Simple model of pointlike spacetime defects and implications for photon propagation

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A model in which pointlike defects are randomly embedded in Minkowski spacetime is considered. The distribution of spacetime defects is constructed to be Lorentz invariant. Since it is based on a sprinkling process, it does not introduce a preferred reference frame. A field-theoretic action for the photon and a fermion is set up, in which the photon is assumed not to couple to the defects directly, but via a scalar field. We are interested in signs for Lorentz violation caused by the spacetime defects, which are expected to reveal themselves in the photon sector. A modification of the photon dispersion relation may result as a quantum effect, and we compute it at leading order perturbation theory. The outcome of the calculation is that the photon dispersion law remains conventional, if the defect distribution is dense, homogeneous, and isotropic. This result sheds some new light on Lorentz violation in the framework of a small-scale structure of spacetime. It shows that Lorentz invariance can be preserved even in the presence of a spacetime structure that is supposed to emerge at the Planck scale. This conclusion has already been drawn on general grounds in other publications, where the current paper delivers a demonstration by a direct computation in a simple model.

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I. INTRODUCTION

Physics at the Planck scale is considered to be the terra incognita in present fundamental research. Up to now it has not been rigorously shown which physical phenomena occur at the Planck scale and how they can be described within a mathematical formalism. For this reason it makes sense to construct simple models in order to mimic effects expected to occur at this scale. The property "simple" means using underlying concepts of established theories such as general relativity and quantum theory, which are well understood and hold for energies much smaller than the Planck energy.

One fundamental concept is Einstein's field equations in general relativity linking energy density to spacetime geometry. Another one is Heisenberg's uncertainty principle in quantum theory saying that two complementary particle properties are always endowed with uncertainties as long as their quantum theoretical operators do not commute. Assuming that both still hold at the Planck scale, energy uncertainty may result in an uncertainty of spacetime geometry or even topology. Thus, under this assumption, spacetime metric coefficients and spacetime curvature will begin to fluctuate at the Planck energy. Such fluctuations are often referred to as spacetime foam or spacetime defects [1–6].

There exist various models for spacetime defects with nontrivial topology. One way in which to obtain a spacetime defect is to cut out an open set of Minkowski spacetime and to impose certain conditions on the remaining boundary. This then leads to a spacetime $\mathcal{M}$ having a different topology. For example, cutting out an open ball of $\mathbb{R}^3$ and identifying antipodal points on the boundary results in $\mathcal{M} = \mathbb{R} \times (\mathbb{R}P^3 - \{\text{point}\})$, where $\mathbb{R}P^3$ is the three-dimensional orientable real projective space [7].

However, the interest not only lies in the defects themselves but also in their influence upon the propagation properties of particles, for example photons. The higher the energy of a photon the smaller its de Broglie wavelength, and the better it can probe the microscopic spacetime structure. This may lead to a modification of the photon dispersion relation, which is an indication of Lorentz invariance violation. Photons are interesting from both an experimental and a theoretical reason. From the experimental point of view very precise and clean experiments are performed in the search for Lorentz violation (see, e.g., [8–12] and references therein). From the theoretical point of view electromagnetism is an Abelian $U(1)$ gauge theory, which is much simpler than the non-Abelian theories the weak and the strong interactions are based on.

In [13], the modification of the photon dispersion law is investigated for certain classes of defects. The method used is to consider the scattering of an electromagnetic wave at one single defect. Certain conditions for the physical fields are then set on the boundary of the defect, and Maxwell's equations are solved by introducing a correction field. Even for a single defect this is a difficult task, and the approach would be more challenging for two defects and impractical for a large number of defects nearby.

Since we are interested in the propagation of photons through a spacetime foam made up of many defects, we proceed with an alternative possibility that was initiated in [14]. Here a CPT anomaly [15,16] is found for a non-Abelian gauge group $SO(10)$ with a chiral representation of left-handed Weyl fermions on two spacetime manifolds with nontrivial topology: a spacetime with a linear defect in its spatial part, $\mathcal{M} = \mathbb{R} \times (\mathbb{R} \times (\mathbb{R}^2 \setminus \{0\})) = \mathbb{R}^4 \setminus \mathbb{R}^2$,

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and a spacetime with two identical open balls removed from $\mathbb{R}^3$ and points on their boundary properly identified (wormhole). This anomaly arises as a topological Chern-Simons term in the effective action of the non-Abelian gauge bosons. It is shown that the CPT anomaly occurs for the Abelian subgroup $U(1) \subset SO(10)$ as well. It gives a contribution to the effective action of the Abelian gauge field as an $F\tilde{F}$ term with the field strength tensor F and its dual $\tilde{F}$. This term still contains characteristics of the original nontrivial manifold $\mathcal{M}$. It is assumed that spacetime at microscopic length scales can be modeled by such defects. However, since it is tremendously difficult to obtain the effective action for several defects of this kind, an accumulation of many defects is described by a background field. This field does not include any microscopic defect properties and, hence, serves as an effective approach for the case when the photon wavelength is much larger than the defect size.

We will follow this idea and describe a single defect as pointlike, where it is assumed to be time dependent—contrary to Ref. [14]. Such defects are distributed randomly in Minkowski spacetime resulting in an effective "random" background field. Furthermore, the distribution of defects is taken as being Lorentz invariant. We study whether and how the dispersion relation of photons is affected by such a time-dependent and Lorentz-invariant background.

The outline of this paper is as follows. In Sec. II we introduce the action of the effective theory that forms the basis of the article. This action describes the interaction between photons and the defects that are mediated via a real scalar field. Section III gives a description of how to treat photon propagation through a distribution of many pointlike defects that are put randomly at distinct points in Minkowski spacetime. In Sec. IV the focus will be on the perturbative solution of the photon field equation. In the first part the solution is obtained by inserting a perturbative *ansatz* into the field equation. In the second part we demonstrate how the same result follows from an approach using Feynman diagrams. As a next step we compute the leading-order solution of the photon field equation in Sec. V, where in this context a renormalization procedure has to be performed. In the follow-up section, VI, the scalar field equation is solved to leading order in perturbation theory as well. Both results are combined to calculate the modified photon dispersion relation in Sec. VII, and the physical meaning of the result is then discussed. Throughout the paper certain assumptions will be made so that the calculation is feasible. In Sec. VIII we make a couple of remarks on how the result may change if certain assumptions are dropped. In the penultimate section, IX, we go on a brief excursion to PT-symmetric quantum field theory in the context of the special model proposed. The last section, X, gives a summary and a conclusion on the results. Here we will also compare our model to alternatives found in the literature. The most important computational steps are recapped in Appendixes A, B, C, and D.

Throughout the paper natural units are used with $\hbar = c = 1$. For some occurrences, $\hbar$ and c will be reinstated for clarity.

II. ACTION OF THE EFFECTIVE THEORY

We wish to describe photon propagation through a Lorentz-invariant distribution of time-dependent, pointlike spacetime defects. By a Lorentz-invariant distribution we mean that certain properties of this distribution are Lorentz invariant. The spacetime coordinate of a single defect is affected by a Lorentz transformation in the standard way (see the discussion in Sec. III A below). Generalizing the result of [14] to a Lorentz-invariant distribution of pointlike defects, the photon field turns out to be described by the modified action

$$S = \int_{\mathbb{R}^4} \mathrm{d}^4x \left[-\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) - \frac{1}{4} g(x) F_{\mu\nu}(x) \tilde{F}^{\mu\nu}(x) \right],$$

$$\tilde{F}^{\mu\nu}(x) \equiv \frac{1}{2} \varepsilon^{\mu\nu\varrho\sigma} F_{\varrho\sigma}(x), \tag{2.1}$$

where $F_{\mu\nu}(x) \equiv \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$ is the field strength tensor of the $U(1)$ gauge field $A_\mu(x)$, $\tilde{F}^{\mu\nu}(x)$ is the dual field strength tensor, and $\varepsilon^{\mu\nu\varrho\sigma}$ is the four-dimensional Levi-Cività tensor. All fields are defined on Minkowski spacetime with metric $g_{\mu\nu}(x) = \eta_{\mu\nu} \equiv \mathrm{diag}(1, -1, -1, -1)$. For the background field $g(x)$ the following *ansatz* is used:

$$g(x) = \lambda \sum_{n=1}^{\mathcal{N}} \varepsilon_n h(x - x_n). \tag{2.2}$$

The sum runs over the total number $\mathcal{N}$ of spacetime defects with the "charge" $\varepsilon = \pm 1$ of an individual defect. The contribution of each defect at the spacetime point x_n is described by $h(x - x_n)$. Note that contrary to [14], $g(x)$ and $h(x)$ now depend on the spacetime coordinate x and not only on the spatial coordinate $\mathbf{x}$. In the latter reference it was possible to derive a modified photon dispersion law by stating some general properties of $h(\mathbf{x})$ and investigating its statistical characteristics. However, unlike for static defects, it is hard to draw conclusions from the general action stated above without specifying the function $h(x)$. The reason is that in the four-dimensional case the possible pole structure of $h(x)$ is crucial.

We resolve this issue by introducing a specific model where the background field does not couple to the photons directly. Instead we use a scalar field $\phi(x)$, which we couple to both the photon and the defects. We describe the latter by four-dimensional pointlike punctures in spacetime. Later on it will turn out that the free solutions of the respective field equations are characterized by functions similar to Eq. (2.2) where $h(x)$ is related to the propagator of the scalar field $\phi(x)$. The advantage is then that we will know about the pole structure of the solutions enabling us to solve the field equations of the interacting theory by applying a perturbative procedure.

Therefore we set up an action that describes the photons, the defects, and the interaction between both sectors as follows [17]:

$$S_{\text{eff}} = \int_{\mathbb{R}^4} \mathrm{d}^4x \Big[-\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) - \frac{1}{2}(\partial_\mu A^\mu(x))^2 + \frac{1}{2(b^{(0)})^2}\Big(\partial_\mu \phi(x) \partial^\mu \phi(x) - \frac{1}{(b^{(0)})^2}\phi(x)^2\Big) + \phi(x) \sum_{n=1}^{\mathcal{N}} \varepsilon_n \delta^{(4)}(x - x_n) - \frac{\lambda^{(0)}}{4} f(\phi(x)) F_{\mu\nu}(x) \tilde{F}^{\mu\nu}(x)\Big]. \tag{2.3}$$

The first term in the action is the standard kinetic term of the free photon field $A_\mu(x)$, and the second fixes the gauge (we use the Feynman gauge). The third contribution contains the kinetic and mass terms of the free real scalar field $\phi(x)$, where $b^{(0)}$ is a parameter with mass dimension -1. The scalar field $\phi(x)$ itself has mass dimension zero, and its bare "mass" is given by $1/b^{(0)}$.

The fourth expression involves $\mathcal{N}$ pointlike spacetime defects sitting at distinct spacetime points x_n. The defects are effectively described as randomly distributed single four-dimensional δ functions, each carrying a random, uniformly distributed "charge" $\varepsilon = \pm 1$. Concretely, this means that each defect appears at a single point in three-space for an infinitesimally short amount of time before disappearing again. This illustrative picture corresponds to what a theorist may have in mind when thinking about a simple spacetime foam. Such a form does not describe the nontrivial spacetime topology[1] (or even topology change) that is supposed to occur at the Planck scale. Hence we assume that for sub-Planckian photons this nontrivial topology is not visible. Besides, only the field $\phi(x)$ is assumed to couple directly to the defects via the charge ε.

Finally the interaction of the photon field with the defects is described by the last term, where the interaction is mediated by the scalar field $\phi(x)$. The latter term involves the dual field strength tensor $\tilde{F}^{\mu\nu}(x)$. The motivation behind such an interaction is that it appears in the effective action in the context of the *CPT* anomaly [14]. The function $f(\phi(x))$ in the last term of Eq. (2.3) can be arbitrary, in principle, but it is assumed to be sufficiently well behaved. To keep the model simple we choose $f(\phi(x)) = \phi(x)$. Note that the fifth and sixth terms explicitly break gauge invariance.[2] To summarize, the field content of the theory is presented in Table I.

[1]Minkowski spacetime endowed with pointlike defects is topologically trivial in the sense that all closed curves can be shrunk to points. This renders the first homotopy group trivial. However, e.g., two-dimensional spheres cannot necessarily be mapped to points. Hence higher homotopy groups may not be trivial.

[2]They are not invariant under a gauge transformation of the field ϕ, namely $\phi(x) \mapsto -\phi(x)$. To be crystal clear, this gauge transformation has nothing to do with the $U(1)$ gauge transformation of the photon field.

TABLE I. Fields and nondynamical objects (defects) appearing in the action of Eq. (2.3) with corresponding mass dimension, bare coupling constant $\lambda^{(0)} \ll 1$, and $\varepsilon = \pm 1$.

Field/object	Mass dimension	Coupling constant (charge) to A_μ	ϕ	Defect
Photon A_μ	1	0	$\lambda^{(0)}$	0
Scalar ϕ	0		0	ε
Defect	4			0

Since we later want to couple photons to a conserved[3] fermionic current $j^\mu(x) = \bar{\psi}(x)\gamma^\mu \psi(x)$ with the standard Dirac field $\psi(x)$ and Dirac matrices γ^μ, the modified theory will be coupled to the Dirac theory of standard spin-1/2 fermions with charge e and mass m_f:

$$S_{\text{Dirac}} = \int_{\mathbb{R}^4} \mathrm{d}^4x \bar{\psi}(x)\{\gamma^\mu[\mathrm{i}\partial_\mu - eA_\mu(x)] - m_f\}\psi(x). \tag{2.4}$$

This makes the complete action of the theory

$$S = S_{\text{eff}} + S_{\text{Dirac}}, \tag{2.5}$$

with S_{eff} given by Eq. (2.3) and S_{Dirac} by Eq. (2.4). The description of the spacetime foam model by the action given is the fundamental assumption of this paper. It will be referred to as Assumption (1).[4]

III. STATISTICAL TREATMENT OF A LARGE NUMBER OF SPACETIME DEFECTS

A. Distribution of defects in Minkowski spacetime (sprinkling)

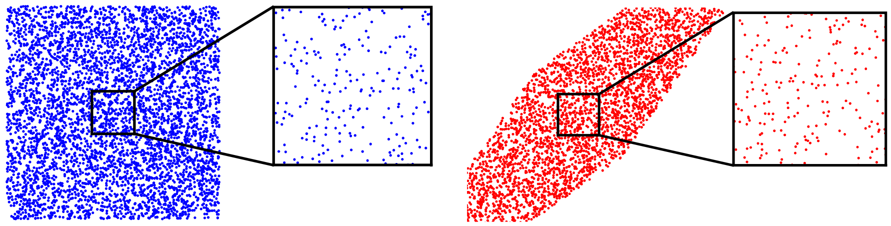
We intend to distribute spacetime defects in four-dimensional Minkowski spacetime $\mathcal{M}$ in a Lorentz-invariant manner. This will be possible if defects are distributed according to a "Poisson process" (i.e., a sprinkling). The result of the Poisson process is a Poisson distribution of defects throughout the spacetime. This means that the probability of observing n defects in a rectangular spacetime region with side length $\mathcal{R}$ and *spacetime volume*

$$\mathcal{V} = \int_{\text{region}} \mathrm{d}^4x \sqrt{-\det(\eta_{\mu\nu})} = \int_{\substack{|x^\mu| \le \mathcal{R}/2 \\ \mu \in \{0,1,2,3\}}} \mathrm{d}^4x = \mathcal{R}^4 \tag{3.1}$$

is given by

[3]By quantum corrections the explicit violation of gauge invariance in the action S_{eff} of Eq. (2.3) may give rise to an anomalous nonconserved current. However, this effect (if it exists at all) is expected to be suppressed by $(\lambda^{(0)})^2$.

[4]In what follows, several further assumptions will be taken. In such a context the word "assumption" is abbreviated as "Asmp." in combination with a number and optional lower case Latin characters.



(a) (b)

FIG. 1 (color online). Example of sprinkling in a finite region of a two-dimensional spacetime (time $\times$ 1D space) as it looks in two different inertial frames. Since the defects are instantaneous, they are illustrated as points. The second frame (b) is boosted along the positive spatial axis with a Lorentz boost factor $\beta = 0.7$ with respect to the first frame (a). This boost changes the shape of the spacetime region. However, the mean density $\langle \varrho_{\mathrm{obs}} \rangle$ of defects is the same in both frames. An illustration compares the two enlarged regions of both distributions. Note that in the boosted distribution not all defects are shown.

$$P_n(\mathcal{V}) = \frac{(\varrho \mathcal{V})^n \exp(-\varrho \mathcal{V})}{n!}. \tag{3.2}$$

Herein, ϱ is at first a parameter that characterizes the distribution. Note that Eq. (3.2) is a valid description for a probability distribution, which is both isotropic and homogeneous. The most natural choice for a distribution of pointlike—i.e., zero-dimensional—spacetime defects is an isotropic one if we do not take into account any mechanism producing defects that make space anisotropic (for example, defects similar to cosmic strings [18]). When the volume of the region approaches zero ($\mathcal{V} \mapsto \delta \mathcal{V}$ with an infinitesimal value $\delta \mathcal{V}$), the probability of finding a single defect in that region is proportional to the volume

$$P_{n=1}(\delta \mathcal{V}) = \varrho \delta \mathcal{V} + \mathcal{O}(\delta \mathcal{V}^2). \tag{3.3}$$

On the other hand, the probability of finding more than one defect is negligible:

$$P_{n>1}(\delta \mathcal{V}) = \mathcal{O}(\delta \mathcal{V}^n). \tag{3.4}$$

An explicit realization of the Poisson process is then described by the following steps [19]:

(1) Divide $\mathcal{V}$ into small boxes with *spacetime volume* ΔV.
(2) Then place a defect into each box with probability $P = \varrho \Delta V$.
(3) The Poisson process is obtained in the limit $\Delta V \mapsto 0$.

The Poisson process or "sprinkling" is invariant under any volume-preserving linear transformation, and in particular it is invariant under Lorentz transformations (this happens because the process only depends on the spacetime volume). Moreover, it has been shown in [20] that the realizations of the Poisson process are Lorentz-invariant individually as well. The Lorentz invariance in this context has the following meaning [19]: *The discrete set of sprinkled points must not, in and of itself, serve to pick out a preferred reference frame.*

That is, the statistical properties of the distribution of defects (e.g., the mean density of defects and the property of being isotropic and homogeneous) do not depend on which reference frame we choose to measure them in (see Fig. 1).

We emphasize that the number of defects contained in different regions of equal (generic) volume V is not constant but fluctuates from region to region,

$$N(V) = \langle N(V) \rangle \pm \delta N(V), \tag{3.5}$$

where the mean number of defects and the standard deviation are, respectively,

$$\langle N(V) \rangle = \sum_{n=0}^{\infty} n P_n(V) = \varrho V, \tag{3.6a}$$

$$\begin{aligned} \delta N(V) &= \sqrt{\sum_{n=0}^{\infty} [n - \langle N(V) \rangle]^2 P_n(V)} \\ &= \sqrt{\langle N(V)^2 \rangle - \langle N(V) \rangle^2} = \sqrt{\varrho V}. \end{aligned} \tag{3.6b}$$

These results can be explicitly obtained by using Eq. (3.2). Therefore we can identify the parameter ϱ with the mean density of defects: $\langle \varrho_{\mathrm{obs}} \rangle = \varrho$. Moreover fluctuations in the number of defects also imply fluctuations in the density ϱ_{obs} from a region to another

$$\varrho_{\mathrm{obs}} = \langle \varrho_{\mathrm{obs}} \rangle \pm \delta \varrho = \frac{\langle N(V) \rangle}{V} \pm \frac{\delta N(V)}{V} = \varrho \pm \sqrt{\frac{\varrho}{V}}. \tag{3.7}$$

Nevertheless, these fluctuations become negligible when the mean volume occupied by a single defect, namely $V_d = 1/\varrho$, is much smaller than the volume V of the region considered,

$$V_d = \frac{1}{\varrho} \ll V \Rightarrow \frac{\delta \varrho}{\varrho} = \frac{1}{\sqrt{\varrho V}} \sim 0. \tag{3.8}$$

Thus we can regard the density to be constant as long as we consider scales that are much larger than the mean separation between defects: $N(V) = \varrho V = \mathrm{const}$ when $V \gg V_d$. Being able to perform computations with a globally constant volume means that the distribution is homogeneous as well. Otherwise the density ϱ would only be defined locally. We will refer to the isotropic and homogeneous distributions of spacetime defects as Asmp. (2a) and Asmp. (2b), respectively.

B. Derivation of auxiliary functions

In this section we intend to derive a set of functions that will be extensively needed for the solution of the scalar and photon field equations. First of all, we investigate the problem of photon propagation through a spacetime with $\mathcal{N}$ defects in the finite subset with volume $\mathcal{V}$ given by Eq. (3.1). The corresponding functions and fields are denoted with indices "$\mathcal{N}$"or "$\mathcal{R}$."We now consider the "free" field equation for ϕ, i.e., neglecting the coupling to the photon but not to the defects. To solve the field equation, the truncated Fourier transform of $\phi(x)$ when restricted to the spacetime region ("box") is necessary:

$$\tilde{g}_{\mathcal{N}}(k) \equiv \int_{\substack{|x^\mu|\leq\mathcal{R}/2 \\ \mu\in\{0,1,2,3\}}} \mathrm{d}^4x \exp(\mathrm{i}k\cdot x)\phi(x). \tag{3.9}$$

Analogously, we define the inverse Fourier transform by

$$\phi(x) = \int_{\substack{|k^\mu|\geq 1/\mathcal{R} \\ \mu\in\{0,1,2,3\}}} \frac{\mathrm{d}^4k}{(2\pi)^4} \exp(\mathrm{i}k\cdot x)\tilde{g}_{\mathcal{N}}(k). \tag{3.10}$$

In configuration space the coordinates are restricted by the side length of the box considered. This corresponds to a minimum value in momentum space, which is manifest in the integration limits. In principle, for a finite volume the Fourier transform would correspond to a Fourier series with discrete coordinates or momenta. However, for simplicity we will assume a continuous spectrum. This does not play any role as we will eventually generalize the results to the whole of Minkowski spacetime anyway. We use Eq. (3.10) as an *ansatz* to obtain the following solution of the field equation of $\phi(x)$ in momentum space:

$$\tilde{g}_{\mathcal{N}}(k) = \sqrt{\mathcal{N}}\tilde{H}(k)\tilde{G}_{\mathcal{N}}(k), \tag{3.11a}$$

$$\tilde{H}(k) \equiv \frac{-(b^{(0)})^2}{k^2 - 1/(b^{(0)})^2 + \mathrm{i}\epsilon},$$

$$\tilde{G}_{\mathcal{N}}(k) \equiv \frac{1}{\sqrt{\mathcal{N}}}\sum_{i=1}^{\mathcal{N}} \varepsilon_i \exp(\mathrm{i}k\cdot x_i). \tag{3.11b}$$

The solution depends on the bare mass of the scalar field and the distribution of spacetime defects. Furthermore it consists of two contributions. The first, $\tilde{G}_{\mathcal{N}}(k)$, solely describes the defects and the second, $\tilde{H}(k)$, serves as a mediator between the defects and the photons. In principle Eq. (3.11a) can be regarded as being a solution of the free field equations restricted to a finite box. Later on we will need to consider the product of two such solutions evaluated at different momenta—especially for the perturbative photon field. The function $\tilde{g}_{\mathcal{N}}(k)$ then serves as an effective background that describes the influence of the spacetime defects on the photons. Therefore we will refer to it as an effective background field in what follows.

Since Eq. (3.11a) gives the background field with respect to the number of defects, we define the corresponding background field with regard to the volume:

$$\tilde{g}_{\mathcal{R}}(k) \equiv \sqrt{\mathcal{V}}\tilde{H}(k)\tilde{G}_{\mathcal{R}}(k),$$

$$\tilde{G}_{\mathcal{R}}(k) \equiv \frac{1}{\sqrt{\mathcal{V}}}\sum_{i=1}^{\mathcal{N}} \varepsilon_i \exp(\mathrm{i}k\cdot x_i). \tag{3.12}$$

We keep in mind that ϕ couples to the photon via the last term of Eq. (2.3). After establishing all intermediate results we compute the limits $\mathcal{N} \mapsto \infty$ and $\mathcal{R} \mapsto \infty$, respectively.

In the subsequent paragraphs we are going to perform a statistical treatment of photon propagation through the background field. In light of this, we will encounter products of functions defined by Eqs. (3.11a) and (3.11b) [or Eq. (3.12)], where each depends on a different four-momentum. Such products will have to be summed over the total number $\mathcal{N}$ of defects distributed in the whole of Minkowski spacetime.

According to Eq. (3.2) the defect density does not change with respect to Lorentz transformations. We are then ready to compute the sum of complex exponential functions (random phases), each evaluated at the spacetime point of a defect. The corresponding result will be needed later:

$$\sum_{n=1}^{\mathcal{N}} \exp(\mathrm{i}k\cdot x_n) \simeq \varrho \int \mathrm{d}^4x \exp(\mathrm{i}k\cdot x) = (2\pi)^4\varrho\delta^{(4)}(k). \tag{3.13}$$

In the second step we have approximated the sum over n as an integral over x:

$$\sum_{n=1}^{\mathcal{N}} \mapsto \int \mathrm{d}n = \varrho \int \mathrm{d}^4x. \tag{3.14}$$

This is possible since the sprinkling procedure ensures the proportionality between the number of defects and the volume: $\mathrm{d}n = \varrho \mathrm{d}^4x$. Furthermore the defect distribution is assumed to be *dense* (i.e., the mean separation between defects is much smaller than the wavelength of the photons considered, $\varrho^{-1/4} \ll \lambda$).

The latter is an important issue not only for calculational but for physical reasons as well. In Ref. [13] a classical spacetime foam with topologically nontrivial defects having a particular size $\bar{b}$ and a mean separation $\bar{l}$ is considered. Bounds obtained from the absence of vacuum Cherenkov radiation lead to the constraint $\bar{b}/\bar{l} \lesssim 10^{-7}$ within the spacetime foam model examined in this reference. Hence, for spacetime defects of the size $10^2 \times L_{\mathrm{Pl}}$ with the Planck length $L_{\mathrm{Pl}} \equiv \sqrt{G\hbar/c^3} \approx 1.62 \times 10^{-35}$m, where a classical approach is supposed to be valid, the defects would be separated by at least $10^9 \times L_{\mathrm{Pl}} \approx 1.62 \times 10^{-26}$ m. Even if their separation is larger by several orders of magnitude, the approximation of a dense distribution still makes sense. We assume that this conclusion can be applied to the spacetime foam model with pointlike defects investigated here. This is fortified by Table VI in Appendix A. The dense distribution of defects will be referred to as Asmp. (2c).

The last assumption in Eq. (3.13), which is taken to evaluate the remaining integral, is an infinite spacetime volume. Current cosmological data implies that we live in a flat universe of finite age. For the curvature radius of the universe, which is related to its size, only upper bounds can be given within the ΛCDM model [21]. Therefore it has not as yet been clarified whether the universe has a finite or infinite volume. However considering photon propagation on time scales that are much smaller than cosmological time scales, the Friedmann-Robertson-Walker metric for a flat universe corresponds to the Minkowski metric to a reasonable approximation. To keep the model as simple as possible, we do not describe cosmological effects and assume the spacetime volume to be infinite. In what follows we will refer to the infinite spacetime volume as Asmp. (2d).

After writing the sum in Eq. (3.13) as an infinite integral over spacetime, the result does not depend on the total number $\mathcal{N}$ of defects any more but only on the density ϱ. The latter acts as a constant of proportionality that is independent of the reference frame. Because of this, the final result is Lorentz invariant.

We can now compute the product of two functions $\tilde{G}_{\mathcal{N}}$ in the limit of large $\mathcal{N}$:

$$\lim_{\mathcal{N}\mapsto\infty} \tilde{G}_{\mathcal{N}}(k)\tilde{G}_{\mathcal{N}}(p) = \lim_{\mathcal{N}\mapsto\infty} \frac{1}{\mathcal{N}}\left[\sum_{i=1}^{\mathcal{N}} \exp[\mathrm{i}(k+p)x_i] + \sum_{m\neq n} P_{mn}\right]. \quad (3.15)$$

The quantity P_{mn} involves the product of charges of different defects:

$$P_{mn} = \varepsilon_m\varepsilon_n \exp(\mathrm{i}k\cdot x_m)\exp(\mathrm{i}p\cdot x_n). \quad (3.16)$$

TABLE II. For a large number of defects we consider a single defect at $x_{E,i}$ with a specific charge. Then the probability of finding a neighboring defect at $x_{E,j}$ with a charge of the opposite sign, where $x_{E,j}$ has a small Euclidian distance to $x_{E,i}$, is close to 1.

Defect index	Sign of charge	Defect position
i	± 1	$x_{E,i}$
$j \neq i$	∓ 1	$x_{E,j} = x_{E,i} + \delta x_{E,i}$

The sum over P_{mn} has to be evaluated for a large number of defects. This will be done in the following few lines. We transform coordinates to Euclidian space via a Wick rotation, where these coordinates will be labeled with an index E. Now consider a small hypercube $H_{a_n}(x_{E,n})$ around a defect at $x_{E,n}$ with side length

$$a_n \equiv \left(\frac{\mathcal{V}}{\mathcal{N}}\right)^{1/4} = \left(\frac{1}{\varrho}\right)^{1/4}, \qquad n \in \{1,\ldots,\mathcal{N}\}. \quad (3.17)$$

On average every defect lies within such a hypercube. For $\mathcal{N} \gg 1$, given a defect at x_n with charge $\varepsilon = \pm 1$, a partner with a charge of opposite sign $\varepsilon = \mp 1$ can be found at a distance $\delta x_{E,n}$ that is of the order of the side length a_n of the hypercube introduced (see Table II). We can therefore perform a Taylor expansion for P_{mn} in the small parameters $\delta x_{E,n}$. For the sum over P_{mn} for $m \neq n$ we consider defects at $x_{E,m}$ and $x_{E,n}$ with charges $\varepsilon = 1$ and their neighbors at the distance $\delta x_{E,m}$ and $\delta x_{E,n}$, respectively, with charges $\varepsilon = -1$. We obtain

$$\begin{aligned}\sum_{m\neq n} P_{mn} &= \sum_{m\neq n} \varepsilon_m\varepsilon_n \exp(\mathrm{i}k\cdot x_{E,m})\exp(\mathrm{i}p\cdot x_{E,n}) \\ &= \sum_{m\neq n}\{(+1)^2\exp(\mathrm{i}k\cdot x_{E,m})\exp(\mathrm{i}p\cdot x_{E,n}) + (-1)^2\exp[\mathrm{i}k\cdot(x_{E,m}+\delta x_{E,m})]\exp[\mathrm{i}p\cdot(x_{E,n}+\delta x_{E,n})] \\ &\quad + (+1)\cdot(-1)\exp(\mathrm{i}k\cdot x_{E,m})\exp[\mathrm{i}p\cdot(x_{E,n}+\delta x_{E,n})] + (-1)\cdot(+1)\exp[\mathrm{i}k\cdot(x_{E,m}+\delta x_{E,m})]\exp(\mathrm{i}p\cdot x_{E,n})\} \\ &= \mathcal{O}(\delta x_{E,m}\cdot\delta x_{E,n}). \end{aligned} \quad (3.18)$$

As a result, the linear term vanishes and all further contributions are suppressed by small distances. Thus for a large number of defects in the sum over all P_{mn}, contributions from neighboring defects with opposite charges compensate each other. Then the second term of Eq. (3.15) averages out,

$$\lim_{\mathcal{N}\mapsto\infty} \frac{1}{\mathcal{N}} \sum_{m\neq n} P_{mn} = 0, \quad (3.19)$$

and we obtain the following result for the product of two functions $\tilde{G}_{\mathcal{N}}$, each evaluated at a different momentum:

$$\lim_{\mathcal{N}\mapsto\infty} \tilde{G}_{\mathcal{N}}(k)\tilde{G}_{\mathcal{N}}(p) = \lim_{\mathcal{N}\mapsto\infty} \frac{1}{\mathcal{N}}(2\pi)^4\varrho\,\delta^{(4)}(k+p). \quad (3.20)$$

Furthermore, the functions $\tilde{G}_{\mathcal{R}}(k)$ and $\tilde{g}_{\mathcal{R}}(k)$ defined by Eq. (3.12) obey a similar relation,

$$\lim_{\mathcal{R}\mapsto\infty} \tilde{G}_{\mathcal{R}}(k)\tilde{G}_{\mathcal{R}}(p) = \lim_{\mathcal{V}\mapsto\infty} \frac{1}{\mathcal{V}}(2\pi)^4\varrho\,\delta^{(4)}(k+p). \quad (3.21)$$

The physical interpretation of the result obtained is as follows. If the scalar field ϕ scatters at a defect, it is supposed to either transfer momentum to the defect or absorb momentum from the defect. Averaging over many defects—implying the limits $\mathcal{N}\mapsto\infty$, $\mathcal{V}\mapsto\infty$, or $\mathcal{R}\mapsto\infty$—leads to zero average momentum transfer at each defect. This means momentum conservation for the ϕ field. Averaging over infinitely many randomly distributed defects results in a translation invariant theory that clearly obeys the property of momentum conservation.

The results obtained depend only on the density of defects and (for some) $\mathcal{N}$ or $\mathcal{V}$. Hence these can be generalized to the whole of Minkowski spacetime. As mentioned this corresponds to blowing up the spacetime region of Eq. (3.1), namely to the limit $\mathcal{R} \mapsto \infty$.

IV. PERTURBATIVE SOLUTION OF THE FIELD EQUATIONS

The dispersion relations of both the scalar field and the photon follow from the appropriate field equations that are modified by the presence of the spacetime defects. We would like to set up the modified field equations at a perturbative level. In the first part of the current section we will follow the lines of Ref. [14]. In the second part we will show that the results obtained can be reproduced with the help of the perturbative Feynman rules. These are given in Appendix B, where Eqs. (B1a)–(B1c) can be directly derived from the action (2.3). The Feynman rules (B1d) and (B1e) follow from the results of the previous section.

A. Perturbative *ansatz* for the photon field

To obtain the modification of the photon field originating from its interaction with spacetime defects via the scalar field ϕ we have to solve the modified photon field equation resulting from the action (2.3). In momentum space it can be written as an integral equation,

$$k^2\tilde{A}^{\nu}_{\mathcal{R}}(k) = -\frac{\lambda^{(0)}}{(2\pi)^4}\int_{\substack{|q^{\mu}|\geq 1/\mathcal{R}\\ \mu\in\{0,1,2,3\}}} \mathrm{d}^4q\,\tilde{g}_{\mathcal{R}}(q)\varepsilon^{\mu\nu\varrho\sigma} \times q_{\mu}(k-q)_{\varrho}\tilde{A}_{\mathcal{R},\sigma}(k-q). \tag{4.1}$$

The index $\mathcal{R}$ denotes that the system is at first considered in a finite rectangular region of side length $\mathcal{R}$. The exact solution to the latter equation is out of reach. Therefore we make the following perturbative *ansatz* for the full solution $\tilde{A}^{\nu}_{\mathcal{R}}$ in powers of the bare coupling constant $\lambda^{(0)}$. This is reasonable if we expect a modified photon dispersion law since current experimental bounds on Lorentz symmetry violation—and thus a modified dispersion relation for the photon—are very tight (see [22] and references therein).

$$\tilde{A}^{\,\nu}_{\mathcal{R}} = \tilde{A}^{(0)\nu} + \lambda^{(0)}\tilde{A}^{(1)\nu}_{\mathcal{R}} + (\lambda^{(0)})^2\tilde{A}^{(2)\nu}_{\mathcal{R}} + \cdots. \tag{4.2}$$

Herein $\tilde{A}^{(0)\nu}$ is a solution of the free-field equation

$$k^2\tilde{A}^{(0)\nu}(k) = 0. \tag{4.3}$$

By successively inserting the power expansion of Eq. (4.2) in Eq. (4.1), we obtain a perturbative expansion of the exact solution. Now the first step is to insert $\tilde{A}^{(0)}$. We identify each perturbative order remembering that $k^2\tilde{A}^{(0)\nu} = 0$. Using the definition

$$\tilde{\Delta} \equiv \frac{1}{k^2 + \mathrm{i}\epsilon}, \tag{4.4}$$

with an infinitesimal real parameter ϵ to avoid the pole at $k^2 = 0$, we find the following first order perturbative solution, where we now take the limit $\mathcal{R} \mapsto \infty$:

$$\lambda^{(0)}\tilde{A}^{(1)\nu}(k) = \lim_{\mathcal{R}\mapsto\infty} -\frac{\lambda^{(0)}}{(2\pi)^4}\tilde{\Delta}(k)\int \mathrm{d}^4q\,\tilde{g}_{\mathcal{R}}(q)\varepsilon^{\mu\nu\varrho\sigma} \times q_{\mu}(k-q)_{\varrho}\tilde{A}^{(0)}_{\sigma}(k-q). \tag{4.5}$$

The first order photon field correction vanishes in the limit considered. This is clear from Eq. (3.21) since $\tilde{g}_{\mathcal{R}\mapsto\infty}(q)$ does not come together with a second background field, thus only producing a contribution for $q = 0$.

The second order solution of the photon field equation reads as follows:

$$\begin{aligned}(\lambda^{(0)})^2\tilde{A}^{(2)\nu}(k) &= \lim_{\mathcal{R}\mapsto\infty} -\frac{\lambda^{(0)}}{(2\pi)^4}\tilde{\Delta}(k)\int \mathrm{d}^4q\,\tilde{g}_{\mathcal{R}}(q)\varepsilon^{\mu\nu\varrho\sigma}q_{\mu}(k-q)_{\varrho}(\lambda^{(0)}\tilde{A}^{(1)}_{\mathcal{R},\sigma}(k-q))\\ &= \lim_{\mathcal{R}\mapsto\infty}\frac{(\lambda^{(0)})^2}{(2\pi)^8}\tilde{\Delta}(k)\int \mathrm{d}^4q\,\tilde{g}_{\mathcal{R}}(q)\varepsilon^{\mu\nu\varrho\sigma}q_{\mu}(k-q)_{\varrho}\tilde{\Delta}(k-q)\int \mathrm{d}^4p\,\tilde{g}_{\mathcal{R}}(k-p-q)\varepsilon_{\alpha\beta\gamma\sigma}(k-p-q)^{\alpha}p^{\beta}\tilde{A}^{(0)\gamma}(p).\end{aligned} \tag{4.6}$$

We contract the Levi-Cività tensors, perform the limit $\mathcal{R} \mapsto \infty$, and use Eq. (3.21) to simplify the latter result, which finally leads to

$$(\lambda^{(0)})^2\tilde{A}^{(2)\nu}(k) = -\mathcal{C}^{(0)}\tilde{\Delta}(k)B^{\nu}{}_{\gamma}(k)\tilde{A}^{(0)\gamma}(k),\quad \mathcal{C}^{(0)} \equiv (b^{(0)})^4(\lambda^{(0)})^2\varrho, \tag{4.7a}$$

$$B^{\nu}{}_{\gamma}(k) = \int\frac{\mathrm{d}^4q}{(2\pi)^4}\left[\frac{1}{(b^{(0)})^2}\tilde{H}(q)\right]^2 \times \frac{1}{(k-q)^2 + \mathrm{i}\epsilon}K^{\nu}{}_{\gamma}, \tag{4.7b}$$

with

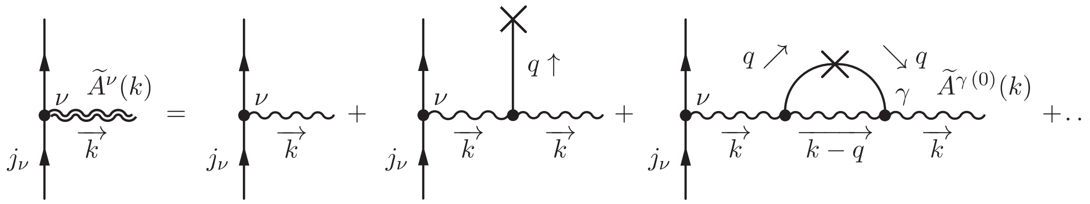
$$K^{\nu}{}_{\gamma} = \delta^{\mu}_{[\alpha}\delta^{\nu}_{\beta}\delta^{\varrho}_{\gamma]}q_{\mu}(k-q)_{\varrho}q^{\alpha}k^{\beta}. \tag{4.7c}$$

In Eq. (4.7c), $[\alpha, \beta, \gamma]$ denotes a totally antisymmetric permutation of the indices α, β, and γ. The tensor $K^{\nu}{}_{\gamma}$ also appears in the modified photon field that is obtained in the context of the effective background field model in [14]. The quantities $(b^{(0)})^4\varrho$ and $\lambda^{(0)}$ have no mass dimension, and hence $\mathcal{C}^{(0)}$ is also a dimensionless parameter.

B. Establishing the perturbation from the Feynman rules of the modified theory

Analogously, the perturbative series of the full solution of the modified photon field equation can be obtained in terms of Feynman diagrams. The corresponding Feynman rules follow from the action of Eq. (2.3) plus Sec. III B and are given in Appendix B.

We couple the photon field to a conserved fermion current j_ν [with $k^\nu j_\nu(k) = 0$] that is represented by a plain line with an arrow, where the scalar field is denoted by a plain line without any arrow. The photon field is drawn as a single wiggly line, and a double wiggly line stands for the full field. Ordinary vertices are represented by dots and the scattering at a defect ("defect vertex") is shown as a cross:



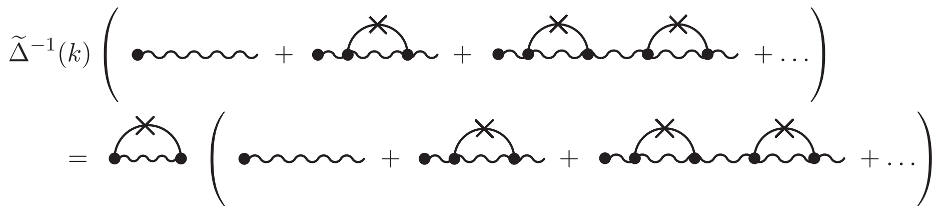
(4.8)

Contributions proportional to odd powers of $\lambda^{(0)}$ (containing a defect vertex connected to only a single ϕ field) vanish because of four-momentum conservation. For example, this is the case for the second diagram on the right-hand side of the diagrammatical equation above; cf. the discussion below Eq. (4.5) in the previous section. The second order perturbative solution $\tilde{A}^{(2)\nu}(k)$—corresponding to the third diagram on the right-hand side of Eq. (4.8)—leads to a nonvanishing correction of the photon field.

A resummation of all one-particle reducible diagrams at one-loop order results in a resummed photon field

(4.9)

Multiplying the inverse standard photon propagator $\tilde{\Delta}^{-1}(k) = k^2$ with $\hat{A}^\mu(k)$ nullifies the zeroth order contribution corresponding to the free photon field $\tilde{A}^{\mu(0)}(k)$. Furthermore, it cancels a propagator in each further term, and therefore an overall contribution can be factored out:

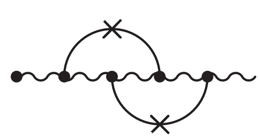


(4.10)

This prefactor is the one-loop diagram in front of the round brackets on the right-hand side of Eq. (4.10). It is given by the contribution $-\mathcal{C}^{(0)}B^\nu{}_\gamma(k)$ with $B^\nu{}_\gamma(k)$ of Eq. (4.7b). Hence, the modified field equation that results from the resummation of all one-loop photon field corrections is

$$k^2\hat{A}^\nu(k) = -\mathcal{C}^{(0)}B^\nu{}_\gamma(k)\hat{A}^\gamma(k). \tag{4.11}$$

From this equation a modified photon dispersion law will be derived later. Note that one-particle irreducible higher-order corrections, for example,



are not covered by the resummed photon field $\hat{A}^\mu$. They are assumed to give a contribution to the modified dispersion relation of the photons as well. However, since these are suppressed by at least one further factor $(\lambda^{(0)})^2$, we will neglect them in our calculations.

V. LEADING-ORDER PERTURBATION OF THE PHOTON FIELD

A. Dimensional regularization

We now want to compute the one-loop contribution $B^\nu{}_\gamma(k)$ to the photon field that was set up in Eq. (4.7b). We begin by contracting the indices in Eq. (4.7c),

$$\begin{aligned}K^{\nu}{}_{\gamma}\tilde{A}^{(0)\gamma}(k) &= \delta^{\mu}_{[\alpha}\delta^{\nu}_{\beta}\delta^{\kappa}_{\gamma]}q_{\mu}(k-q)_{\kappa}q^{\alpha}k^{\beta}\tilde{A}^{(0)\gamma}\\ &= k^{\nu}(k\cdot\tilde{A}^{(0)})q^2 - k^{\nu}k^{\varrho}\tilde{A}^{(0)\sigma}q_{\varrho}q_{\sigma}\\ &\quad - \tilde{A}^{(0)\nu}k^2q^2 + \tilde{A}^{(0)\nu}k^{\varrho}k^{\sigma}q_{\varrho}q_{\sigma}\\ &\quad - q^{\nu}q_{\varrho}k^{\varrho}(k\cdot\tilde{A}^{(0)}) + q^{\nu}q_{\varrho}\tilde{A}^{(0)\varrho}k^2. \end{aligned}\tag{5.1}$$

The second step is to perform the four-dimensional momentum integral over q,

$$\begin{aligned}&(2\pi)^4 B^{\nu}{}_{\gamma}\tilde{A}^{(0)\gamma}(k)\\ &= \int d^4q\,\frac{1}{(q^2-1/b^2+i\epsilon)^2[(k-q)^2+i\epsilon]}K^{\nu}{}_{\gamma}\tilde{A}^{(0)\gamma}(k)\\ &= \hat{I}_{\varrho\sigma}\{k^{\varrho}k^{\sigma}\tilde{A}^{(0)\nu} - k^{\nu}k^{\varrho}\tilde{A}^{(0)\sigma} + \eta^{\nu\sigma}[k^2\tilde{A}^{(0)\varrho}\\ &\quad - k^{\varrho}(k\cdot\tilde{A}^{(0)})]\} + \hat{I}_0[k^{\nu}(k\cdot\tilde{A}^{(0)}) - k^2\tilde{A}^{(0)\nu}],\end{aligned}\tag{5.2}$$

where $\hat{I}_{\varrho\sigma}$ is a tensor one-loop and $\hat{I}_0$ a scalar one-loop integral,

$$\hat{I}_{\varrho\sigma} \equiv \int d^4q\frac{q_{\varrho}q_{\sigma}}{(q^2-1/b^2+i\epsilon)^2[(k-q)^2+i\epsilon]},\tag{5.3a}$$

$$\hat{I}_0 \equiv \hat{I}_{\varrho}{}^{\varrho} = \int d^4q\frac{q^2}{(q^2-1/b^2+i\epsilon)^2[(k-q)^2+i\epsilon]}.\tag{5.3b}$$

By power counting we see that the integrals $\hat{I}_{\varrho\sigma}$ and $\hat{I}_0$ are ultraviolet divergent. Therefore they have to be regularized, and we decide to use dimensional regularization. The basic principle is to analytically continue the integrals to d spacetime dimensions, where $d \neq 4$ is a real number. If we use the convention $d = 4 - 2\hat{\varepsilon}$, with four spacetime dimensions to be recovered in the limit $\hat{\varepsilon} \mapsto 0$, the divergences become manifest as poles in $\hat{\varepsilon}$. Via

$$\begin{aligned}\int d^4q &= (2\pi)^4\int\frac{d^4q}{(2\pi)^4} \mapsto (2\pi)^4\mu^{4-d}\int\frac{d^dq}{(2\pi)^d}\\ &= (2\pi\mu)^{4-d}\int d^dq,\end{aligned}\tag{5.4}$$

the renormalization scale μ, of mass dimension 1, is introduced to conserve the dimension of the integral.

B. Passarino-Veltman decomposition

Equation (5.3a) gives a tensor integral that can be reduced to scalar integrals $\hat{I}_1$ and $\hat{I}_2$ with the following *ansatz*:

$$\hat{I}_{\varrho\sigma} = \eta_{\varrho\sigma}\hat{I}_1 + k_{\varrho}k_{\sigma}\hat{I}_2.\tag{5.5}$$

The integrals $\hat{I}_1$, $\hat{I}_2$ follow from the contractions $K_1 \equiv k^{\varrho}k^{\sigma}\hat{I}_{\varrho\sigma}$, $K_2 \equiv \eta^{\varrho\sigma}\hat{I}_{\varrho\sigma}$ and are given by

$$\hat{I}_1 = \frac{K_1 - k^2K_2}{(1-d)k^2},\qquad \hat{I}_2 = \frac{-dK_1 + k^2K_2}{(1-d)k^4}.\tag{5.6}$$

What remains is the reduction of the contractions K_1 and K_2 to scalar master integrals via a Passarino-Veltman decomposition. This leads to the following result (see Appendix C for a detailed calculation):

$$\begin{aligned}K_1 &= \frac{1}{4}\Big\{-A_0\Big(\frac{1}{(b^{(0)})^2}\Big) + \Big(k^2 + \frac{1}{(b^{(0)})^2}\Big)\Big[2B_0\Big(-k,\frac{1}{(b^{(0)})^2},0\Big) - B_0\Big(0,\frac{1}{(b^{(0)})^2},\frac{1}{(b^{(0)})^2}\Big)\Big]\\ &\quad + \Big[k^4 + \frac{2k^2}{(b^{(0)})^2} + \frac{1}{(b^{(0)})^4}\Big]C_0\Big(-k,0,\frac{1}{(b^{(0)})^2},0,\frac{1}{(b^{(0)})^2}\Big)\Big\},\end{aligned}\tag{5.7a}$$

$$K_2 = \hat{I}_0 = B_0\Big(-k,\frac{1}{(b^{(0)})^2},0\Big) + \frac{1}{(b^{(0)})^2}C_0\Big(-k,0,\frac{1}{(b^{(0)})^2},0,\frac{1}{(b^{(0)})^2}\Big).\tag{5.7b}$$

Here the contractions K_1, K_2 are expressed solely in terms of master integrals given by

$$A_0\Big(\frac{1}{(b^{(0)})^2}\Big) = (2\pi\mu)^{4-d}\int d^dq\,\frac{1}{q^2 - 1/(b^{(0)})^2 + i\epsilon},\tag{5.8a}$$

$$B_0\Big(0,\frac{1}{(b^{(0)})^2},\frac{1}{(b^{(0)})^2}\Big) = (2\pi\mu)^{4-d}\int d^dq\,\frac{1}{(q^2 - 1/(b^{(0)})^2 + i\epsilon)^2},\tag{5.8b}$$

$$B_0\Big(-k,\frac{1}{(b^{(0)})^2},0\Big) = (2\pi\mu)^{4-d}\int d^dq\,\frac{1}{(q^2 - 1/(b^{(0)})^2 + i\epsilon)[(k-q)^2 + i\epsilon]},\tag{5.8c}$$

$$C_0\Big(-k,0,\frac{1}{(b^{(0)})^2},0,\frac{1}{(b^{(0)})^2}\Big) = \int d^4q\,\frac{1}{(q^2 - 1/(b^{(0)})^2 + i\epsilon)^2[(k-q)^2 + i\epsilon]}.\tag{5.8d}$$

Note that the C_0 integral has neither infrared nor ultraviolet divergences so there is no need to regularize it. The integrals can be computed by standard methods such as Feynman parametrization (see for example [23]). The results are as follows:

$$\frac{1}{\mathrm{i}\pi^2}A_0\left(\frac{1}{(b^{(0)})^2}\right)=\frac{1}{(b^{(0)})^2}\left[\frac{1}{\varepsilon}-\ln\left(\frac{1}{(b^{(0)})^2\mu^2}\right)+1\right]+\mathcal{O}(\varepsilon), \tag{5.9a}$$

$$\frac{1}{\mathrm{i}\pi^2}B_0\left(0,\frac{1}{(b^{(0)})^2},\frac{1}{(b^{(0)})^2}\right)=\frac{1}{\varepsilon}-\ln\left(\frac{1}{(b^{(0)})^2\mu^2}\right)+\mathcal{O}(\varepsilon), \tag{5.9b}$$

$$\begin{aligned}\frac{1}{\mathrm{i}\pi^2}B_0\left(-k,\frac{1}{(b^{(0)})^2},0\right)&=\frac{1}{\varepsilon}-\int_0^1\mathrm{d}x\ln\left[\frac{k^2x^2-(k^2+1/(b^{(0)})^2)x+1/(b^{(0)})^2-\mathrm{i}\epsilon}{\mu^2}\right]+\mathcal{O}(\varepsilon)\\&=\frac{1}{\varepsilon}-\ln\left(\frac{1}{(b^{(0)})^2\mu^2}\right)+2-\frac{k^2-1/(b^{(0)})^2}{k^2}\ln[1-(b^{(0)})^2k^2-\mathrm{i}\epsilon]+\mathcal{O}(\varepsilon),\end{aligned} \tag{5.9c}$$

$$\begin{aligned}\frac{1}{\mathrm{i}\pi^2}C_0\left(-k,0,\frac{1}{(b^{(0)})^2},0,\frac{1}{(b^{(0)})^2}\right)&=-\int_0^1\mathrm{d}x\frac{1-x}{k^2x^2-(k^2+1/(b^{(0)})^2)x+1/(b^{(0)})^2-\mathrm{i}\epsilon}\\&=\frac{1}{k^2}\ln[1-(b^{(0)})^2k^2-\mathrm{i}\epsilon].\end{aligned} \tag{5.9d}$$

We have used

$$\frac{1}{\varepsilon}\equiv\frac{1}{\hat{\varepsilon}}-\gamma_E+\ln(4\pi) \tag{5.10}$$

with the Euler-Mascheroni constant $\gamma_E\approx 0.577216$, which is a reasonable redefinition of the regularization parameter. Terms of $\mathcal{O}(\varepsilon)$ have been discarded since they are not needed. An elaborate computation of the scalar integrals is presented in Appendix D.

To summarize, we obtain the following photon field correction at second order in perturbation theory:

$$\begin{aligned}&(2\pi)^4B^{\nu\gamma}(k)\tilde{A}^{(0)}_{\gamma}(k)\\&=(\eta_{\varrho\sigma}\hat{I}_1+k_\varrho k_\sigma\hat{I}_2)\{k^\varrho k^\sigma\tilde{A}^{(0)\nu}-k^\nu k^\varrho\tilde{A}^{(0)\sigma}+\eta^{\nu\sigma}\\&\quad\times[k^2\tilde{A}^{(0)\varrho}-k^\varrho(k\cdot\tilde{A}^{(0)})]\}+\hat{I}_0[k^\nu(k\cdot\tilde{A}^{(0)})\\&\quad-k^2\tilde{A}^{(0)\nu}]\\&=(k^\nu k^\gamma-\eta^{\nu\gamma}k^2)[\hat{I}_0-2\hat{I}_1-k^2\hat{I}_2]\tilde{A}^{(0)}_{\gamma}(k).\end{aligned} \tag{5.11}$$

The scalar integrals $\hat{I}_1$, $\hat{I}_2$ result from the contractions K_1, K_2 via Eq. (5.6), where these are given by Eqs. (5.7a) and (5.7b). The bare correction to the photon field is transverse and contains $1/\varepsilon$ poles. To obtain a physically meaningful result, Eq. (5.11) has to be renormalized.

C. Renormalization procedure

The second order solution of the photon field equation can now be written as follows:

$$\begin{aligned}(\lambda^{(0)})^2\tilde{A}^{(2)\nu}(k)&=-\mathcal{C}^{(0)}\tilde{\Delta}(k)B^{\nu\gamma}(k)\tilde{A}^{(0)}_{\gamma}(k)\\&=-\mathcal{C}^{(0)}\tilde{\Delta}(k)\mathrm{i}\Pi^{\nu\gamma}(k)\tilde{A}^{(0)}_{\gamma}(k),\end{aligned} \tag{5.12a}$$

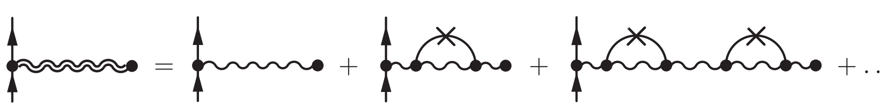
$$\mathrm{i}\Pi^{\nu\gamma}(k)=\mathrm{i}(k^\nu k^\gamma-\eta^{\nu\gamma}k^2)\Pi(k^2), \tag{5.12b}$$

$$\Pi(k^2)=-\mathrm{i}(\hat{I}_0-2\hat{I}_1-k^2\hat{I}_2), \tag{5.12c}$$

with the explicit result

$$\begin{aligned}16\pi^2\Pi(k^2)&=\frac{1}{2\varepsilon}-\frac{1}{2}\ln\left(\frac{1}{(b^{(0)}\mu)^2}\right)+\frac{1}{2}\left(1-\frac{1}{(b^{(0)}k)^2}\right)\\&\quad-\frac{1}{2}\left(1-\frac{2}{(b^{(0)}k)^2}+\frac{1}{(b^{(0)}k)^4}\right)\\&\quad\times\ln[1-(b^{(0)}k)^2-\mathrm{i}\epsilon].\end{aligned} \tag{5.12d}$$

Since the one-loop diagram computed resembles the vacuum polarization contribution of standard quantum electrodynamics (QED), we perform a renormalization of the coupling constant $\lambda^{(0)}$. We construct the modified photon propagator from the modified photon field by successively inserting one-loop corrections. This gives an infinite resummation of one-particle reducible diagrams at one-loop order leading to the full propagator. In this procedure we neglect higher order perturbative corrections $\mathcal{O}((\lambda^{(0)})^4)$ that are one-particle irreducible. Finally the photon propagator is coupled to a conserved current. This procedure will not be performed explicitly, but it is needed to drop all terms proportional to the four-momentum k^μ. In a diagrammatical notation our approach appears as follows:



(5.13)

In terms of equations this corresponds to the vacuum expectation value of the time-ordered product of field operators $\langle T\hat{A}^\mu(k)\hat{A}^\nu(k)\rangle$, which is the resummed Feynman propagator[5]

[5]The coupling to the conserved current is suppressed.

$$\langle T\hat{A}^\mu(k)\hat{A}^\nu(k)\rangle = \frac{-\mathrm{i}\eta^{\mu\nu}}{k^2+\mathrm{i}\epsilon} + \frac{-\mathrm{i}\eta^{\mu\varrho}}{k^2+\mathrm{i}\epsilon}\,\mathrm{i}\Pi_{\varrho\sigma}(k)\frac{-\mathrm{i}\eta^{\sigma\nu}}{k^2+\mathrm{i}\epsilon} + \frac{-\mathrm{i}\eta^{\mu\varrho}}{k^2+\mathrm{i}\epsilon}\,\mathrm{i}\Pi_{\varrho\sigma}(k)\frac{-\mathrm{i}\eta^{\sigma\alpha}}{k^2+\mathrm{i}\epsilon}\,\mathrm{i}\Pi_{\alpha\beta}(k)\frac{-\mathrm{i}\eta^{\beta\nu}}{k^2+\mathrm{i}\epsilon} + \cdots$$
$$= \frac{-\mathrm{i}\eta^{\mu\nu}}{k^2+\mathrm{i}\epsilon} - \frac{\mathrm{i}\Pi^{\mu\nu}(k)}{k^4+\mathrm{i}\epsilon} - \frac{\mathrm{i}\Pi^{\mu}{}_{\sigma}(k)\Pi^{\sigma\nu}(k)}{k^6+\mathrm{i}\epsilon} + \cdots. \tag{5.14}$$

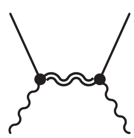
The contraction of two transverse structures $\Pi^{\mu\nu}(k)$ results in

$$\Pi^{\nu\alpha}(k)\Pi_{\alpha}{}^{\varrho}(k) = -k^2\Pi^{\nu\varrho}. \tag{5.15}$$

So the resummation of all one-particle reducible diagrams at one-loop order corresponds to a geometric series and leads to the following result[6]:

$$\langle T\hat{A}^\mu(k)\hat{A}^\nu(k)\rangle = \frac{-\mathrm{i}\eta^{\mu\nu}}{k^2+\mathrm{i}\epsilon}\{1-(\lambda^{(0)})^2\Pi(k^2)+[(\lambda^{(0)})^2\Pi(k^2)]^2 \mp \cdots\} = \frac{-\mathrm{i}\eta^{\mu\nu}}{k^2+\mathrm{i}\epsilon}\frac{1}{1+(\lambda^{(0)})^2\Pi(k^2)}$$
$$= \langle T\hat{A}^{(0)\mu}(k)\hat{A}^{(0)\nu}(k)\rangle\frac{1}{1+(\lambda^{(0)})^2\Pi(k^2)}. \tag{5.16}$$

The full propagator can be expressed via the bare propagator multiplied by a prefactor that contains the bare one-loop correction $\Pi(k^2)$ to the photon field. To perform the renormalization procedure, we consider a physical process that contains a photon propagator, e.g., the scattering of ϕ at a photon and its subsequent emission. We use the resummed propagator for setting up the amplitude of the process with the propagator momentum squared k^2 corresponding to the squared center of mass energy $\sqrt{s}$, namely $k^2 = s$.



$$= (\lambda^{(0)})^2\frac{-\mathrm{i}\eta^{\mu\nu}}{s+\mathrm{i}\epsilon}\frac{1}{1+(\lambda^{(0)})^2\Pi(s)}S_{\mu\nu} \equiv \lambda^2\frac{-\mathrm{i}\eta^{\mu\nu}}{s+\mathrm{i}\epsilon}S_{\mu\nu}\,. \tag{5.17}$$

Here $S_{\mu\nu}$ contains the remainder of the amplitude, which is not important for the current considerations. In the course of renormalization the bare coupling $\lambda^{(0)}$ is replaced by the renormalized coupling λ, such that the renormalized amplitude is finite:

$$\lambda^2 \equiv \frac{(\lambda^{(0)})^2}{1+(\lambda^{(0)})^2\Pi(s)},$$
$$(\lambda^{(0)})^2 = \frac{\lambda^2}{1-\lambda^2\Pi(s)} = \lambda^2 + \mathcal{O}(\lambda^4). \tag{5.18}$$

If the propagator momentum squared k^2 differs from the scale s, we obtain an expression similar to the bare amplitude at order $(\lambda^{(0)})^2$:

$$(\lambda^{(0)})^2\frac{-\mathrm{i}\eta^{\mu\nu}}{k^2+\mathrm{i}\epsilon}\frac{1}{1+(\lambda^{(0)})^2\Pi(k^2)}S_{\mu\nu}$$
$$= \lambda^2\frac{-\mathrm{i}\eta^{\mu\nu}}{k^2+\mathrm{i}\epsilon}\frac{1}{[1-\lambda^2\Pi(s)][1+\lambda^2\Pi(k^2)]}S_{\mu\nu} + \mathcal{O}(\lambda^4)$$
$$= \lambda^2\frac{-\mathrm{i}\eta^{\mu\nu}}{k^2+\mathrm{i}\epsilon}\frac{1}{1+\lambda^2\Pi_{\mathrm{ren}}(k^2)}S_{\mu\nu} + \mathcal{O}(\lambda^4), \tag{5.19}$$

but with $\lambda^{(0)}$ replaced by λ and $\Pi(k^2)$ replaced by the renormalized one-loop correction

$$\Pi_{\mathrm{ren}}(k^2) = \Pi(k^2) - \Pi(s). \tag{5.20}$$

Equation (5.16) shows that the imaginary unit in front of the one-loop correction $\Pi(k^2)$ is put into the propagator. What in fact matters for the photon dispersion relation is the real quantity $\Pi(k^2)$. From the bare one-loop correction we now obtain the renormalized correction according to Eq. (5.20):

$$16\pi^2\Pi_{\mathrm{ren}}(k^2) = \frac{1}{2(b^{(0)})^2}\left(\frac{1}{s}-\frac{1}{k^2}\right) + \frac{1}{2}\Bigg\{-\Bigg[1-\frac{2}{(b^{(0)}k)^2} + \frac{1}{(b^{(0)}k)^4}\Bigg]\ln[1-(b^{(0)}k)^2-\mathrm{i}\epsilon]$$
$$+\Bigg[1-\frac{2}{(b^{(0)})^2 s}+\frac{1}{[(b^{(0)})^2 s]^2}\Bigg] \times \ln[1-(b^{(0)})^2 s-\mathrm{i}\epsilon]\Bigg\}, \tag{5.21}$$

where the renormalization scale μ obviously cancels. The physically important quantity $\Pi_{\mathrm{ren}}(k^2)$ is the difference between the bare self-energy correction evaluated at k^2 and the same quantity evaluated at an arbitrary scale s. In principle s can be chosen such that the last expression of Eq. (5.21) in rectangular brackets vanishes. This holds for $s = 1/(b^{(0)})^2$, which then leads to

[6]In the calculations performed within this section, $(\lambda^{(0)})^2$ is understood to be extracted from the dimensionless constant $\mathcal{C}^{(0)}$ defined in Eq. (4.7a). Consequently, it appears together with the one-loop correction $\Pi(k^2)$ to keep track of all powers of $\lambda^{(0)}$.

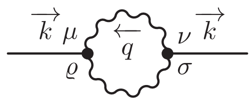
$$16\pi^2\hat{\Pi}_{\text{ren}}(k^2) \equiv 16\pi^2\Pi_{\text{ren}}(k^2)|_{s=1/(b^{(0)})^2}$$
$$= \frac{1}{2}\left(1 - \frac{1}{(b^{(0)}k)^2}\right) - \frac{[1-(b^{(0)}k)^2]^2}{2(b^{(0)}k)^4} \times \ln[1-(b^{(0)}k)^2 - \mathrm{i}\epsilon]. \quad (5.22)$$

The choice $s = 1/(b^{(0)})^2$ is not unreasonable since $1/(b^{(0)})^2$ is the only parameter of the action (2.3) that has the same mass dimension as s.

VI. LEADING ORDER PERTURBATION OF THE SCALAR FIELD

Having obtained the modified photon field we now consider the modification of the scalar field ϕ resulting from the action (2.3). There are two corrections to the bare scalar field. First of all there is a quantum correction involving the photon [see Fig. 2(a)] and, second, the scattering of the scalar field with the defects has to be taken into account [see Fig. 2(b)]. We will now compute the self-energy correction given by the diagram in Fig. 2(a).

$$-\mathrm{i}\Sigma(k^2,m^2) =$$



$$= (\mathrm{i}\lambda)^2(2\pi\mu)^{4-d}\int \mathrm{d}^d q\, \varepsilon^{\alpha\mu\beta\varrho}q_\alpha(-k-q)_\beta\varepsilon^{\gamma\nu\delta\sigma}(-q)_\gamma(k+q)_\delta \times \frac{-\mathrm{i}\eta_{\mu\nu}}{q^2+m^2+\mathrm{i}\epsilon}\frac{-\mathrm{i}\eta_{\varrho\sigma}}{(k+q)^2+m^2+\mathrm{i}\epsilon}$$
$$= 2\lambda^2[k^2\hat{I}_3(m) - \hat{I}_4(m,k)]\,, \quad (6.1a)$$

with

$$\hat{I}_3(k^2,m^2) = (2\pi\mu)^{4-d}\int \mathrm{d}^d q \frac{q^2}{(q^2-m^2+\mathrm{i}\epsilon)[(k+q)^2-m^2+\mathrm{i}\epsilon]}, \quad (6.1b)$$

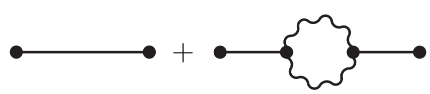
$$\hat{I}_4(k^2,m^2) = (2\pi\mu)^{4-d}\int \mathrm{d}^d q \frac{(k\cdot q)^2}{(q^2-m^2+\mathrm{i}\epsilon)[(k+q)^2-m^2+\mathrm{i}\epsilon]}. \quad (6.1c)$$

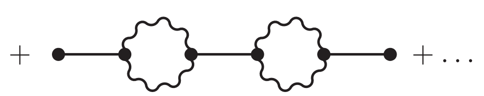
The ultraviolet divergences are again regularized by dimensional regularization with renormalization scale μ. Furthermore a photon mass m has been introduced to regularize possible infrared divergences. The results of the integrals $\hat{I}_3$, $\hat{I}_4$ in the limit $m \mapsto 0$ are given by

$$\hat{I}_3(k^2,0) = 0, \qquad \hat{I}_4(k^2,0) = \mathrm{i}\pi^2\frac{k^4}{4}\left\{\frac{1}{\varepsilon} + \left[2 - \ln\left(-\frac{k^2}{\mu^2} - \mathrm{i}\epsilon\right)\right]\right\}. \quad (6.2)$$

Fortunately the integrals are infrared finite. Hence we consider $\Sigma(k^2,0) \equiv \Sigma(k^2)$ from now on. A resummation of all one-loop photon self-energy corrections leads to a modification of the scalar field propagator:

$$\frac{\mathrm{i}b^2}{k^2 - 1/(b^{(0)})^2 - (b^{(0)})^2\Sigma(k^2,0) + \mathrm{i}\epsilon} =$$





(6.3)

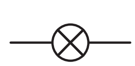
Contrary to Sec. V the renormalization of the correction to the scalar field will be performed with the help of counterterms, which is a more convenient procedure here. We will follow the lines of [24]. Both the scalar mass and the field are renormalized according to

$$\phi = \sqrt{Z_2}\phi_{\text{ren}}, \quad \delta_{1/b} = \frac{Z_2}{(b^{(0)})^2} - \frac{1}{b^2}, \quad \delta_{Z_2} = Z_2 - 1, \quad (6.4)$$

where ϕ is the bare scalar field, ϕ_{ren} the renormalized field, Z_2 the field renormalization constant, δ_{Z_2} the field renormalization counterterm, and $\delta_{1/b}$ the mass counterterm. Furthermore, we employ the renormalization conditions

$$\Sigma(k^2)|_{k^2=1/b^2} = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}k^2}\Sigma(k^2)|_{k^2=1/b^2} = 0. \quad (6.5)$$

From these we obtain the counterterm, which can be used for both the mass and the field renormalization:



$$= \mathrm{i}\left[\delta_{1/b^2}\left(\frac{k^4}{4} - \frac{1}{b^4}\right) + \frac{\delta_{Z_2}}{b^4}\right], \quad (6.6a)$$

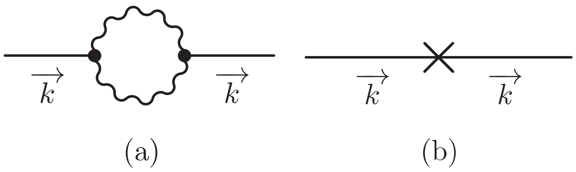


FIG. 2. Possible corrections to the bare scalar field $\phi(k)$. (a) One-loop photon self-energy contribution to the scalar field. (b) Correction to the scalar field originating from the scattering of ϕ at the defects.

with

$$\delta_{1/b} = \pi^2\left\{\frac{2\lambda^2}{\varepsilon} + \lambda^2\left[3 - 2\ln\left(-\frac{1}{b^2\mu^2} - \mathrm{i}\epsilon\right)\right]\right\}, \tag{6.6b}$$

$$\delta_{Z_2} = \pi^2\left\{\frac{2\lambda^2}{\varepsilon} + \frac{\lambda^2}{4}\left[13 - 8\ln\left(-\frac{1}{b^2\mu^2} - \mathrm{i}\epsilon\right)\right]\right\}. \tag{6.6c}$$

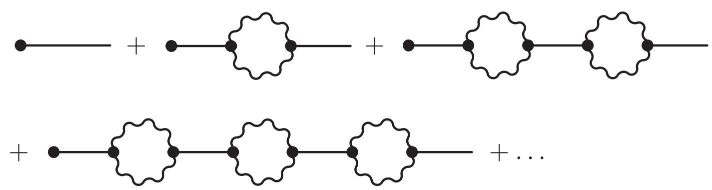
The renormalized photon one-loop self-energy contribution to the scalar field is then

$$-\mathrm{i}\Sigma_{\mathrm{ren}}(k^2, b^2) = -\mathrm{i}\Sigma(k^2) + \mathrm{i}\left[\delta_{1/b^2}\left(\frac{k^4}{4} - \frac{1}{b^4}\right) + \frac{\delta_{Z_2}}{b^4}\right] = \frac{\mathrm{i}\pi^2\lambda^2}{4b^4}\left\{1 - (bk)^4\left[1 + 2\ln\left(\frac{1}{(bk)^2} + \mathrm{i}\epsilon\right)\right]\right\}. \tag{6.7}$$

This correction leads to a renormalization of the ϕ-field mass. The correction is of order λ^2 and depends on both k^2 and the renormalized mass $1/b^2$,

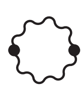
$$\frac{1}{(b^{(0)})^2} \mapsto \frac{1}{b^2} + b^2\Sigma_{\mathrm{ren}}(k^2, b^2). \tag{6.8}$$

As in the case of the photon field we now define the resummation of all one-particle reducible one-loop corrections to the scalar field:

$\widehat{\phi}(k) \equiv$ 

$+$ counterterms $+ \ldots .$ (6.9)

This resummation fulfills a modified field equation with renormalized mass $1/b$ of the ϕ field,

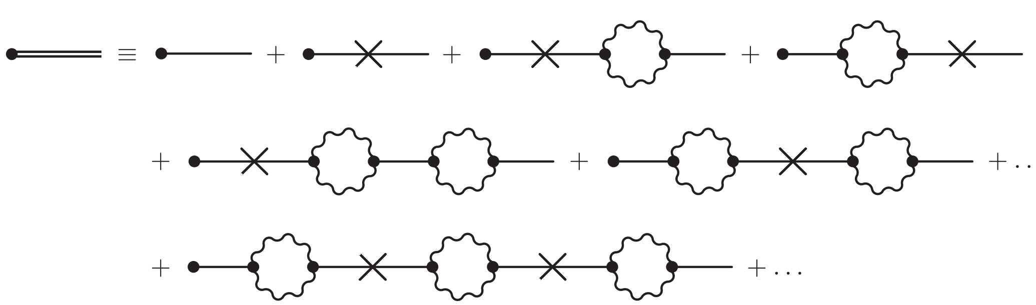
$\frac{1}{b^2}\left(k^2 - \frac{1}{b^2}\right)\widehat{\phi}(k) =$  $\left(\widehat{\phi}(k)\right).$ (6.10)

Inserting this mass correction into the photon field equation leads to $\mathcal{O}(\lambda^4)$ corrections. Note that the imaginary part resulting from both Eq. (6.1a) and (6.7) is given by

$$\mathrm{Im}\,(\Sigma) = \frac{\pi^3}{2}\lambda^2 k^4. \tag{6.11}$$

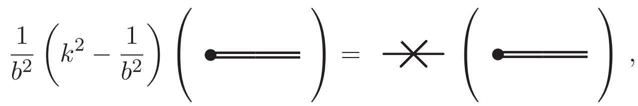
This result does not depend on the renormalization program since by the optical theorem it is linked to the total cross section for the decay of an excitation of the ϕ field (a scalar particle) into two photons.

Finally, we take into account the second contribution in Fig. 2(b), namely the scattering of ϕ at a defect. In so doing we define the full solution $\hat{\phi}_\varrho(k)$ of the corresponding field equation as the resummation of all one-loop corrections with defect vertex insertions according to



$+$ counterterms $+ \ldots .$ (6.12)

This leads to the following field equation for the scalar field:



(6.13a)

or

$$\frac{1}{b^2}\left(k^2-\frac{1}{b^2}\right)\hat{\phi}_\varrho(k)=\varrho\hat{\phi}_\varrho(k). \tag{6.13b}$$

Physically, the interaction of the ϕ field with the defects results in a shift of the mass of ϕ that corresponds to the density of defects as long as momentum transfer to the defects is neglected:

$$\frac{1}{b^2}\mapsto\frac{1}{b^2}+\varrho b^2\equiv\frac{1}{b_\varrho^2}. \tag{6.14}$$

VII. MODIFIED PHOTON DISPERSION RELATION

A. Final result for the modified theory

To summarize, the second order perturbative solution of the photon field equation is

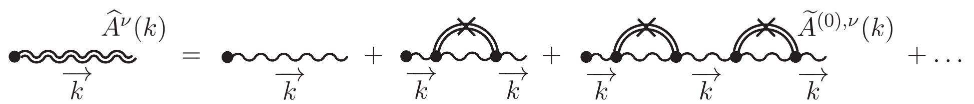
$$\lambda^2\tilde{A}^{(2)\nu}(k)=-\mathcal{C}\tilde{\Delta}(k)B^{\nu\gamma}(k)\tilde{A}^{(0)}_\gamma(k)$$
$$=-\mathcal{C}\tilde{\Delta}(k)\mathrm{i}\Pi^{\nu\gamma}(k)\tilde{A}^{(0)}_\gamma(k), \tag{7.1a}$$

$$\mathrm{i}\Pi^{\nu\gamma}(k)=\mathrm{i}(k^\nu k^\gamma-\eta^{\nu\gamma}k^2)\hat{\Pi}_{\mathrm{ren}}(k^2), \tag{7.1b}$$

$$16\pi^2\hat{\Pi}_{\mathrm{ren}}(k^2)=\frac{1}{2}\left(1-\frac{1}{(b_\varrho k)^2}\right)-\frac{[1-(b_\varrho k)^2]^2}{2(b_\varrho k)^4}\times\ln[1-(b_\varrho k)^2-\mathrm{i}\epsilon], \tag{7.1c}$$

where $\mathcal{C}=b_\varrho^4\lambda^2\varrho$ and $\epsilon=0^+$. In Sec. V we integrated out the scalar field at the one-loop level. Using the results from Sec. VI, this leads to an effective vertex whose Feynman rule is given by Eq. (B1j) in Appendix B.

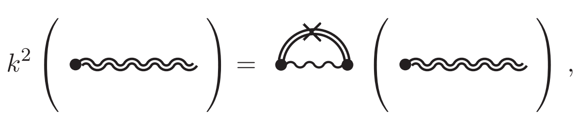
To obtain the modified photon field equation, we define the one-loop resummed photon field $\hat{A}^\nu_\varrho(k)$ by employing the full scalar field solution $\hat{\phi}_\varrho(k)$ from the previous section:



(7.2)

The one-loop contribution to the photon field is transverse, and after renormalizing the coupling constant using Eq. (5.19) it is finite. Furthermore, it becomes imaginary for $k^2=1/b_\varrho^2$. This indicates that an electromagnetic wave is damped when k^2 approaches the mass of the scalar field ϕ leading to a resonance behavior.

Pursuing the discussions at the end of Sec. IV B and below Eq. (5.20) we now consider the modified field equation



(7.3a)

$$k^2\hat{A}^\nu_\varrho(k)=-\mathcal{C}[-\mathrm{i}B^\nu{}_\gamma(k)]\hat{A}^\gamma_\varrho(k)$$
$$=-\mathcal{C}(k^\nu k_\gamma-\delta^\nu{}_\gamma k^2)\hat{\Pi}_{\mathrm{ren}}(k^2)\hat{A}^\gamma_\varrho(k)$$
$$=\mathcal{C}k^2\hat{\Pi}_{\mathrm{ren}}(k^2)\hat{A}^\nu_\varrho(k). \tag{7.3b}$$

Two distinct photon dispersion relations follow from Eq. (7.3b). First of all we obtain the standard dispersion law $k^2=0$. Second, considering $k^2\neq 0$, Eq. (7.3b) results in

$$\hat{A}^\nu_\varrho(k)=-\mathcal{C}\tilde{\Delta}(k)[-\mathrm{i}B^\nu{}_\gamma(k)]\hat{A}^\gamma_\varrho(k)$$
$$=\mathcal{C}\frac{k^2}{k^2+\mathrm{i}\epsilon}\hat{\Pi}_{\mathrm{ren}}(k^2)\hat{A}^\nu_\varrho(k). \tag{7.4}$$

Nontrivial solutions for the photon field will then exist if the following transcendental equation holds:

$$1-\mathcal{C}\hat{\Pi}_{\mathrm{ren}}(k^2)=0. \tag{7.5}$$

Unfortunately, it cannot be solved analytically. By studying the latter equation with a graphical method we observe that no solution is possible for $\mathcal{C}>0$ (for a discussion of the case $\mathcal{C}<0$ see Sec. IX). We can also see this in another way. Since we expect an eventual solution to be a minuscule correction to the standard photon dispersion relation $k^2=0$, we assume $(b_\varrho k)^2\ll 1$. This leads to an approximate equation of (7.5):

$$1+\frac{\mathcal{C}}{64\pi^2}\left[1-\frac{2b_\varrho^2}{3}(k_0^2-|\mathbf{k}|^2)\right]=0. \tag{7.6}$$

The latter can be investigated analytically, and its solution is

$$k_0=\sqrt{|\mathbf{k}|^2+m_\gamma^2},\qquad m_\gamma^2=\frac{3}{2}\left(1+\frac{64\pi^2}{\mathcal{C}}\right)\frac{1}{b_\varrho^2}, \tag{7.7}$$

with the three-momentum $\mathbf{k}$. Thus the photon acquires a mass m_γ. But we have to discard this solution because it is out of range of validity for the approximation $(b_\varrho k)^2\ll 1$, in fact,

$$(b_\varrho k)^2=(b_\varrho m_\gamma)^2=\frac{3}{2}\left(1+\frac{64\pi^2}{\mathcal{C}}\right)>1,\quad\forall\,\mathcal{C}>0. \tag{7.8}$$

Moreover this mass depends inversely on the parameters b_ϱ and $\mathcal{C}$, when instead we expect it to be a correction that vanishes for $b_\varrho\mapsto 0$ or $\mathcal{C}\mapsto 0$. We conclude that the massive photon solution of Eq. (7.7) is spurious and the only possible dispersion relation of this model is the standard one,

$$k^2 = 0. \tag{7.9}$$

B. General remarks on the previous results

As a first step in the current section we intend to choose exemplary values for the parameters of the theory to check the validity of the perturbative expansion performed. Since the scalar field is the mediator between the photon and the spacetime defects, which are Planck-scale effects, the scalar field mass $1/b_\varrho$ is assumed to be large, perhaps some fraction of the Planck mass $M_{\mathrm{Pl}} = \sqrt{\hbar c/G} \approx 1.22 \times 10^{19}\ \mathrm{GeV/c^2}$. Then b_ϱ would be of the order of the Planck length. Assuming the spacetime defects to have an average separation of $10^{10} \times L_{\mathrm{Pl}}$ [see the discussion below Eq. (3.13)] and setting $\lambda = 1$ we obtain

$$\mathcal{C} = \varrho b_\varrho^4 \lambda^2 = 10^{-40}\left(\frac{\varrho}{1/(10^{10}\times L_{\mathrm{Pl}})^4}\right)\left(\frac{b_\varrho^4}{1/M_{\mathrm{Pl}}^4}\right)\left(\frac{\lambda^2}{1^2}\right). \tag{7.10}$$

If the defects have an average distance of multiples of the Planck length, then $\mathcal{C} \ll 1$ even if the coupling λ lies in the order of 1. Hence, the procedure of working within perturbation theory in λ, which in principle corresponds to an expansion in $\mathcal{C}$, is warranted.

The second step now is to understand why the photon dispersion relation remains standard—unlike in the model considered in [14]. The nonexistence of a modified photon dispersion relation is connected to the limit $k^2 \mapsto 0$ of the quantity $\hat{\Pi}_{\mathrm{ren}}$ of Eq. (7.4). The latter is the basis from which a possible modified photon dispersion law could be obtained in principle.[7] For the cases $(b_\varrho k)^2 \ll 1$ and $(b_\varrho k)^2 \gg 1$ we obtain

$$\lim_{k^2\mapsto 0} 16\pi^2 k^2 \hat{\Pi}_{\mathrm{ren}}(k^2)\big|_{\substack{(b_\varrho k)^2 \ll 1\\ \text{fixed}}} = 0, \tag{7.11a}$$

$$\lim_{k^2\mapsto 0} 16\pi^2 k^2 \hat{\Pi}_{\mathrm{ren}}(k^2)\big|_{\substack{(b_\varrho k)^2 \gg 1\\ \text{fixed}}} = \frac{1}{2b_\varrho^2}\left(\frac{1}{1-\mathrm{i}\epsilon} - 1\right) + \lim_{k^2\mapsto 0}\frac{1}{b_\varrho^2}\left(1 - \frac{1}{2b_\varrho^2 k^2}\right)\ln(1-\mathrm{i}\epsilon). \tag{7.11b}$$

Compare these results to the renormalized self-energy correction of the photon (vacuum polarization) $\Pi_{\mathrm{ren}}(k^2)|^{\mathrm{QED}}$ in ordinary QED, for which

$$\lim_{k^2\mapsto 0} k^2 \Pi_{\mathrm{ren}}(k^2)|^{\mathrm{QED}} = 0. \tag{7.12}$$

Equation (7.12) means that gauge invariance is maintained for quantum corrections in QED forcing the photon dispersion law to the standard result $k^2 = 0$. An analogous argument holds for the modified theory defined by the action (2.5). As long as $k^2 \ll 1/b_\varrho^2$, for which the scalar mass is large and a possible modification of the photon dispersion law ought to be a small deviation from $k^2 = 0$, gauge invariance is maintained. Then a photon mass according to Eq. (7.7) cannot appear. The tensor structure $(k^\nu k^\gamma - \eta^{\nu\gamma} k^2)$ of $B^{\nu\gamma}(k)$ in Eq. (7.1) is crucial for this result indicating the conservation of gauge invariance. As long as this structure is conserved by the interaction with the defects, the photon dispersion relation stays $k^2 = 0$ (for $b_\varrho^2 k^2 \ll 1$).

However from Eq. (7.11b) it follows that gauge invariance is violated for large b_ϱ (and small scalar mass $1/b_\varrho$). The technical reason is that $\epsilon = 0^+$ cannot be discarded in this case, since the argument of the logarithm in Eq. (7.1) may be negative. The complex logarithm has a branch cut on the negative real axis and, therefore, a small imaginary part $\epsilon = 0^+$ has to be added to its argument. Thus the limit $k^2 \mapsto 0$ does not exist here and gauge invariance is violated resulting in a photon mass. Because of the infinitesimal imaginary part in the logarithm, $\hat{\Pi}_{\mathrm{ren}}(k^2)$ also has an imaginary part. Physically this corresponds to the damping of electromagnetic waves when the modified photon momentum square approaches the mass square of the scalar field.

Now we would like to argue on a fundamental basis why the photon dispersion relation remains conventional within the simple spacetime foam model proposed. In general, the (modified) photon dispersion relation is obtained as the zero of a scalar function that is sometimes called an off-shell dispersion relation. For the spacetime foam model presented here the latter is given by $1 - \mathcal{C}\hat{\Pi}_{\mathrm{ren}}(k^2)$ according to Eq. (7.5). We recall the following three general modifications of the photon dispersion law that originate from a violation of Lorentz or gauge invariance. They are denoted as cases (1), (2), and (3).

(1) Scaleless Lorentz-violating modification (cf. modified Maxwell theory [25–28]):
In the first case Lorentz invariance is violated by the occurrence of preferred spacetime directions in the action. These are given by four-vectors denoted as α_1, α_2, etc., with dimensionless components κ_1, κ_2, etc. We denote the off-shell dispersion relation by f. Since it is a scalar, it can only contain scalar products of the preferred directions with the momentum four-vector and scalar products among themselves: $f = f(k^2, \alpha_1\cdot k, \alpha_2\cdot k, \alpha_1\cdot\alpha_2, \ldots)$. The physical zero of f for $\kappa_1 \ll 1$, $\kappa_2 \ll 1$, etc., can then be cast in the following form:

$$k_0 = g(\mathbf{k}, \kappa_1, \kappa_2, \ldots) \approx |\mathbf{k}| + g_1(\mathbf{k})\kappa_1 + g_2(\mathbf{k})\kappa_2 + \cdots. \tag{7.13}$$

The functions g_1, g_2, etc., have mass dimension 1, and they only depend on the three-momentum.

[7] In some papers the left-hand side of an equation similar to Eq. (7.5) is called an off-shell dispersion relation.

Since the preferred directions may explicitly point along certain momentum components, the dispersion relation cannot be standard.

(2) Scale-dependent Lorentz-violating modification (cf. Maxwell-Chern-Simons theory [26–29]):
Preferred spacetime directions a_1, a_2, etc., appear in this case as well. They come together with physical scales m_1, m_2, etc.; i.e., a_1, a_2, etc., have dimensionful components. We will denote them using Latin letters in the following. The modified dispersion relation is a zero of the off-shell dispersion relation $\tilde{f} = \tilde{f}(k^2, a_1 \cdot k, a_2 \cdot k, a_1 \cdot a_2, \ldots)$. We perform an expansion in the scales m_1, m_2 of the deformation by considering them to be much smaller than the remaining physical scales. This leads to the following approximate physical zero of $\tilde{f}$:

$$k_0 = \tilde{g}(\mathbf{k}, m_1, m_2, \ldots) \approx |\mathbf{k}| + \tilde{g}_1(\mathbf{k})m_1 + \tilde{g}_2(\mathbf{k})m_2 + \cdots. \tag{7.14}$$

Here the functions $\tilde{g}_1$, $\tilde{g}_2$, etc., are dimensionless, and they again depend on the three-momentum components.

(3) Emergence of a photon mass:
A violation of gauge invariance would result in the following dispersion relation with a photon mass m_γ:

$$k_0 = \sqrt{\mathbf{k}^2 + m_\gamma^2} = |\mathbf{k}| + \frac{m_\gamma^2}{2|\mathbf{k}|} + \cdots. \tag{7.15}$$

Note that in Eq. (7.15) no term that is proportional to m_γ appears—contrary to Eq. (7.14) where linear terms in the dimensionful parameters m_1, m_2, etc., can be found.

A modification of the photon dispersion relation in the context of spacetime foam is expected to emerge according to one of the previous cases. For example, within the spacetime foam model considered in [14] the photon dispersion law is modified according to (1). The reason is that the method of distributing spacetime defects in the latter reference is not Lorentz invariant; i.e., the spatial dimensions are treated differently from the time dimension. This translates to a preferred direction in spacetime, and it results in a modified photon dispersion law of the form of Eq. (7.13), since there is no quantity having a mass dimension.

On the contrary, cases (1) and (2) cannot play a role within the spacetime foam model considered in the current article. In the action (2.5) of the effective theory no preferred spacetime directions appear. Furthermore, no such directions emerge, since the defects are distributed in a Lorentz-invariant way. As a result, the time dimension and the spatial dimensions are treated equally. Then only the dimensionless quantity $b_\varrho k = b_\varrho(k_0^2 - \mathbf{k}^2)^{1/2}$ appears in the off-shell dispersion relation. So the latter must be of the form $h = h(b_\varrho k)$. Assuming $b_\varrho^2 k^2 \ll 1$, which is reasonable within perturbation theory, the equation $h(b_\varrho k) = 0$ can be expanded with respect to its argument $b_\varrho k$ up to quadratic order,

$$h(0) + b_\varrho k h'(0) + \frac{1}{2} b_\varrho^2 k^2 h''(0) \simeq 0 \Rightarrow k_0^2 \simeq \mathbf{k}^2 - \frac{2}{b_\varrho^2}\frac{h(0)}{h''(0)}. \tag{7.16}$$

Here we have used $h'(0) = 0$.[8] The result shows that a nonvanishing photon mass $m_\gamma = -2h(0)/(b_\varrho^2 h''(0))$ appears indicating a violation of gauge invariance. Hence, the only modification may be according to (3). However, the photon mass in Eq. (7.16) is proportional to the scalar mass $1/b_\varrho$, which is not a small perturbation when $1/b_\varrho$ is large. Considering the left-hand side of Eq. (7.5) as the function h we, in fact, obtain the result of Eq. (7.7). So even without a graphical analysis, which is not possible for a general function h, the argument given is still valid. We conjecture that in such a case, where the photon interacts with the defects via a quantum correction with a scalar particle, its dispersion relation remains conventional.

The physical reason for ruling out a modification according to (3) is as follows. Because of $k^2\hat{\Pi}_{\text{ren}}(k^2) = 0$ for $k^2 \mapsto 0$ [see Eq. (7.11a)] gauge invariance is indeed conserved, although the action (2.3) seemed to violate gauge invariance (with respect to the gauge transformation $\phi \mapsto -\phi$) explicitly. However, the sign change of the penultimate term of Eq. (2.3) can be absorbed into the defect charges ε. For an infinite number of defects this does not change anything because there are equally many defects with $\varepsilon = 1$ and $\varepsilon = -1$. Also for the last term in the action the sign can be absorbed into the coupling constant λ. Since the correction computed is proportional to λ^2 this sign change has no physical implications.

VIII. MOMENTUM TRANSFER FROM PARTICLES TO DEFECTS

The final result of Sec. VII was obtained after making a series of assumptions in the course of the calculation. Let us recap these assumptions:

(i) Asmp. (1): The interaction of the photon with the defects is mediated by a scalar field according to the effective theory of Eq. (2.5). This forms the basis of the simple model proposed in this article and is, therefore, the most important assumption.

(ii) Asmp. (2a) and (2b) + (2c) + (2d): Isotropic and homogenous ("random") defect distribution (Sec. III A) + dense defect distribution (Sec. III B)

[8]Avoiding branch cuts by infinitesimal imaginary parts at the appropriate places, perturbative corrections are analytic functions in k^2, and so do not depend on k.

+infinite spacetime volume (Sec. III B). These have been introduced not only to keep the calculation feasible but also for physical reasons. In principle one or several of them can be dropped, leading to a more difficult calculation.

(iii) Asmp. (3): The modified photon momentum squared is supposed to be much smaller than the mass of the scalar field squared, i.e., $k^2 \ll 1/b_\varrho^2$. This assumption has been made only for physical reasons. The mass of the postulated scalar field $1/b_\varrho$ is expected to be large, and the deviation of the photon momentum squared from the standard result $k^2 = 0$ should be a small perturbation.

So far the momentum transfer from the photon to the defects and vice versa has been neglected. The reason for this is Asmp. (2) and Eq. (3.21) that followed from it.

Arguments concerning physics at the Planck scale cannot be rigorous until there is a theory describing such physics. Nevertheless in the following few lines we try to give a simple motivation why the momentum transfer from a defect to a particle and vice versa need not necessarily vanish. A theorist may think of a spacetime defect as a fluctuation of spacetime curvature where energy is associated with it. If spacetime curvature changes as a function of time, energy can be exchanged between the defect and its neighborhood. On the one hand, if a particle travels nearby, it might absorb a part of this energy. On the other hand, if spacetime curvature changes, so will the trajectory of the particle. It may then radiate energy that is absorbed by the defect.

For this reason we would like to investigate how the results obtained change when momentum is transferred from and to the defects. This means that we drop Asmp. (2a), (2b), (2c), or (2d). The technical problem is that the free photon field in Eq. (4.6) then depends on the integration momentum p, which renders the evaluation of the corresponding integral impossible. However, we will stick with Asmp. (1) and treat the propagation of photons through a spacetime foam via the effective theory defined by Eq. (2.3). Thus the external photon momentum is assumed to be much smaller than the Planck scale.

On the one hand, the ϕ field with its renormalized mass $1/b$ directly interacts with the spacetime defects. We assume that ϕ will only probe the defects if $1/b$ is of the order of the Planck scale. For $1/b$ much smaller than the Planck scale a momentum transfer to the defects will be low at most, i.e., suppressed by that scale. As a result, the main contribution of the p integral will come from the region $p \simeq k$. The difference between p and k must be suppressed by a small dimensionless number that can be written as a ratio of two mass scales. The scale where the influence of any spacetime foam may become especially important is the Planck mass M_{Pl}. Since the ϕ field is assumed to interact with the defects, the only other scale is the mass $1/b$.

On the other hand, the interaction of the photon with the spacetime defects is mediated via a quantum correction involving the virtual ϕ field with the new mass $1/b_\varrho$ originating from the interaction of ϕ with the defects. As a result, the photon acquires a size from this quantum correction that is inversely proportional to $1/b_\varrho$. Hence we assume the suppression of a momentum transfer from the photon to the defects to be similar as for the scalar field, but with the mass $1/b$ replaced by $1/b_\varrho$. This behavior will be summarized in the following paragraph.

A. Assumption (4): Momentum transfer suppressed by the Planck scale

To be able to compute the integral over p in Eq. (4.6), we introduce a fourth assumption. We assume that the ratios

$$(k-p)^2 \sim \frac{1}{b^3 M_{\mathrm{Pl}}}, \qquad (k-p)\cdot x \sim \frac{1}{bM_{\mathrm{Pl}}} k\cdot x \tag{8.1}$$

approximately hold for the momentum transfer $k - p$ between a low-energy ϕ field (with initial momentum k and final momentum p) and a spacetime defect. Analogous ratios hold for the momentum transfer of a photon to a defect with the difference that $1/b$ must be replaced by $1/b_\varrho$:

$$(k-p)^2 \sim \frac{1}{b_\varrho^3 M_{\mathrm{Pl}}}, \qquad (k-p)\cdot x \sim \frac{1}{b_\varrho M_{\mathrm{Pl}}} k\cdot x. \tag{8.2}$$

In Eqs. (8.1) and (8.2) x is an arbitrary but fixed four-vector in configuration space.

B. Defect distribution with large separation between individual defects

In Eq. (3.13) we assumed the distribution of spacetime defects to be dense [see Fig. 3(a)]. In this case the defect distribution can be approximated by an effective background field, and the sum over all defects results in a spacetime integral. Discarding Asmp. (2c) means that the sum mentioned can no longer be approximated by such an integral and we cannot define a background field any more. As a result, the δ function in Eq. (3.13), which helps to get rid of the second integral in Eq. (4.6), has to be replaced by a sum again. However, because of Asmp. (4) the momentum transfer is suppressed by the Planck scale, and the integration momentum p does not appear,

$$\int \mathrm{d}^4 p\,\varrho\,\delta(k-p) f(p) \mapsto \frac{1}{\mathcal{V}} \sum_{i=1}^{\mathcal{N}} \exp\left(\mathrm{i}\frac{1}{bM_{\mathrm{Pl}}} k\cdot x_i\right) f(k). \tag{8.3}$$

The mass of the ϕ field given by Eq. (6.14) then changes as follows:

$$\frac{1}{b^2} \mapsto \frac{1}{b^2} + \frac{b^2}{\mathcal{V}} \sum_{i=1}^{\mathcal{N}} \exp\left(\mathrm{i}\frac{1}{bM_{\mathrm{Pl}}} k\cdot x_i\right) \equiv \frac{1}{b(k,x_i)^2}. \tag{8.4}$$

It now explicitly depends on the ϕ-momentum k and the defect positions x_i [see Fig. 3(b)]. In the loop integral of

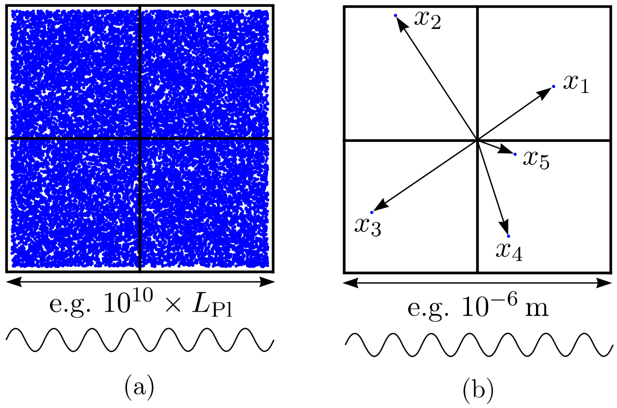


FIG. 3 (color online). Each panel in the current figure shows a distribution of spacetime defects where a photon with a typical wavelength is symbolically illustrated by a wiggly line below each panel. The scales of the panels are chosen to make clear that the distances between the defects in the left square correspond to multiples of the Planck scale (i.e., they are much smaller than the photon wavelength), whereas in the right square they may be in the order of (and even larger than) the wavelength of the photons to be considered. Hence the right panel (b) illustrates a foam in which individual defects are separated by distances that lie many orders of magnitude above the typical distances in the left panel (a).

Eq. (4.7b) the shifted mass has to be inserted. Then the integral depends on the defect positions x_i. This can be stated as

$$B^{\nu}{}_{\gamma}(k) \mapsto B^{\nu}{}_{\gamma}(k, x_i) = \sum_{i=1}^{\mathcal{N}} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \exp\left(\mathrm{i}\frac{1}{b(q,x_i)M_{\rm Pl}} k\cdot x_i\right) \times \left[\frac{1}{b(q,x_i)^2}\tilde{H}(q,x_i)\right]^2 \frac{1}{(k-q)^2+\mathrm{i}\epsilon} K^{\nu}{}_{\gamma}, \quad (8.5a)$$

with

$$\frac{1}{b(q,x_i)^2}\tilde{H}(q,x_i) = \frac{-1}{q^2 - 1/b^2(q,x_i) + \mathrm{i}\epsilon}, \quad (8.5b)$$

and $K^{\nu}{}_{\gamma}$ of Eq. (4.7c).

So the result of the loop integral $B^{\nu\gamma}(k)$ will also depend on the positions x_i of the defects. Because of this its tensor structure may involve terms such as $x_i^{\nu} x_i^{\gamma}$, $x_i^{\nu} k^{\gamma}$, etc., and they will spoil the form of the standard tensor structure $(k^{\nu}k^{\gamma} - \eta^{\nu\gamma}k^2)$. This leads to a dispersion relation of the photon that differs from $k^2 = 0$.

In Sec. III A we saw that the mean observed density of defects in a spacetime region of volume V is $\varrho_{\rm obs} = \varrho \pm \sqrt{\varrho/V}$. The fluctuations of $\varrho_{\rm obs}$ become negligible when $V \gg 1/\varrho$. Here the volume V refers to the region of spacetime probed by the photon with its wavelength ($V \simeq \lambda^4$). We want to investigate the case where $V \simeq 1/\varrho$ and the fluctuations are no longer negligible. For simplicity we consider a two-dimensional spacetime and describe the fluctuations with trigonometric functions (i.e., using periodic fluctuations instead of random fluctuations). In this way we are able to perform some explicit calculations. So we assume

$$\varrho(x) = \varrho[1 + A^2 \cos(\omega_0 x_0)\cos(\omega_1 x_1)]. \quad (8.6)$$

Setting $\omega_0 = \omega_1 = 2\pi/\sqrt{V}$ and $A = [4/(\varrho V)]^{1/4}$ the latter distribution gives the following mean value $\langle\varrho(x)\rangle$ and the standard deviation $\delta\varrho(x)$:

$$\langle\varrho(x)\rangle = \frac{1}{V}\int_V \mathrm{d}V \varrho(x) = \varrho, \quad \delta\varrho(x) = \sqrt{\frac{1}{V}\int_V \mathrm{d}V[\varrho(x) - \langle\varrho(x)\rangle]^2} = \sqrt{\frac{\varrho}{V}}. \quad (8.7)$$

These are the values that we expect from our arguments in Sec. III A. Inserting $\varrho(x)$ of Eq. (8.6) into Eq. (3.21) we obtain

$$\begin{aligned}\lim_{\mathcal{R}\mapsto\infty} \tilde{G}_{\mathcal{R}}(k)\tilde{G}_{\mathcal{R}}(p) &= \lim_{\mathcal{V}\mapsto\infty} \frac{1}{\mathcal{V}}\int_{\mathcal{V}} \mathrm{d}^2 x \varrho(x) \exp[i(k+p)x] \\ &= \lim_{\mathcal{V}\mapsto\infty} \frac{1}{\mathcal{V}}(2\pi)^2 \varrho\Big\{\delta^{(2)}(k+p) + \frac{A^2}{4}[\delta^{(2)}(k+p+\omega) \\ &\quad + \delta^{(2)}(k+p-\omega) + \delta(k_0+p_0+\omega_0)\delta(k_1+p_1+\omega_1) \\ &\quad + \delta(k_0+p_0-\omega_0)\delta(k_1+p_1-\omega_1)]\Big\}, \quad (8.8)\end{aligned}$$

where ω is a vector with components ω_0 and ω_1. We assume for simplicity $\omega_0 = \omega_1 = \omega_* \simeq 1/\lambda \ll 1$ so that we can perform a power expansion around zero. In this way Eq. (4.7b) becomes

$$B^{\nu}{}_{\gamma}(k) = \int \frac{\mathrm{d}^2 q}{(2\pi)^2}\left[\frac{1}{(b^{(0)})^2}\tilde{H}(q)\right]^2 \frac{1}{(k-q)^2 + \mathrm{i}\epsilon} K^{\nu}{}_{\gamma} \times \left\{1 + A^2\left[1 + \frac{4(q_0^2+q_1^2)\omega_*^2}{[q^2 - 1/(b^{(0)})^2]^2} + \cdots\right]\right\}. \quad (8.9)$$

From the latter expression one gets the one-loop correction to the photon field [analogously to Eq. (5.12c)]:

$$\Pi(k) = -\frac{4}{k^2}\left\{I_2(k^2) - \frac{4\varrho A^2 \omega_*^2 (b^{(0)})^4}{k^4}[k^2 I_0(k^2) - 2I_1(k^2)](k_0^2 + k_1^2)\right\}, \quad (8.10)$$

where I_0, I_1, and I_2 are one-loop integrals. Unfortunately these contain divergences that cannot be removed by a renormalization procedure. For this reason their physical meaning is obscure and we do not give the explicit results here. Nevertheless we observe that the photon polarization $\Pi(k)$ of Eq. (8.10) contains a Lorentz-violating term proportional to $(k_0^2 + k_1^2)$, which indicates that the physical photon dispersion relation may be Lorentz violating. The solution of this problem will be relegated to a future paper.

The possible Lorentz violation observed here could be entirely due to the particular periodic structure used to represent the density fluctuations. We also tried alternative ways to model the fluctuations. Another more accurate example is to divide spacetime into boxes of volume $V \simeq \lambda^4$. In each box the density is assumed to be constant, but it varies from box to box according to the Poisson distribution. However, we were not able to perform an explicit calculation within such a model. What we observed numerically is that Eq. (3.21) is modified in this case as well [by some random functions instead of simple δ functions as they appear in Eq. (8.8)].

C. Defect distribution in a finite spacetime volume

Now we stick to Asmp. (2c) but drop Asmp. (2d) and consider defects in a spacetime with finite volume $\mathcal{V}$. For computational reasons the shape of the volume shall be a four-dimensional cube[9] containing $\mathcal{N}$ defects. Because of the finiteness of $\mathcal{V}$ the positions x_i are bounded and the argument of the exponential function can be assumed to be small. So we expand the complex exponential function in Eq. (8.3) with respect to the (small) momentum transfer. In the remainder $f(p)$ of the integral over p in Eq. (4.6) we effectively replace all p by k and drop the integral over p.

This procedure leads to a new replacement rule for the δ function,

$$\int \mathrm{d}^4p\varrho\delta(k-p)f(p)\mapsto\frac{1}{\mathcal{V}}\left(\mathcal{N}+\sum_{n=1}^{\infty}S_n\right)f(k), \tag{8.11a}$$

$$S_n=\frac{\mathrm{i}^n}{n!}(k-p)_{\mu_1}(k-p)_{\mu_2}\ldots(k-p)_{\mu_n}\sum_{i=1}^{\mathcal{N}}x_i^{\mu_1}x_i^{\mu_2}\ldots x_i^{\mu_n}. \tag{8.11b}$$

Since we assume the treatment of the spacetime defect distribution to be a good approximation, the sum in Eq. (8.11b) can be replaced by integrals. These are restricted to the finite spacetime volume $\mathcal{V}$,

$$\int \mathrm{d}^4p\varrho\delta(k-p)f(p)\mapsto\varrho\left(1+\sum_{n=1}^{\infty}I_n\right)f(k), \tag{8.12a}$$

$$I_n=\frac{\mathrm{i}^n}{n!\mathcal{N}\mathcal{V}}(k-p)_{\mu_1}(k-p)_{\mu_2}\ldots(k-p)_{\mu_n}\times\int_{\mathcal{V}}\mathrm{d}^4x x^{\mu_1}x^{\mu_2}\ldots x^{\mu_n}. \tag{8.12b}$$

It is evident that this expansion only makes sense when the integrals run over a finite space; otherwise the integrals would be divergent as the integrands are not suppressed for $x\mapsto\infty$.

If we also keep Asmp. (2a) and (2b) for the moment, we end up with a "random distribution" of spacetime defects. This means that no preferred direction is defined by the distribution. In this case all integrals that are odd in x vanish, $I_{2k-1}=0$ for $k\in\{1,2,\ldots\}$, since there is no counterpart that could make up the index structure of the result.

The tensor structure of the even integrals can be generated by the metric tensor only. That is why the even integrals do not vanish but depend on combinations of metric tensors. For example, in four spacetime dimensions the integral with two indices is given by

$$\int \mathrm{d}^4x x^\mu x^\nu=\frac{\eta^{\mu\nu}}{4}\int\mathrm{d}^4x x^2=-\frac{\eta^{\mu\nu}}{24}\mathcal{R}^2\mathcal{V}. \tag{8.13}$$

Neglecting constant prefactors, the behavior of integrals that are even in x is as follows:

$$I_{2k}\sim\frac{1}{\mathcal{N}}\left(\frac{\sqrt{\mathcal{V}}}{b^3M_{\mathrm{Pl}}}\right)^k,\qquad k\in\{1,2,\ldots\}. \tag{8.14}$$

So further terms in the expansion are suppressed by powers of ratios of the Planck scale and the square root of the spacetime volume. Finally Eq. (6.14) has to be replaced by

$$\frac{1}{b^2}\mapsto\frac{1}{b^2}+\varrho b^2+\frac{b^2}{\mathcal{V}}\sum_{k=1}^{\infty}C_{2k}\left(\frac{\sqrt{\mathcal{V}}}{b^3M_{\mathrm{Pl}}}\right)^k\equiv\frac{1}{\tilde{b}^2}, \tag{8.15}$$

where C_{2k} are mere numbers. Since the new mass of ϕ neither involves any preferred directions nor any defect positions x_i, the resulting $B^{\nu\gamma}$ of Eq. (4.7b) will still have the gauge-invariant tensor structure $(k^\nu k^\gamma-\eta^{\nu\gamma}k^2)$. Although the shape of a finite spacetime volume is not Lorentz invariant, the photon is not affected by the finiteness of the spacetime. The reason is that we do not set any boundary conditions on the photon field in the framework of the simple spacetime foam model considered. Thus the dispersion relation of the photon stays $k^2=0$.

D. Anisotropic or inhomogeneous distribution

If we additionally drop Asmp. (2a), the spacetime defect distribution may be anisotropic and, therefore, define preferred directions in spacetime (see the first two panels of Fig. 4). Let us assume that there is one such direction: $(\zeta^\mu)=(\zeta^0,\zeta^1,\zeta^2,\zeta^3)$. Then the result of all spatial integrals will involve ζ. For example, the result for the integral with two indices is then made up of the metric tensor and the tensor product of preferred directions:

$$\int \mathrm{d}^4x x^\mu x^\nu=-\frac{\mathcal{R}^2\mathcal{V}}{36}(C_2^{(0)}(\zeta)\eta^{\mu\nu}+C_2^{(2)}(\zeta)\zeta^\mu\zeta^\nu), \tag{8.16a}$$

$$C_2^{(0)}(\zeta)=\frac{\zeta^2+2(\zeta^0)^2}{\zeta^2},\qquad C_2^{(2)}(\zeta)=\frac{2[\zeta^2-4(\zeta^0)^2]}{\zeta^4}. \tag{8.16b}$$

Using Asmp. (4), the second integral in the expansion can be written as follows:

[9] With side length $\mathcal{R}$ and centered at the origin.

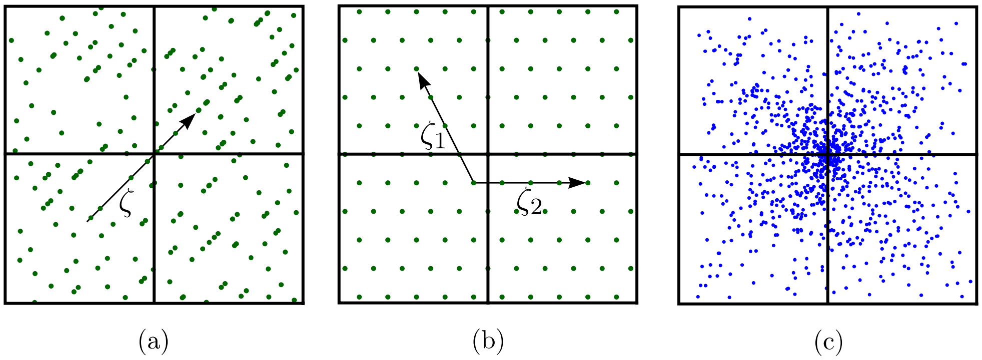


FIG. 4 (color online). The first panel (a) illustrates an anisotropic distribution defining a single preferred direction ζ, whereas the second panel (b) contains a regular lattice of defects defining two preferred directions ζ_1 and ζ_2. The third panel (c) depicts a section of an inhomogeneous defect distribution, where in the center of the region shown the density is higher than near the margin.

$$I_2 \sim \frac{\sqrt{\mathcal{V}}}{\mathcal{N}}(C_2^{(0)}(\zeta)(k-p)^2 + C_2^{(2)}(\zeta)[(k-p)\cdot\zeta]^2) \simeq \frac{1}{\mathcal{N}}\frac{\sqrt{\mathcal{V}}}{b^3 M_{\mathrm{Pl}}}\left[C_2^{(0)}(\zeta) + C_2^{(2)}(\zeta)(k\cdot\zeta)^2\frac{b}{M_{\mathrm{Pl}}}\right]. \tag{8.17}$$

It can be shown that despite the existence of a preferred spacetime direction the integrals I_{2k+1} still vanish because they are odd with respect to the integration variable. Since the tensor structure of each even integral can involve at the most $2k$ preferred directions ζ, we obtain the following general behavior for the new mass of the ϕ field:

$$\frac{1}{b^2}\mapsto\frac{1}{b^2} + \varrho b^2 + \frac{b^2}{\mathcal{V}}\sum_{k=1}^{\infty}\left(\frac{\sqrt{\mathcal{V}}}{b^3 M_{\mathrm{Pl}}}\right)^k\sum_{l=0}^{k}C_{2k}^{(l)}(\zeta)(k\cdot\zeta)^{2l}\left(\frac{b}{M_{\mathrm{Pl}}}\right)^l \equiv \frac{1}{b(k,\zeta)^2}, \tag{8.18}$$

where $C_{2k}^{(l)}(\zeta)$ are functions with respect to the preferred direction ζ. The loop integral now depends on ζ, and its tensor structure may involve terms such as $\zeta^\nu\zeta^\gamma$, $\zeta^\nu k^\gamma$, etc., which destroy the standard structure $(k^\nu k^\gamma - \eta^{\nu\gamma}k^2)$. This case may also produce a deviation from $k^2 = 0$ in the dispersion relation of the photon. For example, the appearance of a preferred spacetime direction is the reason for the modification computed in [14].

Although the calculation presented here has been performed for a finite volume, there is nothing to suggest that the physical effect originates from this fact. We have just seen that a finite volume does not have any effect on the dispersion relation of the photon so long as the defect distribution is isotropic and homogeneous.

Keeping Asmp. (2a), but dropping (2b) would result in an inhomogeneous defect distribution (see the third panel of Fig. 4). This means that the defect density cannot be considered as constant and, therefore, it cannot be pulled in front of the integral in Eq. (3.13). There is no physical input for a varying density function $\varrho = \varrho(x)$, since we are not aware of a mechanism leading to an inhomogeneous distribution of defects. The resulting integral in Eq. (3.13) will certainly be much more complicated if ϱ is a function of x. If there are regions with a high (constant) density ϱ_h and regions with a low (constant) density ϱ_l such that $\varrho_h \gg \varrho_l$, the main contributions from the integral will come from the regions with $\varrho(x) = \varrho_h$. Effectively, this means that the integration volume is reduced. How the photon dispersion relation is affected by such changes will not be further investigated in this paper.

IX. *PT*-SYMMETRIC EXTENSION AND THE PERCOLATION OF DEFECTS

In this section we would like to deliver a brief discussion on an interesting issue. Replacing the real coupling constant $\lambda^{(0)}$ in the action (2.3) by an imaginary one we find that the equation

$$k^2[1 + \mathrm{i}^2\mathcal{C}\hat{\Pi}_{\mathrm{ren}}(k^2)] = 0 \tag{9.1}$$

has a second physically acceptable solution different from the standard dispersion relation,

$$k^2 = \alpha(\gamma)\frac{1}{b_\varrho^2}, \tag{9.2}$$

with a function $\alpha(\gamma)$ and $\gamma \propto \lambda^2\varrho$. Equation (9.2) means that the photon becomes massive.

This solution will exist if γ is larger than a critical value γ_c. Furthermore α seems to lie in the interval [0, 1]. The replacement $\lambda^{(0)} \to \mathrm{i}\lambda^{(0)}$ (with $\lambda^{(0)} \in \mathbb{R}$) makes the interacting part of the Lagrangian non-Hermitian. However, this may not be a problem so long as the Lagrangian is symmetric under a combined parity transformation P and time reversal transformation T: $(PT)\mathcal{L} = \mathcal{L}$. Non-Hermitian but PT-symmetric quantum mechanics has been thoroughly studied over the past few years [30]. It was found that it can serve as a description of real physical systems; see, e.g., [31–33]. PT-symmetric quantum field theories have not been investigated that profoundly,

TABLE III. Transformation properties of the vector potential A^μ, field strength tensor $F^{\mu\nu}$, dual field strength tensor $\tilde{F}^{\mu\nu}$, (pseudo) scalar ϕ with phases η_P plus η_T, and the imaginary unit with respect to P, C, T, and their combinations PT and CPT.

Object	C	P	T	PT	CPT
Vector potential A^μ	$-A^\mu$	A_μ	A_μ	A^μ	$-A^\mu$
Field strength tensor $F^{\mu\nu}$	$-F^{\mu\nu}$	$F_{\mu\nu}$	$-F_{\mu\nu}$	$-F^{\mu\nu}$	$F^{\mu\nu}$
Dual field strength tensor $\tilde{F}^{\mu\nu}$	$-\tilde{F}^{\mu\nu}$	$-\tilde{F}_{\mu\nu}$	$\tilde{F}_{\mu\nu}$	$-\tilde{F}^{\mu\nu}$	$\tilde{F}^{\mu\nu}$
(Pseudo) scalar ϕ	ϕ	$\eta_P\phi$	$\eta_T\phi$	$\eta_P\eta_T\phi$	$\eta_P\eta_T\phi$
Imaginary unit i	i	i	−i	−i	−i

though. Nevertheless there are indications that such extensions of ordinary Hermitian theories are physically meaningful [34]. According to Table III the last term of the action (2.3) is indeed PT symmetric for an imaginary coupling constant $\lambda^{(0)}$ (and the scalar field replaced by a (pseudo) scalar field with appropriate transformation properties, i.e., with the phase choices $\eta_P = \pm 1$, $\eta_T = \mp 1$).

Observing the relation between γ and ϱ we speculate that this solution could be related to the percolation of defects. Percolation theory studies the formation and properties of clusters of objects randomly distributed over a lattice or a continuous space [35,36]. This theory exhibits a phase transition. As the density of the objects increases so does the mean size of the clusters, and it becomes infinitely large at some critical density ϱ_c. For densities smaller than ϱ_c only clusters of finite size exist, while for densities larger than ϱ_c clusters of infinite size also appear. At the critical point the system is scale invariant.

The assumption that defect percolation may play a role here is supported by the observation that the behavior of

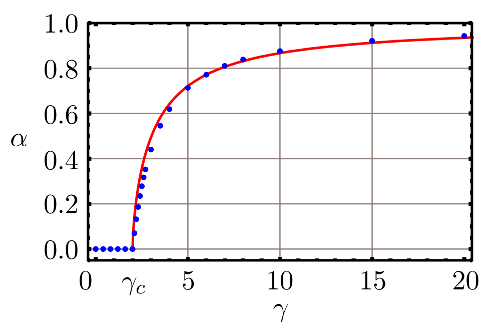


FIG. 5 (color online). Several data points obtained for the coefficient $\alpha(\gamma)$ in Eq. (9.2) are compared with the expected behavior for the order parameter: $\alpha(\gamma) \propto (1 - \gamma_c/\gamma)^\beta$. The exponent used is $\beta = 0.64$, which is also obtained in the context of a four-dimensional lattice percolation. (See Table 2 on p. 52 in [35] and Table 1.2 on p. 81 in [36] for a slightly different value. Besides, in [37] continuous percolation is studied, and it is observed that the critical exponents are the same for continuous and lattice percolation, at least in two and three dimensions.) From the plot it becomes evident that $\gamma_c \approx 2$. The latter point depends on the renormalization scale we choose.

$\alpha(\gamma)$ is well described by the critical exponent β of the percolation phase transition in four dimensions (cf. Fig. 5). This issue deserves a further study.

The general argument about the occurrence of a photon mass given at the end of Sec. VII B does not hold any more. This is because the argument relied on a photon mass continuously depending on the model parameters, which is not the case here (as can be seen in Fig. 5).

X. SUMMARY AND CONCLUSIONS

To summarize, we have investigated photon propagation through a spacetime foam made up of a Lorentz-invariant distribution of time-dependent, pointlike defects. According to the action of Eq. (2.5), photons do not interact directly with the defects but via a scalar mediator field.

If the spacetime volume is assumed to be infinite and the distribution of defects is dense, isotropic, and homogeneous, the only effect on the scalar mediator is that its mass increases. The defects then effectively act as a modified background analogous to the pictorial representation of a particle getting its mass via the interaction with the vacuum expectation value of the Higgs field. So for the scalar field the defects have been "integrated out," and the photon interacts with a scalar having a mass that is larger compared to the case without defects.

The outcome is that the photon dispersion law remains standard: $k^2 = k_0^2 - |\mathbf{k}|^2 = 0$. Hence the photon does not "feel" the random background field that mimics the defects as an effective theory—at least at leading order in the interaction between the photon and the scalar field. The reason for this is that, on the one hand, no preferred spacetime directions will appear if the defects are distributed in a Lorentz-invariant manner. On the other hand, the photon does not become massive as gauge invariance is not violated.

The result obtained had already been anticipated by, e.g., [19,20] based on general arguments. Nevertheless it always makes sense to test physical arguments by a direct calculation, which was our motivation for the work presented here. In fact, the realm of spacetime foam is not flooded by physical models. Furthermore the properties of a quantum field theory based on an action of the form that we proposed are not well-known facts. For this reason it makes sense to thoroughly investigate and understand such theories.

The outcome of the calculation sheds some new light on the very restrictive bounds on Lorentz-violating parameters of the standard model extension [22]. If a nontrivial spacetime structure is assumed to be the underlying cause for a possible Lorentz violation, low-energy experiments are unlikely to detect Lorentz violation if at energies much smaller than the Planck energy this structure can be described by the effective theory considered.

TABLE IV. The fate of Lorentz invariance where, in short, the assumptions taken are listed as follows: (1) effective theory of Eq. (2.5); (2a)/(2b) homogeneous/isotropic defect distribution; (2c) dense defect distribution; (2d) infinite spacetime volume; (3) modified photon momentum square $\ll$ mass of scalar field square; (4) momentum transfer from particles to defects suppressed by the Planck scale.

Keep	(1)	(2a), (2b)	(2c)	(2d)	(3)	(4)	Lorentz violation
	√	√	√	√	√	Not needed	Absent
	√		√	√	√	√	√
	√	√		√	√	√	√
	√	√	√		√	√	(?)
	√	√	√	√		Not needed	Massive photon

However, one also has to keep in mind that the photon dispersion relation stays conventional because of several idealized assumptions taken to ensure a feasible computation. Using physical principles we have tried to predict the effect on the photon dispersion relation when one of these assumptions is dropped. Discarding the assumption of

(i) the action of Eq. (2.5): The model is no longer an appropriate description as the defect structure itself is expected to become important. Furthermore, the theory could be modified such that the scalar field is discarded altogether and the photon is made to interact directly with the defects. Then the photon would probe the defects with its wavelength and the photon dispersion relation can be assumed to depend on the photon energy k_0. Because of $k_0 = \xi \cdot k$ with $\xi = (1, 0, 0, 0)$, a preferred spacetime direction comes into play indicating a violation of Lorentz invariance.

(ii) an infinite spacetime volume: In this case the dispersion relation of the photon stays the same since gauge invariance is not violated. Note that in the simple model proposed no boundary conditions are set on any field and, therefore, the photon does not feel the finite volume. However, if the sprinkling procedure changes because of the finite volume, the photon dispersion relation might also change. Because of calculational difficulties this case is a challenging task.

(iii) a dense distribution of defects: If the defects are assumed to be separated by a large distance, their distribution can no longer be considered to be continuous. The defect positions x_i will appear explicitly in the tensor structure of the photon field equation. This may lead to a modified dispersion relation of the photon that involves these positions.

(iv) an isotropic and homogeneous (random) defect distribution: If the distribution is dense but anisotropic, it defines a preferred direction ζ in spacetime. Then ζ also shows up in the tensor structure of the photon field equation. This may lead to a modified dispersion relation of the photon and, therefore, Lorentz violation.

(v) the photon momentum squared being much smaller than the mass of scalar particle squared: The photon gets a mass and electromagnetic waves are damped. We discard this case as it does not correspond to the physical reality for photons with an energy much smaller than the Planck energy. Otherwise we would not be able to observe light from distant galaxies.

It has not been possible to derive the modified photon dispersion relation for the more complicated cases mentioned above. This may be done in a future research project. Besides that, we did not rigorously demonstrate the influence of a finite spacetime volume as we did not impose any boundary conditions on the photon field. The final results are summarized in Table IV.

Finally, let us compare our model to alternative realizations of a spacetime foam. In [38] a toy model for

TABLE V. Comparison of various characteristics of certain spacetime foam models to the properties of the model introduced in this paper. Here QFT means quantum field theory and EM/QM abbreviates (classical) electrodynamics/quantum mechanics. (*) A possible topological defect structure is considered in the first chapters of the paper but not in the effective model. (**) The main results are obtained for scattering at one single defect. Under certain approximations they are generalized to many defects.

	[38]	[14]	[13]	[39]	Our model
Spacetime	2 + 1	3 + 1	2(3) + 1	3 + 1	3 + 1
Framework	QFT	QFT	EM/QM	QFT	QFT
Connection to electrons, photons		√	√	√	√
Internal defect structure	√	(*)	√		
Momentum conserved	√	√	√		√
Gauge invariance conserved	⋯	√	√		√
Number of defects considered	One	Many	One (**)	Many (sprinkling)	Many (sprinkling)
Background field replacing defects	⋯	√	⋯	√	√
Connection to experiments		√	√	√	

spacetime foam is proposed. Contrary to the model presented here it treats defects with a topological structure. This is done in a $(2+1)$-dimensional spacetime. A defect is constructed by removing a disk from two-dimensional flat space and identifying points on the remaining boundary. Dependent on the procedure of identification, both orientable and nonorientable topological spaces may emerge. After performing the identification, connected sums of the individual spaces are considered as well. The stationary wave equation is solved for the scattering of a scalar field at such a defect from which the cross section of the respective process can be calculated. It was then shown that the cross section decreases with increasing topological complexity (e.g., the genus of the topological space) or deformation of the defect considered.

Reference [39] has appeared recently and proposes another alternative approach to spacetime defects where the first article deals with nonlocal and the second with local defects. Since the model presented here ought to describe local defects, we especially refer to the second of these papers. An internal defect structure is neglected as was done in our model. A sprinkling of defects according to a Poisson process is introduced. The scattering of particles at defects is characterized by momentum violation where the momentum change of the initial particle is described by a Gaussian distribution. The defects are then minimally coupled to standard model particles such as the electron and the photon. This leads to modified particle-physics processes serving as a basis for bounds on the model parameters.

See Table V for a summary of the commonalities and differences of the spacetime foam models mentioned in comparison to the model presented within the current paper.

As a final remark, the physical understanding of spacetime defects is at its infancy. Currently there are not many models on the market. To our best knowledge we have referred extensively to work that is directly related to ours. Among the existing models proposed there exists a lot of controversy. A better understanding of defect scattering from a theoretical point of view may help to merge some ideas of each approach to obtain a consistent description of spacetime foam—at least for energies much below the Planck scale.

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TABLE VI. Different values Δ of dimensionless defect distances are considered. For specific values the numbers of Eq. (A1) are computed and compared to each other.

Δ	w_1/w_2
1	0.701693
0.1	0.966576
0.01	0.996615
0.004	0.998645

APPENDIX A: HOW GOOD IS THE ASSUMPTION OF A DENSE DEFECT DISTRIBUTION?

In this section we consider four-dimensional Minkowski spacetime with dimensionless coordinates t^μ. The numbers

$$w_1 = \sum_{k=1}^{n} \exp(\mathrm{i}t_k^\mu 1_\mu), \qquad w_2 = \varrho \int_{\mathbb{H}} \exp(\mathrm{i}t^\mu 1_\mu) \tag{A1}$$

shall be computed, where $(1^\mu) = (1, 1, 1, 1)$. The sum in w_1 runs over n defects contained in a four-dimensional unit-hypercube $\mathbb{H}$ whose edges are supposed to lie parallel to the axes of the coordinate system. The space diagonal of the hypercube runs from the point $(0, 0, 0, 0)$ to $(1, 1, 1, 1)$. The defects are assumed to lie at equal distances Δ. Hence, the hypercube contains $(\Delta^{-1}+1)^4$ defects leading to the density $\varrho = (\Delta^{-1}+1)^4$. The integral in w_2 runs over the same hypercube. We compare the values w_1 and w_2 for different defect spacings Δ. In principle w_1 corresponds to w_2 for infinitesimal defect separation. In Table VI we see that the integral is already a good approximation for the dimensionless distance $\Delta = 0.01$.

We can now replace the dimensionless scalar product $t^\mu 1_\mu$ by the scalar product of two dimensionful quantities: the wave vector k^μ and the spatial four-vector x_μ. Then Δ corresponds to a product of a wave vector K and a distance ΔX in configuration space. Assuming the photon energy $E = 1$ TeV the dimensionless distance Δ is in accordance with the following ΔX[10]:

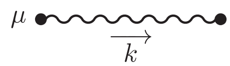
$$\Delta X = \frac{\Delta}{K} = \frac{1/100}{10^{12}\ \mathrm{eV} \times 1.602 \times 10^{-19}\ \mathrm{J/eV}} \hbar c \approx 2 \times 10^{-21}\ \mathrm{m}. \tag{A2}$$

Thus the approximation we use is already very good even if the defect separation lies many orders of magnitude above the Planck length.

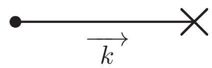
[10]We assume the standard dispersion relation of the photon.

APPENDIX B: PERTURBATIVE FEYNMAN RULES OF THE MODIFIED THEORY

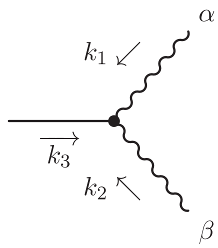
The Feynman rules (B1a)–(B1c) directly follow from the action given by Eq. (2.3):



$$= -\mathrm{i}\eta^{\mu\nu}\widetilde{\Delta}(k)\,, \quad \widetilde{\Delta}(k) = \frac{1}{k^2+\mathrm{i}\epsilon}\,, \tag{B1a}$$



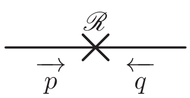
$$= -\mathrm{i}\widetilde{H}(k)\,, \quad \widetilde{H}(k) = \frac{-(b^{(0)})^2}{k^2-1/(b^{(0)})^2+\mathrm{i}\epsilon}\,, \tag{B1b}$$



$$= \mathrm{i}\lambda^{(0)}\varepsilon^{\mu\alpha\varrho\beta}k_{1,\mu}k_{2,\varrho} = -\mathrm{i}\lambda^{(0)}\varepsilon^{\mu\alpha\varrho\beta}k_{1,\mu}k_{3,\varrho}\,. \tag{B1c}$$

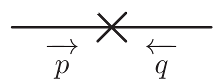
The first Feynman rule gives the photon propagator of the modified theory in the Feynman gauge, which corresponds to the photon propagator of standard QED (in this gauge). The second gives the propagator of the scalar field ϕ, which has to be connected to a single defect. The third describes the interaction between ϕ and the photon.

The fourth Feynman rule, i.e., the "defect vertex" for a finite number $\mathcal{N}$ of defects in a box with side length $\mathcal{R}$, follows from the general considerations in III B. It reads



$$= \widetilde{G}_{\mathscr{R}}(p)\widetilde{G}_{\mathscr{R}}(q)\,. \tag{B1d}$$

Performing the limit $\mathcal{N}\mapsto\infty$ [consider Asmp. (2a)–(2d)] leads to the fifth Feynman rule that we use to compute the one-loop correction in this limit:



$$= \varrho\,\delta^{(4)}(p+q)\,. \tag{B1e}$$

So each ϕ line has to begin or end at one defect. If this is not the case, one of the momenta has to be set to zero.

In fact, there exist additional Feynman rules for the scattering of the scalar field at a defect. This can be seen by computing, e.g., the third-order perturbative solution of Eq. (4.1). It follows from inserting the first-order perturbative solution into the second-order solution:

$$\begin{aligned}(\lambda^{(0)})^3\tilde{A}^{(3)\nu}(k) &= \lim_{\mathcal{R}\mapsto\infty}\frac{(\lambda^{(0)})^2}{(2\pi)^8}\tilde{\Delta}(k)\int \mathrm{d}^4q\,\tilde{g}_{\mathcal{R}}(q)\varepsilon^{\mu\nu\varrho\sigma}q_\mu(k-q)_\varrho\tilde{\Delta}(k-q)\\ &\quad\times\int\mathrm{d}^4p\,\tilde{g}_{\mathcal{R}}(k-p-q)\varepsilon_{\alpha\beta\gamma\sigma}(k-p-q)^\alpha p^\beta(\lambda^{(0)}\tilde{A}^{(1)\gamma}_{\mathcal{R}}(p))\\ &= -\lim_{\mathcal{R}\mapsto\infty}\frac{(\lambda^{(0)})^3}{(2\pi)^{12}}(\varepsilon^{\mu\nu\kappa\lambda}\varepsilon_{\alpha\beta\gamma\lambda}\varepsilon^{\delta\gamma\varrho\sigma})\tilde{\Delta}(k)\int\mathrm{d}^4q\int\mathrm{d}^4p\\ &\quad\times\int\mathrm{d}^4l\,\tilde{\Delta}(k-q)\tilde{\Delta}(p)q_\mu(k-q)_\kappa(k-p-q)_\alpha p_\beta l_\delta(p-l)_\varrho\tilde{g}_{\mathcal{R}}(q)\tilde{g}_{\mathcal{R}}(k-p-q)\tilde{g}_{\mathcal{R}}(l)\tilde{A}^{(0)}_\sigma(p-l).\end{aligned} \tag{B1f}$$

Now the product of three functions $\tilde{g}_{\mathcal{R}}(k)$ occurring in Eq. (B1f) has to be evaluated. According to Sec. III B and Eq. (3.15) this expression leads to a product of three $\tilde{G}_{\mathcal{N}}$:

$$\lim_{\mathcal{N}\mapsto\infty}\tilde{G}_{\mathcal{N}}(k)\tilde{G}_{\mathcal{N}}(p)\tilde{G}_{\mathcal{N}}(q) = \lim_{\mathcal{N}\mapsto\infty}\frac{1}{\mathcal{N}}\left[\sum_{i=1}^{\mathcal{N}}\exp[\mathrm{i}(k+p+q)x_i] + \sum_{l\neq m\neq n}P_{lmn}\right], \tag{B1ga}$$

$$P_{lmn} = \varepsilon_l\varepsilon_m\varepsilon_n\exp(\mathrm{i}k\cdot x_l)\exp(\mathrm{i}p\cdot x_m)\exp(\mathrm{i}q\cdot x_n). \tag{B1gb}$$

Analogously to Eq. (3.18) it can be shown that the sum over the P_{lmn} is suppressed due to the existence of neighboring defects with opposite charge. Hence we obtain the following result, which is similar to Eq. (3.21):

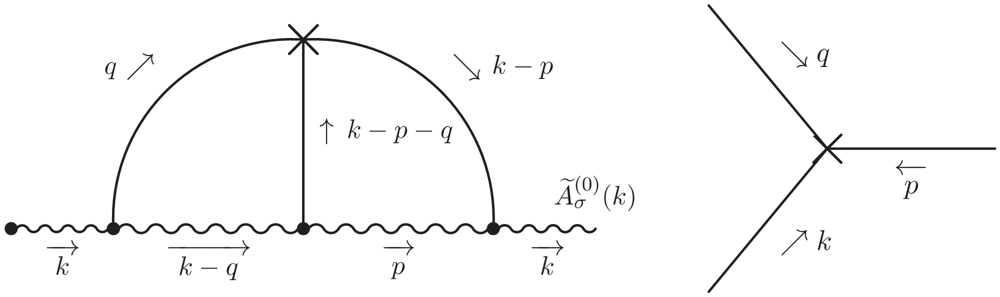


FIG. 6. Additional contribution to the photon field with more than two scalar field lines attached to a defect vertex [see (a)]. From this and Eq. (B1h) we obtain the Feynman rule where three scalar fields are attached to such a vertex, corresponding to the mathematical expression $\varrho\delta^{(4)}(k+q+p)$ [see (b)]. Additional Feynman rules with even more external scalars can be derived analogously.

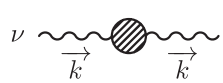
$$\lim_{\mathcal{R}\mapsto\infty}\tilde{G}_{\mathcal{R}}(k)\tilde{G}_{\mathcal{R}}(p)\tilde{G}_{\mathcal{R}}(q)=\lim_{\mathcal{V}\mapsto\infty}\frac{1}{\mathcal{V}}(2\pi)^4\varrho\delta^{(4)}(k+p+q). \tag{B1h}$$

Using this, one of the integrals in Eq. (B1f) is eliminated by the δ function:

$$\begin{aligned}(\lambda^{(0)})^3\tilde{A}^{(3)\nu}(k)=&-\frac{(\lambda^{(0)})^3\varrho}{(2\pi)^8}(\varepsilon^{\mu\nu\kappa\lambda}\varepsilon_{\alpha\beta\gamma\lambda}\varepsilon^{\delta\gamma\varrho\sigma})\tilde{\Delta}(k)\int\mathrm{d}^4q\int\mathrm{d}^4p\,\tilde{\Delta}(k-q)\tilde{\Delta}(p)\tilde{H}(q)\tilde{H}(k-p-q)\tilde{H}(p-k)q_\mu(k-q)_\kappa\\&\times(k-p-q)_\alpha p_\beta(p-k)_\delta k_\varrho\tilde{A}^{(0)}_\sigma(k).\end{aligned} \tag{B1i}$$

The latter contribution is a two-loop integral, and it is associated with the Feynman diagram in Fig. 6(a). Therefore it is evident that not only defect vertices exist with two scalar fields attached. In principle there are such vertices where an arbitrary number of scalar fields can come together such that momentum is conserved [see Fig. 6(b) for such a vertex with three external scalar fields]. However, diagrams involving such vertices are at least of order $(\lambda^{(0)})^3$; i.e., we neglect these contributions in our calculation.

Respecting Asmp. (2a)–(2d) the following effective vertex can be introduced for the interaction of a photon with the defect via the ϕ field. The latter has been integrated out at the one-loop level resulting in



$$=-\mathcal{C}\mathrm{i}\Pi^{\nu\gamma}(k)\,,\quad \mathcal{C}=b_\varrho^4\lambda^2\varrho\,, \tag{B1j}$$

with $\Pi^{\nu\gamma}(k)$ given by Eq. (7.1b).

APPENDIX C: DERIVATION OF THE PASSARINO-VELTMAN REDUCTION

In this section the Passarino-Veltman reduction of the tensor integral $I_{\varrho\sigma}$ given by Eq. (5.3a) in Sec. V B will be presented in detail.

$$I_{\varrho\sigma}=\int\mathrm{d}^dq\frac{q_\varrho q_\sigma}{[q^2-1/(b^{(0)})^2+\mathrm{i}\epsilon]^2[(k-q)^2+\mathrm{i}\epsilon]}. \tag{C1}$$

Performing the reduction we can neglect all $\mathrm{i}\epsilon$. First, we contract $I_{\varrho\sigma}$ with $-2k^\varrho$, and this leads to

$$-2k^\varrho I_{\varrho\sigma}=\int\mathrm{d}^dq\frac{-(2k\cdot q)q_\sigma}{[q^2-1/(b^{(0)})^2]^2(k-q)^2}=\underbrace{\int\mathrm{d}^dq\frac{q_\sigma}{[q^2-1/(b^{(0)})^2]^2}}_{=0}-\int\mathrm{d}^dq\frac{(k^2+q^2)q_\sigma}{[q^2-1/(b^{(0)})^2]^2(k-q)^2}. \tag{C2}$$

Second, by contracting the latter result again with $-2k^\sigma$ we obtain the quantity $4K_1$ defined in Sec. V B,

$$4K_1 = 4k^{\varrho}k^{\sigma}I_{\varrho\sigma} = -\int \mathrm{d}^d q \frac{(k^2+q^2)(-2k\cdot q)}{[q^2-1/(b^{(0)})^2]^2(k-q)^2} = -\underbrace{\int \mathrm{d}^d q \frac{k^2+q^2}{[q^2-1/(b^{(0)})^2]^2}}_{\equiv \hat{I}_3} + \underbrace{\int \mathrm{d}^d q \frac{(k^2+q^2)^2}{[q^2-1/(b^{(0)})^2]^2(k-q)^2}}_{\equiv \hat{I}_4}. \tag{C3}$$

Now we still have to decompose the integrals $\hat{I}_3$ and $\hat{I}_4$ into the master integrals of Eq. (5.8),

$$\begin{aligned}\hat{I}_3 &= \int \mathrm{d}^d q \frac{k^2+q^2}{[q^2-1/(b^{(0)})^2]^2} = k^2\int \mathrm{d}^d q \frac{1}{[q^2-1/(b^{(0)})^2]^2} + \int \mathrm{d}^d q \frac{q^2}{[q^2-1/(b^{(0)})^2]^2} \\ &= \left(k^2+\frac{1}{(b^{(0)})^2}\right)\int \mathrm{d}^d q \frac{1}{[q^2-1/(b^{(0)})^2]^2} + \int \mathrm{d}^d q \frac{1}{q^2-1/(b^{(0)})^2}.\end{aligned} \tag{C4}$$

$$\begin{aligned}\hat{I}_4 &= \int \mathrm{d}^d q \frac{k^4+2k^2q^2+q^4}{[q^2-1/(b^{(0)})^2]^2(k-q)^2} \\ &= k^4\int \mathrm{d}^d q \frac{1}{[q^2-1/(b^{(0)})^2]^2(k-q)^2} + 2k^2\underbrace{\int \mathrm{d}^d q \frac{q^2}{[q^2-1/(b^{(0)})^2]^2(k-q)^2}}_{\equiv \hat{I}_5} + \underbrace{\int \mathrm{d}^d q \frac{q^4}{[q^2-1/(b^{(0)})^2]^2(k-q)^2}}_{\equiv \hat{I}_6}.\end{aligned} \tag{C5}$$

$$\hat{I}_5 = \int \mathrm{d}^d q \frac{1}{[q^2-1/(b^{(0)})^2](k-q)^2} + \frac{1}{(b^{(0)})^2}\int \mathrm{d}^d q \frac{1}{[q^2-1/(b^{(0)})^2]^2(k-q)^2}. \tag{C6}$$

$$\begin{aligned}\hat{I}_6 &= \underbrace{\int \mathrm{d}^d q \frac{1}{(k-q)^2}}_{=0} + \frac{2}{(b^{(0)})^2}\int \mathrm{d}^d q \frac{q^2}{[q^2-1/(b^{(0)})^2]^2(k-q)^2} - \frac{1}{(b^{(0)})^4}\int \mathrm{d}^d q \frac{1}{[q^2-1/(b^{(0)})^2]^2(k-q)^2} \\ &= \frac{2}{(b^{(0)})^2}\int \mathrm{d}^d q \frac{1}{[q^2-1/(b^{(0)})^2](k-q)^2} + \frac{1}{(b^{(0)})^4}\int \mathrm{d}^d q \frac{1}{[q^2-1/(b^{(0)})^2]^2(k-q)^2}.\end{aligned} \tag{C7}$$

The contraction of $I_{\varrho\sigma}$ with the metric tensor leads to K_2, which was also defined in Sec. V B,

$$\eta^{\varrho\sigma}I_{\varrho\sigma} = \int \mathrm{d}^d q \frac{q^2}{[q^2-1/(b^{(0)})^2]^2(k-q)^2} = \int \mathrm{d}^d q \frac{1}{[q^2-1/(b^{(0)})^2](k-q)^2} + \frac{1}{(b^{(0)})^2}\int \mathrm{d}^d q \frac{1}{[q^2-1/(b^{(0)})^2]^2(k-q)^2}. \tag{C8}$$

APPENDIX D: COMPUTATION OF SCALAR INTEGRALS

Finally we evaluate the scalar integrals of Eq. (5.8) in Sec. V B. We use dimensional regularization with $d = 4 - 2\hat{\varepsilon}$, and later on we employ the reasonable redefinition

$$\frac{1}{\varepsilon} \equiv \frac{1}{\hat{\varepsilon}} - \gamma_E + \ln(4\pi). \tag{D1}$$

The simplest integral without any external momenta can be computed as follows:

$$\begin{aligned}A_0\left(\frac{1}{(b^{(0)})^2}\right) &= -\mathrm{i}(2\pi\mu)^{4-d}\int \mathrm{d}\Omega_d \int_0^\infty \mathrm{d}q \frac{q^{d-1}}{q^2+1/(b^{(0)})^2} = -\mathrm{i}(2\mu)^{4-d}\frac{2\pi^{4-d/2}}{\Gamma(d/2)}(b^{(0)})^{2-d}\frac{\Gamma(1-d/2)\Gamma(d/2)}{2} \\ &= -\mathrm{i}(2\mu)^{4-d}\pi^{4-d/2}(b^{(0)})^{2-d}\Gamma\left(1-\frac{d}{2}\right) = -\mathrm{i}(2\mu)^{2\hat{\varepsilon}}\pi^{2+\hat{\varepsilon}}(b^{(0)})^{2(\hat{\varepsilon}-1)}\Gamma(\hat{\varepsilon}-1) \\ &= \mathrm{i}\pi^2\frac{1}{(b^{(0)})^2}\left[\frac{1}{\varepsilon} - \ln\left(\frac{1}{(b^{(0)})^2\mu^2}\right) + 1\right] + \mathcal{O}(\varepsilon).\end{aligned} \tag{D2}$$

The remaining integrals can be evaluated in the same way,

$$\begin{aligned} B_0\left(0, \frac{1}{(b^{(0)})^2}, \frac{1}{(b^{(0)})^2}\right) &= \mathrm{i}(2\pi\mu)^{4-d} \int \mathrm{d}\Omega_d \int_0^\infty \mathrm{d}q \frac{q^{d-1}}{(q^2 + 1/(b^{(0)})^2)^2} \\ &= \mathrm{i}(2\pi\mu)^{4-d} \frac{2\pi^{d/2}}{\Gamma(d/2)} (b^{(0)})^{4-d} \frac{\Gamma(2 - d/2)\Gamma(d/2)}{2\Gamma(2)} \\ &= \mathrm{i}(2\mu)^{4-d} \pi^{4-d/2} (b^{(0)})^{4-d} \Gamma\left(2 - \frac{d}{2}\right) = \mathrm{i}(2\mu)^{2\hat{\varepsilon}} \pi^{2+\hat{\varepsilon}} (b^{(0)})^{2\hat{\varepsilon}} \Gamma(\hat{\varepsilon}) \\ &= \mathrm{i}\pi^2 \left[\frac{1}{\varepsilon} - \ln\left(\frac{1}{(b^{(0)})^2\mu^2}\right)\right] + \mathcal{O}(\varepsilon). \end{aligned} \tag{D3}$$

$$\begin{aligned} B_0\left(-k, \frac{1}{(b^{(0)})^2}, 0\right) &= (2\pi\mu)^{4-d} \int \mathrm{d}^d q \frac{1}{(q^2 - 1/(b^{(0)})^2 + \mathrm{i}\epsilon)[(k-q)^2 + \mathrm{i}\epsilon]} \\ &= (2\pi\mu)^{4-d} \int_0^1 \mathrm{d}x \int \mathrm{d}^d q \left[q^2 - \frac{1}{(b^{(0)})^2} + \left(k^2 - 2k\cdot q + \frac{1}{(b^{(0)})^2}\right)x + \mathrm{i}\epsilon\right]^{-2} \\ &= (2\pi\mu)^{4-d} \int_0^1 \mathrm{d}x \int \mathrm{d}^d q [(q - kx)^2 - M^2]^{-2}, \end{aligned} \tag{D4}$$

where

$$-M^2 \equiv -k^2x^2 + \left(k^2 + \frac{1}{(b^{(0)})^2}\right)x - \frac{1}{(b^{(0)})^2} + \mathrm{i}\epsilon. \tag{D5}$$

This then leads to

$$B_0\left(-k, \frac{1}{(b^{(0)})^2}, 0\right) = \mathrm{i}\pi^2 \left\{\frac{1}{\varepsilon} - \int_0^1 \mathrm{d}x \ln\left[\frac{k^2x^2 - (k^2 + 1/(b^{(0)})^2)x + 1/(b^{(0)})^2 - \mathrm{i}\epsilon}{\mu^2}\right]\right\} + \mathcal{O}(\varepsilon) \equiv \mathrm{i}\pi^2\left(\frac{1}{\varepsilon} - I_{B_{0,2}}\right) + \mathcal{O}(\varepsilon). \tag{D6}$$

As already mentioned, the C_0 integral is both infrared and ultraviolet convergent. Hence we can set $d = 4$ at the beginning, and we obtain

$$\begin{aligned} C_0\left(-k, 0, \frac{1}{(b^{(0)})^2}, 0, \frac{1}{(b^{(0)})^2}\right) &= \int \mathrm{d}^4 q \frac{1}{(q^2 - 1/(b^{(0)})^2 + \mathrm{i}\epsilon)[(k-q)^2 + \mathrm{i}\epsilon]} \\ &= \frac{\Gamma(2+1)}{\Gamma(2)\Gamma(1)} \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \delta(1 - x_1 - x_2) x_1 \left\{\left(q^2 - \frac{1}{(b^{(0)})^2} + \mathrm{i}\epsilon\right)x_1 + [(k-q)^2 + \mathrm{i}\epsilon]x_2\right\}^{-3} \\ &= 2\int_0^1 \mathrm{d}x_2 \frac{1 - x_2}{[(q - kx_2)^2 - M^2]^3}, \end{aligned} \tag{D7}$$

with M^2 given by Eq. (D5). Finally this results in

$$C_0\left(-k, 0, \frac{1}{(b^{(0)})^2}, 0, \frac{1}{(b^{(0)})^2}\right) = -\mathrm{i}\pi^2 \int_0^1 \mathrm{d}x \frac{1 - x}{k^2x^2 - (k^2 + 1/(b^{(0)})^2)x + 1/(b^{(0)})^2} \equiv -\mathrm{i}\pi^2 I_{C_0}. \tag{D8}$$

Now we want to compute the remaining one-dimensional integrals

$$\begin{aligned} I_{B_{0,2}} &= \int_0^1 \mathrm{d}x \ln\left[\frac{k^2x^2 - (k^2 + 1/(b^{(0)})^2) + 1/(b^{(0)})^2 - \mathrm{i}\epsilon}{\mu^2}\right] \overset{x \mapsto 1-x}{=} \int_0^1 \mathrm{d}x \ln\left[\frac{k^2x^2 - (k^2 - 1/(b^{(0)})^2)x - \mathrm{i}\epsilon}{\mu^2}\right] \\ &= -\ln(\mu^2) + \int_0^1 \mathrm{d}x \ln(x) + \int_0^1 \mathrm{d}x \ln[k^2 x - (k^2 - 1/(b^{(0)})^2) - \mathrm{i}\epsilon] \\ &= -\ln(\mu^2) - 1 + \frac{1}{k^2}[y\ln(y) - y]_{-k^2 + 1/(b^{(0)})^2 - \mathrm{i}\epsilon}^{1/(b^{(0)})^2 - \mathrm{i}\epsilon} \\ &= -\ln(\mu^2) - 1 + \frac{1}{k^2}\left[\frac{1}{(b^{(0)})^2}\ln\left(\frac{1}{(b^{(0)})^2} - \mathrm{i}\epsilon\right) - \frac{1}{(b^{(0)})^2} + \left(k^2 - \frac{1}{(b^{(0)})^2}\right)\ln\left(-k^2 + \frac{1}{(b^{(0)})^2} - \mathrm{i}\epsilon\right) - k^2 + \frac{1}{(b^{(0)})^2}\right] \\ &= \ln\left(\frac{1}{(b^{(0)})^2\mu^2}\right) - 2 + \frac{k^2 - 1/(b^{(0)})^2}{k^2}\ln\left(\frac{-k^2 + 1/(b^{(0)})^2}{1/(b^{(0)})^2} - \mathrm{i}\epsilon\right). \end{aligned} \tag{D9}$$

$$k^2 I_{C_0} = \int_0^1 \mathrm{d}x \frac{1-x}{k^2x^2 - (k^2 + 1/(b^{(0)})^2)x + 1/(b^{(0)})^2 - \mathrm{i}\epsilon} \overset{x\mapsto 1-x}{=} \int_0^1 \mathrm{d}x \frac{1}{x - 1 + 1/[(b^{(0)})k]^2 - \mathrm{i}\epsilon}$$
$$= \left[\ln\left(x - 1 + \frac{1}{[(b^{(0)})k]^2} - \mathrm{i}\epsilon\right)\right]_0^1 = \ln\left(\frac{1}{[(b^{(0)})k]^2} - \mathrm{i}\epsilon\right) - \ln\left(\frac{1}{[(b^{(0)})k]^2} - 1 - \mathrm{i}\epsilon\right) = -\ln[1 - (b^{(0)})^2k^2 - \mathrm{i}\epsilon]. \tag{D10}$$

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