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Disentangling the decay observables in $B^- \to \pi^+\pi^-\ell^-\bar{\nu}_\ell$

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We study the semileptonic $b \to u$ transition in the decay mode $B^- \to \pi^+\pi^-\ell^-\bar{\nu}_\ell$. We define $B \to \pi\pi$ form factors in the helicity basis, and study their properties in various kinematic limits, including form factor relations in the heavy-mass and large-energy limits, the decomposition into partial waves of the dipion system, and the resonant contribution of vector and scalar mesons. We show how angular observables in $B^- \to \pi^+\pi^-\ell^-\bar{\nu}_\ell$ can be used to measure dipion form factors or to perform null tests of the Standard Model.

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I. INTRODUCTION

The decay $B \to \pi\pi\ell^-\bar{\nu}_\ell$ is interesting for several reasons. At quark level, it is generated by the semileptonic $b \to u\ell\nu_\ell$ transition which, in the Standard Model (SM), is induced by tree-level W-boson exchange but proportional to the small element V_{ub} of the Cabibbo-Kobayashi-Maskawa matrix. For a while the common paradigm has been to search for new physics (NP) in rare *loop-induced* flavor transitions. Meanwhile, in light of the small tension observed between the determinations of $|V_{ub}|$ from inclusive $B \to X_u\ell^-\bar{\nu}_\ell$ or exclusive semileptonic $B \to \{\pi,\rho\}\ell^-\bar{\nu}_\ell$ decays [1,2], systematic tests of $b \to u$ transitions in the SM and beyond appear timely. In this context, the dipion system in the hadronic final state not only provides an independent decay channel, but, more importantly, offers the possibility to explore a number of angular observables that are sensitive to the spin structure of the underlying short-distance operators responsible for the decay in the SM or NP. The situation here is similar to the analysis of rare $b \to s$ transitions in $B \to (K\pi)_{S,P}\ell^+\ell^-$ decays, see for example [3–11].

Moreover, the phase space associated with the kinematics of the four-body decay covers various limiting cases for which specific theoretical approaches to handle the strong-interaction effects in quantum chromodynamics (QCD) are applicable. In particular, this includes expansions in small light-quark or large heavy-quark masses based on effective-field theory methods. For instance, the case of two pions recoiling against each other with a large energy can be used to assess the reliability of theoretical predictions in the QCD factorization (QCDF) approach which has been frequently used for nonleptonic $B \to \pi\pi$ decays [12,13]. The decay $B^- \to \pi^+\pi^-\ell^-\bar{\nu}_\ell$ also involves the resonant channel $B^- \to \rho^0(\to \pi^+\pi^-)\ell^-\bar{\nu}_\ell$ decay which is one of the aforementioned exclusive modes where the $|V_{ub}|$ extraction is not in perfect agreement with the inclusive determination. The theoretical exploration of the various corners of (nonresonant) phase space will therefore also help to better understand the proper description of the $B \to \rho\ell\bar{\nu}_\ell$ decay beyond the approximation of narrow width and flat nonresonant background.

Our paper is organized as follows. In the following Sec. II, we provide the basic definitions for $B \to \pi\pi$ form factors that are most convenient for the angular analysis and for the theoretical description of the decay in certain kinematic limits. In Sec. III we consider the dipion form factors for two kinematic limits giving rise to symmetry relations in heavy-quark effective theory (HQET), and soft-collinear effective theory (SCET), respectively. Further form factor properties in specific kinematic situations, namely the perturbative factorization in the limit of almost back-to-back energetic pions, on the one hand, and the description of hadronic resonances in the dipion channel, on the other hand, are the subject of Sec. IV. The phenomenology of the angular distributions of the decay $B^- \to \pi^+\pi^-\ell^-\bar{\nu}_\ell$ in the SM is worked out in Sec. V. In Sec. VI we combine knowledge of the form factor limits with the angular distribution to derive relations between the angular observables that do not depend on a hadronic model. We conclude in Sec. VII.

II. $B \to \pi\pi$ FORM FACTORS

A. Kinematics

Let us begin with the definition of the kinematics. In the following, $p^\mu = M_B v^\mu$ will denote the 4-momentum of the decaying B meson. The projection with its four-velocity v^μ defines the energy of the final-state particles in the B-meson rest frame (B-RF), $p^0 = (v \cdot p)$. The momenta of the decay products will be denoted as k_1^μ, k_2^μ for the two pions, and q_1^μ, q_2^μ for the two leptons, with the specific charge assignment according to

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$$B^-(p) \to \pi^+(k_1)\pi^-(k_2)\bar{\nu}(q_1)\ell^-(q_2).$$

We define the sum and difference of hadronic and leptonic momenta as

$$q = q_1 + q_2, \qquad k = k_1 + k_2,$$
$$\bar{q} = q_1 - q_2, \qquad \bar{k} = k_1 - k_2. \tag{2.1}$$

The hadronic system is then described by three kinematic Lorentz invariants: the momentum transfer q^2, the dipion invariant mass k^2, and the scalar product $q \cdot \bar{k}$. The latter defines the polar angle θ_π of the π^+ in the dipion rest frame

$$q \cdot \bar{k} = \frac{\beta_\pi}{2}\sqrt{\lambda}\cos\theta_\pi, \tag{2.2}$$

where $\beta_\pi^2 = (k^2 - 4M_\pi^2)/k^2 = -\bar{k}^2/k^2$, and $\lambda = \lambda(M_B^2, q^2, k^2)$ is the Källén function

$$\lambda(a,b,c) = a^2 + b^2 + c^2 - 2(ab + bc + ca). \tag{2.3}$$

The relative orientation between the leptons and hadrons in the final state is further characterized by the Lorentz invariants

$$k \cdot \bar{q} = \frac{1}{2}\sqrt{\lambda}\cos\theta_\ell,$$
$$\bar{k} \cdot \bar{q} = \frac{\beta_\pi}{2}((M_B^2 - k^2 - q^2)\cos\theta_\ell\cos\theta_\pi - 2\sqrt{q^2k^2}\sin\theta_\ell\sin\theta_\pi\cos\varphi), \tag{2.4}$$

where θ_ℓ is the polar angle of the negatively charged lepton in the dilepton rest frame, and φ is the azimuthal angle between the dilepton and dipion decay plane. Here and in the following lepton masses are set to zero. More details can be found in the Appendix.

In the following, it will be convenient to construct an orthogonal basis of momentum vectors,

$$q^\mu,$$
$$k^\mu_{(0)} = k^\mu - \frac{k \cdot q}{q^2}q^\mu,$$
$$\bar{k}^\mu_{(\parallel)} = \bar{k}^\mu - \frac{4(k\cdot q)(q\cdot\bar{k})}{\lambda}k^\mu + \frac{4k^2(q\cdot\bar{k})}{\lambda}q^\mu,$$
$$\bar{q}^\mu_{(\perp)} = 2\epsilon^{\mu\alpha\beta\gamma}\frac{q_\alpha k_\beta \bar{k}_\gamma}{\sqrt{\lambda}}. \tag{2.5}$$

Properly normalized, using

$$k^2_{(0)} = -\frac{\lambda}{4q^2}, \qquad \bar{k}^2_{(\parallel)} = \bar{q}^2_{(\perp)} = -\beta_\pi^2 k^2\sin^2\theta_\pi, \tag{2.6}$$

the vectors in Eq. (2.5) represent an orthonormal basis of timelike and spacelike polarization vectors associated with the leptonic currents, see also Eq. (A4) in the Appendix,

$$\varepsilon^\mu(t) = \frac{1}{\sqrt{q^2}}q^\mu, \qquad \varepsilon^\mu(0) = -\frac{2\sqrt{q^2}}{\sqrt{\lambda}}k^\mu_{(0)},$$
$$\varepsilon^\mu(\pm) = -\frac{1}{\sqrt{2k^2}\beta_\pi\sin\theta_\pi}(\bar{k}^\mu_{(\parallel)} \mp i\bar{q}^\mu_{(\perp)})e^{\mp i\varphi}, \tag{2.7}$$

which will be used to project onto helicity form factors.

B. Vector and axial-vector form factors

In the SM, the $B \to \pi\pi\ell\nu$ decay amplitudes are characterized by the transition form factors for vector and axial-vector $b \to u$ currents between a B meson and two pions. Using the definitions of the previous subsection, we parametrize the hadronic matrix elements in terms of one vector form factor $F_\perp$,

$$\langle\pi^+(k_1)\pi^-(k_2)|\bar{u}\gamma^\mu b|B^-(p)\rangle = iF_\perp\frac{1}{\sqrt{k^2}}\bar{q}^\mu_{(\perp)}, \tag{2.8}$$

and three axial-vector form factors F_t, F_0, $F_\parallel$,

$$-\langle\pi^+(k_1)\pi^-(k_2)|\bar{u}\gamma^\mu\gamma_5 b|B^-(p)\rangle = F_t\frac{q^\mu}{\sqrt{q^2}} + F_0\frac{2\sqrt{q^2}}{\sqrt{\lambda}}k^\mu_{(0)} + F_\parallel\frac{1}{\sqrt{k^2}}\bar{k}^\mu_{(\parallel)}. \tag{2.9}$$

Note here, that the apparent divergence of the hadronic matrix elements in the limit $q^2 \to 0$ is compensated by an appropriate phase space factor, see Eqs. (5.2) and (5.3). Here, each form factor depends on the three independent Lorentz invariants q^2, k^2, and $q\cdot\bar{k}$. It is also to be noted that, in general, the dipion form factors are complex functions above threshold $k^2 > 4m_\pi^2$. The prefactors in Eqs. (2.8) and (2.9) are chosen in such a way that the form factors correspond to particular helicity amplitudes which can be simply obtained by contraction

$$H_\lambda \equiv \langle\pi^+\pi^-|\bar{u}\gamma^\mu(1-\gamma_5)b|B^-\rangle\varepsilon^\dagger_\mu(\lambda), \tag{2.10}$$

with the polarization vectors as defined in Eq. (2.7). We obtain

$$H_t = F_t, \qquad H_0 = F_0, \qquad H_\pm = (F_\parallel \pm F_\perp)\frac{\beta_\pi}{\sqrt{2}}\sin\theta_\pi e^{\pm i\varphi}. \tag{2.11}$$

In terms of the so-defined "helicity form factors," one obtains simple expressions for the differential decay widths in the angular analysis and simple relations between form factors in HQET or SCET, which have also been emphasized for other decay modes [5,14–17].

C. Partial waves

The $B \to \pi\pi$ helicity amplitudes can be expanded in terms of associated Legendre polynomials $P_\ell^{(m)}(\cos\theta_\pi)$, with $\ell = (0, 1, 2, \ldots)$ corresponding to $(S, P, D, \ldots)$ partial waves. For the helicity amplitudes H_0 and H_t one obtains

$$H_{0,t} = \sum_{\ell=0}^{\infty} \sqrt{2\ell+1}\, H_{0,t}^{(\ell)}(q^2, k^2) P_\ell^{(0)}(\cos\theta_\pi) = H_{0,t}^{(S)}(q^2, k^2) + \sqrt{3} H_{0,t}^{(P)}(q^2, k^2)\cos\theta_\pi + \cdots, \tag{2.12}$$

and for $H_\pm$ one gets

$$H_\pm = \sum_{\ell=1}^{\infty} \sqrt{2\ell+1}\, H_\pm^{(\ell)}(q^2, k^2) P_\ell^{(\pm 1)}(\cos\theta_\pi) e^{\pm i\varphi} = \mp \frac{\sqrt{3}}{\sqrt{2}} H_\pm^{(P)}(q^2, k^2)\sin\theta_\pi e^{\pm i\varphi} + \cdots, \tag{2.13}$$

which contains no S-wave contribution. For the form factors F_0 and F_t this directly translates into the partial-wave expansion

$$F_{0,t} = \sum_{\ell=0}^{\infty} \sqrt{2\ell+1}\, F_{0,t}^{(\ell)}(q^2, k^2) P_\ell^{(0)}(\cos\theta_\pi) = F_{0,t}^{(S)}(q^2, k^2) + \sqrt{3} F_{0,t}^{(P)}(q^2, k^2)\cos\theta_\pi + \cdots \tag{2.14}$$

so that $H_{0,t}^{(\ell)} = F_{0,t}^{(\ell)}$. For the form factors $F_\parallel$ and $F_\perp$ we define

$$F_{\parallel,\perp} = -\sum_{\ell=1}^{\infty} \sqrt{2\ell+1}\, F_{\parallel,\perp}^{(\ell)}(q^2, k^2) \frac{P_\ell^{(1)}(\cos\theta_\pi)}{\sin\theta_\pi} = \frac{\sqrt{3}}{\sqrt{2}} F_{\parallel,\perp}^{(P)}(q^2, k^2) + \cdots, \tag{2.15}$$

such that $H_\pm^{(\ell)} = \mp \frac{\beta_\pi}{\sqrt{2}}(F_\parallel^{(\ell)} \pm F_\perp^{(\ell)})$.

III. FORM FACTOR RELATIONS

In certain kinematic limits, the form factors will obey approximate symmetry relations which would become exact in the limit of infinitely heavy b-quark mass. Similar to what is known from $B \to K^{(*)}\ell^+\ell^-$ decays, the form factor relations allow relatively robust predictions for angular observables which are independent of hadronic matrix elements in these limits. Note that the form factor relations are valid for each partial wave separately.

A. HQET limit

If the energy transfer to the hadronic final state is small, i.e. $(v \cdot k) \sim \Lambda_{\rm had} \ll m_b$, the heavy b quark acts as a quasistatic source of color, and the techniques of HQET are applicable. For the kinematic invariants in the $\pi\pi$ system this implies

$$q^2 \sim m_b^2, \qquad k^2 \sim \Lambda_{\rm had}^2, \qquad (q \cdot k) \sim \Lambda_{\rm had} m_b, \tag{3.1}$$

with $\Lambda_{\rm had}$ being a typical hadronic scale of order of a few hundred MeV; see also Fig. 1 for a sketch of the phase space. In particular, the general set of heavy-to-light form factors for arbitrary Dirac structures can be related to a smaller set of Isgur-Wise functions [18,19]. To this end, the dipion system is represented by the most general Dirac structure that can be constructed from the two pion momenta and the heavy-quark velocity. We define the following parametrization,

$$\mathcal{M}_{\pi\pi}(k, \bar{k}, v) \equiv \Xi_1 \frac{1}{\sqrt{q^2}} \not{q} + \Xi_2 \frac{2\sqrt{q^2}}{\sqrt{\lambda}} \not{k}_{(0)} + \Xi_3 \frac{1}{\sqrt{k^2}} \not{\bar{k}}_{(\parallel)} + i\Xi_4 \frac{1}{\sqrt{k^2}} \not{\bar{q}}_{(\perp)}\gamma_5, \tag{3.2}$$

which introduces four independent Isgur-Wise functions $\Xi_i = \Xi_i(v \cdot k, k^2, \cos\theta_\pi)$. For a given decay current, the form factors can then be obtained in terms of Clebsch-Gordan coefficients given by the Dirac trace,

$$\langle \pi^+(k_1)\pi^-(k_2)|\bar{u}\Gamma h_v^{(b)}|B^-(p)\rangle = \frac{1}{2}\mathrm{Tr}\left[\mathcal{M}_{\pi\pi}\Gamma \frac{1+\not{v}}{2}(-\gamma_5)\right], \tag{3.3}$$

where Γ is the Dirac matrix of the underlying current. For left-handed SM currents this yields one-to-one relations between the four helicity form factors F_i and the Isgur-Wise functions,

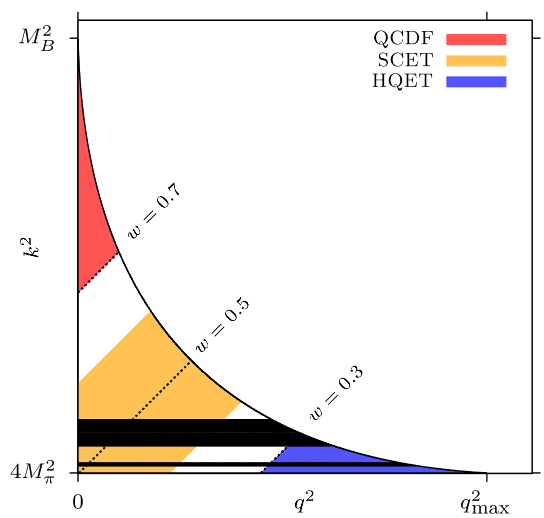

FIG. 1 (color online). Sketch of the q^2-k^2 phase space with $w = (v \cdot k)/M_B$ isolines for $w = 0.3, 0.5, 0.7$ (dashed lines). The typical regions of applicability for the different theory approaches, labeled QCDF ($w \geq 0.7$, red), SCET ($0.6 \geq w \geq 0.4$, orange), and HQET ($0.3 \geq w$, blue), are highlighted. The ρ resonances $\rho(770)$, $\rho(1450)$, $\rho(1700)$ are overlaid as horizontal grey bands for illustrative purpose.

$$F_t = \Xi_1, \qquad F_0 = \Xi_2, \qquad F_\parallel = \Xi_3, \qquad F_\perp = \Xi_4. \tag{3.4}$$

In the presence of NP, other $b \to u\ell\nu$ operators may contribute, and the corresponding form factors for pseudoscalar, tensor, or pseudotensor currents would be given by the same set of Isgur-Wise functions $\Xi_{1\text{–}4}$.

Explicit theoretical expressions for the Isgur-Wise functions $\Xi_{1\text{–}4}$ can be obtained in the limit where the two pions are soft, $v \cdot k_i \sim M_\pi$, in which case the methods of heavy-meson chiral perturbation theory [20] are applicable.

B. SCET limit

If the energy transfer to the hadronic final state is large, $(v \cdot k) \sim m_b/2 \gg \Lambda_{\rm had}$, while the invariant mass is small, $k^2 \ll m_b^2$, which also implies $q^2 \ll m_b^2$, the hadronic dynamics can be treated in SCET [21,22]. The phase space region associated with this limit is sketched in Fig. 1. Similar to the HQET case, this yields new form factor symmetry relations which have already been established for single light pseudoscalars or vector mesons in the final state [23,24] (analogous relations for baryonic decays can be found in [15,25]). These can be conveniently derived by introducing lightlike vectors $n_\pm^\mu$ in the k-q plane, according to

$$n_\pm^\mu = \left(1 \mp \frac{1}{\eta}\right) v^\mu \pm \frac{1}{|\vec{q}|} q^\mu, \tag{3.5}$$

in terms of the rapidity η and the three-momentum $|\vec{q}|$ of the lepton pair in the B-meson rest frame:

$$\eta = \frac{\sqrt{\lambda}}{M_B^2 - k^2 + q^2}, \qquad |\vec{q}| = \frac{\sqrt{\lambda}}{2M_B}. \tag{3.6}$$

These vectors satisfy the relations

$$n_\pm^2 = 0, \qquad n_+ \cdot n_- = 2, \qquad n_+^\mu + n_-^\mu = 2v^\mu, \tag{3.7}$$

and can be used to construct Dirac projectors

$$P_\pm = \frac{\not{n}_\pm \not{n}_\mp}{4}. \tag{3.8}$$

In the large-energy limit, only the P_+ projection of the energetic u quark in the $b \to u\ell\nu$ transition contributes. The trace in Eq. (3.3) then simplifies further, because the terms with $\not{q}$ and $\not{k}_{(0)}$ [$(\bar{k}_{(\parallel)}$ and $\vec{q}_{(\perp)}$] in $\mathcal{M}_{\pi\pi}$ yield the same contribution,

$$\langle \pi^+(k_1)\pi^-(k_2)|\bar{u}\Gamma h_v^{(b)}|B^-(p)\rangle = \mathrm{Tr}\left[\left(\frac{\not{q}}{\sqrt{q^2}}\xi_L + \frac{\bar{\not{k}}_{(\parallel)}}{\sqrt{k^2}}\xi_T\right)P_+\Gamma\frac{1+\not{v}}{2}(-\gamma_5)\right], \tag{3.9}$$

which implies the large-recoil form factor relations

$$F_t = F_0 = \Xi_1 = \Xi_2 \equiv \xi_L,$$
$$F_\parallel = F_\perp = \Xi_3 = \Xi_4 \equiv \xi_T. \tag{3.10}$$

]Theoretical approaches to predict the form factors ξ_L and ξ_T in the SCET limit depend on the distribution of the large energy/momentum among the two pions:

(i) If both pions are energetic and move collinear with a small invariant mass $k^2 \sim \Lambda_{\rm had}^2$, the two-pion state could be described by generalized distribution amplitudes, i.e. two-pion light cone distribution amplitudes (2πLCDAs) [26–29]. The 2πLCDAs contain the timelike pion form factors and the contributing hadronic resonances (notably $\rho \to \pi\pi$) as a limiting case.

(ii) If only one pion is energetic and the other soft, a combination of SCET/QCDF and chiral perturbation theory should apply, similar to [30] where this combination was studied in the context of $\bar{B} \to \bar{K}\pi\ell^+\ell^-$ decays.

IV. FORM FACTOR PROPERTIES

In this section we briefly comment on further generic properties of the dipion form factors that are characteristic in certain regions of the $|\pi\pi\rangle$ phase space.

A. QCD factorization for large dipion masses

Let us consider the kinematic regime where—in the B-meson rest frame—the two pions in the hadronic final state move almost back to back, each with large energy, such that their invariant mass is large, $k^2 \sim \mathcal{O}(m_b^2)$; see Fig. 1 for an illustration.[1] In this case, we face a similar situation as in nonleptonic $B \to \pi\pi$ decays, and thus expect that the QCD factorization approach from [12,13] should also be applicable.

Note that in nonleptonic decays, the short-distance quark transitions are dominantly described by four-quark operators that can directly induce the leading partonic Fock states required for $B \to \pi\pi$ transitions in the "naive" factorization approach. For nonleptonic decays, the radiative effects from additional gluons with virtualities of $\mathcal{O}(m_b^2)$ (hard) or $\mathcal{O}(\Lambda_{\rm had} m_b)$ (hard collinear) thus provide corrections to naive factorization.

In $B \to \pi\pi\ell\nu$ the situation is different, as the first nonvanishing contribution already requires the exchange of a hard gluon in order to produce the additional $q\bar{q}$ pair in the final state, see Fig. 2. This situation corresponds to the perturbative limit of the 2πLCDAs discussed in [31] which can then be expressed in terms of the conventional pion LCDAs.

[1] It is to be noted that the QCDF approach for two back-to-back pions also makes use of SCET techniques for the resummation of large logarithms in higher-order perturbation theory. In the same way, radiative corrections to form factor symmetry relations in SCET can be calculated within the QCDF approach.

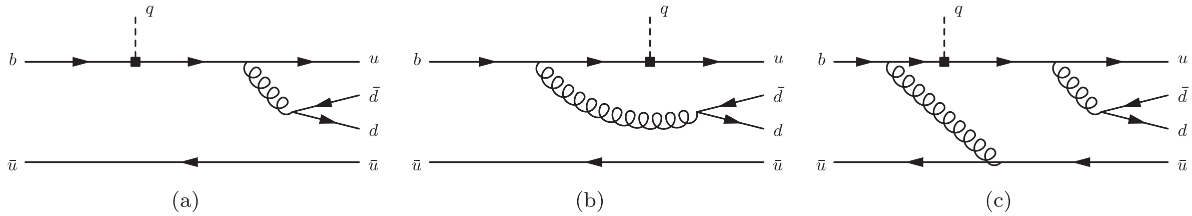

FIG. 2. Sketch of QCD factorization in $B \to \pi\pi\ell\nu$ decays at large dipion masses: (a),(b) Leading contributions from hard gluon exchange; (c) sample diagram for hard-collinear spectator scattering corrections.

We thus expect the QCD factorization formula for the dipion form factors in the considered kinematic limit (and for $m_b \gg \Lambda_{\rm had}$) to take an analogous form as for nonleptonic $B \to \pi\pi$ decays. Here at leading term all dipion form factors would be expressed in terms of a universal $B \to \pi$ form factor, the first inverse moment of the pion LCDA, and simple kinematic factors. The measurement of the dipion form factors would thus provide an independent test of the QCD factorization approach, respectively an independent determination of the relevant hadronic input parameters. Radiative corrections from hard and hard-collinear gluon exchange could be calculated perturbatively, see Fig. 2. More details will be provided in [32].

B. Resonance contributions

Formally, a resonance contribution to $B \to \pi\pi$ form factors can be obtained using hadronic dispersion relations in the variable k^2,

$$\langle\pi\pi|J^\mu_{V-A}|\bar{B}\rangle = \frac{1}{\pi}\int_{4M_\pi^2}^\infty \mathrm{d}s\,\frac{\mathrm{Im}\langle\pi\pi|J^\mu_{V-A}|\bar{B}\rangle}{s-k^2-i\varepsilon} + \text{subtractions}, \tag{4.1}$$

with the current $J^\mu_{V-A} = \bar{u}\gamma^\mu(1-\gamma_5)b$. Insertion of all possible intermediate states yields a unitarity relation

$$2\,\mathrm{Im}\langle\pi\pi|J^\mu_{V-A}|\bar{B}\rangle = \sum_H \int \mathrm{d}\tau_H \langle\pi\pi|H\rangle\langle H|J^\mu_{V-A}|\bar{B}\rangle, \tag{4.2}$$

with integration over the phase space τ_H and summation over the helicity states of the intermediate hadronic state H. We single out in this relation $H = R$, with a resonant one-particle intermediate state R, so that the right-hand side contains the strong coupling $\langle\pi\pi|R\rangle$ of R with two pions multiplied by the form factors for $B \to R$ transitions.

At this point we must carefully identify the resonances that emerge in the k^2 spectrum, according to the isospin quantum numbers of the dipion. In the decay $B^- \to \pi^+\pi^-\ell^-\bar{\nu}_\ell$ the dipion system is a superposition of the isoscalar $I^G = 0^+$ and isovector $(I^G, I_3) = (1^+, 0)$ states. In the analogous decay $\bar{B}^0 \to \pi^+\pi^0\ell^-\bar{\nu}_\ell$ and $B^- \to \pi^0\pi^0\ell^-\bar{\nu}_\ell$, however, the pions are purely in the isovector $(I^G, I_3) = (1^+, +1)$ and isoscalar state, respectively. Altogether, the three hadronic matrix elements for $B \to \pi\pi$ are expressed in terms of two independent isospin amplitudes. From this we obtain in the isospin symmetry limit the relation

$$\langle\pi^+\pi^-|J^\mu_{V-A}|B^-\rangle + \frac{1}{\sqrt{2}}\langle\pi^+\pi^0|J^\mu_{V-A}|\bar{B}^0\rangle = \langle\pi^0\pi^0|J^\mu_{V-A}|B^-\rangle. \tag{4.3}$$

We consider only resonant contributions due to the isovector vector mesons $\rho(n)$, as well as the isoscalar scalar mesons $f_0(n)$, where n denotes the quantum number of radial excitation. We sketch the region of phase space where the $\rho(n)$ dominate in Fig. 1. Since we consider only dipion states up to angular momentum 1, we discard resonances with spin larger than 1. Hereafter, we will proceed with the more general case of $B^- \to \pi^+\pi^-\ell^-\bar{\nu}_\ell$. The B^0 decay can be recovered by omitting the f_0 contributions and adding a relevant isospin factor.

Continuing with Eq. (4.2), we obtain for the contribution of the ρ intermediate states

$$\mathrm{Im}\langle\pi\pi|J^\mu_{V-A}|\bar{B}\rangle = -\pi g_{\rho\pi\pi}\delta(M_\rho^2 - s)\sum_{a=0,+,-}[\bar{k}\cdot\eta(a)] \times \langle\rho(k,\eta(a))|J^\mu_{V-A}|\bar{B}(p)\rangle, \tag{4.4}$$

with η being the polarization vector for the vector state associated with the four-momentum k. In the B-RF

$$\eta(\pm)^\mu|_{B\text{-RF}} = \varepsilon(\mp)^\mu|_{B\text{-RF}}, \quad \eta(0)^\mu|_{B\text{-RF}} = (|\vec{q}|, 0, 0, M_B - q_0)/M_V, \tag{4.5}$$

see the Appendix for details. For the f_0 state we obtain

$$\mathrm{Im}\langle\pi\pi|J^\mu_{V-A}|\bar{B}\rangle = \pi g_{f_0\pi\pi}\delta(M_{f_0}^2 - s)M_{f_0}\langle f_0(k)|J^\mu_{V-A}|\bar{B}(p)\rangle. \tag{4.6}$$

For both ρ and f_0, the above formulas still employ the narrow-width approximation. The strong couplings are fixed via

$$\langle\pi\pi|f_0\rangle = g_{f_0\pi\pi}M_{f_0}, \qquad \langle\pi\pi|\rho(a)\rangle = -(\bar{k}\cdot\eta(a))g_{\rho\pi\pi}, \tag{4.7}$$

for the f_0, and for the ρ helicity states $a = \pm, 0$. Note that $g_{\rho\pi\pi} = g_{\rho^0\pi^+\pi^-} = -g_{\rho^+\pi^+\pi^0}$ due to isospin.

We use the helicity decomposition of $B\to R$, $R = S, V$ form factors as in [14], adjusted to our notation and phase convention. By $S(k)$ and $V(k,\eta)$ we shall denote a hadronic scalar and vector state with momentum k and polarization vector η, respectively. We define for the vector resonances

$$\kappa\frac{\sqrt{q^2}}{\sqrt{\lambda_V}}\langle V(k,\eta(\pm))|\bar{u}\gamma^\mu b|\bar{B}(p)\rangle = \pm F_\perp^{B\to V}(q^2)\varepsilon^\mu(\pm), \tag{4.8}$$

as well as

$$-\kappa\frac{\sqrt{q^2}}{\sqrt{\lambda_V}}\langle V(k,\eta(0))|\bar{u}\gamma^\mu\gamma_5 b|\bar{B}(p)\rangle = F_t^{B\to V}(q^2)\varepsilon^\mu(t) - F_0^{B\to V}(q^2)\varepsilon^\mu(0), \tag{4.9}$$

$$-\kappa\frac{\sqrt{q^2}}{\sqrt{\lambda_V}}\langle V(k,\eta(\pm))|\bar{u}\gamma^\mu\gamma_5 b|\bar{B}(p)\rangle = F_\parallel^{B\to V}(q^2)\varepsilon^\mu(\pm), \tag{4.10}$$

and for the scalar resonances

$$-\frac{\sqrt{q^2}}{\sqrt{\lambda_S}}\langle S(k)|\bar{u}\gamma^\mu\gamma_5 b|\bar{B}(p)\rangle = F_t^{B\to S}(q^2)\varepsilon^\mu(t) - F_0^{B\to S}(q^2)\varepsilon^\mu(0), \tag{4.11}$$

where we abbreviate $\lambda_R \equiv \lambda(M_B^2, M_R^2, q^2)$ and use an isospin factor $\kappa = \sqrt{2}$ for $B^-\to\rho^0$ transitions, and $\kappa = 1$ for $\bar{B}^0\to\rho^+$ transitions.

We express the resonant pole contributions to the $B\to\pi\pi$ form factors in terms of the $B\to V$ and $B\to S$ form factors. In this way we obtain for all final-state polarizations the P-wave contributions

$$\frac{\sqrt{3}}{\sqrt{2}}\,\mathrm{Res}\,F_{\parallel,\perp}^{(P)}(q^2,k^2)|_{k^2=P_V} = \frac{g_{V\pi\pi}}{\kappa}\frac{\sqrt{\lambda_V}M_V}{\sqrt{q^2}}F_{\parallel,\perp}^{B\to V}(q^2), \tag{4.12}$$

$$\sqrt{3}\mathrm{Res}\,F_{0,t}^{(P)}(q^2,k^2)|_{k^2=P_V} = \frac{g_{V\pi\pi}\beta_\pi}{\kappa}\frac{\sqrt{\lambda_V}M_V}{\sqrt{q^2}}F_{0,t}^{B\to V}(q^2). \tag{4.13}$$

For the S-wave contributions we find

$$\mathrm{Res}\,F_{0,t}^{(S)}(q^2,k^2)|_{k^2=P_S} = g_{S\pi\pi}\frac{\sqrt{\lambda_S}M_S}{\sqrt{q^2}}F_{0,t}^{B\to S}(q^2). \tag{4.14}$$

The total decay width Γ_R was added to the pole $P_R = M_R^2 - iM_R\Gamma_R$, thus yielding standard Breit-Wigner factors

$$BW_R(k^2) = \frac{1}{[M_R^2 - k^2 - iM_R\Gamma_R]}, \tag{4.15}$$

which govern the resonance behavior in the variable k^2 close to $k^2 = M_R^2$, $R = \rho(n), f_0(n)$. Note that the widths can be interpreted as contribution of multihadron states to the imaginary part of $\langle\pi\pi|\bar{B}\rangle$ in k^2. For more details we refer to [33] where the origin of the ρ width in the pion form factor was discussed in detail.

V. DECAY RATE AND ANGULAR ANALYSIS

In terms of the vector and axial-vector form factors, the amplitude for $B\to\pi\pi\ell\nu$ in the SM can be expressed as

$$i\mathcal{M} = i\frac{G_F V_{ub}}{\sqrt{2}}\left[F_0\varepsilon^\mu(0) + \frac{F_\parallel + F_\perp}{\sqrt{2}}\beta_\pi\,\sin\theta_\pi e^{+i\varphi}\varepsilon^\mu(+) + \frac{F_\parallel - F_\perp}{\sqrt{2}}\beta_\pi\,\sin\theta_\pi e^{-i\varphi}\varepsilon^\mu(-)\right][\bar{u}_\ell\gamma_\mu(1-\gamma_5)v_\nu], \tag{5.1}$$

where the helicity form factor for timelike polarization F_t does not contribute in the limit of massless leptons. In the following, we find it convenient to express our result in terms of normalized partial-wave amplitudes defined from the corresponding partial-wave expansion of the form factors,

$$A_n^{(k)} = NF_n^{(k)} \quad (\text{with } n = 0, \parallel, \perp), \tag{5.2}$$

where the normalization factor absorbs kinematic and coupling parameters,

$$N = G_F|V_{ub}|\frac{\sqrt{q^2\beta_\ell\beta_\pi\sqrt{\lambda}}}{\sqrt{3\cdot 2^{10}\pi^5 M_B^3}}, \tag{5.3}$$

and we restrict our analysis to $k = S$, P waves in the following.

The fivefold differential decay width for $\bar{B}\to\pi^+\pi^0\ell^-\bar{\nu}$ then takes a similar form as for the rare flavor-changing neutral current decay $\bar{B}\to\bar{K}\pi\ell^+\ell^-$, which has received a lot of attention recently [6–10]. Choosing q^2, k^2, $\cos\theta_\pi$, $\cos\theta_\ell$, and φ as the five independent kinematic variables, we obtain

$$\frac{8\pi}{3}\frac{d^5\Gamma}{dq^2 dk^2 d\cos\theta_\pi d\cos\theta_\ell d\varphi} \equiv J \equiv \sum_n J_n f_n, \tag{5.4}$$

where $J(q^2, k^2, \cos\theta_\pi, \cos\theta_\ell, \varphi)$ is decomposed into the angular functions $f_n \equiv f_n(\cos\theta_\pi, \cos\theta_\ell, \varphi)$ and angular observables $J_n \equiv J_n(q^2, k^2)$. This notation has been introduced in [3] for $B \to K^*(\to K\pi)\ell^+\ell^-$ decays, originally restricted to pure P-wave contributions and not taking into account scalar or pseudoscalar operators (which could be relevant in certain NP models). The general case, including a general basis of $b \to u$ operators and interference effects between S- and P-wave contributions, can be worked out following [4,10] and reads

$$\begin{aligned} J = {}& (J_{1s}\sin^2\theta_\pi + J_{1c}\cos^2\theta_\pi + J_{1sc}\cos\theta_\pi) + (J_{2s}\sin^2\theta_\pi + J_{2c}\cos^2\theta_\pi + J_{2sc}\cos\theta_\pi)\cos 2\theta_\ell \\ & + J_3\sin^2\theta_\pi\sin^2\theta_\ell\cos 2\varphi + (J_4\sin 2\theta_\pi + J_{4i}\sin\theta_\pi)\sin 2\theta_\ell\cos\varphi \\ & + (J_5\sin 2\theta_\pi + J_{5i}\sin\theta_\pi)\sin\theta_\ell\cos\varphi + (J_{6s}\sin^2\theta_\pi + J_{6c}\cos^2\theta_\pi)\cos\theta_\ell \\ & + (J_7\sin 2\theta_\pi + J_{7i}\sin\theta_\pi)\sin\theta_\ell\sin\varphi + (J_8\sin 2\theta_\pi + J_{8i}\sin\theta_\pi)\sin 2\theta_\ell\sin\varphi \\ & + J_9\sin^2\theta_\pi\sin^2\theta_\ell\sin 2\varphi. \end{aligned} \tag{5.5}$$

Comparing with Eq. (5.4) in the SM, we obtain

$$\begin{aligned} \frac{4}{3}J_{1s} &= \frac{3}{4}\beta_\pi^2(|A_\perp^{(P)}|^2 + |A_\parallel^{(P)}|^2) + \frac{1}{3}|A_0^{(S)}|^2, \\ \frac{4}{3}J_{1c} &= |A_0^{(P)}|^2 + \frac{1}{3}|A_0^{(S)}|^2 = -\frac{4}{3}J_{2c}, \\ \frac{4}{3}J_{1sc} &= \frac{2}{\sqrt{3}}\mathrm{Re}\{A_0^{(P)}A_0^{(S)*}\} = -\frac{4}{3}J_{2sc}, \\ \frac{4}{3}J_{2s} &= \frac{1}{4}\beta_\pi^2(|A_\perp^{(P)}|^2 + |A_\parallel^{(P)}|^2) - \frac{1}{3}|A_0^{(S)}|^2, \\ \frac{4}{3}J_3 &= \frac{1}{2}\beta_\pi^2(|A_\perp^{(P)}|^2 - |A_\parallel^{(P)}|^2), \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} \frac{4}{3}J_4 &= \frac{1}{\sqrt{2}}\beta_\pi\mathrm{Re}\{A_0^{(P)}A_\parallel^{(P)*}\}, \\ \frac{4}{3}J_{4i} &= \frac{\sqrt{2}}{\sqrt{3}}\beta_\pi\mathrm{Re}\{A_0^{(S)}A_\parallel^{(P)*}\}, \\ \frac{4}{3}J_5 &= \sqrt{2}\beta_\pi\mathrm{Re}\{A_0^{(P)}A_\perp^{(P)*}\}, \\ \frac{4}{3}J_{5i} &= \frac{2\sqrt{2}}{\sqrt{3}}\beta_\pi\mathrm{Re}\{A_0^{(S)}A_\perp^{(P)*}\}, \\ \frac{4}{3}J_{6s} &= 2\beta_\pi^2\mathrm{Re}\{A_\parallel^{(P)}A_\perp^{(P)*}\}, \\ \frac{4}{3}J_{6c} &= 0, \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} \frac{4}{3}J_7 &= \sqrt{2}\beta_\pi\mathrm{Im}\{A_0^{(P)}A_\parallel^{(P)*}\}, \\ \frac{4}{3}J_{7i} &= \frac{2\sqrt{2}}{\sqrt{3}}\beta_\pi\mathrm{Im}\{A_0^{(S)}A_\parallel^{(P)*}\}, \\ \frac{4}{3}J_8 &= \frac{1}{\sqrt{2}}\beta_\pi\mathrm{Im}\{A_0^{(P)}A_\perp^{(P)*}\}, \\ \frac{4}{3}J_{8i} &= \frac{\sqrt{2}}{\sqrt{3}}\beta_\pi\mathrm{Im}\{A_0^{(S)}A_\perp^{(P)*}\}, \\ \frac{4}{3}J_9 &= \beta_\pi^2\mathrm{Im}\{A_\perp^{(P)}A_\parallel^{(P)*}\}. \end{aligned} \tag{5.8}$$

Our result for the functions J_i takes an analogous form as found for $\bar{B} \to (\bar{K}\pi)_{S,P}\ell^+\ell^-$ decays in e.g. [7,9]. Note that the relative strong phases of the dipion form factors can be sizeable, and we thus keep all the angular observables that involve an imaginary part in Eq. (5.8).

VI. MODEL-INDEPENDENT RESULTS

The large number of observables J_n in the angular distribution allows us to infer certain information from experimental data, search for physics beyond the SM, and test various theoretical approaches to QCD.

A. Null tests in and of the SM

The V-A nature of the weak interaction in $b \to u$ transitions can be probed in $B \to \pi\pi\ell^-\bar{\nu}_\ell$ decays through two independent, experimental sets of null tests.

The first set is given by the theory prediction that

$$J_{6c} = 0, \tag{6.1}$$

and

$$J_{1c} + J_{2c} = 0, \qquad J_{1sc} + J_{2sc} = 0, \tag{6.2}$$

$$J_{1sc} - \frac{J_5 J_{5i} + 4J_8 J_{8i}}{J_{1s} + J_{2s} + 2J_3} = 0, \quad (6.3)$$

$$J_{6s} - \frac{8J_4 J_5 + 8J_7 J_8}{4J_{1c} - J_{1s} + 3J_{2s}} = 0, \quad (6.4)$$

$$J_9 - \frac{2J_5 J_7 - 8J_4 J_8}{4J_{1c} - J_{1s} + 3J_{2s}} = 0, \quad (6.5)$$

$$(-4J_{2c} - (J_{1s} - 3J_{2s}))(J_{1s} + J_{2s} - 2J_3) - (16J_4^2 + 4J_7^2) = 0, \quad (6.6)$$

$$(-4J_{2c} - (J_{1s} - 3J_{2s}))(J_{1s} + J_{2s} + 2J_3) - (4J_5^2 + 16J_8^2) = 0, \quad (6.7)$$

$$(J_{1s} - 3J_{2s})(J_{1s} + J_{2s} - 2J_3) - (4J_{4i}^2 + J_{7i}^2) = 0, \quad (6.8)$$

$$(J_{1s} - 3J_{2s})(J_{1s} + J_{2s} + 2J_3) - (J_{5i}^2 + 4J_{8i}^2) = 0, \quad (6.9)$$

$$4J_9(J_{5i}J_{4i} + J_{8i}J_{7i}) + J_{6s}(4J_{8i}J_{4i} - J_{7i}J_{5i}) = 0, \quad (6.10)$$

in the absence of D-wave or higher partial-wave contributions.[2] Any deviation from Eq. (6.1) would indicate NP effects of both scalar and tensor nature, compare [10] in the context of $\bar{B} \to \bar{K}\pi\ell^+\ell^-$. Breaking of Eq. (6.2) can be achieved by less exotic models which introduce $V + A$ interactions. The relations Eqs. (6.6)–(6.10) hold in the absence of contributions from either scalar or tensor operators. The above relations are similar to those obtained for the decay $\bar{B} \to \bar{K}^*(\to \bar{K}\pi)\ell^+\ell^-$ in [34].

The second set of test only holds in the SCET limit. In that limit

$$J_3 = \mathcal{O}(\Lambda_{\rm had}/m_b), \qquad J_9 = \mathcal{O}(\Lambda_{\rm had}/m_b), \quad (6.11)$$

as well as

$$J_{1s} + J_{2s} - J_{6s} = \mathcal{O}(\Lambda_{\rm had}/m_b), \quad (6.12)$$

$$\frac{J_7}{2J_4} - \frac{2J_8}{J_5} = \mathcal{O}(\Lambda_{\rm had}/m_b), \quad (6.13)$$

$$\frac{J_{7i}}{2J_{4i}} - \frac{2J_{8i}}{J_{5i}} = \mathcal{O}(\Lambda_{\rm had}/m_b), \quad (6.14)$$

since the form factors fulfill $F_\perp^{(k)} = F_\parallel^{(k)} + \mathcal{O}(\Lambda_{\rm had}/m_b)$ for all partial waves k. Breaking of the relations (6.11)–(6.14) in the SCET limit can only be achieved through either (a) subleading corrections to the form factor relation or (b) NP effects in $b \to u$ transitions, such as $V + A$ transitions.

B. Accessing form factor ratios and phase differences

We write each form factor $F_i^{(l)}$ in polar form,

$$F_i^{(l)} = r_i^{(l)} e^{i\varphi_i^{(l)}}, \quad (6.15)$$

using the moduli $r_i^{(l)}$ and phases $\varphi_i^{(l)}$. Given the explicit $V - A$ nature of $b \to u$ transitions in the SM, we can access five phase differences through ratios of angular observables,

$$\frac{-2J_9}{J_{6s}} = \tan(\varphi_\parallel^{(P)} - \varphi_\perp^{(P)}), \qquad \frac{J_7}{2J_4} = \tan(\varphi_0^{(P)} - \varphi_\parallel^{(P)}),$$
$$\frac{J_{7i}}{2J_{4i}} = \tan(\varphi_0^{(S)} - \varphi_\parallel^{(P)}), \qquad \frac{2J_8}{J_5} = \tan(\varphi_0^{(P)} - \varphi_\perp^{(P)}),$$
$$\frac{2J_{8i}}{J_{5i}} = \tan(\varphi_0^{(S)} - \varphi_\perp^{(P)}), \quad (6.16)$$

where we employ ten independent angular observables. Moreover, we can access four ratios of moduli $r_i^{(l)}/r_j^{(k)}$

$$\frac{J_{2sc}}{J_{2c}} = \frac{2\sqrt{3} r_0^{(S)}/r_0^{(P)}}{3 + (r_0^{(S)}/r_0^{(P)})^2} \cos(\varphi_0^{(P)} - \varphi_0^{(S)}), \quad (6.17)$$

and

$$\frac{J_{1s} + J_{2s} + 2J_3}{J_{1s} + J_{2s} - 2J_3} = \left(\frac{r_\perp^{(P)}}{r_\parallel^{(P)}}\right)^2,$$
$$\frac{3\beta_\pi^2 (J_{1s} - 3J_{2s})}{2(J_{1s} + J_{2s} - 2J_3)} = \left(\frac{r_0^{(S)}}{r_\parallel^{(P)}}\right)^2,$$
$$\frac{3\beta_\pi^2 (J_{1s} - 3J_{2s})}{2(J_{1s} + J_{2s} + 2J_3)} = \left(\frac{r_0^{(S)}}{r_\perp^{(P)}}\right)^2 \quad (6.18)$$

using four further independent observables. Overall this amounts to nine constraints on the form factors that arise from 14 angular observables. Together with J_{6c} (vanishing

[2] We expect sizable contributions when the dipion mass approaches the mass of the f_2 meson or its radial excitations.

in the SM), $J_{1c,1sc}$ (not independent from $J_{2c,2sc}$ in the SM), and the differential decay width,

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}q^2} = J_{1c} - \frac{1}{3}J_{2c} + 2J_{1s} - \frac{2}{3}J_{2s} \propto |V_{ub}|^2[3(r_\perp^{(P)})^2 + 3(r_\parallel^{(P)})^2 + 3(r_0^{(P)})^2 + (r_0^{(S)})^2], \tag{6.19}$$

we arrive at 18 angular observables. Thus, the determination of form factor ratios, form factor phases, and the product of form factor moduli and $|V_{ub}|$ as described in Eqs. (6.16)–(6.19) extracts the maximum amount of information from the angular distribution.

VII. CONCLUSION

In this paper we have considered the semileptonic decay $B \to \pi\pi\ell\bar{\nu}_\ell$ in SM and analyzed the complete set of angular observables describing the four-body final state. Detailed quantitative predictions for these observables require genuinely nonperturbative information, which is encoded in hadronic $B \to \pi\pi$ form factors. In turn, as we have explored, a full-fledged angular analysis of the decay will allow one to extract form factor ratios and relative strong phases from experimental data. We have also shown that in the soft or collinear limit, the number of independent form factors is reduced due to heavy-quark symmetries in HQET or SCET, respectively.

The tension in the determination of $|V_{ub}|$ has lead to speculations about possible nonstandard contributions in $b \to u$ transitions. As we have discussed in this paper, the chiral structure of weak interactions can be used to identify null tests of the SM in $B \to \pi\pi\ell\bar{\nu}_\ell$ decay observables; i.e. any violation of the $V - A$ structure in $b \to u$ transitions will show up in modifications of Eqs. (6.1) and (6.2).

Our observations can also be useful for interpolation between different corners of phase space, where the resonance structure of the $\pi\pi$ system is described by phenomenological models, or theoretical calculations based on QCD factorization, heavy-hadron chiral perturbation theory, or QCD sum rules are applicable. Detailed analyses of these kinds go beyond the scope of the present paper and are left for future work.

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APPENDIX: DETAILS ON THE KINEMATICS

This appendix shall elaborate on the definitions of kinematic variables in the course of our calculations, starting with general remarks.

First, we choose the z axis along the flight direction of the dipion system, and consequently the dilepton system moves along the negative z axis. We also put the dilepton system into the x-z plane.

Second, we make use of a set of virtual polarization vectors $\varepsilon(n)^\mu$, $n = t, \pm, 0$ that fulfill the completeness relations

$$\varepsilon(n)\cdot q = 0, \qquad n = \pm, 0,$$
$$\varepsilon(n)\cdot\varepsilon^\dagger(n') = g_{nn'}, \qquad \varepsilon(n)_\mu \varepsilon^\dagger(n')_\nu g_{nn'} = g_{\mu\nu}, \tag{A1}$$

where $g_{nn'} = \mathrm{diag}(+1, -1, -1, -1)$ for n, $n' = t, +, -, 0$.

In the following we will discuss the explicit expressions for the various momenta and polarization vectors in the three frames that are relevant to the decay analysis.

1. The dilepton rest frame

We describe the dilepton system through its invariant mass q^2 as well as the lepton helicity angle θ_ℓ, i.e., the angle between the ℓ^- direction of flight and the z axis in the dilepton rest frame. We choose the x-z plane as the decay plane of the dilepton system. Thus, we write in the $\ell\nu$ rest frame ($\ell\nu$-RF)

$$q^\mu_{1,2}|_{\ell\nu\text{-RF}} = \frac{\sqrt{q^2}}{2}(1, \mp\sin\theta_\ell, 0, \mp\cos\theta_\ell), \tag{A2}$$

and correspondingly

$$q^\mu|_{\ell\nu\text{-RF}} = \sqrt{q^2}(1,0,0,0),$$
$$\bar{q}^\mu|_{\ell\nu\text{-RF}} = -\sqrt{q^2}(0, \sin\theta_\ell, 0, \cos\theta_\ell). \tag{A3}$$

The polarization vectors $\varepsilon^\mu(n)$ take the explicit form

$$\varepsilon^\mu(t)|_{\ell\nu\text{-RF}} = (1,0,0,0),$$
$$\varepsilon^\mu(\pm)|_{\ell\nu\text{-RF}} = (0,1,\mp i,0)/\sqrt{2},$$
$$\varepsilon^\mu(0)|_{\ell\nu\text{-RF}} = (0,0,0,-1). \tag{A4}$$

Comments are due on the choice of the polarization vectors, especially the signs of $\varepsilon^z(0)$ as well as $\varepsilon^y(\pm)$. These have been adopted to obtain longitudinal and right-handed/left-handed polarization of the $\ell\nu$ system, which moves along the negative z axis.

2. The B-meson rest frame

In the rest frame of the $\bar{B}$ meson (B-RF) we write explicitly

$$p^\mu|_{B\text{-RF}} = (M_B, 0, 0, 0), \qquad q^\mu|_{B\text{-RF}} = (q^0, 0, 0, -|\vec{q}|),$$
$$k^\mu|_{B\text{-RF}} = (M_B - q^0, 0, 0, +|\vec{q}|). \tag{A5}$$

Since we chose to describe the decay through the invariants q^2 and k^2, we use

$$q^0|_{B\text{-RF}} = \frac{M_B^2 - k^2 + q^2}{2M_B}, \qquad |\vec{q}||_{B\text{-RF}} = \frac{\sqrt{\lambda}}{2M_B}. \tag{A6}$$

Application of a Lorentz boost along the z axis from the $\ell\nu$-RF to the B-RF leaves $\varepsilon(\pm)$ invariant, while $\varepsilon(t)$ and $\varepsilon(0)$ are transformed:

$$\varepsilon^\mu(t)|_{B\text{-RF}} = (q^0, 0, 0, -|\vec{q}|)/\sqrt{q^2},$$
$$\varepsilon^\mu(0)|_{B\text{-RF}} = (|\vec{q}|, 0, 0, -q^0)/\sqrt{q^2}. \tag{A7}$$

3. The dipion rest frame

We describe the dipion system through its invariant mass k^2 as well as the pion helicity angle θ_π, i.e., the angle between the π^+ direction of flight and the z axis in the dipion rest frame ($\pi\pi$-RF). In addition, there is an azimuthal angle φ between the dipion and the dilepton decay planes. The planes' normal vectors are defined in the B-RF as $\vec{e}_\pi = (\vec{k}_1 \times \vec{k}_2)/|\vec{k}_1 \times \vec{k}_2|$ and $\vec{e}_\ell = (\vec{q}_2 \times \vec{q}_1)/|\vec{q}_2 \times \vec{q}_1|$, respectively. Since the angle φ depends only on the x and y components of k_1, k_2, q_1, and q_2— which are invariant under z-axis boosts between the $\bar{B}$ rest frame, the dipion rest frame, and the dilepton rest frame—we find that φ is the same in all considered frames of reference laid out in this section. We fix the x axis by requiring $(q_2)_x > 0$, which implies $\vec{e}_\ell = \vec{e}_y$. From $\varphi = 0$ then follows $\vec{e}_\pi = \vec{e}_y$ and further $(\vec{k}_1)_x < 0$ as well as $(\vec{k}_1)_y = 0$. The spatial components of k_1 therefore point in the negative x direction for $\varphi = 0$. Furthermore, we use $\sin\varphi \equiv (\vec{e}_\ell \times \vec{e}_\pi)\cdot\vec{e}_z$ as in [3], from which we infer that φ is the azimuthal angle of the momentum k_2. The $\pi^+\pi^-$ decay plane is therefore rotated with regard to the dilepton (x-z) plane by the angle $-\varphi$ around the z axis. From this, one obtains in the dipion rest frame

$$k_1^\mu|_{\pi\pi\text{-RF}} = \begin{pmatrix} E_\pi \\ -|\vec{k}_{\rm RF}|\sin\theta_\pi\cos\varphi \\ -|\vec{k}_{\rm RF}|\sin\theta_\pi\sin\varphi \\ +|\vec{k}_{\rm RF}|\cos\theta_\pi \end{pmatrix}, \tag{A8}$$

$$k_2^\mu|_{\pi\pi\text{-RF}} = \begin{pmatrix} E_\pi \\ +|\vec{k}_{\rm RF}|\sin\theta_\pi\cos\varphi \\ +|\vec{k}_{\rm RF}|\sin\theta_\pi\sin\varphi \\ -|\vec{k}_{\rm RF}|\cos\theta_\pi \end{pmatrix}, \tag{A9}$$

and consequently

$$k^\mu|_{\pi\pi\text{-RF}} = \begin{pmatrix} \sqrt{k^2} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{A10}$$

$$\bar{k}^\mu|_{\pi\pi\text{-RF}} = \begin{pmatrix} 0 \\ -2|\vec{k}_{\rm RF}|\sin\theta_\pi\cos\varphi \\ -2|\vec{k}_{\rm RF}|\sin\theta_\pi\sin\varphi \\ 2|\vec{k}_{\rm RF}|\cos\theta_\pi \end{pmatrix}, \tag{A11}$$

with

$$|\vec{k}_{\rm RF}| \equiv \frac{\beta_\pi}{2}\sqrt{k^2}, \qquad E_\pi \equiv \frac{\sqrt{k^2}}{2}, \tag{A12}$$

where $\beta_\pi^2 = (k^2 - 4M_\pi^2)/k^2$.

4. Frame-independent quantities

For convenience we present here the scalar products and Levi-Cività contractions that were used in our calculations, expressed in terms of the five kinematic variables q^2, k^2 and the three angles θ_π, θ_ℓ, and φ. The scalar products read

$$\varepsilon(t)\cdot\bar{q} = 0, \tag{A13}$$

$$\varepsilon(t)\cdot k_{(0)} = \varepsilon(t)\cdot\bar{k}_{(\|)} = 0, \tag{A14}$$

$$\varepsilon(\pm)\cdot\bar{q} = +\frac{\sqrt{q^2}}{\sqrt{2}}\sin\theta_\ell, \tag{A15}$$

$$\varepsilon^\dagger(\pm)\cdot\bar{k}_{(\|)} = \frac{\beta_\pi\sqrt{k^2}}{\sqrt{2}}\sin\theta_\pi\exp(\pm i\varphi), \tag{A16}$$

$$\varepsilon(0)\cdot\bar{q} = -\sqrt{q^2}\cos\theta_\ell, \tag{A17}$$

$$\varepsilon(0)\cdot k_{(0)} = \frac{\sqrt{\lambda}}{2\sqrt{q^2}}, \tag{A18}$$

$$\varepsilon(0) \cdot \bar{k}_{(\parallel)} = 0. \tag{A19}$$

For the contractions with the Levi-Cività we obtain

$$\varepsilon(\varepsilon^\dagger(t), q, k, \bar{k}) = \varepsilon(\varepsilon^\dagger(0), q, k, \bar{k}) = 0, \tag{A20}$$

$$\varepsilon(\varepsilon^\dagger(\pm), q, k, \bar{k}) = \mp i\beta_\pi \frac{\sqrt{\lambda}\sqrt{k^2}}{2\sqrt{2}} \sin\theta_\pi \exp(\pm i\varphi), \tag{A21}$$

$$\varepsilon(q, k, \bar{k}, \mu)^2 = -\frac{\beta_\pi^2}{4} k^2 \lambda \sin^2\theta_\pi, \tag{A22}$$

where we abbreviate

$$\varepsilon(a, b, c, d) \equiv a^\mu b^\nu c^\rho d^\sigma \varepsilon_{\mu\nu\rho\sigma}, \tag{A23}$$

and use $\varepsilon^{0123} = -\varepsilon_{0123} = +1$.

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