$\alpha^2(Z\alpha)^4 m$ contributions to the Lamb shift and the fine structure in light muonic atoms

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Corrections to energy levels in light muonic atoms are investigated in order $\alpha^2(Z\alpha)^4 m$. We pay attention to corrections which are specific for muonic atoms and include the electron vacuum polarization loop. In particular, we calculate relativistic and relativistic-recoil two-loop electron vacuum polarization contributions. The results are obtained for the levels with $n = 1$, 2 and in particular for the Lamb shift $(2p_{1/2} - 2s_{1/2})$ and fine-structure intervals $(2p_{3/2} - 2p_{1/2})$ in muonic hydrogen, deuterium, and muonic helium ions.

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I. INTRODUCTION

Precision studies of light muonic atoms allow a determination of nuclear structure with an accuracy not accessible otherwise. A recent result of the CREMA collaboration on two lines for the Lamb shift muonic hydrogen [1], their current evaluation of the Lamb shift in muonic deuterium and their project on muonic helium Lamb shift necessitate a clarification of the related theory.

The problem becomes of special importance due to a discrepancy of the value of the proton radius [1] derived from the results on hydrogen and deuterium spectroscopy (see, e.g., [2]) and from electron-proton scattering (see, e.g., [3]). The situation is reviewed, e.g., in [4,5].

A comprehensive compilation of the present theoretical situation on muonic-hydrogen Lamb shift can be found in recent overviews [1,6–9] (see, also, [10–13]).

A theoretical expression for the Lamb shift in muonic hydrogen comprises a number of terms of a few clearly distinguishable types. Indeed, there are pure QED corrections and corrections which involve proton structure. The QED corrections may be of the same type as in ordinary hydrogen and those need only a rescaling with a substitution of the electron mass for the muon one. (Since the muon-proton mass ratio is about 1/9, while the electron-proton mass ratio is about 1/2000, one has to remember, indeed, higher importance of the recoil corrections in muonic hydrogen, as well as various reduced-mass effects.) A review on the Lamb shift in ordinary hydrogen can be found in [4,13,14].

In addition to those rescaled terms, there is a number of specific muonic-hydrogen contributions, which are summarized in Table I. They come from Feynman diagrams with closed electron loops.

The results obtained up to date for muonic hydrogen include contributions of the one-loop, two-loop [10], and three-loop [15–18] electronic vacuum polarization (eVP) as well as various contributions of the electronic block of the light-by-light scattering (LbL) [19,20]. Except for the one-loop eVP contributions, the results are available only for the leading terms. For the one-loop contribution additionally to the leading nonrelativistic term [21,22], also a relativistic nonrecoil [10,12,23,24] and recoil [25,26] terms are known. The results are summarized in Table I. The n-loop results are complete in a sense that they include all possible contributions of the related order with n'-eVP potentials ($n' \le n$) and their iterations. For example, the eVP2 result in Table I consists of a contribution of the Källen-Sabry potential and of a double-iteration term with the Uehling potential.

TABLE I. Specific contributions to the Lamb shift $\Delta E(2p_{1/2} - 2s_{1/2})$ in light muonic atoms up to the order $\alpha^5 m$: hydrogen, deuterium, helium-3, and helium-4 ions. The results concern one-loop, two-loop, and three-loop electronic vacuum polarization (eVP) contributions as well as the contribution of the light-by-light scattering block (Fig. 1). The results marked with an asterisk are obtained in this paper.

	$\Delta E(2p_{1/2} - 2s_{1/2})$ [meV]			
Term	μH	μD	μ^3He	μ^4He
eVP1	205.026 12	227.656 45	1642.3954	1666.2940
eVP2	1.658 85	1.838 04	13.0843	13.2769
eVP3	0.007 52	0.008 42(7)	0.073(3)*	0.074(3)
LbL	−0.000 89(2)	−0.000 96(2)	−0.0134(6)*	−0.0136(6)

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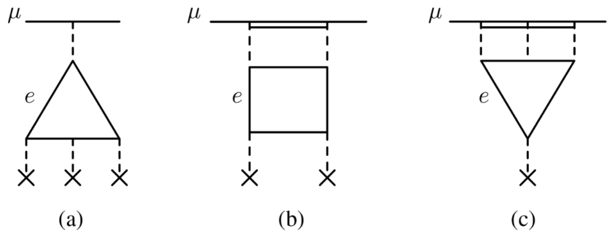

FIG. 1. Leading-order diagrams, which include the electronic (e) light-by-light scattering block. The horizontal double line is for the reduced Green function of a muon (μ) at the Coulomb field.

Most of the results mentioned are calculated in the leading nonrelativistic approximation and thus do not contribute to the fine structure. The only correction among them, relevant for the fine structure, is the one-loop relativistic contribution [10,12,23,24].

The two-loop eVP corrections, as mentioned, are known only in the leading order, which is $\alpha^2(Z\alpha)^2m$, where Z is the nuclear charge and m is the muon mass, and here we consider relativistic corrections to them. They are of the order of $\alpha^2(Z\alpha)^4m$. In muonic atoms a ratio of the muon and nuclear mass is small, but not very small and in particular in muonic hydrogen $m/M \sim 0.1$. That means that any more or less accurate calculation should also involve recoil corrections. Here we consider them exactly in m/M, which is the ratio of the muon and nuclear masses.

While the main purpose of this paper is to calculate two-loop relativistic and relativistic recoil eVP contributions, we also analyze all other sources of corrections of order of $\alpha^2(Z\alpha)^4m$ and $\alpha^2(Z\alpha)^4m^2/M$.

In principle, some of the $\alpha^2(Z\alpha)^4m$ contributions can appear from the higher-order LbL contributions. The leading LbL term, presented in Table I, includes the Wichmann-Kroll contribution [Fig. 1(a)] in order $\alpha(Z\alpha)^3m$ [13,19,20,27–31], the virtual-Delbrück-scattering contribution [Fig. 1(b)] [12,19,20,28,29,32,33], and the third contribution, which does not have a specific "common" name [Fig. 1(c)].

Higher-order corrections due to the addition of a radiative correction to the electron loop or the eVP to either line will add an extra factor of α. However, the LbL term is so uncertain that such a correction should be below uncertainty. Besides, it is rather substantially smaller than the two-loop eVP $\alpha^2(Z\alpha)^4m$ contribution studied in this paper.

II. TWO-LOOP EVP RELATIVISTIC RECOIL CONTRIBUTION

A calculation of eVP nonrelativistic contributions to the energy levels of a two-body muonic atom can be performed in terms of the nonrelativistic perturbation theory (NRPT). The only potentials in such a calculation are the Coulomb and eVP potentials. While the Coulomb problem is considered nonperturbatively, all the eVP potentials (see Fig. 2) are considered as a perturbation. Nonrelativistic two-loop [10] and three-loop [15–18] eVP terms were found some time ago within such an NRPT framework.

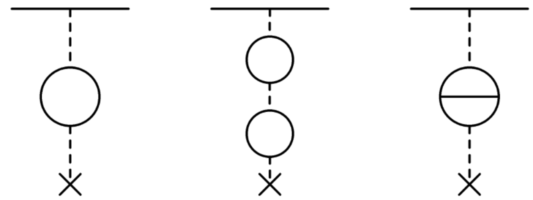

FIG. 2. The eVP potentials are required for a nonrelativistic calculation of the eVP two-loop contribution. They include the Uehling potential and a reducible and irreducible part of the Källen-Sabry potential [38].

In case of the relativistic problem, one can apply a Breit-type approach and also use an NRPT-type calculation, where in addition to the eVP potentials one has to take into account various perturbations of the nonrelativistic Hamiltonian that describe the relativistic corrections. That is applicable for the leading [in $(Z\alpha)$] relativistic term, but not for higher-order corrections. Such a leading [in $(Z\alpha)$] term can be found exactly in m/M.

In particular, such a Breit-type approach to the one-loop eVP was developed in [10,25,26,34].

Indeed, for the one-loop eVP correction one can directly calculate the matrix element over the Dirac-Coulomb wave functions, however, such a purely relativistic calculation is difficult to generalize to recoil effects and in particular to the $\alpha(Z\alpha)^4m^3/M^2$ term as well as the relativistic two-loop terms.

Here we apply the NRPT based on the Breit-type Hamiltonian to the evaluation of two-loop relativistic corrections and obtain below the $\alpha^2(Z\alpha)^4m$ term in all orders in m/M.

To arrive at an NRPT consideration, one has first to consider two particles which exchange with photons. The NRPT approach eventually assumes only instantaneous one-photon exchange. Once a Hamiltonian with instantaneous one-photon exchange is obtained, one can rely on the perturbation theory,

$$\begin{aligned}\Delta E = {}& \langle\Psi_{nl}|\delta V|\Psi_{nl}\rangle + \langle\Psi_{nl}|\delta V G'_{nl}\delta V|\Psi_{nl}\rangle \\ & + \langle\Psi_{nl}|\delta V G'_{nl}\delta V G'_{nl}\delta V|\Psi_{nl}\rangle \\ & - \langle\Psi_{nl}|\delta V|\Psi_{nl}\rangle \times \langle\Psi_{nl}|\delta V G'_{nl}G'_{nl}\delta V|\Psi_{nl}\rangle, \qquad (1)\end{aligned}$$

where Ψ_{nl} is the nonrelativistic Coulomb wave function of the nl state in hydrogenic atom (see, e.g., [35]), n is the principal quantum number, and l is the orbital quantum number. Here, G'_{nl} stands for the nonrelativistic reduced Coulomb Green function.

The expression is valid for any central potential. In case of the spin-orbit and spin-spin interactions, the identity needs some corrections. The interaction of the muon spin and orbit with the nuclear spin is neglected, since it vanishes after we average over the hyperfine structure. When necessary, the hyperfine effects can be studied separately.

As for the spin-orbit interaction, we have to apply the wave functions the radial part of which is the same as that of Ψ_{nl}, while the angular and spin part is chosen to realize the physical basis with eigenstates of the muon angular momentum and its projections. Indeed, the matrix elements for the energy do not depend on the projection.

Strictly speaking, the contributions to the perturbation of the nonrelativistic Hamiltonian, denoted as δV, are not necessary potentials, since they may include momentum (see below) and thus be nonlocal. That does not change the equations and for simplicity we still use for then a term "effective potentials."

For the one-loop eVP contribution the derivation of the NRPT equations was done in detail in [26]. A proper choice of the gauge of the photon propagator $D_{\mu\nu}$ allows one to avoid retardation effects in the D_{00} component of the one-photon exchange and neglect those effects in the D_{ij} component, since the retardation effects produce there only corrections in the higher order in $(Z\alpha)$. Meanwhile the two-photon exchange contributions lead to $(Z\alpha)^5 m^2/M$ terms only. Thus, the application of the NRPT approach to calculate $\alpha(Z\alpha)^4 m$ exactly in m/M is validated.

The evaluation is based on the eVP correction to the photon propagator, which is proportional to the dispersion integral (see [26] for details),

$$D^{\rm VP}_{\mu\nu}(k) \propto \int_0^1 dv \rho_e(v) \frac{1}{k^2-\lambda^2}, \qquad (2)$$

where the dispersion parameter serves as an effective photon mass

$$\lambda^2 = \frac{4m_e^2}{1-v^2} \qquad (3)$$

and the dispersion function ρ_e depends on the contribution we are to study. In particular, for the one-loop eVP calculation the dispersion density is

$$\rho_1(v) = \left(\frac{\alpha}{\pi}\right)\frac{v^2(1-v^2/3)}{1-v^2}. \qquad (4)$$

The effective potentials at order α^0 are determined for a Coulomb-bound two-body system by the standard Breit equation [36,37],

$$\begin{aligned} V_{\rm Br}(\mathbf{r}) = &-\left(\frac{1}{m^3}+\frac{1}{M^3}\right)\frac{\mathbf{p}^4}{8} + \frac{Z\alpha}{8}\left(\frac{1}{m^2}+\frac{1}{M^2}\right)4\pi\delta^3(\mathbf{r}) \\ &+ Z\alpha\left(\frac{1}{4m^2}+\frac{1}{2mM}\right)\frac{\mathbf{L}\cdot\boldsymbol{\sigma}}{r^3} + \frac{Z\alpha}{2mM}4\pi\delta^3(\mathbf{r}) \\ &+ \frac{Z\alpha}{2mM}\left[\frac{1}{r^3}\mathbf{L}^2 - \mathbf{p}^2\frac{1}{r} - \frac{1}{r}\mathbf{p}^2\right] \end{aligned} \qquad (5)$$

and considered as a perturbation of the unperturbed problem of the nonrelativistic Schrödinger equation with the Coulomb potential

$$V_C(\mathbf{r}) = -\frac{Z\alpha}{r}. \qquad (6)$$

Here M stands for the nuclear mass, m is for the muon mass m_μ, $\mathbf{s} = \boldsymbol{\sigma}/2$ and $\mathbf{L}$ are spin and orbital moments of muon, $\mathbf{p}$ is the momentum operator and the relativistic units in which $c = \hbar = 1$ are applied. Here Z is the nuclear change and M is the nuclear mass and the final expression is valid for the nuclear spin $1/2$, assuming that we average over the nuclear spin (i.e. over the hyperfine structure).

Those in order α^1 are [34]

$$\begin{aligned} V^{\rm VP}_{\rm Br}(\mathbf{r}) = &\left(\frac{1}{8m^2}+\frac{1}{8M^2}\right)\nabla^2 V_U + \left(\frac{1}{4m^2}+\frac{1}{2mM}\right)\frac{V'_U}{r}\mathbf{L}\cdot\boldsymbol{\sigma} \\ &+ \frac{1}{2mM}\nabla^2\left[V_U - \frac{1}{4}(rV_U)'\right] + \frac{1}{2mM}\left[\frac{V'_U}{r}\mathbf{L}^2\right. \\ &\left.+ \frac{\mathbf{p}^2}{2}(V_U - rV'_U) + (V_U - rV'_U)\frac{\mathbf{p}^2}{2}\right], \end{aligned} \qquad (7)$$

where V_U is the Uehling potential,

$$V_U(\mathbf{r}) = -Z\alpha\int_0^1 dv \rho_1(v)\frac{e^{-\lambda r}}{r}. \qquad (8)$$

Graphically, the related effective potential is presented in Fig. 3 and the diagrams for the calculation of the relativistic recoil corrections in order $\alpha(Z\alpha)^4 m$ [exactly in m/M] are depicted in Fig. 4.

The relativistic recoil eVP correction of order $\alpha(Z\alpha)^4 m$ originates from terms of the first and second order of NRPT (1) with the effective potential defined in Eqs. (5) and (7). To generalize the result and calculate relativistic recoil two-loop eVP corrections, we have to calculate terms of the second [see Fig. 5(c1)] and third [see Figs. 5(c2) and (c3)] order with the same potentials and the first and the second order with the effective two-loop potential [see Figs. 5(a) and 5(b)]. The latter can be easily obtained

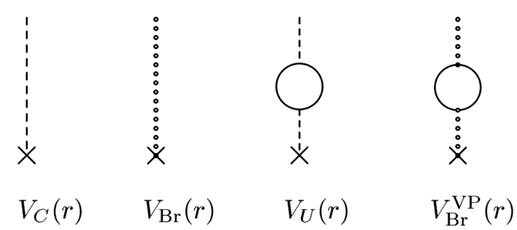

FIG. 3. Effective potentials for the NRPT calculation of the relativistic recoil one-loop eVP contribution.

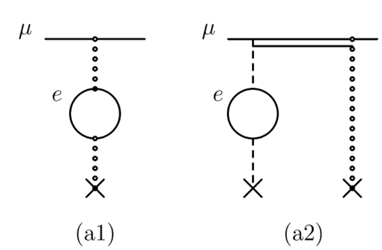

FIG. 4. Diagrams for the relativistic recoil one-loop eVP correction.

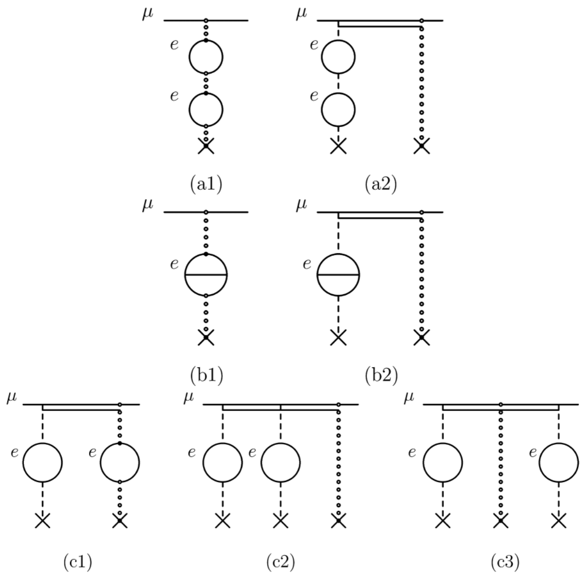

FIG. 5. Diagrams of the nonrelativistic perturbation theory for the calculation of the relativistic recoil two-loop eVP contributions.

from the related one-loop potentials (7) by a substitution of the two-loop eVP dispersion density for the one-loop one.

The two-loop eVP dispersion function for the reducible part is [38,39]

$$\rho_{1\cdot 1}(v) = -\frac{1}{9}\left(\frac{\alpha}{\pi}\right)^2 \frac{v^2(1-v^2/3)}{1-v^2} \times \left\{16 - 6v^2 + 3v(3-v^2)\ln\left(\frac{1-v}{1+v}\right)\right\}, \tag{9}$$

and for the irreducible one it takes the form [38,40,41]

$$\begin{aligned}\rho_2(v) = &\frac{2}{3}\left(\frac{\alpha}{\pi}\right)^2 \frac{v}{1-v^2} \times \left\{(3-v^2)(1+v^2)\left[\mathrm{Li}_2\left(-\frac{1-v}{1+v}\right)\right.\right.\\ &\left. + 2\mathrm{Li}_2\left(\frac{1-v}{1+v}\right) + \ln\left(\frac{1+v}{1-v}\right)\left(\frac{3}{2}\ln\left(\frac{1+v}{2}\right) - \ln(v)\right)\right]\\ &+ \left(\frac{11}{16}(3-v^2)(1+v^2) + \frac{1}{4}v^4\right)\ln\left(\frac{1+v}{1-v}\right)\\ &+ \frac{3}{2}v(3-v^2)\ln\left(\frac{1-v^2}{4}\right) - 2v(3-v^2)\ln(v)\\ &\left. + \frac{3}{8}v(5-3v^2)\right\},\end{aligned} \tag{10}$$

where Li_2 is the Euler dilogarithm [42].

The evaluation of the contributions in Figs. 5(a) and 5(b) of the reducible and irreducible parts of the two-loop eVP is similar to the related one-loop eVP contributions in Fig. 4 and immediately leads to a result. The nonrecoil results of order $\alpha^2(Z\alpha)^4 m_r c^2$ for muonic hydrogen are summarized in Table II, while the recoil corrections are presented in Table III.

The evaluation of contributions related to Fig. 5(c), which presents terms of the third order of NRPT (1), is somewhat more complicated. The related calculation involves integrations with the radial parts of the reduced Green function of the nonrelativistic Coulomb problem.

We use two representations of the reduced Coulomb Green function G'_{nl} which allow us to provide a crosscheck of our calculations. The most fruitful is a representation for the Coulomb Green function developed in [43]. The expressions we applied for the radial part of the reduced Coulomb Green functions are [44,45] (see also [10])

$$\tilde{G}_{1s}(r,r') = 4Z\alpha m_r^2 \exp\left(\frac{z_> + z_<}{2}\right)\left\{\frac{1}{z_>} + \frac{1}{z_<} + \frac{7}{2} - \frac{z_> + z_<}{2} + \mathrm{Ei}(z_<) - 2C - \ln(z_> z_<) - \frac{e^{z_<}}{z_<}\right\}, \tag{11}$$

$$\begin{aligned}\tilde{G}_{2s}(r,r') = Z\alpha m_r^2 \frac{\exp\left(\frac{z_> + z_<}{2}\right)}{4z_> z_<}\{&8z_< - 4z_<^2 + 8z_>\\ &+ 12z_> z_< - 26z_> z_<^2 + 2z_> z_<^3 - 4z_>^2\\ &- 26z_>^2 z_< + 23z_>^2 z_<^2 - z_>^2 z_<^3 + 2z_>^3 z_< - z_>^3 z_<^2\\ &+ 4(z_> - 2)z_>(1 - z_<)e^{z_<} + 4(z_> - 2)z_>\\ &\times (z_< - 2)z_<[-2C + \mathrm{Ei}(z_<) - \ln(z_> z_<)]\},\end{aligned} \tag{12}$$

TABLE II. Relativistic eVP corrections for the low-lying levels in muonic hydrogen in the external field approximation ($m/M \to 0$). The result is for the Schrödinger problem with the reduced mass. The contributions are labeled as in Fig. 5. The units are $(\alpha/\pi)^2(Z\alpha)^4 m_r c^2$.

	$\Delta E(nl_j)$			
Diagram	$1s_{1/2}$	$2s_{1/2}$	$2p_{1/2}$	$2p_{3/2}$
(a1) + 2 (a2)	−0.187	−0.0307	−0.000 957	0.000 205
(b1) + 2 (b2)	−0.586	−0.103	−0.0311	−0.003 75
2 (c1)	1.16	0.129	−0.000 995	0.000 201
2 (c2) + (c3)	−1.46	−0.166	−0.000 758	−0.000 351
Total	−1.07	−0.171	−0.0338	−0.00370

TABLE III. Relativistic recoil eVP corrections for the low-lying levels in muonic hydrogen in order $\alpha^2(Z\alpha)^4 m_r c^2$. The contributions are labeled as in Fig. 5. The units are $(\alpha/\pi)^2 (Z\alpha)^4 m_r c^2 (m_r/M)$.

	$\Delta E(nl_j)$			
Diagram	$1s_{1/2}$	$2s_{1/2}$	$2p_{1/2}$	$2p_{3/2}$
(a1) + 2 (a2)	0.454	0.0648	0.000 198	0.000 0805
(b1) + 2 (b2)	0.125	0.0518	0.004 43	0.001 67
2 (c1)	−3.12	−0.349	−0.000 413	−0.000 534
2 (c2) + (c3)	3.60	0.409	0.000 796	0.000 755
Total	1.05	0.176	0.005 01	0.001 97

$$\tilde{G}_{2p}(r,r') = Z\alpha m_r^2 \frac{\exp\left(\frac{z_> + z_<}{2}\right)}{36(z_> z_<)^2}\{(24 z_<^3 + 36 z_> z_<^3 + 36 z_>^2 z_<^3 + 24 z_>^3 + 36 z_>^3 z_< + 36 z_>^3 z_<^2 + 49 z_>^3 z_<^3 - 3 z_>^3 z_<^4 - 3 z_>^4 z_<^3 - 12 z_>^3 (2 + z_< + z_<^2) e^{z_<} + 12 z_>^3 z_<^3 \times [-2C + \mathrm{Ei}(z_<) - \ln(z_> z_<)]\}, \qquad (13)$$

where

$$z_> = \frac{2Z\alpha m_r}{n}\max(r,r'), \qquad z_< = \frac{2Z\alpha m_r}{n}\min(r,r'),$$

$C = 0.577216\ldots$ is the Euler constant, and

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{e^t}{t} dt$$

is the exponential integral.

This representation is especially useful in case of contact potentials, proportional to the δ function, which sets the smaller radius to zero [for a general expression for $\tilde{G}_{nl}(r,0)$ for an arbitrary state see [46]].

The other representation of the reduced Coulomb Green function we used is the Sturmian one [47]. The radial part of the reduced Coulomb Green function is of the form [47]

$$\tilde{G}_{nl}(r,r') = \frac{n^2}{(Z\alpha)^2 m_r}\left\{\sum_{\substack{k=l+1\\k\neq n}}^{\infty} \frac{k}{k-n} R_{kl}(n;r) R_{kl}(n;r') + \frac{3}{2} R_{nl}(n;r) R_{nl}(n;r') + r R'_{nl}(n;r) R_{nl}(n;r') + r' R'_{nl}(n;r') R_{nl}(n;r)\right\}, \qquad (14)$$

and

$$R_{kl}(n;r) = \left(\frac{k}{n}\right)^{3/2} R_{kl}\left(\frac{k}{n} r\right),$$

and $R_{nl}(r)$ stands for the radial part of the standard wave function of the nonrelativistic Coulomb problem (see, e.g., [35]).

An evaluation of the relativistic corrections to the Hamiltonian (5) and (7) involves various differentiations and we consider them in the Appendix. The Laplacian of the Uehling potential is considered in Appendix A, the differentiation of the Green function in Sturmian representation is discussed in Appendix B and the differentiation procedure applied to the Green function with $r_{>,<}$ is summarized in Appendix C. Such a special treatment of derivatives allows us to simplify the evaluation. The final results for relativistic nonrecoil and recoil eVP contributions in order $\alpha^2(Z\alpha)^4m$ for the low-lying states in muonic hydrogen are summarized in Tables II and III.

As for the other light muonic atoms, our results in order $\alpha^2(Z\alpha)^4m$ and $\alpha^2(Z\alpha)^4m_rc^2(m_r/M)$ are presented in Tables IV and V.

TABLE IV. Relativistic eVP corrections (Fig. 5) for the low-lying levels in muonic hydrogen in the external field approximation. The result is for the Schrödinger problem with the reduced mass. The units are $(\alpha/\pi)^2\,(Z\alpha)^4 m_r c^2$.

	$\Delta E(nl_j)$			
Atom	$1s$	$2s$	$2p_{1/2}$	$2p_{3/2}$
μH	−1.07	−0.171	−0.0338	−0.003 70
μD	−1.13	−0.180	−0.0370	−0.004 15
μ^3He	−2.21	−0.347	−0.113	−0.0163
μ^4He	−2.22	−0.350	−0.115	−0.0165

TABLE V. Relativistic recoil corrections in order $\alpha^2(Z\alpha)^4m_rc^2$ (Fig. 5) for the low-lying levels in muonic hydrogen. The units are $(\alpha/\pi)^2\,(Z\alpha)^4m_rc^2(m_r/M)$.

	$\Delta E(nl_j)$			
Atom	$1s$	$2s$	$2p_{1/2}$	$2p_{3/2}$
μH	1.05	0.176	0.005 01	0.001 97
μD	1.13	0.191	0.004 53	0.002 78
μ^3He	1.39	0.276	0.008 96	0.005 44
μ^4He	1.40	0.280	0.008 52	0.005 81

We have performed our calculations applying two different representations of the reduced Coulomb Green function described above. The calculations were done also with and without the trick with the operator $\mathbf{p}^4$, considered in Appendix C. Calculations without the trick are possible but require more time and are less accurate. All the results are consistent.

The evaluation based on the Breit-type approach allows one to obtain recoil effects in order $\alpha^2(Z\alpha)^4m$ exactly in m/M, and we have done here such a calculation. However, we have also performed another evaluation, applying an alternative technique, which allows terms linear in m_r/M only. The details will be published elsewhere [48]. The results obtained within these two approaches are consistent. We thus consider our results on the relativistic recoil two-loop corrections as well established.

III. CONCLUSIONS

Indeed, the most interesting are not the shifts of energy of any level by itself, but rather two intervals, namely, the Lamb-shift ($2p_{1/2} - 2s_{1/2}$) and the fine-structure ($2p_{3/2} - 2p_{1/2}$) intervals. The results for the two-loop eVP contributions [including the previously known leading term of order $\alpha^2(Z\alpha)^2m$ [10]] are summarized in Tables VI and VII.

The results for different muonic atoms are obtained by the same method, however, following [49,50] the so-called Zitterbewegung term is not included for the muonic deuterium and helium-4 ion (cf. [25,26]).

We note that the recoil effects in order $\alpha^2(Z\alpha)^4m$ are very small for the fine structure. That is because the

TABLE VI. The second-order eVP contributions to the Lamb shift $2p_{1/2} - 2s_{1/2}$ in light muonic atoms. The units are meV. The results marked with an asterisk are obtained in this paper.

	$\Delta E(2p_{1/2} - 2s_{1/2})$ [meV]			
Atom	μH	μD	μ^3He	μ^4He
$\alpha^2(Z\alpha)^2m$	1.658 85	1.838 04	13.0843	13.2769
$\alpha^2(Z\alpha)^4m^*$	0.000 199	0.000 218	0.005 82	0.005 90
$\alpha^2(Z\alpha)^4m$ (recoil)*	−0.000 0251	−0.000 0131	−0.000 242	−0.000 174
Total	1.659 02	1.838 24	13.0899	13.2826

TABLE VII. The second-order eVP contributions to the fine-structure interval $2p_{3/2} - 2p_{1/2}$ in light muonic atoms. The units are meV. The results marked with an asterisk are obtained in this paper.

	$\Delta E(2p_{3/2} - 2p_{1/2})$ [meV]			
Atom	μH	μD	μ^3He	μ^4He
$\alpha^2(Z\alpha)^2m$	0	0	0	0
$\alpha^2(Z\alpha)^4m^*$	0.000 0438	0.000 0502	0.002 42	0.002 47
$\alpha^2(Z\alpha)^4m$ (recoil)*	-4.5×10^{-7}	-1.4×10^{-7}	-3.2×10^{-6}	-1.9×10^{-6}
Total	0.000 0433	0.000 0501	0.002 42	0.002 47

correction, linear in m/M, vanishes (cf. [23,51]) and the remaining term is of order of $(m/M)^2$.

It is interesting to compare the obtained above two-loop eVP relativistic contributions with other contributions of the same order, i.e. of order $\alpha^2(Z\alpha)^4m$. To conclude let us briefly overview such contributions.

Indeed, first of all there are rescaled contributions of the electronic Lamb shift which are well known (see, e.g., [4,13,14]). Additional specific contributions to the Lamb shift in muonic atoms in order $\alpha^2(Z\alpha)^4m$ are presented in Fig. 6.

As we mention in the Introduction, one may also consider radiative corrections to the block of the light-by-light scattering, which modify the Wichmann-Kroll potential, and various Uehling corrections to the leading Wichmann-Kroll contribution. Since the leading Wichmann-Kroll contributions is very small and the uncertainty of the complete light-by-light scattering-scattering contribution is not small, we expect that the α corrections to the leading Wichmann-Kroll contribution are negligible and below that uncertainty. The related diagrams are not presented in Fig. 6. All of the others are. They are split into several classes.

We remind that there are nonspecific contributions of order $\alpha(Z\alpha)^4m$ which are obtained by the rescaling [4,13,14]. Meantime certain corrections to them with an additional factor of α are already specific. The typical diagrams are depicted in Figs. 6(a) and 6(b).

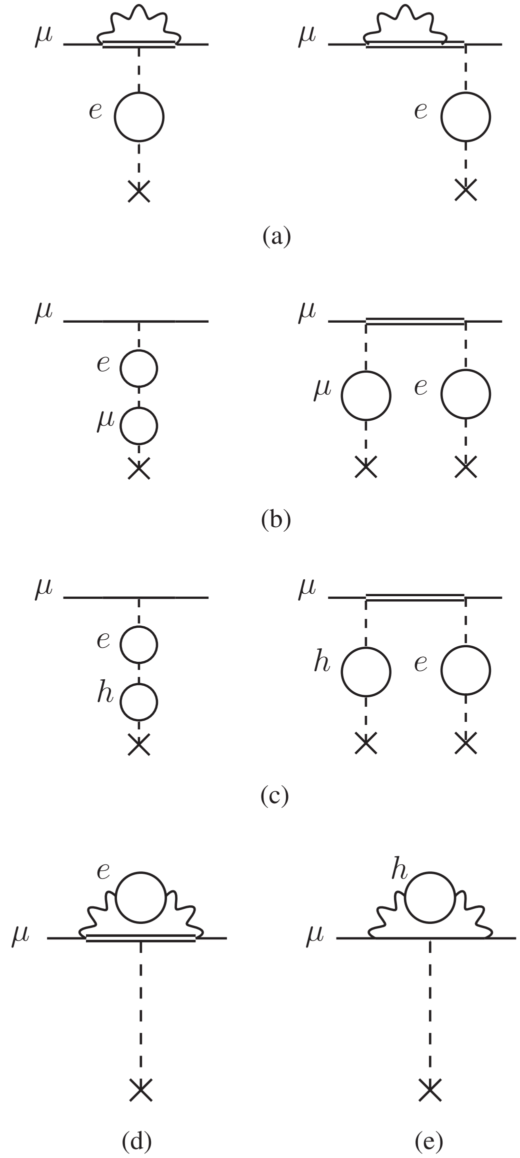

FIG. 6. Diagrams for various classes of specific corrections, contributing at order $\alpha^2(Z\alpha)^4m$. h stands for hadronic vacuum polarization.

The b-type contributions are due to Uehling corrections to the contribution of the muon VP. A similar contribution comes from the Uehling correction to the hadronic vacuum polarization contribution [see Fig. 6(c)].

We note that rescaling should include a substitution of the mass (electron → muon), but it keeps the same expression in terms of α and $Z\alpha$. Technically, that means that we keep the same expressions for the radiative corrections and in particular for the anomalous magnetic moment of the muon. The effects which contribute to the difference in the values of the anomalous magnetic moments, a_e and a_μ, should be considered separately. The contributions to $a_\mu - a_e$ appear in order α^2 and the most important of them are due to electronic or hadronic VP (on various

TABLE VIII. Various $\alpha^2(Z\alpha)^4 m_\mu$ contributions the Lamb shift $(2p_{1/2} - 2s_{1/2})$ in light muonic atoms. The units are μeV. Notation follows Fig. 6. The eVP2 term is the correction of order $\alpha^2(Z\alpha)^4 m$ of Table VI. The results marked with an asterisk are obtained in this paper.

	$\Delta E(2p_{1/2} - 2s_{1/2})$ [μeV]			
Atom	μH	μD	μ^3He	μ^4He
(a)	−2.54	−3.06	−62.69	−64.62
(b)	0.128	0.054*	3.83*	3.95*
(c)	0.081(8)*	0.097(10)*	2.4(2)*	2.5(2)*
(d)	−1.52	−1.77*	−29.92*	−30.73*
(e)	−0.020(2)5*	−0.024(2)*	−0.40(4)*	−0.41(4)*
eVP2	0.173*	0.203*	5.58*	5.72*
Total	−3.70(2)	−4.40(2)	−81.2(2)	−83.6(2)

contributions to the anomalous magnetic moment of muon see [52,53]). The characteristic diagrams are presented in Figs. 6(d) and 6(e). Those diagrams are also responsible for a specific contribution to the slope of the Dirac form factor in order α^2 and thus for the related contribution to the Lamb shift.

The related contributions are summarized in Tables VIII and IX. Most of the contributions have been known before.

The type-a contributions to the Lamb shift and fine structure were considered in [54]. The b contributions were found in [55] (see also [9,13]) (see also [6,17,39]).

The contribution, which involves the hadronic vacuum polarization in the Coulomb photon, [Fig. 6(c)] is calculated in this paper. In particular, we found

$$\Delta E(2s, \mu\mathrm{H}) = \frac{\alpha}{\pi} 3.2248\ldots \Delta E^{(0)}_{\mathrm{hadr}}(2s, \mu\mathrm{H}),$$

where $\Delta E^{(0)}_{\mathrm{hadr}}(\mu\mathrm{H})$ is the leading hadronic contribution, considered in Appendix D.

The contributions d and e are considered for muonic hydrogen in [10,13,56,57]. In particular, there is a result of [58] for the e contributions. We have recalculated it and our result is different from that in [58]. The details of our calculations as well as a comparison with the earlier result are presented in Appendix E.

TABLE IX. Various $\alpha^2(Z\alpha)^4 m$ contributions to the fine-structure interval $2p_{3/2} - 2p_{1/2}$ in light muonic atoms. The units are μeV. Notation follows Fig. 6. The eVP2 term is the correction of order $\alpha^2(Z\alpha)^4 m$ of Table VII. The results marked with an asterisk are obtained in this paper.

	$\Delta E(2p_{3/2} - 2p_{1/2})$ [μeV]			
Atom	μH	μD	μ^3He	μ^4He
(a)	0.0105	0.0127	0.606	0.624
(d)*	0.0893	0.0991	1.64	1.67
(e)*	0.0010(1)	0.0012(1)	0.019(2)	0.020(2)
eVP2*	0.0433	0.0501	2.42	2.47
Total	0.144	0.164	4.69	4.78

The corrections in Tables VIII and IX are leading nonrelativistic corrections in $Z\alpha$ the corresponding order. They are calculated by means of the nonrelativistic atomic physics, i.e. the related wave functions and Coulomb Green functions are nonrelativistic. That means that all recoil effects are covered by the reduced mass. We note that the internal integration of the radiative loops for the anomalous magnetic moment and the slope of the Dirac form factor are relativistic. In principle, additionally to those diagrams, one has to take into account diagrams similar to those in Figs. 6(a), 6(d), and 6(e) which are radiative corrections to the nuclear line. However, they are incorporated into the proton form factors and should be considered separately.

The complete result of the $\alpha^2(Z\alpha)^4 m$ (see Table VIII) is comparable with the theoretical and experimental uncertainty for the Lamb shift in muonic hydrogen [1] and has to be taken into account.

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APPENDIX A: DIFFERENTIATION OF THE UEHLING POTENTIAL

Calculation of the Laplacian of the Uehling and Källen-Sabry potentials involves singularities and may cost certain troubles.

Following [10,39,59], we apply the identity

$$\nabla^2 V(\mathbf{r}) = \int_0^\infty dv \rho(v) \left(4\pi\delta^3(\mathbf{r}) - \frac{\lambda^2}{r} e^{-\lambda r} \right).$$

APPENDIX B: DIFERENTIATION FOR STURMIAN BASIS FUNCTIONS

While calculating the third order of the NRPT (1) we have to deal with $\Psi_{nlm}\mathbf{p}^4\tilde{G}$, $\tilde{G}\mathbf{p}^4\tilde{G}$, $\Psi_{nlm}A\mathbf{p}^2\tilde{G}$, and $\tilde{G}A\mathbf{p}^2\tilde{G}$, where Ψ_{nlm} is the hydrogen wave function and A is an operator, diagonal in coordinate space. All these expressions require only a calculation of $\mathbf{p}^2\tilde{G}$ in closed form, since for $\mathbf{p}^4$ we can consider one $\mathbf{p}^2$ as acting on the right, while the other as acting on the left.

Differentiation of the Coulomb wave function is obvious:

$$\mathbf{p}^2\Psi_{nlm}(\mathbf{r}) = 2m_r\left(\frac{Z\alpha}{r} + E_n\right)\Psi_{nlm}(\mathbf{r}). \quad \text{(B1)}$$

A Sturmian basis function,

$$\Phi_{klm}(n,\mathbf{r}) = R_{kl}(n;r)Y_{lm}(\Omega),$$

is a solution of the related Sturm-Liouville problem, differential equation which can be rewritten as

$$\mathbf{p}^2\Phi_{klm}(n,\mathbf{r}) = 2m_r\left(\frac{k}{n}\frac{Z\alpha}{r} + E_n\right)\Phi_{klm}(n,\mathbf{r}). \quad \text{(B2)}$$

These identities allow one to carry out any differentiation of the Green function, presented in terms of the Sturmian basis (14), required for the calculation in the third order of NRPT.

APPENDIX C: DIFFERENTIATION OF THE GREEN FUNCTION (11)

The representation of the Green function (11) in terms of $r_>$ and $r_<$ has certain advantages, however its differentiation is somewhat complicated. To avoid it we used a trick described below. It may be applied to any representation of the Green function.

First we note that the NRPT expression (1) was previously applied in a certain order of the expansion. Having in mind a calculation of operator $\mathbf{p}^4/m_r^3$, we consider now not a single term of the required order (α^2), but the sum of all the terms up to the second order in α. The sum of all the terms, which include $\mathbf{p}^4/m_r^3$, is

$$\begin{aligned}\Delta E_4 = {}& \left\langle\Psi_{nl}\left|\frac{\mathbf{p}^4}{m_r^3}\right|\Psi_{nl}\right\rangle + \left\langle\Psi_{nl}\left|\frac{\mathbf{p}^4}{m_r^3}G'_{nl}V_U\right|\Psi_{nl}\right\rangle + \left\langle\Psi_{nl}\left|V_UG'_{nl}\frac{\mathbf{p}^4}{m_r^3}\right|\Psi_{nl}\right\rangle + \left\langle\Psi_{nl}\left|\frac{\mathbf{p}^4}{m_r^3}G'_{nl}V_UG'_{nl}V_U\right|\Psi_{nl}\right\rangle \\ & - \left\langle\Psi_{nl}|V_U|\Psi_{nl}\right\rangle\cdot\left\langle\Psi_{nl}\left|\frac{\mathbf{p}^4}{m_r^3}G'_{nl}G'_{nl}V_U\right|\Psi_{nl}\right\rangle + \left\langle\Psi_{nl}\left|V_UG'_{nl}V_UG'_{nl}\frac{\mathbf{p}^4}{m_r^3}\right|\Psi_{nl}\right\rangle \\ & - \left\langle\Psi_{nl}|V_U|\Psi_{nl}\right\rangle\cdot\left\langle\Psi_{nl}\left|V_UG'_{nl}G'_{nl}\frac{\mathbf{p}^4}{m_r^3}\right|\Psi_{nl}\right\rangle + \left\langle\Psi_{nl}\left|V_UG'_{nl}\frac{\mathbf{p}^4}{m_r^3}G'_{nl}V_U\right|\Psi_{nl}\right\rangle \\ & - \left\langle\Psi_{nl}\left|\frac{\mathbf{p}^4}{m_r^3}\right|\Psi_{nl}\right\rangle\cdot\left\langle\Psi_{nl}|V_UG'_{nl}G'_{nl}V_U|\Psi_{nl}\right\rangle. \end{aligned} \quad \text{(C1)}$$

To find ΔE_4 one can consider a solution of the Coulomb-Uehling problem,

$$\left(\frac{\mathbf{p}^2}{2m_r} + V_C + V_U\right)|\Psi_{nl}^{CU}\rangle = E_{CU}|\Psi_{nl}^{CU}\rangle. \quad \text{(C2)}$$

The energy and wave function can be presented in terms of series

$$E_{CU} = E^{(0)} + \alpha E^{(1)} + \alpha^2 E^{(2)} + \cdots \quad \text{(C3)}$$

and

$$\Psi_{nl}^{CU} = \Psi_{nl}^{(0)} + \alpha\Psi_{nl}^{(1)} + \alpha^2\Psi_{nl}^{(2)} + \cdots. \quad \text{(C4)}$$

Indeed, we can find E_{CU} and Ψ_{nl}^{CU} only using a perturbation theory with the related leading terms that are the result of solving a pure Coulomb problem.

Meantime, we note that ΔE_4 has in these terms a simple form,

$$\Delta E_4 = \langle\Psi_{nl}^{CU}\left|\frac{\mathbf{p}^4}{m_r^3}\right|\Psi_{nl}^{CU}\rangle, \quad \text{(C5)}$$

which after applying identity (C2) can be rewritten as

$$\Delta E_4 = 4\left\langle\Psi_{nl}^{CU}\left|\frac{(E_{CU} - V_C - V_U)^2}{m_r}\right|\Psi_{nl}^{CU}\right\rangle. \quad \text{(C6)}$$

To obtain ΔE_4, one still has to apply the perturbative expressions for E_{CU} and Ψ_{nl}^{CU}, however, the further evaluation does not include any derivatives anymore.

A calculation of contributions of $A\mathbf{p}^2$ can be done similarly.

APPENDIX D: LEADING CONTRIBUTION OF THE HADRONIC VACUUM POLARIZATION

In the leading order the hadronic vacuum polarization contribution (see Fig. 7) is determined by the value of polarizability at zero momentum transfer,

$$\left.\frac{\mathcal{P}_{\rm hadr}(-\mathbf{k}^2)}{-\mathbf{k}^4}\right|_{\mathbf{k}^2=0} = -\int_{(2m_\pi)^2}^{\infty} ds\,\frac{\rho_{\rm hadr}(s)}{s},$$

where the dispersion density function can be directly obtained from experiment by measuring, e.g., the cross section of e^+e^- annihilation into hadrons. The leading contribution has roughly order $\alpha(Z\alpha)^4m$, but it is additionally suppressed by a factor $4m_\mu^2/m_\rho^2$. It was calculated previously for a number of occasions [13] (see [58,60,61] for details). Here we recalculate it.

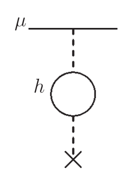

FIG. 7. The leading hadronic VP contribution to the Lamb shift.

For the calculation we use a model of the dispersion function applied in [62]. Indeed, we have to update parameters [63] for the hadronic resonances. Following [62], we estimate the uncertainty at the level of few percents. The result for the leading hadronic vacuum polarization contribution for the ns state is

$$\Delta E^{(0)}_{\rm hadr}(nl) = -0.169(16)\frac{\alpha}{\pi}\frac{(Z\alpha)^4}{n^3}\frac{m_r^3}{m_\mu^2}\delta_{l0},$$

which corresponds to 0.0106(11) meV for the $2p_{1/2} - 2s_{1/2}$ splitting in muonic hydrogen, which is consistent with the previous calculations [58,60].

The dominant contribution, which is roughly two thirds of the total one, comes from the pion contributions, which is sufficiently described by the ρ meson (see, e.g. [64]). The situation is very similar to that for the hadronic contribution to the anomalous magnetic moment of a muon and for the muonium hyperfine splitting.

If necessary, the leading term can be calculated with accuracy comparable with that for the anomalous magnetic moment of muon (see, e.g., [53,65]) or the muonium hyperfine interval (see, e.g., [66]). That should provide uncertainty below 1%. However, to calculate higher-order effects, related to diagrams in Figs. 6(d) and 6(e), the model considered here is sufficient.

APPENDIX E: MUON ELECTRIC FORM FACTOR WITH INSERTION OF THE HADRONIC VACUUM POLARIZATION

The insertion of the hadronic vacuum polarization into the muon vertex [see Fig. 6(e)] on the mass shell affects both Dirac (F_1) and Pauli (F_2) form factors. Those induce the contributions to the energy. The former is determined by the slope of the Dirac form factor $\partial F_1(q^2)/\partial q^2$ at $q = 0$, and the latter is determined by the value $F_2(0)$, which is the anomalous magnetic moment of the muon.

While we agree with [58] on the calculation of the F_2 contribution, we do not agree on the F_1 contribution. Any vacuum polarization contribution into the slope can be described by integrating the Dirac form factor with a nonzero photon mass $\sqrt{s}$ with a dispersion density function. The slope is of the form [cf. (11.3.25) in [67]]

$$\partial_{q^2} F_1(q^2)|_{q^2=0} = \frac{\alpha}{2\pi}\frac{1}{m^2}\int_0^1 dz\left(\frac{1-z^3}{3D} + \frac{(1-z)^3(1-4z+z^2)}{6D^2}\right), \quad \text{(E1)}$$

where

$$D = (1-z)^2 + z\frac{s}{m^2}. \quad \text{(E2)}$$

This expression does not agree with [58]. Actually in each reference of [58] a different expression for the slope is presented and ours does not agree with any of them.

To check (E1) and alternative expressions from [58] we performed several tests. First, we reproduced the well-known infrared logarithm in the Dirac form factor with $s \to 0$. Only one of three expressions in [58] reproduced it. Next, we considered a contribution of insertion of the muon VP into the muon vertex. It is indeed well known and we reproduced the known result [68] from (E1), but not from the expressions in [58]. Our expression (E1) is consistent with (11.3.25) in [67].

After those checks we calculated the contribution into the slope of the Dirac form factor from diagrams in Fig. 6(e) using the model of the hadronic VP density presented in Appendix D. Our result is presented in Table VIII. It disagrees with results published in [58] as well as with those obtained by us from their expressions for the slope of the Dirac form factor. We believe we have performed a sufficient number of tests to rely on our results.

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