\begin{aligned}C(x, \hat{m}, \Lambda, L) &= b^{-(\gamma^*_{O_1}+\gamma^*_{O_2})}C(x, b^{y^*_m}\hat{m}, \Lambda/b, L), \\ y^*_m &= 1 + \gamma^*_m.\end{aligned} \tag{B1}$$

The underlying assumption in the equation above is that the volume is large enough, such that a blocking transformation does not change the volume dependence. When all dimensionful quantities are rescaled by the corresponding power of the reference mass m_0, Eq. (5) becomes:

$$C(x,\hat{m},\Lambda,L) = (\hat{x}^2)^{-\alpha}(m_0)^{d_{O_1}+d_{O_2}+d_{\varphi_a}+d_{\varphi_b}}F(\hat{x}^{y^*_m}\hat{m}, \hat{\Lambda}, \hat{x}/\hat{L}). \tag{B2}$$

In the thermodynamic limit, $\hat{L}\to\infty$

$$F(\hat{x}^{y^*_m}\hat{m}, \hat{\Lambda}, \hat{x}/\hat{L}) \to F(\hat{x}^{y^*_m}\hat{m}, \hat{\Lambda}) + \kappa\frac{\hat{x}}{\hat{L}} + \cdots, \tag{B3}$$

where κ is a number.

2. Charge radius

We discuss the modifications of the scaling laws of the form factor (7), relevant to the charge radius, due to finite-size effects. The form factor depends on the fermion mass, the UV cutoff, and the physical size of the lattice:

$$f(q^2) = f(q^2; \hat{m}, \Lambda, L), \tag{B4}$$

where the hat indicates that dimensionful quantities have been rescaled by the appropriate powers of the reference mass m_0. Keeping Λ unchanged, and performing the standard RG analysis that we used above, yields:

$$f(q^2; \hat{m}, \Lambda, L) = \tilde{f}\left(\frac{\hat{q}^2}{\hat{m}^{2/y_m}}, \hat{L}\hat{m}^{1/y_m}\right). \tag{B5}$$

Expanding Eq. (B4) in powers of $\hat{q}^2\hat{m}^{-2/y_m}$:

$$f(q^2; \hat{m}, \Lambda, L) = \tilde{f}(0, \hat{L}\hat{m}^{1/y_m}) + \tilde{f}'(0, \hat{L}\hat{m}^{1/y_m})\frac{\hat{q}^2}{\hat{m}^{2/y_m}} + \cdots. \tag{B6}$$

This is the same expansion obtained in Eq. (10), but now the coefficients of the expansion depend on the physical size of the lattice L. Denoting the nth derivative of the form factor by $\tilde{f}_{,n}$, and introducing the dimensionless finite-size scaling variable $\ell = \hat{L}\hat{m}^{1/y_m}$, we obtain:

$$\tilde{f}_{,n}(0,\ell) = \frac{1}{\hat{L}^{y_m\eta_n}}\ell^{\eta_n y_m}\left(1+\frac{\kappa}{\ell}+\cdots\right) \quad \text{(B7)}$$

with κ a number and the dots denote the finite volume corrections. The dependence in Eq. (B7) reproduces the expected mass scaling discussed before in the large-volume limit.

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