Three-loop Standard Model effective potential at leading order in strong and top Yukawa couplings

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(Received 7 November 2013; published 8 January 2014)

I find the three-loop contribution to the effective potential for the Standard Model Higgs field, in the approximation that the strong and top Yukawa couplings are large compared to all other couplings, using dimensional regularization with modified minimal subtraction. Checks follow from gauge invariance and renormalization group invariance. I also briefly comment on the special problems posed by Goldstone boson contributions to the effective potential, and on the numerical impact of the result on the relations between the Higgs vacuum expectation value, mass, and self-interaction coupling.

DOI: 10.1103/PhysRevD.89.013003 PACS numbers: 14.80.Bn, 12.38.Bx

I. INTRODUCTION

The discovery [1–4] of the Standard Model Higgs boson with a mass near 126 GeV, and so far no other new fundamental physics, implies that a new era of precision analyses of the minimal electroweak symmetry breaking dynamics has begun. The relation between the Higgs field vacuum expectation value (VEV) and the underlying Lagrangian parameters, as well as the issue of vacuum stability, can be analyzed precisely using the effective potential approach [5–7]. At present, the Standard Model Higgs effective potential has been evaluated at two-loop order in [8]. (The extension to more general models, including supersymmetric ones, is given in [9].) An intriguing aspect of the observed Higgs mass is that the resulting potential is in the metastable region near the critical value associated with a very small Higgs self-interaction at very high energy scales. Analyses of the vacuum stability condition before the Higgs discovery were given in Refs. [7,10–20], and some of the more detailed analyses since after the Higgs mass became known are given in [21–27].

The purpose of this paper is to find the three-loop contributions to the effective potential of the Standard Model, in the approximation that the QCD coupling g_3 and the top-quark Yukawa coupling y_t are large compared to the Higgs self-interaction λ and the electroweak gauge couplings g and g' and the other quark and lepton Yukawa couplings. In this approximation, the three-loop part of the effective potential is proportional to m_t^4, multiplied by terms g_3^4, $g_3^2 y_t^2$, and y_t^4, and up to cubic logarithms. Here m_t is the field-dependent tree-level top quark mass. The effective potential $V_{\rm eff}(\phi)$ is found as the sum of one-particle-irreducible vacuum Feynman diagrams, using couplings and masses obtained in the presence of a classical background field ϕ whose value at the minimum of the effective potential coincides with the Higgs VEV. The effective potential will be calculated in dimensional regularization [28–32] with modified minimal subtraction [33,34]. The result may be used to improve the accuracy and/or theoretical error estimates for analyses of vacuum stability and Lagrangian parameter determination for the Standard Model.

To establish conventions for the present paper, consider the Higgs Lagrangian

$$\mathcal{L} = -\partial^\mu \Phi^\dagger \partial_\mu \Phi - \Lambda - m^2 \Phi^\dagger \Phi - \lambda(\Phi^\dagger \Phi)^2, \quad (1.1)$$

where m^2 is the (negative) Higgs squared mass parameter, and λ is the self-coupling in the normalization to be used in this paper, and I use the metric with signature $(-+++)$. The field-independent vacuum energy term Λ must be included in order to maintain renormalization scale invariance of the full effective potential and a proper treatment of renormalization group improvement [35–38], [12], but will play no direct role in the present paper. The complex Higgs doublet field is written

$$\Phi(x) = \begin{pmatrix} \frac{1}{\sqrt{2}}[\phi + H(x) + iG^0(x)] \\ G^+(x) \end{pmatrix}, \quad (1.2)$$

where ϕ is the real background field, and H is the real Higgs quantum field, while G^0 and $G^+ = G^{-*}$ are the real neutral and complex charged Goldstone boson fields. The effective potential is then a function of ϕ, with a minimum that equals the vacuum expectation value.

The rest of this paper is organized as follows. Section II reviews the integrals that are necessary for the calculation. Section III calculates the three-loop effective potential in terms of bare quantities, with individual diagram contributions provided in the Appendix. Section IV performs the reexpression of the effective potential in terms of renormalized quantities, to obtain the form that can be used for phenomenological analyses. Section V comments briefly on the special problems posed by Goldstone boson contributions to the effective potential, and Sec. VI briefly discusses the numerical impact of the three-loop effective potential on the relations between the Higgs VEV, mass, and self-interaction coupling.

II. THE NECESSARY INTEGRALS

In this section, I review the results for Feynman integrals that are necessary for the calculations in the rest of the paper. Euclidean momentum integrals in

$$d = 4 - 2\epsilon \tag{2.1}$$

dimensions are written using the notation

$$\int_p \equiv \int \frac{d^d p}{(2\pi)^d}. \tag{2.2}$$

Consider first the integrals that depend only on one squared mass scale, to be denoted x below. In this paper, x will be the (bare) field-dependent squared mass of the top quark. The one-loop scalar vacuum master integral is

$$A \equiv \int_p \frac{1}{p^2 + x} = \frac{\Gamma(1 - d/2)}{(4\pi)^{d/2}} x^{d/2-1}. \tag{2.3}$$

For the two-loop integrals with one mass scale that are relevant below, there are two master integrals:

$$A^2 = \int_p \int_q \frac{1}{(p^2 + x)(q^2 + x)}, \tag{2.4}$$

$$B \equiv x \int_p \int_q \frac{1}{(p^2 + x)q^2(p+q)^2} = \frac{\Gamma(3-d)\Gamma(d/2)}{\Gamma(2-d/2)} A^2. \tag{2.5}$$

Here, the term "master integral" is taken to mean one of the minimal set of integrals at a given loop order to which all others can be reduced, with coefficients that are ratios of polynomials in d, by elementary algebra or integration by parts [39] identities. Thus, B and A^2 are considered distinct two-loop one-scale master integrals by this criterion, but

$$\int_p \int_q \frac{1}{(p^2 + x)(q^2 + x)(p+q)^2} = \frac{2-d}{2(d-3)} \frac{A^2}{x} \tag{2.6}$$

is not a master integral.

The three-loop one-scale Feynman integrals encountered below can be reduced by elementary algebra to integrals of the types

$$C(n_1, n_2, n_3, n_4, n_5, n_6) = \int_p \int_q \int_k \frac{1}{[(p-k)^2 + x]^{n_1} [p^2 + x]^{n_2} [q^2 + x]^{n_3} [(q+k)^2 + x]^{n_4} [(p+q)^2]^{n_5} [k^2]^{n_6}}, \tag{2.7}$$

$$D(n_1, n_2, n_3, n_4, n_5, n_6) = \int_p \int_q \int_k \frac{1}{[(p-k)^2 + x]^{n_1} [(q+k)^2 + x]^{n_2} [k^2 + x]^{n_3} [p^2]^{n_4} [q^2]^{n_5} [(p+q)^2]^{n_6}}, \tag{2.8}$$

$$E(n_1, n_2, n_3, n_4, n_5, n_6) = \int_p \int_q \int_k \frac{1}{[(p-k)^2 + x]^{n_1} [(q+k)^2 + x]^{n_2} [k^2]^{n_3} [p^2]^{n_4} [q^2]^{n_5} [(p+q)^2]^{n_6}}, \tag{2.9}$$

$$F(n_1, n_2, n_3, n_4, n_5, n_6) = \int_p \int_q \int_k \frac{1}{[(p+q)^2 + x]^{n_1} [k^2 + x]^{n_2} [(p-k)^2]^{n_3} [(q+k)^2]^{n_4} [q^2]^{n_5} [p^2]^{n_6}}, \tag{2.10}$$

illustrated in Fig. 1, and studied in [40–43]. Here the exponents n_i are integers, which can be positive, negative, or zero. Some of the integrals with some nonpositive exponents n_i vanish trivially due to the dimensional regularization identity $\int_p 1/(p^2)^n = 0$. Expressions involving the remaining integrals can then be systematically simplified using the following identities. First, relabeling the momenta gives the symmetry identities:

$$C(n_1, n_2, n_3, n_4, n_5, n_6) = C(n_2, n_3, n_4, n_1, n_6, n_5) = C(n_4, n_3, n_2, n_1, n_5, n_6), \tag{2.11}$$

$$D(n_1, n_2, n_3, n_4, n_5, n_6) = D(n_1, n_3, n_2, n_6, n_5, n_4) = D(n_2, n_3, n_1, n_6, n_4, n_5), \tag{2.12}$$

$$E(n_1, n_2, n_3, n_4, n_5, n_6) = E(n_2, n_1, n_3, n_5, n_4, n_6), \tag{2.13}$$

$$F(n_1, n_2, n_3, n_4, n_5, n_6) = F(n_1, n_2, n_6, n_5, n_4, n_3) = F(n_2, n_1, n_5, n_4, n_3, n_6), \tag{2.14}$$

and others obtained by repeated application of those. Also, dimensional analysis yields

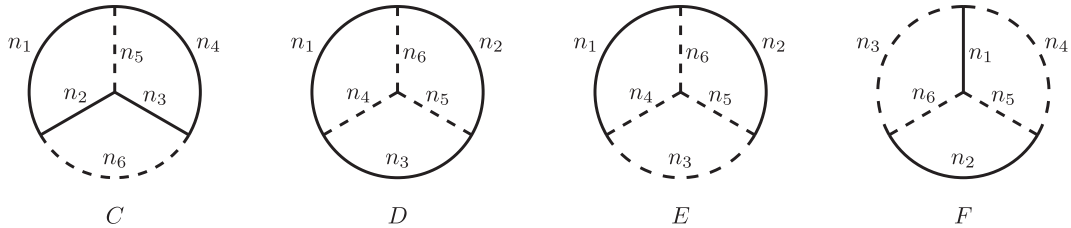

FIG. 1. From left to right, the topologies for the Feynman integrals C–F defined in Eqs. (2.7)–(2.10), respectively. Solid lines represent propagators with squared mass x, and dashed lines are for massless propagators. The integers n_i represent the powers to which the propagators are raised, and can be positive, zero, or negative.

$$0 = \left[3d/2 - \sum_{j=1}^{6} n_j + x\sum_{j=1}^{4} n_j \mathbf{j}^+\right] C(n_1,n_2,n_3,n_4,n_5,n_6), \tag{2.15}$$

$$0 = \left[3d/2 - \sum_{j=1}^{6} n_j + x\sum_{j=1}^{3} n_j \mathbf{j}^+\right] D(n_1,n_2,n_3,n_4,n_5,n_6), \tag{2.16}$$

$$0 = \left[3d/2 - \sum_{j=1}^{6} n_j + x\sum_{j=1}^{2} n_j \mathbf{j}^+\right] E(n_1,n_2,n_3,n_4,n_5,n_6), \tag{2.17}$$

$$0 = \left[3d/2 - \sum_{j=1}^{6} n_j + x\sum_{j=1}^{2} n_j \mathbf{j}^+\right] F(n_1,n_2,n_3,n_4,n_5,n_6), \tag{2.18}$$

where here and below the notation for boldface raising and lowering operators is the standard one such that, for each integer $j = 1,\ldots,6$, we have $\mathbf{j}^\pm C(\ldots,n_j,\ldots) \equiv C(\ldots,n_j \pm 1,\ldots)$ and similarly for D, E, and F. Finally, integration by parts [39] gives the identities:

$$0 = [d - n_3 - n_4 - 2n_5 + n_3\mathbf{3}^+(\mathbf{2}^- - \mathbf{5}^-) + n_4\mathbf{4}^+(\mathbf{1}^- - \mathbf{5}^-)]C, \tag{2.19}$$

$$0 = [d - n_1 - 2n_2 - n_5 + n_1\mathbf{1}^+(\mathbf{6}^- - \mathbf{2}^- + 2x) + 2xn_2\mathbf{2}^+ + n_5\mathbf{5}^+(\mathbf{3}^- - \mathbf{2}^-)]C, \tag{2.20}$$

and

$$0 = [d - n_3 - n_4 - 2n_5 + n_3\mathbf{3}^+(\mathbf{2}^- - \mathbf{5}^-) + n_4\mathbf{4}^+(\mathbf{6}^- - \mathbf{5}^-)]D, \tag{2.21}$$

$$0 = [d - n_1 - 2n_2 - n_3 + n_1\mathbf{1}^+(\mathbf{6}^- - \mathbf{2}^- + 2x) + 2xn_2\mathbf{2}^+ + n_3\mathbf{3}^+(\mathbf{5}^- - \mathbf{2}^- + 2x)]D, \tag{2.22}$$

$$0 = [d - 2n_2 - n_5 - n_6 + 2xn_2\mathbf{2}^+ + n_5\mathbf{5}^+(\mathbf{3}^- - \mathbf{2}^-) + n_6\mathbf{6}^+(\mathbf{1}^- - \mathbf{2}^-)]D, \tag{2.23}$$

and

$$0 = [d - n_2 - n_5 - 2n_6 + n_2\mathbf{2}^+(\mathbf{1}^- - \mathbf{6}^-) + n_5\mathbf{5}^+(\mathbf{4}^- - \mathbf{6}^-)]E, \tag{2.24}$$

$$0 = [d - 2n_3 - n_4 - n_5 + n_4\mathbf{4}^+(\mathbf{1}^- - \mathbf{3}^- - x) + n_5\mathbf{5}^+(\mathbf{2}^- - \mathbf{3}^- - x)]E, \tag{2.25}$$

$$0 = [d - 2n_2 - n_5 - n_6 + 2xn_2\mathbf{2}^+ + n_5\mathbf{5}^+(\mathbf{3}^- - \mathbf{2}^- + x) + n_6\mathbf{6}^+(\mathbf{1}^- - \mathbf{2}^-)]E, \tag{2.26}$$

$$0 = [d - n_3 - 2n_4 - n_5 + n_3\mathbf{3}^+(\mathbf{1}^- - \mathbf{4}^- - x) + n_5\mathbf{5}^+(\mathbf{6}^- - \mathbf{4}^-)]E, \tag{2.27}$$

$$0 = [d - n_1 - n_2 - 2n_3 + n_1\mathbf{1}^+(\mathbf{4}^- - \mathbf{3}^- + x) + n_2\mathbf{2}^+(\mathbf{5}^- - \mathbf{3}^- + x)]E, \tag{2.28}$$

$$0 = [d - n_1 - 2n_2 - n_3 + n_1\mathbf{1}^+(\mathbf{6}^- - \mathbf{2}^- + 2x) + 2xn_2\mathbf{2}^+ + n_3\mathbf{3}^+(\mathbf{5}^- - \mathbf{2}^- + x)]E, \tag{2.29}$$

$$0 = [d - n_1 - 2n_4 - n_6 + n_1\mathbf{1}^+(\mathbf{3}^- - \mathbf{4}^- + x) + n_6\mathbf{6}^+(\mathbf{5}^- - \mathbf{4}^-)]E, \tag{2.30}$$

and

$$0 = [d - 2n_2 - n_5 - n_6 + 2xn_2\mathbf{2}^+ + n_5\mathbf{5}^+(\mathbf{4}^- - \mathbf{2}^- + x) + n_6\mathbf{6}^+(\mathbf{3}^- - \mathbf{2}^- + x)]F, \tag{2.31}$$

$$0 = [d - n_2 - 2n_5 - n_6 + n_2\mathbf{2}^+(\mathbf{4}^- - \mathbf{5}^- + x) + n_6\mathbf{6}^+(\mathbf{1}^- - \mathbf{5}^- - x)]F. \tag{2.32}$$

In Eqs. (2.19)–(2.32), the arguments $(n_1, n_2, n_3, n_4, n_5, n_6)$ are implicit for C, D, E, and F, and were omitted for the sake of simplicity.

Repeated applications of the identities in Eqs. (2.11)–(2.32) allows [40,41] all of the one-scale integrals of the types C, D, E, and F to be reduced (with many redundant checks) to just the seven master integrals depicted in Fig. 2:

$$J \equiv C(0,1,1,1,0,0)/x = D(1,1,1,0,0,0)/x = A^3/x, \tag{2.33}$$

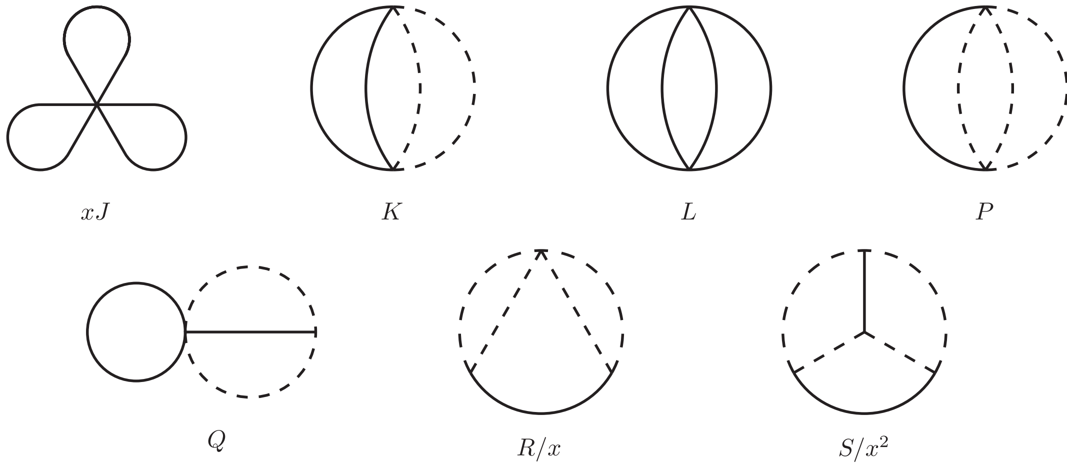

FIG. 2. The three-loop master integrals J, K, L, P, Q, R, and S defined in Eqs. (2.33)–(2.39). Solid lines represent scalar propagators with squared mass x, and dashed lines are for massless propagators. All of the integrals in Fig. 1 can be reduced to linear combinations of these, with coefficients that are ratios of polynomials in the number of spacetime dimensions, d.

$$K \equiv C(0,1,0,1,1,1) = D(0,1,1,1,0,1) = E(1,1,0,1,1,0) = F(1,1,0,1,0,1), \quad (2.34)$$

$$L \equiv C(1,1,1,1,0,0), \quad (2.35)$$

$$P \equiv E(0,1,1,1,0,1), \quad (2.36)$$

$$Q \equiv E(1,1,1,0,1,0) = F(1,1,0,1,1,0) = AB/x, \quad (2.37)$$

$$R \equiv xF(0,1,1,1,1,1), \quad (2.38)$$

$$S \equiv x^2 F(1,1,1,1,1,1). \quad (2.39)$$

The expressions for J, K, P, Q, and R are known exactly in terms of gamma functions, and for the remaining integrals L and S the results are known as expansions in ϵ. Actually, the integral S will not be needed in the present paper. The integrals J, L, and P are needed here to order ϵ^2, while K and Q are needed to order ϵ^1, and R to order ϵ^0. Writing each of the other master integrals in terms of A^3/x, the expansions to these orders (and beyond) are found from Refs. [40,42]:

$$J = A^3/x, \quad (2.40)$$

$$K = \frac{A^3}{x}\left(-\frac{1}{3} - \frac{\epsilon}{6} + \frac{5\epsilon^2}{12} + \left[\frac{79}{24} - \frac{8\zeta(3)}{3}\right]\epsilon^3 + \left[\frac{685}{48} + \frac{2\pi^4}{15} - \frac{4\zeta(3)}{3}\right]\epsilon^4 + \cdots\right), \quad (2.41)$$

$$L = \frac{A^3}{x}\left(-2 - \frac{5}{3}\epsilon - \frac{1}{2}\epsilon^2 + \frac{103}{12}\epsilon^3 + [1141/24 - 112\zeta(3)/3]\epsilon^4 + [9055/48 + 136\pi^4/45 + 32\ln^2(2)\times[\pi^2 - \ln^2(2)]/3 - 168\zeta(3) - 256Li_4(1/2)]\epsilon^5 + \cdots\right), \quad (2.42)$$

$$P = \frac{A^3}{x}\left(\frac{\epsilon}{12} + \frac{3\epsilon^2}{8} + [67/48 + \pi^2/12]\epsilon^3 + [457/96 + 3\pi^2/8 - 5\zeta(3)/6]\epsilon^4 + [2971/192 + 67\pi^2/48 + 23\pi^4/360 - 15\zeta(3)/4]\epsilon^5 + \cdots\right), \quad (2.43)$$

$$Q = \frac{A^3}{x}\left(-\frac{1}{2} - \frac{\epsilon}{2} + \left[-1 - \frac{\pi^2}{6}\right]\epsilon^2 + \left[\zeta(3) - 2 - \frac{\pi^2}{6}\right]\epsilon^3 + \left[\zeta(3) - 4 - \frac{\pi^2}{3} - \frac{\pi^4}{20}\right]\epsilon^4 + \cdots\right), \quad (2.44)$$

$$R = \frac{A^3}{x}\left(\frac{1}{3} + \frac{2\epsilon}{3} + [5/3 + \pi^2/3]\epsilon^2 + [4 + 2\pi^2/3 - 4\zeta(3)/3]\epsilon^3 + \cdots\right). \quad (2.45)$$

A particularly useful and systematic compendium of these and many other one-scale multiloop vacuum integral results can be found in Ref. [43].

In the following, we will also need certain integrals that depend on two squared mass scales: x (which, as above, will be the top-quark squared mass) and y (which will be either the Higgs or Goldstone bare squared mass). Because of the approximation used in this paper, it is sufficient to have these integrals to first order in y, and to order ϵ^1 for two-loop integrals and ϵ^0 for three-loop integrals. The two-scale integrals needed are

$$I_{xxy} \equiv \int_p \int_q \frac{1}{(p^2+x)(q^2+x)[(p+q)^2+y]}, \tag{2.46}$$

$$I_{x0y} \equiv \int_p \int_q \frac{1}{(p^2+x)q^2[(p+q)^2+y]}, \tag{2.47}$$

$$I_{xxxxy} \equiv \int_p \int_q \int_k \frac{1}{[(p-k)^2+x][(q+k)^2+x][p^2+x][q^2+x][k^2+y]}, \tag{2.48}$$

$$I_{xx00y} \equiv \int_p \int_q \int_k \frac{1}{[(p-k)^2+x][(q+k)^2+x]p^2q^2[k^2+y]}. \tag{2.49}$$

Using integration by parts and dimensional analysis, these integrals are found to obey the differential equations

$$y(4x-y)\frac{d}{dy}I_{xxy} = (d-3)(2x-y)I_{xxy} + (d-2)[1-(y/x)^{d/2-1}]A^2, \tag{2.50}$$

$$(x-y)^2\frac{d}{dy}I_{x0y} = (3-d)(x-y)I_{x0y} + (1-d/2)(1-y/x)(y/x)^{d/2-1}A^2, \tag{2.51}$$

$$y(4x-y)\frac{d}{dy}I_{xxxxy} = (2dx-8x+5y-3dy/2)I_{xxxxy} + (3d/2-4)L + (4-2d)AI_{xxy}, \tag{2.52}$$

$$2y(y-x)\frac{d}{dy}I_{xx00y} = (dx-2x+3dy-10y)I_{xx00y} + (2d-4)AI_{x0y} + (8-3d)K. \tag{2.53}$$

It follows that, to the order needed below,

$$I_{xxy} = \frac{A^2}{x}\frac{2-d}{2(d-3)}\left\{1 + r\left(\frac{1}{2} - \epsilon\ln(r) + \epsilon^2\left[\frac{1}{2}\ln^2(r) + 2\ln(r) - 4\right] + \epsilon^3\left[-\frac{1}{6}\ln^3(r) - \ln^2(r) + 8\right] + \mathcal{O}(\epsilon^4)\right) + \mathcal{O}(r^2)\right\}, \tag{2.54}$$

$$I_{x0y} = \frac{B}{x}\left\{1 + r\left(1 - 2\epsilon\ln(r) + \epsilon^2[\ln^2(r) + 2\ln(r) - 2 - 2\pi^2/3] + \epsilon^3\left[-\frac{1}{3}\ln^3(r) - \ln^2(r) + (2+2\pi^2/3)\ln(r) - 2 + 2\pi^2/3 + 4\zeta(3)\right] + \mathcal{O}(\epsilon^4)\right) + \mathcal{O}(r^2)\right\}, \tag{2.55}$$

$$I_{xxxxy} = \frac{1}{4(4-d)}\left[(3d-8)\frac{L}{x} + \frac{2(d-2)^2}{d-3}\frac{A^3}{x^2}\right] + r\frac{A^3}{x^2}\left\{\frac{1}{3} + \epsilon[2/3 - \ln(r)] + \epsilon^2\left[\frac{1}{2}\ln^2(r) + 2\ln(r) - \frac{19}{3}\right] + \epsilon^3\left[-\frac{1}{6}\ln^3(r) - \ln^2(r) - \ln(r) + 16 + \frac{14}{3}\zeta(3)\right] + \mathcal{O}(\epsilon^4)\right\} + \mathcal{O}(r^2), \tag{2.56}$$

$$I_{xx00y} = \left(\frac{3d-8}{d-2}\right)\frac{K}{x} - 2\frac{AB}{x^2} + r\frac{A^3}{x^2}\left\{\frac{1}{3} + \epsilon\left[\frac{2}{3} - \ln(r)\right] + \epsilon^2\left[\frac{1}{2}\ln^2(r) - \frac{\pi^2}{3} - \frac{1}{3}\right]\right.$$
$$\left. + \epsilon^3\left[-\frac{1}{6}\ln^3(r) - \frac{1}{2} + \frac{\pi^2}{2} + \frac{14}{3}\zeta(3)\right] + \mathcal{O}(\epsilon^4)\right\} + \mathcal{O}(r^2), \quad (2.57)$$

where $r = y/x$. The above expansions for I_{xxy} and I_{x0y} can also be obtained as special cases of results in [44], and the expansion for I_{xxxxy} can be obtained as a special case of Eq. (3.27) in [45].

III. EFFECTIVE POTENTIAL IN TERMS OF BARE QUANTITIES

Consider the effective potential written in terms of the bare external scalar field ϕ_B and the bare coupling parameters including the Yukawa coupling y_{tB}, the strong coupling g_{3B}, and the Higgs self-coupling λ_B. This is calculated in $d = 4 - 2\epsilon$ dimensions in terms of the bare parameters in the Lagrangian, without including any counterterms. The conversion to $\overline{MS}$ parameters will be done in the next section. The expansion in terms of the loop order ℓ reads

$$V_{\rm eff} = \sum_{\ell=0}^{\infty} V_B^{(\ell)}, \quad (3.1)$$

where the tree-level potential in this expansion is

$$V_B^{(0)} = \Lambda_B + \frac{m_B^2}{2}\phi_B^2 + \frac{\lambda_B}{4}\phi_B^4. \quad (3.2)$$

The bare field-dependent squared masses of the top quark, Higgs scalar H^0, and the Goldstone bosons G^0, $G^\pm$ (in Landau gauge), are denoted by

$$x = y_{tB}^2\phi_B^2/2, \quad (3.3)$$

$$x_H = m_B^2 + 3\lambda_B\phi_B^2, \quad (3.4)$$

$$x_G = m_B^2 + \lambda_B\phi_B^2, \quad (3.5)$$

and the integrals A, B, J, K, L, P, Q, and R of the previous section are taken to be functions of x. In order to maximize the generality of results below, and allow more informative checks, they are written in terms of the group theory quantities

$$C_G = N_c = 3, \quad (3.6)$$

$$C_F = \frac{N_c^2 - 1}{2N_c} = 4/3, \quad (3.7)$$

$$T_F = 1/2, \quad (3.8)$$

$$N_q = 6, \quad (3.9)$$

where C_G is the Casimir invariant and Dynkin index of the $SU(3)_c$ gauge group; C_F, T_F, and N_c are the Casimir invariant, Dynkin index, and dimension of the fundamental representation, respectively; and N_q is the number of quarks in the theory.

The well-known one-loop order top, Higgs, and Goldstone contributions are then

$$V_B^{(1)} = \frac{xA}{d}[-4N_c + (x_H/x)^{d/2} + 3(x_G/x)^{d/2}]. \quad (3.10)$$

Note that even though the aim of this paper is to neglect terms proportional to the Higgs and Goldstone masses in the three-loop order part of the renormalized result, they do need to be included in the one-loop bare contribution. This is because when λ_B is expressed in terms of renormalized couplings, it includes a counterterm proportional to y_t^4 with no λ. The other one-loop contributions involving electroweak vector bosons and lighter fermions are not written here, because after expressing bare quantities in terms of renormalized quantities, they do not affect the determination of the three-loop contribution at leading order in the QCD and top-quark Yukawa couplings.

At two-loop order, the pertinent contributions are from the diagrams shown in Fig. 3. (Here, and below, each figure is taken to represent diagrams with all helicities and mass insertions consistent with the topology shown.) The gluon is treated with an arbitrary gauge-fixing parameter ξ, with propagator

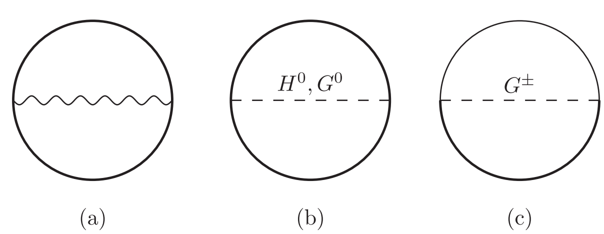

FIG. 3. The two-loop vacuum Feynman diagrams involving the top quark, neglecting electroweak interactions. Wavy lines are gluons, heavy solid lines are top quarks, lighter solid lines are bottom quarks, and dashed lines are scalars as labeled.

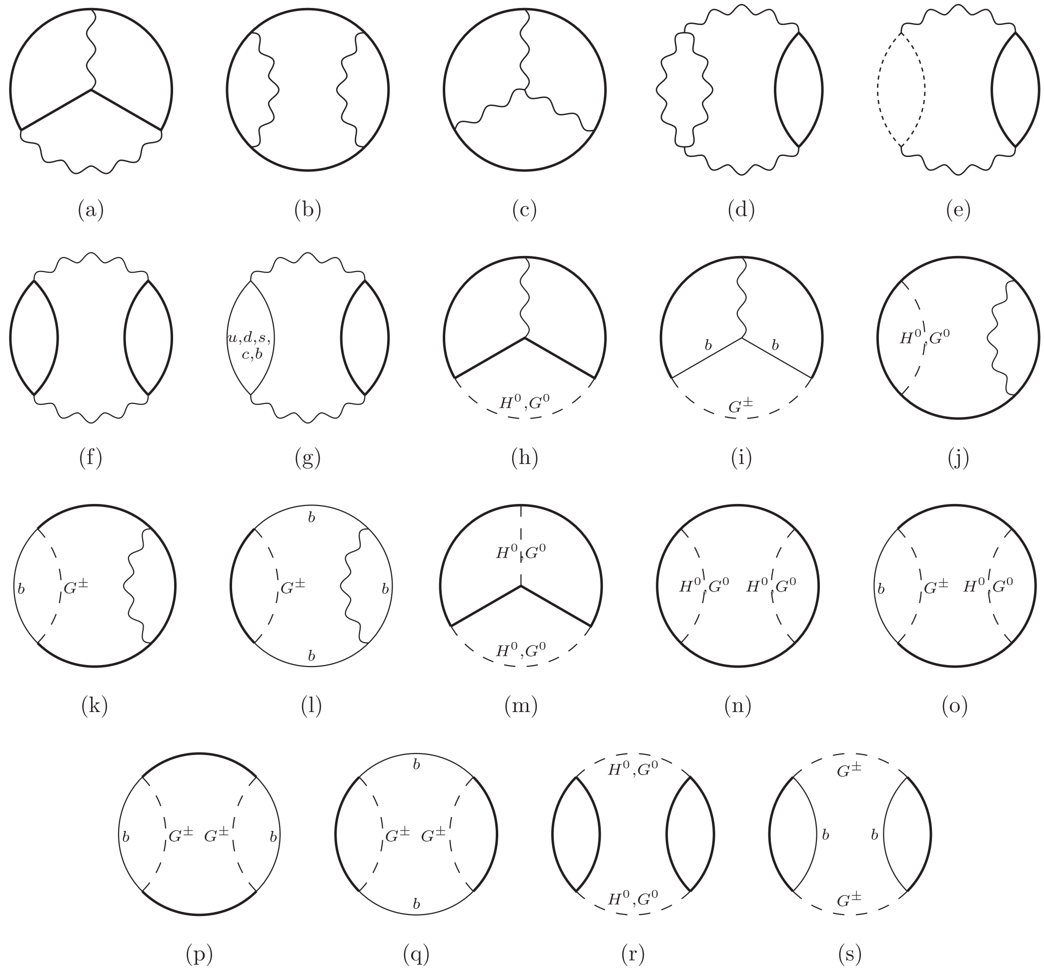

FIG. 4. The three-loop vacuum Feynman diagrams involving the top quark and strong interactions. Wavy lines are gluons; heavy solid lines are top quarks; lighter solid lines are other quarks as labeled; dashed lines are H^0, G^0 and $G^\pm$ as labeled; and dotted lines are ghosts.

$$-i[g^{\mu\nu}/p^2 - (1-\xi)p^\mu p^\nu/(p^2)^2], \tag{3.11}$$

where $\xi = 0$ for Landau gauge and $\xi = 1$ for Feynman gauge. The dependence on the QCD ξ cancels in the effective potential, providing a useful check. The combined two-loop order result is

$$V_B^{(2)} = g_{3B}^2 N_c C_F \frac{(d-1)(d-2)}{(d-3)} A^2 + y_{tB}^2 N_c \{A^2[1 - (x_H/x)^{d/2-1} - 2(x_G/x)^{d/2-1}] + (2x - x_H/2)I_{xxx_H} - x_G I_{xxx_G}/2 + (x - x_G)I_{x0x_G}\}. \tag{3.12}$$

Note that here one must include terms up to linear order in λ_B and first order in ϵ from the diagrams of Figs. 3(b)–(c) again because λ_B written in terms of renormalized couplings will contain a term proportional to y_t^4 with no λ.

The pertinent three-loop order diagrams at leading order in the strong and top-quark Yukawa couplings are shown in Fig. 4. All except diagrams 4(r) and 4(s) are evaluated by first writing them in terms of the functions C, D, E, and F defined in the previous section, and then using the identities in Eqs. (2.11)–(2.32) to reduce the result to the six master integrals J, K, L, P, Q, and R. For diagrams 4(r) and 4(s), it is necessary to also make use of the two-scale integrals in Eqs. (2.46)–(2.57), because of the “doubled” Higgs and Goldstone propagators. I find

$$
\begin{aligned}
V_B^{(3)} = {}& g_{3B}^4 N_c C_F \Bigg\{ C_G \left[\frac{(2-d)^3}{2(d-4)^2(d-3)} J + \frac{(d-2)^2(2d^2-17d+32)}{2(d-4)(2d-7)} K + \frac{(3-d)(d^3-13d^2+50d-48)}{4(d-4)^2} L \right] \\
& + C_F \left[\frac{(d-2)^2(-d^5+13d^4-67d^3+181d^2-274d+188)}{2(d-4)^2(d-3)^2} J \right. \\
& \left. + \frac{(2-d)(2d^3-21d^2+67d-68)}{(d-4)(d-3)} K + \frac{(d-6)(d-3)(d^2-7d+8)}{2(d-4)^2} L \right] \\
& + T_F \left[\frac{2(5-d)(d-2)^3}{(d-6)(d-4)(d-3)} J + \frac{d^3-7d^2+6d+16}{(d-6)(4-d)} L + (N_q - 1) \frac{4(d-3)(d-2)}{7-2d} K \right] \Bigg\} \\
& + g_{3B}^2 y_{tB}^2 N_c C_F \Bigg\{ \frac{2(2-d)(2d^3-17d^2+48d-46)}{(d-4)^2(d-3)^2} J + \frac{2(2-d)(2d-5)}{(d-4)(d-3)} K \\
& + \frac{(3-d)(d^2+2d-16)}{2(d-4)^2} L + \frac{(d-2)(d^3-d^2-16d+32)}{2(d-4)^2(d-3)} P + \frac{(2-d)(2d^3-11d^2+9d+16)}{2(d-4)(d-3)} Q \Bigg\} \\
& + y_{tB}^4 N_c \Bigg\{ \frac{-d^4+13d^3-52d^2+68d-8}{4(d-4)^2(d-3)^2} J + \frac{3d^4-34d^3+182d^2-480d+480}{2d(d-2)(d-3)(d-4)} K \\
& + \frac{(6-d)(3d-8)}{8(d-4)^2} L + \frac{4d^2-21d+28}{2(d-4)(d-3)} P + \frac{3(2-d)}{8} R + \frac{2d^2-d-12}{2d(d-3)} Q \Bigg\} \\
& + y_{tB}^4 N_c^2 \Bigg\{ \left[\frac{2(d-2)}{(d-4)(d-3)} + \frac{d-2}{2} (x_H/x)^{d/2-2} + \frac{3(d-2)}{4} (x_G/x)^{d/2-2} \right] J + \frac{18-7d}{2(d-2)} K \\
& + \frac{d-2}{d-4} L + 3Q + 4x \frac{d}{dx_H} [x I_{xxxxx_H} - A I_{xxx_H}] + x \frac{d}{dx_G} \left[\frac{x}{2} I_{xx00x_G} - A I_{x0x_G} \right] \Bigg\}. \qquad (3.13)
\end{aligned}
$$

The individual diagram contributions, exhibiting the separate dependences on ξ, are shown in the Appendix. The task of the next section is to reexpress these results in terms of $\overline{MS}$ renormalized quantities.

IV. EFFECTIVE POTENTIAL IN TERMS OF RENORMALIZED QUANTITIES

The effective potential in the $\overline{MS}$ renormalization scheme is obtained by reexpressing the bare quantities in terms of renormalized quantities. Write

$$\phi_B = \mu^{-\epsilon} \phi \sqrt{Z_\phi}, \qquad (4.1)$$

$$Z_\phi = 1 + \sum_{\ell=1}^{\infty} \sum_{n=1}^{\ell} \frac{c_{\ell,n}^{\phi}}{(16\pi^2)^\ell \epsilon^n}, \qquad (4.2)$$

$$z_{kB} = \mu^{\rho_k \epsilon} \left(z_k + \sum_{\ell=1}^{\infty} \sum_{n=1}^{\ell} \frac{c_{\ell,n}^{k}}{(16\pi^2)^\ell \epsilon^n} \right), \qquad (4.3)$$

where the subscript B indicates bare quantities, the absence of a subscript B indicates an $\overline{MS}$ renormalized quantity, ℓ is the loop order, and k is an index that runs over the list of Lagrangian parameters, including $z_k = \lambda$, y_t, g_3, m^2, Λ with[1] $\rho_\lambda = 2$, $\rho_{g_3} = \rho_{y_t} = 1$, $\rho_{m^2} = 0$, $\rho_\Lambda = -2$. The mass scale μ is the (arbitrary) dimensional regularization scale, introduced so that $\int d^d x V$ is dimensionless, and so that g_3 and y_t are also dimensionless, and the field-dependent top-quark mass $y_t\phi/\sqrt{2}$ has mass dimension 1, for any ϵ. The regularization scale μ is related to the $\bar{M}S$ renormalization scale Q by [33,34]

$$Q^2 = 4\pi e^{-\gamma_E} \mu^2, \qquad (4.4)$$

where $\gamma_E = 0.5772\ldots$ is the Euler-Mascheroni constant. The counterterm quantities $c_{\ell,n}^{\phi}$ and $c_{\ell,n}^{k}$ are polynomials in the $\bar{\text{M}}$S renormalized parameters z_k, and are independent of ϵ and ϕ. They are determined by the requirement that the full effective potential and other physical quantities have no poles in ϵ when expressed in terms of $\overline{\text{MS}}$ quantities.

The $\bar{M}S$ beta functions and the scalar anomalous dimension are defined by

$$\beta_k \equiv Q \frac{dz_k}{dQ}\bigg|_{\epsilon=0} = Q \frac{dz_k}{dQ} + \epsilon \rho_k z_k, \qquad (4.5)$$

[1] As a simplifying notation, in subscripts and superscripts a specific parameter z_k is used interchangeably with the corresponding index k, so that $\rho_k \equiv \rho_{z_k}$ and $c_{\ell,n}^{k} \equiv c_{\ell,n}^{z_k}$ and $\beta_k \equiv \beta_{z_k}$.

$$\gamma \equiv -Q\frac{d\ln\varphi}{dQ}\bigg|_{\epsilon=0} = -Q\frac{d\ln\varphi}{dQ} + \epsilon = \frac{1}{2}Q\frac{d}{dQ}\ \ln(Z_\varphi). \tag{4.6}$$

It is useful to write these as loop expansions:

$$\beta_k = \sum_{\ell=1}^{\infty}\frac{1}{(16\pi^2)^\ell}\beta_k^{(\ell)}, \tag{4.7}$$

$$\gamma = \sum_{\ell=1}^{\infty}\frac{1}{(16\pi^2)^\ell}\gamma^{(\ell)}. \tag{4.8}$$

Now, by using the fact that the bare quantities ϕ_B and z_{kB} cannot depend on Q (or μ), one obtains the beta functions and anomalous dimension in terms of the simple pole counterterms:

$$\beta_k^{(\ell)} = 2\ell c^k_{\ell,1}, \tag{4.9}$$

$$\gamma^{(\ell)} = -\ell c^\phi_{\ell,1}, \tag{4.10}$$

as well as the consistency conditions for higher pole counterterms with $\ell \ge n \ge 2$:

$$2\ell c^k_{\ell,n} = \sum_{\ell'=1}^{\ell-n+1}\sum_j \beta_j^{(\ell')}\frac{\partial}{\partial z_j}c^k_{\ell-\ell',n-1}, \tag{4.11}$$

$$\ell c^\phi_{\ell,n} = \sum_{\ell'=1}^{\ell-n+1}\left(-\gamma^{(\ell')} + \frac{1}{2}\sum_j \beta_j^{(\ell')}\frac{\partial}{\partial z_j}\right)c^\phi_{\ell-\ell',n-1}. \tag{4.12}$$

The identities

$$\sum_j \rho_j z_j\frac{\partial}{\partial z_j}c^k_{\ell,n} = (2\ell + \rho_k)c^k_{\ell,n}, \tag{4.13}$$

$$\sum_j \rho_j z_j\frac{\partial}{\partial z_j}c^\phi_{\ell,n} = 2\ell c^\phi_{\ell,n} \tag{4.14}$$

have been used to simplify the preceding expressions. [Note that Eq. (2.11) in Ref. [46] has a missing factor of $-\rho_k$ on the left side.]

Equations (4.9–4.12) allow the coefficients $c^k_{\ell,n}$ and $c^\phi_{\ell,n}$ to be determined from the known results for the beta functions and scalar anomalous dimension. The ones that are needed for this paper are [46–49], [8], [50] (see also [51,52])

$$c^\lambda_{1,1} = -N_c y_t^4 + \lambda y_t^2(2N_c) + 12\lambda^2 + \cdots, \tag{4.15}$$

$$c^\lambda_{2,1} = g_3^2 y_t^4(-2N_cC_F) + y_t^6(5N_c/2) + \lambda g_3^2 y_t^2(5N_cC_F) + \lambda y_t^4(-N_c/4) + \cdots, \tag{4.16}$$

$$c^\lambda_{2,2} = g_3^2 y_t^4(6N_cC_F) + y_t^6(-2N_c^2 - 3N_c/2) + \lambda g_3^2 y_t^2(-6N_cC_F) + \lambda y_t^4(3N_c^2 - 21N_c/2) + \cdots \tag{4.17}$$

$$\begin{aligned} c^\lambda_{3,1} = {}& g_3^4 y_t^4 N_c C_F\{[8\zeta(3) - 109/6]C_G + [131/6 - 16\zeta(3)]C_F + (16 + 10N_q/3)T_F\} \\ &+ g_3^2 y_t^6 N_c C_F[20\zeta(3) - 19/6] + y_t^8 N_c[-4\zeta(3) + 13/6 - 65N_c/8] + \cdots, \end{aligned} \tag{4.18}$$

$$c^\lambda_{3,2} = g_3^4 y_t^4 N_c C_F(24C_G + 10C_F - 16T_FN_q/3) + g_3^2 y_t^6 N_c C_F(-25 - 9N_c) + y_t^8 N_c(11/4 + 107N_c/12) + \cdots, \tag{4.19}$$

$$c^\lambda_{3,3} = g_3^4 y_t^4 N_c C_F(-22C_G/3 - 24C_F + 8T_FN_q/3) + g_3^2 y_t^6 N_c C_F(15 + 18N_c) + y_t^8 N_c(-9/4 - N_c - 3N_c^2) + \cdots, \tag{4.20}$$

$$c^{y_t}_{1,1} = g_3^2 y_t(-3C_F) + y_t^3(N_c/2 + 3/4) + \cdots, \tag{4.21}$$

$$c^{y_t}_{2,1} = g_3^4 y_t C_F\left(-\frac{97}{12}C_G - \frac{3}{4}C_F + \frac{5}{3}T_FN_q\right) + g_3^2 y_t^3 C_F\left(3 + \frac{5}{4}N_c\right) + y_t^5\left(\frac{3}{8} - \frac{9}{8}N_c\right) + \cdots, \tag{4.22}$$

$$c^{y_t}_{2,2} = g_3^4 y_t C_F\left(\frac{11}{2}C_G + \frac{9}{2}C_F - 2T_FN_q\right) + g_3^2 y_t^3 C_F(-3N_c - 9/2) + y_t^5\left(\frac{3}{8}N_c^2 + \frac{9}{8}N_c + \frac{27}{32}\right) + \cdots, \tag{4.23}$$

$$c^{g_3}_{1,1} = g_3^3\left(-\frac{11}{6}C_G + \frac{2}{3}T_FN_q\right), \tag{4.24}$$

$$c^\varphi_{1,1} = -y_t^2 N_c + \cdots, \tag{4.25}$$

$$c^\varphi_{2,1} = g_3^2 y_t^2(-5N_cC_F/2) + y_t^4(9N_c/8) + \cdots, \tag{4.26}$$

$$c_{2,2}^{\varphi} = g_3^2 y_t^2 (3N_c C_F) + y_t^4(-3N_c/4) + \cdots. \quad (4.27)$$

Here the ellipses refer to contributions that are known, but are suppressed by couplings other than y_t or g_3 to a sufficient extent that they are not pertinent to this paper.

Now, plugging Eqs. (4.1–4.3) into the results (3.1), (3.2), (3.10), (3.12) and (3.13) gives the effective potential in terms of renormalized quantities. This can be written in a loop expansion as

$$V_{\rm eff} = \sum_{\ell=0}^{\infty} \frac{1}{(16\pi^2)^{\ell}} V^{(\ell)}. \quad (4.28)$$

Note that, as a convention, here the loop factors of $1/(16\pi^2)^{\ell}$ have been extracted, unlike the corresponding loop expansion in terms of bare parameters, Eq. (3.1). In this section, I write

$$T = y_t^2 \phi^2/2, \quad (4.29)$$

$$H = m^2 + 3\lambda\phi^2, \quad (4.30)$$

$$G = m^2 + \lambda\phi^2, \quad (4.31)$$

for the $\overline{\rm MS}$ field-dependent squared masses of the top quark, Higgs boson H, and Goldstone bosons $G^0, G^{\pm}$, respectively, and define

$$\overline{\ln}(X) \equiv \ln(X/Q^2) \quad (4.32)$$

for $X = T, H, G$. Retaining terms quadratic in λ and m^2 in the one-loop part, and linear in λ and m^2 in the two-loop part, and taking the limit $\epsilon \to 0$, now gives

$$V^{(0)} = \Lambda + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4, \quad (4.33)$$

$$V^{(1)} = -N_c T^2[\overline{\ln}(T) - 3/2] + \frac{H^2}{4}[\overline{\ln}(H) - 3/2] + \frac{3G^2}{4}[\overline{\ln}(G) - 3/2], \quad (4.34)$$

$$\begin{aligned} V^{(2)} = {}& g_3^2 N_c C_F T^2[6\overline{\ln}^2(T) - 16\overline{\ln}(T) + 18] + y_t^2 N_c T^2\left[-\frac{3}{2}\overline{\ln}^2(T) + 8\overline{\ln}(T) - \frac{23}{2} - \frac{\pi^2}{6}\right] \\ &+ y_t^2 N_c T H\left[\frac{9}{2} + 4\overline{\ln}(T) - \frac{3}{2}\overline{\ln}^2(T) + \{1 - 3\overline{\ln}(T)\}\ln(H/T)\right] \\ &+ y_t^2 N_c T G\left[\frac{3}{2} + \frac{\pi^2}{3} + 2\overline{\ln}(T) - \frac{3}{2}\overline{\ln}^2(T) + 3\{1 - \overline{\ln}(T)\}\ln(G/T)\right], \end{aligned} \quad (4.35)$$

which agrees with the relevant limits of Ref. [8], and the new result:

$$\begin{aligned} V^{(3)} = {}& g_3^4 N_c C_F T^2\Bigg\{C_G\left[-\frac{22}{3}\overline{\ln}^3(T) + \frac{185}{3}\overline{\ln}^2(T) + \left(24\zeta(3) - \frac{1111}{6}\right)\overline{\ln}(T)\right. \\ &\left. + \frac{2609}{12} + \frac{44}{45}\pi^4 - \frac{232}{3}\zeta(3) + \frac{16}{3}\ln^2(2)[\pi^2 - \ln^2(2)] - 128 Li_4(1/2)\right] + C_F\left[-24\overline{\ln}^3(T) + 63\overline{\ln}^2(T)\right. \\ &\left. - \left(48\zeta(3) + \frac{121}{2}\right)\overline{\ln}(T) + \frac{85}{12} - \frac{88}{45}\pi^4 + 192\zeta(3) - \frac{32}{3}\ln^2(2)[\pi^2 - \ln^2(2)] + 256 Li_4(1/2)\right] \\ &+ T_F\left[48\overline{\ln}(T) - \frac{232}{3} + 96\zeta(3)\right] + T_F N_q\left[\frac{8}{3}\overline{\ln}^3(T) - \frac{52}{3}\overline{\ln}^2(T) + \frac{142}{3}\overline{\ln}(T) - \frac{161}{3} - \frac{64}{3}\zeta(3)\right]\Bigg\} \\ &+ g_3^2 y_t^2 N_c C_F T^2\Bigg\{15\overline{\ln}^3(T) - 90\overline{\ln}^2(T) + [407/2 + 3\pi^2 + 60\zeta(3)]\overline{\ln}(T) - 54\zeta(3) - \frac{2393}{12} - \frac{29}{6}\pi^2 + \frac{31}{15}\pi^4 \\ &+ \frac{32}{3}\ln^2(2)[\pi^2 - \ln^2(2)] - 256 Li_4(1/2)\Bigg\} + y_t^4 N_c T^2\Bigg\{-\frac{9}{4}\overline{\ln}^3(T) + \frac{57}{4}\overline{\ln}^2(T) + \left[-\frac{3}{4}\pi^2 - \frac{121}{4} - 12\zeta(3)\right]\overline{\ln}(T) \\ &+ \frac{529}{24} + \frac{23}{12}\pi^2 - \frac{22}{45}\pi^4 + \frac{93}{2}\zeta(3) - \frac{8}{3}\ln^2(2)[\pi^2 - \ln^2(2)] + 64 Li_4(1/2)\Bigg\} \\ &+ y_t^4 N_c^2 T^2\Bigg\{\frac{7}{2}\overline{\ln}^3(T) + [17/4 + 9\ln(H/T) + 3\ln(G/T)]\overline{\ln}^2(T) \\ &+ \left[-\frac{659}{8} - \frac{5}{6}\pi^2 - 6\ln(H/T) - 6\ln(G/T)\right]\overline{\ln}(T) + \frac{4903}{48} + \frac{3}{4}\pi^2 - 64\zeta(3) + \ln(H/T) + 3\ln(G/T)\Bigg\}. \end{aligned} \quad (4.36)$$

Note that poles in ϵ are absent from Eqs. (4.34–4.36); this is a nontrivial check on the calculation, showing agreement between the counterterm quantities $c^{k}_{\ell,n}$ and $c^{\phi}_{\ell,n}$ as extracted from the known beta functions and anomalous dimension in the literature, and as obtained from the diagrams calculated here. This is equivalent to the check of renormalization group scale independence of the effective potential:

$$0 = Q\frac{\partial}{\partial Q}V^{(\ell)} + \sum_{\ell'=1}^{\ell}\left[\sum_k \beta_k^{(\ell')}\frac{\partial}{\partial z_k} - \gamma^{(\ell')}\phi\frac{\partial}{\partial\phi}\right]V^{(\ell-\ell')}, \tag{4.37}$$

which follows from $dV_{\rm eff}/{\rm dQ} = 0$. In fact, Eq. (4.37) could have been used to infer all of the terms in $V^{(3)}$ that contain $\bar{\ln}(T)$, just from knowledge of the two-loop effective potential and the beta functions and scalar anomalous dimension. I have checked this.

Plugging the Standard Model group theory constants of Eqs. (3.6–3.9) into Eq. (4.36) gives

$$\begin{aligned} V^{(3)} = {} & g_3^4 T^2\Big\{-184\bar{\ln}^3(T) + 868\,\bar{\ln}^2(T) + \left(32\zeta(3) - \frac{5642}{3}\right)\bar{\ln}(T) + \frac{16633}{9} + \frac{176}{135}\pi^4 + 32\zeta(3) \\ & + \frac{64}{9}\ln^2(2)[\pi^2 - \ln^2(2)] - \frac{512}{3}Li_4(1/2)\Big\} \\ & + g_3^2 y_t^2 T^2\Big\{60\,\bar{\ln}^3(T) - 360\,\bar{\ln}^2(T) + \left[814 + 12\pi^2 + 240\zeta(3)\right]\bar{\ln}(T) - \frac{2393}{3} - \frac{58}{3}\pi^2 + \frac{124}{15}\pi^4 \\ & + \frac{128}{3}\ln^2(2)[\pi^2 - \ln^2(2)] - 216\zeta(3) - 1024 Li_4(1/2)\Big\} \\ & + y_t^4 T^2\Big\{\frac{99}{4}\bar{\ln}^3(T) + [81 + 81\ \ln(H/T) + 27\ \ln(G/T)]\bar{\ln}^2(T) \\ & + \left[-\frac{6657}{8} - \frac{39}{4}\pi^2 - 36\zeta(3) - 54\ \ln(H/T) - 54\ \ln(G/T)\right]\bar{\ln}(T) + \frac{15767}{16} + \frac{25}{2}\pi^2 - \frac{22}{15}\pi^4 - 8\pi^2\ \ln^2(2) \\ & + 8\ \ln^4(2) - \frac{873}{2}\zeta(3) + 192 Li_4(1/2) + 9\ \ln(H/T) + 27\ \ln(G/T)\Big\}, \end{aligned} \tag{4.38}$$

or, numerically,

$$\begin{aligned} V^{(3)} \approx {} & g_3^4 T^2\{-184\,\bar{\ln}^3(T) + 868\,\bar{\ln}^2(T) - 1842.2\,\bar{\ln}(T) + 1957.3\} \\ & + g_3^2 y_t^2 T^2\{60\,\bar{\ln}^3(T) - 360\,\bar{\ln}^2(T) + 1220.9\,\bar{\ln}(T) - 780.3\} \\ & + y_t^4 T^2\{24.75\,\bar{\ln}^3(T) + [81 + 81\ \ln(H/T) + 27\ \ln(G/T)]\bar{\ln}^2(T) \\ & + [-971.6 - 54\ \ln(H/T) - 54\ \ln(G/T)]\bar{\ln}(T) + 504.5 + 9\ \ln(H/T) + 27\ \ln(G/T)\}. \end{aligned} \tag{4.39}$$

Equations (4.38) or (4.39)may be consistently added to the full two-loop effective potential as given in Ref. [8].

V. THE GOLDSTONE BOSON CATASTROPHE

Because of the doubled Goldstone boson propagators in diagrams (r) and (s) of Fig. 4, the three-loop effective potential has a logarithmic singularity in the limit $G = m^2 + \lambda\phi^2 = 0$, which corresponds to ϕ being at the minimum of the tree-level renormalized potential. In fact, the situation becomes progressively worse at higher loop orders, as these diagrams are part of a family that also includes the one-loop Goldstone contributions and the two-loop diagrams (b) and (c) in Fig. 3, and more generally, ℓ-loop vacuum diagrams consisting of a ring of $\ell - 1$ Goldstone boson propagators (all carrying the same momentum) punctuated by $\ell - 1$ top (for G^0) or top/bottom (for $G^{\pm}$) one-loop subdiagrams. These diagrams give rise to contributions to the ℓ-loop effective potential of the form[2]

[2]This sort of contribution has been noted before in Refs. [12,53] in the context of non-Goldstone scalars with small field-dependent squared masses.

$$V^{(\ell)} \sim (N_c y_t^2)^{\ell-1} T^2 (G/T)^{3-\ell}[\ln(G/T) + \cdots]. \quad (5.1)$$

The ellipsis in Eq. (5.1) includes constant terms. At least for $\ell = 1, 2, 3$, higher powers of $\ln(G/T)$ are absent; from Eqs. (4.34)–(4.36)we see that at those loop orders one has specifically

$$V^{(1)} \sim \frac{3}{4} G^2 \ln(G/T), \quad (5.2)$$

$$V^{(2)} \sim -3N_c y_t^2 T(\overline{\ln}\, T - 1) G \ln(G/T), \quad (5.3)$$

$$V^{(3)} \sim 3[N_c y_t^2 T(\overline{\ln}\, T - 1)]^2 \ln(G/T). \quad (5.4)$$

Equation (5.1) means that at four-loop order and higher, the singularity in $V_{\rm eff}$ as $G \to 0$ will be power law, going like $1/G^{\ell-3}$ multiplied by terms constant and logarithmic in G. Moreover, the first derivative of the effective potential with respect to ϕ diverges logarithmically in the $G = 0$ limit even at two-loop order, and the second derivative already at one-loop order.

For a generic choice of renormalization scale, at the minimum of the full radiatively corrected effective potential, G will be small (compared to T), but nonzero, and there is no true singularity. Nevertheless, the numerical effect can be nontrivial and can be quite important if one happens to choose a renormalization scale where G is very close to 0.

The behavior of the effective potential for small G that is illustrated in Eqs. (5.1)–(5.4) seems quite troubling. Unlike similar situations where renormalization group improvement has been employed to study the behavior in the presence of small field-dependent masses in toy models, the fact that G can be small in magnitude (and negative) is not associated with any real or apparent near-instability of the vacuum. Rather, small G is just the expected and inevitable result for any spontaneously broken weakly gauged symmetry, even in a clearly stable vacuum. One might even have naively imagined that a particularly *good* choice of renormalization scale would be one that makes G as small as possible (and positive), given that the Goldstone boson masses should be 0 when computed exactly (and the imaginary parts of the effective potential from negative G do not really correspond to any instability in the theory). But, instead, a choice of renormalization scale that makes G very small will actually provoke unphysically large contributions to the perturbatively computed effective potential and especially to its derivatives, and so apparently should be avoided. It would be interesting to see in explicit detail how renormalization group improvement (or some other resummation or trick) can mitigate this behavior in the Standard Model case. However, I declare this to exceed the scope of the present paper.

VI. NUMERICAL IMPACT

A full numerical study is also beyond the scope of this paper, but a few remarks about the practical impact of the results obtained above are in order. Consider, as a template, the central values of model parameters given in Ref. [25]:

$$Q = M_t = 173.35 \text{ GeV}, \quad (6.1)$$

$$\lambda(M_t) = 0.12710, \quad (6.2)$$

$$y_t(M_t) = 0.93697, \quad (6.3)$$

$$g_3(M_t) = 1.1666, \quad (6.4)$$

$$m^2(M_t) = -(93.36 \text{ GeV})^2, \quad (6.5)$$

$$g(M_t) = 0.6483, \quad (6.6)$$

$$g'(M_t) = 0.3587. \quad (6.7)$$

Now, minimizing the (real part of the) full two-loop effective potential of [8], I obtain the Landau gauge $\overline{\rm MS}$ VEV:

$$v(M_t)_{\text{2-loop}} = 247.25 \text{ GeV}. \quad (6.8)$$

[At this minimum, one has $G = -(30.76 \text{ GeV})^2$, so that the effective potential computed in perturbation theory has an imaginary part due to $\ln(G)$ factors.] If the three-loop contribution found above in Eq. (4.39) is included, I obtain instead

$$v(M_t)_{\text{3-loop}} = 246.91 \text{ GeV} \quad (6.9)$$

for the same set of Lagrangian parameters. The majority of this shift comes from the g_3^4 contribution to $V^{(3)}$; if only those contributions were included, the VEV would be 246.84 GeV. However, beyond the observation that the effect of $V^{(3)}$ is to reduce the VEV by about 0.34 GeV when all $\overline{\rm MS}$ Lagrangian parameters are held fixed, this way of assessing the impact is of somewhat limited interest, because in the real world the Lagrangian parameters m^2 and λ are not directly accessible.

Another exercise is to consider the relation between the physical Higgs mass M_H and λ. Writing $V_{\rm eff} = V^{(0)} + \Delta V$, the minimum of the potential $v \equiv \phi_{\rm min}$ is determined by $\partial V_{\rm eff}/\partial\phi = 0$, which allows us to eliminate m^2 according to

$$m^2 = -\lambda v^2 - \frac{1}{\phi}\frac{\partial(\Delta V)}{\partial\phi}\bigg|_{\phi=v}. \quad (6.10)$$

The pole squared mass of the Higgs boson is determined from

$$M_H^2 = m^2 + 3\lambda v^2 + \Pi_{HH}(M_H^2), \quad (6.11)$$

where $\Pi_{HH}(s)$ is the self-energy function of the external momentum squared $s = -p^2$. When evaluated at $s = 0$, Π_{HH} coincides with the second derivative of the radiative part of the effective potential. Thus we can write

$$M_H^2 = m^2 + 3\lambda v^2 + \frac{\partial^2(\Delta V)}{\partial \phi^2}\bigg|_{\phi=v} + [\Pi_{HH}(M_H^2) - \Pi_{HH}(0)] \tag{6.12}$$

$$= 2\lambda v^2 + \left(\left[-\frac{1}{\phi}\frac{\partial}{\partial\phi} + \frac{\partial^2}{\partial\phi^2}\right]\Delta V\right)\bigg|_{\phi=v} + [\Pi_{HH}(M_H^2) - \Pi_{HH}(0)]. \tag{6.13}$$

Now if we consider M_H^2 and v as fixed inputs, and treat $\Pi_{HH}(M_H^2) - \Pi_{HH}(0)$ as small, then we can estimate the change in λ coming from inclusion of a new contribution to the effective potential δV (e.g. three-loop effects) as

$$\Delta\lambda \approx -\frac{1}{2v^2}\Delta M_H^2 \approx -\frac{1}{2v^2}\left(\left[-\frac{1}{\phi}\frac{\partial}{\partial\phi} + \frac{\partial^2}{\partial\phi^2}\right]\delta V\right)\bigg|_{\phi=v}. \tag{6.14}$$

The neglect of $\Pi_{HH}(M_H^2) - \Pi_{HH}(0)$ here is not entirely justified, even for diagrams that involve the top mass as the only other scale, because the expansion parameter $M_H^2/M_t^2 \approx 0.53$ is not so small. For some of the diagrams contributing to $\Pi_{HH}(s)$, the expansion in s/M_t^2 tends to have powers of the expansion variable with numerical coefficients that are smaller than 1, and the expansion in M_H^2/M_t^2 converges fairly quickly. However, terms of first order in M_H^2/M_t^2 can be quite significant. Furthermore, diagrams contributing to $\Pi_{HH}(M_H^2)$ in which the external momentum can be routed through the diagram in such a way as to miss all top-quark propagators will not be approximated well by $\Pi_{HH}(0)$ at all.

In particular, this is true of some of the diagrams involving the Goldstone bosons, notably the ones obtained from the vacuum diagrams described in the previous section by attaching two external H^0 legs. Those contributions are not just wrong, but potentially very large. The naive estimates from Eq. (6.14) and Eqs. (5.2)–(5.4) for the most singular contribution as $G \to 0$ from each loop order $\ell = 1, 2, 3$ are

$$\Delta M_H^2|_{\text{1-loop}} \sim \frac{6\lambda^2\phi^2}{16\pi^2}\ \ln(G/T), \tag{6.15}$$

$$\Delta M_H^2|_{\text{2-loop}} \sim -\frac{12\lambda^2\phi^2}{(16\pi^2)^2}\left[\frac{N_c y_t^2 T(\overline{\ln}\ T - 1)}{G}\right] \tag{6.16}$$

$$\Delta M_H^2|_{\text{3-loop}} \sim -\frac{12\lambda^2\phi^2}{(16\pi^2)^3}\left[\frac{N_c y_t^2 T(\overline{\ln}\ T - 1)}{G}\right]^2, \tag{6.17}$$

and for higher loop orders, using Eq. (5.1),

$$\Delta M_H^2|_{\ell\text{-loop}} \sim \frac{\lambda^2\phi^2}{(16\pi^2)^\ell}\left[\frac{N_c y_t^2 T}{G}\right]^{\ell-1}, \tag{6.18}$$

where the multiplicative numerical factors and logarithms are unknown. These apparent singularities as $G \to 0$ are unphysical nonsense, and they cannot appear in the correct expression for M_H^2. The resolution is that they are canceled by contributions to $\Pi_{HH}(M_H^2) - \Pi_{HH}(0)$, as one can check explicitly at two-loop order.

Since we do not yet have $\Pi_{HH}(s)$ at three-loop order, we should certainly not attempt to estimate $\Delta\lambda$ (even roughly) using the part of $V^{(3)}$ involving y_t^4, because it includes the offensive $\ln(G/T)$ [and $\ln(H/T)$] factors. However, we can still make estimates of the contributions proportional to g_3^4 and $g_3^2 y_t^2$, since at three-loop order these are not singular for $G \to 0$. These should be taken only as estimates because, as mentioned above, corrections from $[\Pi_{HH}(M_H^2) - \Pi_{HH}(0)]$ that go like M_H^2/M_t^2 can be significant, even when Goldstone boson shenanigans are absent. With this caveat, using Eq. (6.14), one obtains with the model parameters listed above

$$\Delta\lambda|_{\text{3-loop } g_3^4 \text{ terms}} = -0.000014 \tag{6.19}$$

$$v|_{\text{3-loop } g_3^2 y_t^2 \text{ terms}} = -0.000153 \tag{6.20}$$

for a total of $\Delta\lambda = -0.000167$. This can be compared to the theoretical error estimate used in Ref. [25] of ± 0.00030, and the parametric error from the uncertainty on the Higgs mass of $0.000206(\Delta M_H/(100\ \text{MeV}))$.

It might seem somewhat surprising that the estimated shift in λ from the g_3^4 contribution to $V^{(3)}$ is so much smaller than the $g_3^2 y_t^2$ effect, given that $g_3 > y_t$ and the numerical coefficients are larger in the g_3^4 terms than in the $g_3^2 y_t^2$ terms. This is due to an accidental cancellation. To see how this works, consider a generic contribution to V_{eff} of the form

$$\delta V = T^2[a_0 + a_1\overline{\ln}(T) + a_2\overline{\ln}^2(T) + a_3\overline{\ln}^3(T)]. \tag{6.21}$$

From Eq. (6.14), one obtains the estimate for the corresponding shift in M_H^2:

$$\Delta M_H^2 = 2y_t^2 T[(2a_0 + 3a_1 + 2a_2) + (2a_1 + 6a_2 + 6a_3)\overline{\ln}(T) + (2a_2 + 9a_3)\overline{\ln}^2(T) + 2a_3\overline{\ln}^3(T)]. \tag{6.22}$$

Having chosen $Q = M_t$, the logarithms are small, $\overline{\ln}(T) = -0.11315$, and so the largest contribution might, naively, be expected to come from the term that does not have $\overline{\ln}(T)$ in it, which is proportional to $2a_0 + 3a_1 + 2a_2$. However, for the one-loop contribution,

$$(a_0, a_1, a_2, a_3)_{\text{1-loop}} = \frac{1}{16\pi^2}(9/2, -3, 0, 0), \tag{6.23}$$

so that $2a_0 + 3a_1 + 2a_2$ happens to vanish. At two loops, for the leading order in QCD:

$$(a_0, a_1, a_2, a_3)_{\text{2-loop}, g_3^2} = \frac{g_3^2}{(16\pi^2)^2}(72, -64, 24, 0), \tag{6.24}$$

and again $2a_0 + 3a_1 + 2a_2$ happens to vanish. At three loops, for the leading order in QCD,

$$(a_0, a_1, a_2, a_3)_{\text{3-loop},g_3^4} = \frac{g_3^4}{(16\pi^2)^3}(1957.3, -1842.2, 868, -184). \quad (6.25)$$

Here the cancellation is not quite complete, but still

$$(2a_0 + 3a_1 + 2a_2)_{\text{3-loop},g_3^4} = \frac{g_3^4}{(16\pi^2)^3}(124.1), \quad (6.26)$$

which is well over an order of magnitude smaller than either a_0 or a_1 individually. Furthermore, the $\overline{\ln}(T)$ term has the opposite sign, and cancels about 40% of this.

In contrast, for the three-loop $g_3^2 y_t^2$ contribution, the individual coefficients are smaller,

$$(a_0, a_1, a_2, a_3)_{\text{3-loop},g_3^2 y_t^2} = \frac{g_3^2 y_t^2}{(16\pi^2)^3} \times (-780.3, 1220.9, -360, 60), \quad (6.27)$$

but there is no efficient accidental cancellation in the term independent of $\overline{\ln}(T)$:

$$(2a_0 + 3a_1 + 2a_2)_{\text{3-loop},g_3^2 y_t^2} = \frac{g_3^2 y_t^2}{(16\pi^2)^3}(1382.2). \quad (6.28)$$

The preceding discussion points to an amusing fact. Suppose we took the "new" contribution to the effective potential δV to consist of only the three-loop g_3^4 and $g_3^2 y_t^2$ contributions that do not include $\overline{\ln}(T)$, on the grounds that the terms that do have $\overline{\ln}(T)$ were all in principle known before this paper from the two-loop effective potential and renormalization group invariance, by virtue of Eq. (4.37). In other words, consider as the "new" contribution

$$\delta V = \frac{1}{(16\pi^2)^3} T^2 [1957.3 g_3^4 - 780.3 g_3^2 y_t^2]. \quad (6.29)$$

From that point of view, we would find, instead of Eqs. (6.19) and (6.20) above,

$$\Delta\lambda|_{\text{3-loop } g_3^4 \text{ terms}} = -0.000710 \quad (6.30)$$

$$\Delta\lambda|_{\text{3-loop } g_3^2 y_t^2 \text{ terms}} = 0.000182 \quad (6.31)$$

for a total of $\Delta\lambda = -0.000527$. The difference between this and the value $\Delta\lambda = -0.000167$ obtained above is due to the subset of $V^{(3)}$ terms dependent on $\overline{\ln}(T)$. Therefore, a well-meaning attempt to include three-loop effects by using renormalization group invariance to obtain the $\overline{\ln}(T)$ terms in $V^{(3)}$ would have produced a spuriously large estimate for the shift in λ, because it does not capture the accidental cancellations present in the more complete calculation. In any case, the shift in λ should really be calculated using the full M_H^2 pole squared mass following from the three-loop $\Pi_{HH}(s)$. The effective potential found in this paper will allow a partial check of such a calculation through comparison with the three loop $\Pi_{HH}(0) = \partial^2(\Delta V)/\partial\phi^2|_{\phi=v}$.

VII. OUTLOOK

The main new result of this paper is Eq. (4.36) [or Eqs. (4.38) or (4.39)], which contains the three-loop contributions to the effective potential in the Standard Model proportional to m_t^4 and to g_3^4, $g_3^2 y_t^2$, or y_t^4. In principle, this allows an improved determination of the relation between the $\overline{\text{MS}}$ Lagrangian parameters and the VEV, although in practice one must deal with the fact that m^2 is not directly accessible. The estimates of the numerical impact of the result, described in the previous section, seem to suggest that the effects are not large compared to the present parametric and other theoretical uncertainties, although there is some accidental cancellation at work. While this is not unexpected, it is always a worthwhile goal to, if possible, reduce all theoretical errors far below the level where experimental errors can compete with them, so that all uncertainties can be reliably blamed on experimentalists. Hopefully, the results above are one step in this direction.

ACKNOWLEDGMENTS

This work was supported in part by the National Science Foundation Grant No. PHY-1068369. This research was supported in part by the National Science Foundation under Grant No. NSF PHY11-25915.

APPENDIX: INDIVIDUAL DIAGRAM CONTRIBUTIONS

The individual contributions to Eq. (3.13) from the diagrams in Fig. 4 are

$$V^{(3,a)} = g_{3B}^4 N_c C_F (C_F - C_G/2)\left\{\frac{(7-2d)(d-5)(d-2)^3}{(d-4)^2(d-3)^2} J + \left[\frac{2000 - 3656d + 2643d^2 - 939d^3 + 163d^4 - 11d^5}{2(d-4)(d-3)(2d-7)} + \frac{(d-3)(3d^2-11d+8)}{2d-7}\xi + \frac{(d-4)(d-3)(d-1)}{2(2d-7)}\xi^2\right] K + \frac{(d-6)(d-3)(d^2-7d+8)}{2(d-4)^2} L\right\}, \quad (A1)$$

$$V^{(3,b)} = g_{3B}^4 N_c C_F^2 \left\{\frac{(d-2)^2(d-1)^2}{2(3-d)} J + \left[\frac{(d-1)^2(3d-8)}{2(2d-7)} + \frac{(1-d)(3d^2-17d+24)}{2d-7}\xi + \frac{(1-d)(d^2-7d+12)}{2(2d-7)}\xi^2\right] K\right\}, \quad (A2)$$

$$V^{(3,c)} = g_{3B}^4 N_c C_F C_G \left\{ \frac{(2-d)^3}{(d-3)^2} J + \left[\frac{7d^4 - 67d^3 + 237d^2 - 373d + 226}{2(3-d)(2d-7)} + \frac{3(d-1)(d-3)^2}{2d-7} \xi + \frac{(d-4)(d-3)(d-1)}{2(2d-7)} \xi^2 \right] K \right\}, \tag{A3}$$

$$V^{(3,d)} = g_{3B}^4 N_c C_F C_G \frac{d-3}{4(2d-7)} [7d^2 - 19d + 14 + 2(1-d)(3d-10)\xi + (4-d)(d-1)\xi^2] K, \tag{A4}$$

$$V^{(3,e)} = g_{3B}^4 N_c C_F C_G \frac{d-3}{2(2d-7)} K, \tag{A5}$$

$$V^{(3,f)} = g_{3B}^4 N_c C_F T_F \left[\frac{2(5-d)(d-2)^3}{(d-6)(d-4)(d-3)} J + \frac{d^3 - 7d^2 + 6d + 16}{(d-4)(6-d)} L \right], \tag{A6}$$

$$V^{(3,g)} = g_{3B}^4 N_c C_F T_F (N_q - 1) \frac{4(3-d)(d-2)}{2d-7} K, \tag{A7}$$

$$\begin{aligned} V^{(3,h)} = g_{3B}^2 y_{tB}^2 N_c C_F \bigg\{ & \frac{(2-d)(d^4 - 8d^3 + 17d^2 + 8d - 44)}{(d-4)^2(d-3)^2} J + \frac{(3-d)(d^2 + 2d - 16)}{2(d-4)^2} L \\ & + \left[\frac{5d^4 - 60d^3 + 283d^2 - 618d + 520}{(4-d)(d-3)(2d-7)} + \frac{(d-6)(d-3)}{2d-7} \xi \right] K \bigg\}, \end{aligned} \tag{A8}$$

$$V^{(3,i)} = g_{3B}^2 y_{tB}^2 N_c C_F \frac{d-3}{d-4} \left\{ 4(d-3)K + (4-2d)Q + (2d-4) \left[\frac{d}{d-4} - 2\xi \right] P \right\}, \tag{A9}$$

$$V^{(3,j)} = g_{3B}^2 y_{tB}^2 N_c C_F \left\{ \frac{(d-2)(d-1)}{d-3} J + \left[\frac{(1-d)(3d-8)}{2d-7} + \frac{(6-d)(d-3)}{2d-7} \xi \right] K \right\}, \tag{A10}$$

$$V^{(3,k)} = g_{3B}^2 y_{tB}^2 N_c C_F \left\{ \left[\frac{(2-d)(d-1)(3d-8)}{2(d-4)(d-3)} + \frac{2(d-3)(d-2)}{d-4} \xi \right] P + \frac{(2-d)(d-1)(2d-5)}{2(d-3)} Q \right\}, \tag{A11}$$

$$V^{(3,l)} = g_{3B}^2 y_{tB}^2 N_c C_F \frac{2(d-3)(d-2)}{d-4} \xi P, \tag{A12}$$

$$V^{(3,m)} = y_{tB}^4 N_c \left[-\frac{(d-2)^2(d^2 - 11d + 26)}{4(d-4)^2(d-3)^2} J + \frac{(3d-8)(d^2 - 4d + 2)}{2(d-4)(d-3)(2d-7)} K + \frac{(6-d)(3d-8)}{8(d-4)^2} L \right], \tag{A13}$$

$$V^{(3,n)} = y_{tB}^4 N_c \left[\frac{1}{2(3-d)} J + \frac{d^2 - 12d + 26}{2(2-d)(2d-7)} K \right], \tag{A14}$$

$$V^{(3,o)} = y_{tB}^4 N_c \left[\frac{4d^2 - 21d + 28}{2(d-4)(d-3)} P + \frac{2d-5}{2(d-3)} Q \right], \tag{A15}$$

$$V^{(3,p)} = y_{tB}^4 N_c \frac{3(2-d)}{8} R, \tag{A16}$$

$$V^{(3,q)} = y_{tB}^4 N_c \left[\frac{d^2 - 10d + 20}{d(d-2)} K + \frac{2}{d} Q \right], \tag{A17}$$

$$V^{(3,r)} = y_{tB}^4 N_c^2 \left\{ \left[\frac{2(d-2)}{(d-4)(d-3)} + \frac{d-2}{2}(x_H/x)^{d/2-2} + \frac{d-2}{2}(x_G/x)^{d/2-2} \right] J + \frac{d-2}{d-4} L + 4x \frac{d}{dx_H}\left[x I_{xxxxx_H} - A I_{xxx_H} \right] \right\}, \tag{A18}$$

$$V^{(3,s)} = y_{tB}^4 N_c^2 \left\{ \frac{d-2}{4}(x_G/x)^{d/2-2} J + \frac{18-7d}{2(d-2)} K + 3Q + x \frac{d}{dx_G}\left[\frac{x}{2} I_{xx00x_G} - A I_{x0x_G} \right] \right\}. \tag{A19}$$

The sums of these contributions gives Eq. (3.13). The cancellation of the dependence on the QCD gauge-fixing parameter ξ provides a useful check.

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