

Superembedding methods for current superfields

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We extend the superembedding formalism for 4D $\mathcal{N} = 1$ superconformal field theory to the case of fields in arbitrary representations of the superconformal group $SU(2, 2|1)$. For applications we obtain manifestly superconformally covariant expressions for two- and three-point functions involving conserved currents, e.g., the supercurrent multiplet or global symmetry current superfields. The embedding space results are presented in a compact form by employing an index-free formalism. Our expressions are consistent with the literature, but the manifestly covariant forms of correlators presented here are new.

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I. INTRODUCTION

Four-dimensional conformal symmetry imposes stringent constraints on the form of quantum field theory correlators as well as restrictions on scaling dimensions of certain operators [1]. However, the full implications of conformal invariance in four dimensions are not yet completely known. Recently, there has been some progress in using general principles such as unitarity, the operator product expansion (OPE) and its conformal block decomposition, and crossing symmetry to derive constraints on four-dimensional conformal field theories (CFTs). See, for instance, Refs. [2] for recent results in this direction.

A useful tool for exploring the consequences of conformal invariance in four dimensions is the embedding space formalism [3,4], in which four-dimensional Minkowski space is identified with the projective light cone in a flat six-dimensional space with signature (4, 2) metric, which we will refer to as the “embedding space.” The conformal group $SO(4, 2)$ acts linearly on the embedding space, so working in this framework makes conformal symmetry manifest at the level of the correlation functions. The embedding space language was first applied to field theory in Refs. [3] to derive manifestly covariant free wave equations and employed in the context of general interacting CFTs in Ref. [4]. From the point of view of constraining CFTs, applications of embedding space methods include a conformally covariant formulation of the OPE [5] and the derivation of closed-form expressions for the conformal partial wave decomposition of four-point functions in four and six dimensions [6]. More recent results can be found, e.g., in Refs. [7,8].

In this paper, we consider a supersymmetric version of the projective light-cone formalism, which is appropriate to describe four-dimensional (4D) $\mathcal{N} = 1$ superconformally invariant field theory (SCFT), following the recent work of Ref. [9]. One motivation for focusing on supersymmetric conformal theories is that they provide a large sample of interacting CFTs with often tractable dynamics, which could be used to explicitly test the recent ideas discussed in Refs. [2]. Indeed, most nontrivial 4D CFTs

with a known microscopic realization (apart from perturbative Banks–Zaks-type fixed point theories) are in fact SCFTs.

Reference [9] showed how to realize the $SU(2, 2|1)$ $\mathcal{N} = 1$ superconformal symmetry on an embedding superspace for which the coordinates transform linearly under $SU(2, 2|1)$. These coordinates are spanned by a set of supermatrices that decompose into seven bosonic and four Grassmann components and transform in 11-dimensional irreducible representations of $SU(2, 2|1)$. Four-dimensional Minkowski superspace is realized in terms of a set of covariant quadratic constraints on these coordinates. Related work on supersymmetric generalizations of embedding space methods include Ref. [10], which makes $SO(4, 2)$ rather than $SU(2, 2|1)$ invariance manifest, and Refs. [11,12], which employ supertwistor techniques. The equivalence of superembedding methods and the supertwistor approach was established in Refs. [13,14]. The $\mathcal{N} > 1$ extension of the construction discussed in Ref. [9] was introduced in Ref. [15] and fully developed in Ref. [13].

Here, we further develop the realization of superconformal fields in the language of Ref. [9]. In particular, we establish a correspondence between superfields of arbitrary spin on Minkowski superspace and superfields in the embedding space and use it to work out the implications for two-point and three-point correlators. To illustrate our methods, we focus on the physically relevant cases of global symmetry current multiplets, described in four dimensions by real scalar superfields, and the supercurrent multiplet. Our results are presented in a compact index-free notation analogous to the one developed in Refs. [7,8] for nonsupersymmetric CFTs. When written in four-dimensional language, our results agree with the existing literature [16–18].

The paper is organized as follows. In the next section, we review the superembedding formalism of Ref. [9] and establish our notation. In Sec. III, we establish a correspondence between superfields in arbitrary Lorentz representations and their superembedding space counterparts. In Sec. IV, we apply our formalism to examples of two-point

and three-point correlators. Emphasis is placed on correlators involving the various symmetry currents of the SCFT. In particular, we recover the known superconformal relations between possible anomalies of global currents. Construction of manifestly covariant correlators reduces to the problem of enumerating $SU(2, 2|1)$ invariants built from products of embedding space supercoordinates with a fixed number of supertwistors and their complex conjugates. We conclude in Sec. V.

II. SUPEREMBEDDING FORMALISM

In order to establish our notation, we briefly review the “superembedding formalism” developed in Ref. [9]. The four-dimensional $\mathcal{N} = 1$ Minkowski superspace, $\mathcal{M} = \mathbb{R}^{4|4}$, is embedded in a higher-dimensional superspace \mathcal{E} , on which the superconformal group $SU(2, 2|1)$ acts linearly. The reduction from the embedding space \mathcal{E} to the four-dimensional superspace \mathcal{M} is accomplished by a set of covariant constraints, which we review in this section.

A. Superconformal group

Our notation for the $\mathcal{N} = 1$ superconformal group $SU(2, 2|1)$ follows that of Ref. [9]. The supergroup $SU(2, 2|1)$ consists of 5×5 supermatrices¹ of the form

$$U_A{}^B = \begin{pmatrix} U_\alpha{}^\beta & \phi_\alpha \\ \psi^\beta & z \end{pmatrix}, \quad (1)$$

with bosonic (c -number) entries $U_\alpha{}^\beta$ and z and (anticommuting) fermionic entries ϕ_α and ψ^β . These matrices act on a fundamental (defining) five-dimensional representation V_A , where we assign V_α to be fermionic and V_5 is bosonic, as $V_A \rightarrow U_A{}^B V_B$. On the conjugate representation, $V_{\dot{A}} \equiv V_A^\dagger$, $SU(2, 2|1)$ acts as $V_{\dot{A}} \rightarrow V_{\dot{B}} U_{\dot{A}}{}^{\dot{B}}$, with $U_{\dot{A}}{}^{\dot{B}} = (U_A{}^B)^\dagger$. A supermatrix $U_A{}^B$ belongs in $SU(2, 2|1)$ if it satisfies the “unitarity” constraint

$$A^{\dot{A}\dot{A}} = U_{\dot{B}}^{\dot{A}} A^{\dot{B}B} U_B{}^A, \quad (2)$$

where the $SU(2, 2|1)$ invariant metric is given by

$$A^{\dot{A}\dot{B}} = \begin{pmatrix} A^{\dot{\alpha}\dot{\beta}} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \delta^{\dot{\alpha}\dot{\beta}} & 0 \\ \delta_a{}^b & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3)$$

as well as the “unimodular” constraint

$$[\text{s det } U]^{-1} = \frac{\det(U_\alpha{}^\beta - z^{-1} \phi_\alpha \psi^\beta)}{z} = 1. \quad (4)$$

It is most often convenient to work with infinitesimal generators rather than with finite group elements. Near the identity,

$$U_A{}^B = \delta_A{}^B + iT_A{}^B, \quad (5)$$

where the generators take the form

$$T_A{}^B = \begin{pmatrix} T_\alpha{}^\beta + \frac{1}{4} \delta_\alpha{}^\beta \phi & \phi_\alpha \\ \bar{\phi}^\beta & \phi \end{pmatrix}. \quad (6)$$

The traceless tensor $T_\alpha{}^\beta$ is a generator of $SU(2, 2) \subset SU(2, 2|1)$ [the invariant $SU(2, 2)$ metric is the matrix $A^{\dot{\alpha}\dot{\beta}}$ defined above], and $\bar{\phi}^\beta \equiv \phi_{\dot{\alpha}} A^{\dot{\alpha}\beta}$. The action of the generators on fundamental and antifundamental representations is then

$$\delta V_A = iT_A{}^B V_B, \quad (7)$$

and

$$\delta \bar{V}^A = -i\bar{V}^B T_B{}^A, \quad (8)$$

where we have introduced the notation $\bar{V}^A \equiv V_{\dot{A}} A^{\dot{A}A}$.

Tensor products of fundamental and antifundamental V_A and \bar{V}^A representations generally have mixed symmetry properties. To keep track of minus signs that arise when permuting such objects, we employ the notation

$$\sigma(AB) \equiv \begin{cases} -1 & \text{if both } A \text{ and } B \text{ are fermionic indices } \alpha, \beta, \\ +1 & \text{otherwise,} \end{cases} \quad (9)$$

$$\sigma(A) \equiv \sigma(AA). \quad (10)$$

For instance,

$$V_A W_B = \sigma(AB) W_B V_A, \quad (11)$$

$$V_A T_B{}^C = \sigma(AB)\sigma(AC) T_B{}^C V_A. \quad (12)$$

Note that the indices inside $\sigma()$ do not obey the standard repeated index sum convention; i.e., no sum is implied on the right-hand sides of Eqs. (11) and (12). However, when an index is repeated, not including the arguments of $\sigma()$ ’s in the count, then a single sum is implicit over such an index. Equations (7) and (8) imply that

$$W_A \bar{V}^A \sigma(A) = \bar{V}^A W_A \quad (13)$$

is an $SU(2, 2|1)$ singlet, and as we already mentioned, a sum over the index A is implicit on both sides. This is a specific case of a general rule for covariantly contracting $SU(2, 2|1)$ indices when α is fermionic and 5 is bosonic. From left to right, an upper index contracts with a lower index without any σ factor, and from left to right, a lower index contracts with an upper index with a σ factor. Extensive discussions of $SU(2, 2|\mathcal{N})$ representations can be found in Refs. [19–21], and a discussion of tensor product representations and the super-Young tableaux can be found in Refs. [22,23], although for our purposes, the properties summarized above will suffice.

¹Capital indices A, B, \dots , run over $A = \alpha = 1, \dots, 4$ or $A = 5$.

B. Superembedding Space

We introduce an embedding superspace, which we will refer to as the superembedding space \mathcal{E} , which contains four-dimensional superspace $\mathbb{R}^{4|4}$ (denoted by \mathcal{M}) and transforms linearly under superconformal transformations. It is the analog of the embedding space for 4D CFTs \mathbb{R}^6 with coordinates X^m (with m taking the values $+$, $-$, or $\mu = 0, 1, 2, 3$) that transforms linearly under $SO(4, 2)$ and contains the conformal compactification of four-dimensional Minkowski space, realized as the projective light cone $0 = X^2 = \eta_{mn}X^mX^n = \eta_{\mu\nu}X^\mu X^\nu + X^+X^-$ [3,4].

To do so, we need a ‘‘coordinate supermultiplet’’ X_{AB} defined to possess identical $SU(2, 2|1)$ transformation properties and index exchange symmetry as the tensor product $V_A V_B$, i.e.,

$$X_{AB} = \sigma(AB)X_{BA}, \quad (14)$$

with infinitesimal $SU(2, 2|1)$ transformation

$$\delta X_{AB} = iT_A{}^{B'}X_{B'B} + i\sigma(AB)T_B{}^A X_{A'A}. \quad (15)$$

The multiplet X_{AB} contains the bosonic components $X_{\alpha\beta} = -X_{\beta\alpha}$, $X_{55} \equiv \varphi$ and fermionic coordinates $X_{5\alpha} = X_{\alpha 5} \equiv \theta_\alpha$. The antisymmetric $SU(2, 2)$ tensor $X_{\alpha\beta}$ can be equivalently written in $SO(4, 2)$ notation as the six-dimensional vector X^m . The explicit correspondence is

$$X^m = \frac{1}{2}X_{\alpha\beta}\Gamma^{m\alpha\beta}, \quad X_{\alpha\beta} = \frac{1}{2}X_m\tilde{\Gamma}^m_{\alpha\beta}, \quad (16)$$

where the matrices $\Gamma^{m\alpha\beta}$ and $\tilde{\Gamma}^m_{\alpha\beta}$ and their properties are given in Appendix A of Ref. [9].

Because there is no covariant reality condition in $SU(2, 2|1)$, X_{AB} is in a complex representation. We introduce an additional coordinate \bar{X}^{AB} with the same properties as the tensor product $\bar{V}^A \bar{V}^B$, i.e., $\bar{X}^{AB} = \sigma(AB)\bar{X}^{BA}$, and

$$\delta \bar{X}^{AB} = -i\bar{X}^{AA'}T_{A'}{}^B - i\sigma(AB)\bar{X}^{BB'}T_{B'}{}^A. \quad (17)$$

The superembedding space \mathcal{E} consists of the space $\mathbb{C}^{7|4}$ spanned by the pair (X_{AB}, \bar{X}^{AB}) .

The real four-dimensional Minkowski superspace \mathcal{M} is recovered as the subset of \mathcal{E} obtained by projective identification $(X, \bar{X}) \sim (\lambda X, \bar{\lambda} \bar{X})$, and by imposing the relation

$$\bar{X}^{AB} = A^{AA}A^{BB}X_{\bar{A}\bar{B}}, \quad \text{where } X_{\bar{A}\bar{B}} \equiv (X_{BA})^\dagger, \quad (18)$$

between X_{AB} and \bar{X}^{AB} , together with the constraints [9,24]

$$[X_{AB}X_{CD}]_{16} = 0, \quad [\bar{X}^{AB}\bar{X}^{CD}]_{16} = 0, \quad (19)$$

$$[X_{AB}\bar{X}^{BC}]_{24} = 0, \quad (20)$$

where the boldface subscripts denote the dimensions of the irreducible $SU(2, 2|1)$ representation that we project onto. For example, the (adjoint) **24** representation consists of supermatrices $M_A{}^B$ with zero supertrace,

$$\text{str } M \equiv -\sigma(A)M_A{}^A = 0, \quad (21)$$

while $[X_{AB}X_{CD}]_{16} = 0$ is equivalent to the cyclic constraint,

$$\begin{aligned} X_{\underline{A}\underline{B}}X_{\underline{C}\underline{D}} &\equiv X_{AB}X_{CD} + \sigma(AC)\sigma(AB)X_{BC}X_{AD} \\ &\quad + \sigma(AC)\sigma(BC)X_{CA}X_{BD} \\ &= 0, \end{aligned} \quad (22)$$

where appropriate σ factors are inserted to ensure $SU(2, 2|1)$ covariance. Solutions of the constraint equations, (19) and (20), also automatically satisfy $[X_{AB}\bar{X}^{AB}]_1 = 0$. Therefore, $X_{AB}\bar{X}^{BC} = 0$ for any values of A and C .

To see that four-dimensional superspace \mathcal{M} corresponds to the subspace of \mathcal{E} defined by these equations, we note that solutions can be generated, at least locally, by applying all possible $SU(2, 2|1)$ to any single point obeying the constraints [9]. For example, one may start from ‘‘the origin’’

$$\hat{X}_{AB} = \begin{pmatrix} \frac{i}{2}\epsilon_{ab}X^+ & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (23)$$

which is (projectively) invariant under $SO(3, 1) \subset SU(2, 2|1)$, special conformal transformations, special superconformal transformations, dilatations, and by a global $U(1)_R \subset SU(2, 2|1)$. It follows that the space of solutions to the constraints can be identified with the coset $SU(2, 2|1)/H$, with H the isotropy group of the origin. This $(4+4)$ -dimensional space is Minkowski superspace \mathcal{M} . A convenient parametrization near the origin is $(X^+, \bar{X}^+, x^\mu, \theta_a, \bar{\theta}^a)$, where $a, \dot{a} = 1, 2$ are two-component $SL(2, \mathbb{C})$ indices (our conventions for two-component spinors are those of Wess and Bagger [25]), and

$$X_{AB} = X^+ \begin{pmatrix} \frac{i}{2}\epsilon_{ab} & \frac{1}{2}(y\sigma\epsilon)_a{}^{\dot{b}} & \theta_a \\ -\frac{1}{2}(y\bar{\sigma}\epsilon)^{\dot{a}}{}_b & -\frac{i}{2}y^2\epsilon^{\dot{a}\dot{b}} & i(y\bar{\sigma}\theta)^{\dot{a}} \\ \theta_b & i(y\bar{\sigma}\theta)^{\dot{b}} & 2i\theta^2 \end{pmatrix}, \quad (24)$$

$$\bar{X}^{AB} = \bar{X}^+ \begin{pmatrix} -\frac{i}{2}\bar{y}^2\epsilon^{ab} & -\frac{1}{2}(\epsilon\sigma\bar{y})^a{}_b & -i(\bar{\theta}\bar{\sigma}\bar{y})^a \\ \frac{1}{2}(\epsilon\bar{\sigma}\bar{y})^{\dot{a}}{}_b & \frac{i}{2}\epsilon_{\dot{a}\dot{b}} & \bar{\theta}_{\dot{a}} \\ -i(\bar{\theta}\bar{\sigma}\bar{y})^{\dot{b}} & \bar{\theta}_{\dot{b}} & -2i\bar{\theta}^2 \end{pmatrix}. \quad (25)$$

The four-dimensional coordinates satisfy $y^\mu - \bar{y}^\mu = 2i\theta\sigma^\mu\bar{\theta}$ on account of the $[X\bar{X}]_{24} = 0$ constraint, and $x^\mu = \frac{1}{2}(y^\mu + \bar{y}^\mu)$. It is straightforward to verify [9] that $(x^\mu, \theta_a, \bar{\theta}^a)$ transform in the standard way under superconformal transformations (for instance in the form given in Ref. [26]). Note that the upper 4×4 block in Eq. (24) is $\frac{1}{2}X_m\tilde{\Gamma}^m_{\alpha\beta}$ with $X^m = (X^+, X^\mu = X^+y^\mu, X^- = -X^+y^2)$ on the (complexified) $SO(4, 2)$ light cone. Similar results hold for the conjugate coordinates.

When X_{AB} is restricted to \mathcal{M} as in Eq. (24), its components are linearly dependent. The second and third columns of the matrix X_{AB} can be given in terms of the first column:

$$X_A{}^{\dot{b}} = iy\bar{\sigma}^{\dot{b}b}X_{Ab} \quad \text{and} \quad X_{A5} = 2i\theta^bX_{Ab}\sigma(A). \quad (26)$$

Similar relations for \bar{X}^{AB} relate the first and third rows to the second one:

$$\bar{X}^{bA} = -i\bar{X}_b{}^A\bar{\sigma}^{bb} \quad \text{and} \quad \bar{X}^{5A} = -2i\bar{X}_b{}^A\bar{\theta}^b\sigma(A). \quad (27)$$

The parametrization of \mathcal{M} given above describes points near the origin $x^\mu = 0$. Points near infinity, with $z^\mu = x^\mu/x^2$ close to $z^\mu = 0$, can be described by applying all possible $SU(2, 2|1)$ transformations to

$$\check{X}_{AB} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{i}{2}\epsilon_{\dot{a}\dot{b}}X^- & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (28)$$

which is left invariant by $SL(2, \mathbb{C})$ Lorentz transformations, translations, and Poincare supersymmetry transformations. Given two points in \mathcal{M} , there exists an $SU(2, 2|1)$ transformation that simultaneously sends one of the points to the origin, Eq. (23), and the other to ‘‘infinity’’ given in Eq. (28).

III. SUPERFIELDS

We now develop the correspondence between superfields on \mathcal{M} and their counterparts in \mathcal{E} . A generic primary superfield $\Phi_{\mathcal{M}}$ on \mathcal{M} is specified by its $SL(2, \mathbb{C})$ Lorentz quantum numbers, by its scaling dimension Δ , and by the $U(1)_R$ charge of its lowest component field [17]. It is given by an $SL(2, \mathbb{C})$ multispinor,

$$\Phi_{\mathcal{M}a_1\dots a_{2j}}{}^{\dot{b}_1\dots\dot{b}_{2\bar{j}}}(x, \theta, \bar{\theta}), \quad (29)$$

where irreducibility requires complete symmetry under the interchange of pairs of dotted or undotted indices. Transformation properties of superfields under the superconformal transformations were discussed in Ref. [26]. It is useful to label this superfield by its quantum numbers (j, \bar{j}, q, \bar{q}) , where

$$q \equiv \frac{1}{2}\left(\Delta + \frac{3}{2}R\right), \quad (30)$$

$$\bar{q} \equiv \frac{1}{2}\left(\Delta - \frac{3}{2}R\right). \quad (31)$$

The correspondence for $j = \bar{j} = 0$ was developed in Ref. [9]: A $(0, 0, q, \bar{q})$ primary operator $\Phi_{\mathcal{M}}(x, \theta, \bar{\theta})$ in Minkowski spacetime maps into an embedding space $SU(2, 2|1)$ scalar superfield $\Phi_{\mathcal{E}}(X, \bar{X}) = (X^+)^{-q} \times (\bar{X}^+)^{-\bar{q}}\Phi_{\mathcal{M}}(x, \theta, \bar{\theta})$ homogeneous in its arguments. We now extend this result to other representations.

First, we recall the mapping between primary operators of $SO(4, 2)$ in spinor representations and their projective

light-cone counterparts. Recent detailed discussions can be found in Refs. [8,27]. Starting with primary $\psi_a(x^\mu)$ transforming in the $(1/2, 0)$ representation of $SL(2, \mathbb{C})$, one constructs a projective light-cone field

$$\psi_\alpha(X) = (X^+)^{-\Delta-1/2}X_{\alpha b}\psi^b(x^\mu), \quad (32)$$

which is a homogeneous function, $\psi_\alpha(\lambda X) = \lambda^{-\Delta+1/2}\psi_\alpha(X)$, transforming in the fundamental representation of $SU(2, 2) \sim SO(4, 2)$. Points on the projective light cone automatically obey the stronger constraint $X_{\alpha\lambda}X^{\lambda\beta} = 0$, and thus $\psi_\alpha(X)$ defined by Eq. (32) satisfies the relation

$$X^{\alpha\beta}\psi_\beta(X) = 0. \quad (33)$$

Conversely, given a spinor $\psi_\alpha(X)$ on the projective light cone satisfying the constraint in Eq. (33), it is possible to project onto a spinor primary field $\psi_a(x^\mu)$ in Minkowski space,²

$$\psi_a(x^\mu) = (X^+)^{\Delta-1/2}\psi_{\alpha=a}(X). \quad (34)$$

This correspondence generalizes to fields in other representations in the obvious way.

To establish the analogous correspondence for superconformal fields, we need to supersymmetrize the constraint in Eq. (33). The generalization of $\psi_\alpha(X)$ is an embedding space superfield $\Phi_{\mathcal{E}A}(X, \bar{X})$ in the fundamental representation of $SU(2, 2|1)$ satisfying the scaling property

$$\Phi_{\mathcal{E}A}(\lambda X, \bar{\lambda}\bar{X}) = \lambda^{-(q-\frac{1}{2})}\bar{\lambda}^{-\bar{q}}\Phi_{\mathcal{E}A}(X, \bar{X}). \quad (35)$$

To generalize Eq. (33), we must find a supermultiplet of linear constraints constructed from the product of $\Phi_{\mathcal{E}A}$ with either X_{AB} or \bar{X}^{AB} . The two possibilities are either $\bar{X}^{AB}\Phi_{\mathcal{E}B} = 0$, which reduces to Eq. (33) at the point $\bar{X}^{5\alpha} = \bar{X}^{55} = 0$ or $[X_{AB}\Phi_{\mathcal{E}C}]_{15} = 0$, which can be written explicitly as

$$\begin{aligned} X_{\underline{A}\underline{B}}\Phi_{\underline{E}\underline{C}}(X, \bar{X}) &\equiv X_{AB}\Phi_{\mathcal{E}C} + \sigma(AC)\sigma(AB)X_{BC}\Phi_{\mathcal{E}A} \\ &\quad + \sigma(AC)\sigma(BC)X_{CA}\Phi_{\mathcal{E}B} \\ &= 0, \end{aligned} \quad (36)$$

or componentwise

$$X^{\alpha\beta}\Phi_{\mathcal{E}\beta} = 0, \quad (37)$$

$$X_{\alpha\beta}\Phi_{\mathcal{E}5} + \theta_\alpha\Phi_{\mathcal{E}\beta} - \theta_\beta\Phi_{\mathcal{E}\alpha} = 0, \quad (38)$$

$$2\theta_\alpha\Phi_{\mathcal{E}5} + \varphi\Phi_{\mathcal{E}\alpha} = 0, \quad (39)$$

²Note that there is some redundancy in this construction, as the light-cone constraint on $X_{\alpha\beta}$ means that the solution to Eq. (33) is not unique. Given any solution to Eq. (33), one may generate others by adding terms of the form $X_{\alpha\beta}\chi^\beta(X)$. While this freedom changes the explicit expression for $\psi_a(x^\mu)$ in Eq. (34), it does not change results for correlation functions obtained via the embedding space formalism; see Ref. [8] for details.

$$\varphi\Phi_{\mathcal{E}5} = 0. \quad (40)$$

In particular the $A = \alpha$, $B = \beta$, $C = \gamma$ component of Eq. (36), given in Eq. (37), implies that $\Phi_{\mathcal{E}\alpha}$ obeys Eq. (33) at all points of \mathcal{E} . Furthermore, it can be checked that imposing $[X_{AB}\Phi_{\mathcal{E}C}]_{15} = 0$ on \mathcal{E} also implies that $\Phi_{\mathcal{E}A}$ obeys the antifundamental constraint $\bar{X}^{AB}\Phi_{\mathcal{E}B} = 0$ restricted to \mathcal{M} .

From these observations, we conclude that $[X_{AB}\Phi_{\mathcal{E}C}]_{15} = 0$ yields the constraint necessary to recover the correct four-dimensional superfield $\Phi_{\mathcal{M},a}(x^\mu, \theta, \bar{\theta})$. Indeed, for fixed $\Phi_{\mathcal{E}a=a}$ and X_{AB} restricted to \mathcal{M} , using the parametrization in Eqs. (24) and (39) implies

$$\Phi_{\mathcal{E}5} = 2i\theta^a\Phi_{\mathcal{E}a}, \quad \Phi_{\mathcal{E}}^{\dot{a}} = iy^\mu\bar{\sigma}_{\mu}^{\dot{a}a}\Phi_{\mathcal{E}a}. \quad (41)$$

The remaining constraints, Eqs. (37), (38), and (40), are then automatically satisfied given Eq. (41).

Since only the $A = a$ components of $\Phi_{\mathcal{E}A}$ are left independent by the constraint in Eq. (36), we define $\Phi_{\mathcal{M}a}(x^\mu, \theta, \bar{\theta})$ as

$$\Phi_{\mathcal{M}a}(x, \theta, \bar{\theta}) = (X^+)^{q-\frac{1}{2}}(\bar{X}^+)^{\bar{q}}\Phi_{\mathcal{E}A=a}(X, \bar{X}), \quad (42)$$

where the coordinates on the right-hand side are restricted to \mathcal{M} . It is also possible to uplift the superfield $\Phi_{\mathcal{M}a}$ from \mathcal{M} to \mathcal{E} via

$$T_A^B = \begin{pmatrix} \frac{i}{2}\log\lambda\delta_a^b + \frac{i}{2}\omega_{\mu\nu}(\sigma^{\mu\nu})_a^b + \frac{1}{4}\phi\delta_a^b & b_\mu\sigma_{ab}^\mu & -2\eta_a \\ a_\mu\bar{\sigma}^{\mu\dot{a}b} & -\frac{i}{2}\log\lambda\delta_{\dot{b}}^{\dot{a}} + \frac{i}{2}\omega_{\mu\nu}(\bar{\sigma}^{\mu\nu})_{\dot{b}}^{\dot{a}} + \frac{1}{4}\phi\delta_{\dot{b}}^{\dot{a}} & 2\bar{\tau}^{\dot{a}} \\ 2\tau^b & -2\bar{\eta}_{\dot{b}} & \phi \end{pmatrix}, \quad (45)$$

we obtain

$$\delta X^+ = \left[-\log\lambda + \frac{i}{2}\phi - 4\eta^a\theta_a + 2(b \cdot y) \right] X^+, \quad (46)$$

$$\delta \bar{X}^+ = \left[-\log\lambda - \frac{i}{2}\phi - 4\bar{\eta}_{\dot{a}}\bar{\theta}^{\dot{a}} + 2(b \cdot \bar{y}) \right] \bar{X}^+, \quad (47)$$

and, using Eq. (41),

$$\delta_s\Phi_{\mathcal{E}a} = -\frac{1}{2}\omega_{\mu\nu}(\sigma^{\mu\nu})_a^b\Phi_{\mathcal{E}b} - \frac{1}{2}\log\lambda\Phi_{\mathcal{E}a} + (b \cdot y)\Phi_{\mathcal{E}a} - 2b_\mu y_\nu(\sigma^{\mu\nu})_a^b\Phi_{\mathcal{E}b} + \frac{i}{4}\phi\Phi_{\mathcal{E}a} + 4\eta_a\theta^b\Phi_{\mathcal{E}b}. \quad (48)$$

Note that this field has zero variation under translations and Poincare supersymmetry transformations. Putting together these results, we finally get

$$\begin{aligned} \delta_s\Phi_{\mathcal{M}a} &= -\frac{1}{2}\omega_{\mu\nu}(\sigma^{\mu\nu})_a^b\Phi_{\mathcal{M}b} - \log\lambda(q + \bar{q})\Phi_{\mathcal{M}a} + 2q(b \cdot y)\Phi_{\mathcal{M}a} + 2\bar{q}(b \cdot \bar{y})\Phi_{\mathcal{M}a} - 2b_\mu y_\nu(\sigma^{\mu\nu})_a^b\Phi_{\mathcal{M}b} \\ &\quad - 4\left(q - \frac{1}{2}\right)(\eta\theta)\Phi_{\mathcal{M}a} + 4\eta_a\theta^b\Phi_{\mathcal{M}b} - 4\bar{q}(\bar{\eta}\bar{\theta})\Phi_{\mathcal{M}a} + \frac{i}{2}\phi(q - \bar{q})\Phi_{\mathcal{M}a}, \end{aligned} \quad (49)$$

which is the correct transformation rule for a $(1/2, 0)$ spinor multiplet; see, e.g., Ref. [26]. The parameter ϕ does not correspond to the standard R -symmetry charge assignment in four dimensions but differs by a factor of $\frac{4}{3}$; see Ref. [9].

$$\Phi_{\mathcal{E}A}(X, \bar{X}) = (X^+)^{-(q+\frac{1}{2})}(\bar{X}^+)^{-\bar{q}}X_A^c\Phi_{\mathcal{M}c}(x, \theta, \bar{\theta}), \quad (43)$$

which follows by using Eqs. (41), (42), and (26). Of course, it is understood that the field $\Phi_{\mathcal{E}A}(X, \bar{X})$ is not defined for all values of X and \bar{X} in the superembedding space \mathcal{E} since $\Phi_{\mathcal{M}c}(x, \theta, \bar{\theta})$ is only defined on \mathcal{M} .

We now show that $\Phi_{\mathcal{M}a}$ has the correct transformation property of an undotted spinor with the label $(\frac{1}{2}, 0, q, \bar{q})$. Equation (42) gives

$$\begin{aligned} \delta_s\Phi_{\mathcal{M}a} &= (X^+)^{q-\frac{1}{2}}(\bar{X}^+)^{\bar{q}}\delta_s\Phi_{\mathcal{E}A=a} + \left(q - \frac{1}{2}\right)\left(\frac{\delta X^+}{X^+}\right)\Phi_{\mathcal{M}a} \\ &\quad + \bar{q}\left(\frac{\delta \bar{X}^+}{\bar{X}^+}\right)\Phi_{\mathcal{M}a}, \end{aligned} \quad (44)$$

where the variation δ_s on a field $\Phi(X)$ is defined by $\delta_s\Phi(X) = \Phi'(X'(X)) - \Phi(X)$ [in other words, we omit the action of $SU(2, 2|1)$ acting on the coordinates in Eq. (24)]. Parametrizing the $SU(2, 2|1)$ generator T_A^B in Eq. (6) in terms of the more familiar four-dimensional superconformal transformations, the translation a_μ , $SO(3, 1)$ transformation $\omega_{\mu\nu}$, dilatation λ , special conformal transformation b_μ , Poincare supersymmetry τ_a , special superconformal transformation η_a , and $U(1)_R$ charge ϕ ,

To summarize, a field $\Phi_{\mathcal{E}A}(X, \bar{X})$ on \mathcal{E} obeying the scaling relation (i) $\Phi_{\mathcal{E}A}(\lambda X, \bar{\lambda} \bar{X}) = \lambda^{-(q+\frac{1}{2})}\bar{\lambda}^{-\bar{q}}\Phi_{\mathcal{E}A}(X, \bar{X})$ and the covariant constraint (ii) $[X_{AB}\Phi_{\mathcal{E}C}]_{15} = 0$ defines a superconformal multiplet $\Phi_{\mathcal{M}a}(x, \theta, \bar{\theta})$ in the $(1/2, 0, q, \bar{q})$ representation via the relation in Eq. (42). Conversely,

given $\Phi_{\mathcal{M}a}(x, \theta, \bar{\theta})$ we can construct an operator $\Phi_{\mathcal{E}A}(X, \bar{X})$ that transforms linearly under $SU(2, 2|1)$.

This construction generalizes easily to other representations (j, \bar{j}, q, \bar{q}) . A superfield $\Phi_{\mathcal{M}\hat{a}}$ transforming in the $(0, \frac{1}{2}, q, \bar{q})$ representation corresponds in \mathcal{E} to a superfield $\Phi_{\mathcal{E}}^A$ that obeys the scaling property

$$\Phi_{\mathcal{E}}^A(\lambda X, \bar{\lambda} \bar{X}) = \lambda^{-q} \bar{\lambda}^{-(\bar{q}-\frac{1}{2})} \Phi_{\mathcal{E}}^A(X, \bar{X}) \quad (50)$$

and the constraint $[\Phi_{\mathcal{E}}^A \bar{X}^{BC}]_{\bar{1}\bar{5}} = 0$, or

$$\Phi_{\mathcal{E}}^A \bar{X}^{BC} = 0, \quad (51)$$

which implies the additional constraint $\Phi_{\mathcal{E}}^A X_{AB} = 0$ on \mathcal{M} . One obtains $\Phi_{\mathcal{M}\hat{a}}$ from $\Phi_{\mathcal{E}}^A$ by

$$\Phi_{\mathcal{M}\hat{a}}(x, \theta, \bar{\theta}) = (X^+)^q (\bar{X}^+)^{\bar{q}-\frac{1}{2}} \Phi_{\mathcal{E}}^{A=\hat{a}}(X, \bar{X}), \quad (52)$$

and conversely

$$\Phi_{\mathcal{E}}^A(X, \bar{X}) = (X^+)^{-q} (\bar{X}^+)^{-(\bar{q}+\frac{1}{2})} \Phi_{\mathcal{M}\hat{c}}(x, \theta, \bar{\theta}) \bar{X}^{\hat{c}A} \quad (53)$$

defines a field $\Phi_{\mathcal{E}}^A(X, \bar{X})$ that transforms linearly under superconformal transformations.

For the more general case of $\Phi_{\mathcal{M}a_1 \dots a_{2j}}^{b_1 \dots b_{2\bar{j}}}(x, \theta, \bar{\theta})$ in the (j, \bar{j}, q, \bar{q}) representation, we introduce a field $\Phi_{\mathcal{E}A_1 \dots A_{2j}}^{B_1 \dots B_{2\bar{j}}}(X, \bar{X})$, which we assign to have identical $SU(2, 2|1)$ transformation property to that of the tensor product of $2j$ fundamentals $V_{A_1}^{(1)} \dots V_{A_{2j}}^{(2j)}$ and $2\bar{j}$ antifundamentals $\bar{V}_{(1)}^{B_1} \dots \bar{V}_{(2\bar{j})}^{B_{2\bar{j}}}$,

$$\Phi_{\mathcal{E}A_1 \dots A_{2j}}^{B_1 \dots B_{2\bar{j}}} \sim V_{A_1}^{(1)} \dots V_{A_{2j}}^{(2j)} \bar{V}_{(1)}^{B_1} \dots \bar{V}_{(2\bar{j})}^{B_{2\bar{j}}}. \quad (54)$$

We pick the highest-weight (j, \bar{j}) representation by imposing a symmetry under the exchange of adjacent indices $\Phi_{\mathcal{E} \dots A_i A_{i+1} \dots}^{B_1 \dots B_{2\bar{j}}} = -\sigma(A_i A_{i+1}) \Phi_{\mathcal{E} \dots A_{i+1} A_i \dots}^{B_1 \dots B_{2\bar{j}}}$. The fields are homogenous

$$\begin{aligned} \Phi_{\mathcal{E}A_1 \dots A_{2j}}^{B_1 \dots B_{2\bar{j}}}(\lambda X, \bar{\lambda} \bar{X}) \\ = \lambda^{-(q-j)} \bar{\lambda}^{-(\bar{q}-\bar{j})} \Phi_{\mathcal{E}A_1 \dots A_{2j}}^{B_1 \dots B_{2\bar{j}}}(X, \bar{X}) \end{aligned} \quad (55)$$

and satisfy the constraints

$$0 = X_{CD} \Phi_{\mathcal{E}A_1 A_2 \dots A_{2j}}^{B_1 \dots B_{2\bar{j}}}, \quad (56)$$

$$0 = \Phi_{\mathcal{E}A_1 \dots A_{2j}}^{B_1 \dots B_{2\bar{j}-1} B_{2\bar{j}}} \bar{X}^{\bar{C}D}. \quad (57)$$

Using the quadratic equations satisfied by points (X_{AB}, \bar{X}^{AB}) on \mathcal{M} , these constraints also imply that

$$\bar{X}^{AA_1} \Phi_{\mathcal{E}A_1 A_2 \dots A_{2j}}^{B_1 \dots B_{2\bar{j}}} = 0, \quad (58)$$

$$\Phi_{\mathcal{E}A_1 A_2 \dots A_{2j}}^{B_1 \dots B_{2\bar{j}}} X_{B_{2\bar{j}}A} = 0. \quad (59)$$

The superfield $\Phi_{\mathcal{M}a_1 \dots a_{2j}}^{b_1 \dots b_{2\bar{j}}}$ is recovered through

$$\begin{aligned} \Phi_{\mathcal{M}a_1 \dots a_{2j}}^{b_1 \dots b_{2\bar{j}}}(x, \theta, \bar{\theta}) \\ = (X^+)^{q-j} (\bar{X}^+)^{\bar{q}-\bar{j}} \Phi_{\mathcal{E}A_1=a_1 \dots A_{2j}=a_{2j}}^{B_1=b_1 \dots B_{2\bar{j}}=b_{2\bar{j}}}, \end{aligned} \quad (60)$$

and one can also easily generalize Eq. (43) to this case.

It is sometimes possible to further reduce the superfield by imposing holomorphy or antiholomorphy, i.e., functional dependence only on X_{AB} or \bar{X}_{AB} , respectively. Given the constraints in Eqs. (56) and (57), holomorphic superfields $\Phi_{\mathcal{E}}(X)$ on \mathcal{E} project onto chiral superfields $\Phi_{\mathcal{M}}(y, \theta)$ only for the special values $\bar{j} = \bar{q} = 0$ of the quantum numbers. In particular this implies the standard relation $\Delta = \frac{3}{2}R$ between the scaling dimension and R charge in the chiral sector [in a normalization with $R(\theta_a) = 1$]. Likewise, antichiral fields $\Phi_{\mathcal{M}}(\bar{y}, \bar{\theta})$ on \mathcal{M} correspond to antiholomorphic fields on \mathcal{E} with $j = q = 0$, and consequently $\Delta = -\frac{3}{2}R$.

In what follows, we will focus on special cases of (j, \bar{j}, q, \bar{q}) that have particular relevance to physical applications. In addition to chiral/antichiral fields with $j = \bar{j} = 0$, we will consider the real scalar multiplet $V_{\mathcal{M}}$ transforming in the representation $(0, 0, q, q)$. The case $q = 1$, i.e., $\Delta = 2, R = 0$, usually denoted by $L_{\mathcal{M}}$, contains a dimension $\Delta = 3$ conserved current $j^\mu(x)$. This multiplet can be obtained from a real multiplet $L_{\mathcal{M}}$ by imposing the constraints [17] $D^a L_{\mathcal{M}} = \bar{D}^{\dot{a}} L_{\mathcal{M}} = 0$, where $D_a, \bar{D}^{\dot{a}}$ are the $\mathcal{N} = 1$ Poincare supercovariant derivatives, which restrict $L_{\mathcal{M}}$ to

$$\begin{aligned} L_{\mathcal{M}}(x, \theta, \bar{\theta}) &= C(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) - \theta\sigma^\mu\bar{\theta}j_\mu(x) \\ &+ \frac{1}{2}\theta^2\bar{\theta}\bar{\sigma}^\mu\partial_\mu\chi(x) - \frac{1}{2}\bar{\theta}^2\theta\sigma^\mu\partial_\mu\bar{\chi}(x) \\ &- \frac{1}{4}\theta^2\bar{\theta}^2\Box C(x). \end{aligned} \quad (61)$$

Finally, we will also consider the supercurrent multiplet [28], a real superfield $\mathcal{T}_{\mathcal{M}ab}(x, \theta, \bar{\theta})$ transforming in the $(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2})$ representation and obeying the conservation law $D^a \mathcal{T}_{\mathcal{M}ab} = \bar{D}^{\dot{b}} \mathcal{T}_{\mathcal{M}ab} = 0$ with component field expansion

$$\begin{aligned} \mathcal{T}_{\mathcal{M}\mu}(x, \theta, \bar{\theta}) &= j_\mu^R(x) + \theta^a S_{\mu a}(x) + \bar{\theta}_{\dot{a}} \bar{S}_{\dot{a}\mu}(x) \\ &+ 2\theta\sigma^\nu\bar{\theta}T_{\nu\mu}(x) + \dots, \end{aligned} \quad (62)$$

where j_μ^R is the $U(1)_R$ current, $S_{\mu a}$ the supercurrent, and $T_{\mu\nu} = T_{\nu\mu}$ the energy-momentum tensor. In the embedding approach, $\mathcal{T}_{\mathcal{M}ab}$ gets lifted to a superfield $\mathcal{T}_{\mathcal{E}A}^B$ in the **24** (adjoint) representation, satisfying the constraints of Eqs. (56) and (57), $\text{str } \mathcal{T} = 0$, together with the scaling law

$$\mathcal{T}_{\mathcal{E}A}^B(\lambda X, \bar{\lambda} \bar{X}) = \lambda^{-1} \bar{\lambda}^{-1} \mathcal{T}_{\mathcal{E}A}^B(X, \bar{X}) \quad (63)$$

and the reality condition

$$\mathcal{T}_{\mathcal{E}A}^B(X, \bar{X}) = \bar{\mathcal{T}}_{\mathcal{E}A}^B(X, \bar{X}) \equiv A_{AB} \mathcal{T}_{\mathcal{E}}^{\dagger\dot{B}\dot{A}}(X, \bar{X}) A^{\dot{A}B}. \quad (64)$$

The relation between $\mathcal{T}_{\mathcal{M}}$ and $\mathcal{T}_{\mathcal{E}}$ is

$$\mathcal{T}_{\mathcal{M}ab}(x, \theta, \bar{\theta}) = X^+ \bar{X}^+ \mathcal{T}_{\mathcal{E}A=a}^{B=b}(X, \bar{X}), \quad (65)$$

and

$$\mathcal{T}_{\mathcal{E}\mathcal{A}}^B(X, \bar{X}) = (X^+)^{-2} (\bar{X}^+)^{-2} X_A^a \mathcal{T}_{ab}(x, \theta, \bar{\theta}) \bar{X}^{bB}. \quad (66)$$

IV. CORRELATORS

We now use the superembedding formalism to construct manifestly covariant expressions for SCFT correlation functions.

A. Two-point functions

Given a set of coordinates $Z_i = (X_i, \bar{X}_i)$, an overcomplete set of $SU(2, 2|1)$ invariants is given by the supertraces [29], e.g.,

$$\langle 1\bar{2}3\bar{4}\dots \rangle \equiv \text{str} X_1 \sigma \bar{X}_2 X_3 \dots, \quad (67)$$

where the rules for constructing the tensor $(X_1 \sigma \bar{X}_2 X_3 \dots)_A^B$ were given in Sec. II. For two independent points, the only invariant is $\langle 1\bar{2} \rangle$ and its complex conjugate, as can be readily seen by going to the frame in which X_1 is at the origin and X_2 at infinity. It follows from the scaling relation, Eq. (55), that, up to normalization,

$$\langle \Phi_{\mathcal{E}1}(Z_1) \Phi_{\mathcal{E}2}(Z_2) \rangle = \frac{\delta_{q_1 \bar{q}_2} \delta_{\bar{q}_1 q_2}}{\langle 1\bar{2} \rangle^{q_1} \langle 2\bar{1} \rangle^{\bar{q}_1}}. \quad (68)$$

The four-dimensional correlator immediately follows upon inserting the expression

$$\langle 1\bar{2} \rangle = -\frac{1}{2} (X_1^+)(\bar{X}_2^+) (\bar{y}_2 - y_1 + 2i\theta_1 \sigma \bar{\theta}_2)^2. \quad (69)$$

Equation (68) contains as a special case the two-point function of a chiral ($\bar{q} = 0$) scalar with an antichiral ($q = 0$) scalar, and the two-point function of the current superfield $L_{\mathcal{E}} \sim (0, 0, 1, 1)$. In the normalization of Ref. [30], with τ a real constant,

$$\langle L_{\mathcal{E}}^I(Z_1) L_{\mathcal{E}}^J(Z_2) \rangle = \frac{\delta^{IJ}}{64\pi^4} \frac{\tau}{\langle 1\bar{2} \rangle \langle 2\bar{1} \rangle}, \quad (70)$$

where I, J are the adjoint indices of the symmetry group. This result holds up to contact terms, which the embedding formalism does not account for [9]. It is straightforward to check that upon projection to \mathcal{M} , the correlator satisfies $D^2 \langle L_{\mathcal{E}} L_{\mathcal{E}} \rangle = \bar{D}^2 \langle L_{\mathcal{E}} L_{\mathcal{E}} \rangle = 0$. Thus, current conservation is automatic at the level of the two-point function.

To construct correlators for higher-spin supermultiplets, we employ a variant of the index-free notation introduced in Ref. [7] for $SO(4, 2)$ tensors and in Ref. [8] for $SU(2, 2)$ multitwistors. We use fundamental and antifundamental representations W_A, \bar{W}^A for which the components are now W_α and \bar{W}^α c numbers, while components W_5, \bar{W}^5 are Grassmann variables. We write

$$\begin{aligned} \Phi(W, \bar{W}, Z) &= \sum_{j, \bar{j}} \sigma(\dots) \bar{W}^{A_1} \dots \bar{W}^{A_{2j}} \\ &\times \Phi_{\mathcal{E}A_1 \dots A_{2j}}^{B_1 \dots B_{2j}}(Z) W_{B_1} \dots W_{B_{2j}}, \end{aligned} \quad (71)$$

which is a superconformal scalar under simultaneous transformations of $\Phi_{\mathcal{E}A_1 \dots A_{2j}}^{B_1 \dots B_{2j}}, X_{AB}, \bar{X}^{AB}, W_A, \bar{W}^A$. [The symbol $\sigma(\dots)$ denotes factors of Eq. (9) inserted to ensure $SU(2, 2|1)$ invariance].

Correlators of $\Phi(W, \bar{W}, Z)$ are functions of invariants constructed from insertions of the objects W, \bar{W}, X , and \bar{X} . Note that due to the constraints in Eqs. (58) and (59), there is an additional ‘‘gauge invariance’’ under the shifts

$$W_A \rightarrow W_A + \bar{S}^B X_{BA}, \quad (72)$$

$$\bar{W}^A \rightarrow \bar{W}^A + \bar{X}^{AB} S_B, \quad (73)$$

with arbitrary S_A, \bar{S}^A , which can be used to reduce the number of invariants. For the two-point function $\langle \Phi_1 \Phi_2 \rangle$, a complete set of invariants on \mathcal{M} consists of $\langle 1\bar{2} \rangle$ and its conjugate, together with

$$\bar{W}_1(1\bar{2})W_2, \quad (74)$$

$$\bar{W}_2(2\bar{1})W_1, \quad (75)$$

where $\bar{W}_1(1\bar{2})W_2 \equiv \bar{W}_1(X_1 \sigma \bar{X}_2)W_2$, etc. In the following discussion, we will abbreviate X_1 as 1, \bar{X}_1 as $\bar{1}$, and so on and omit the σ 's as their position is uniquely specified by $SU(2, 2|1)$ invariance. The gauge transformations on W_A, \bar{W}^A introduced above forbid $SU(2, 2|1)$ invariants like $\bar{W}_1(2\bar{1})W_2$ and others, while the exchange symmetries $\bar{W}^A \bar{W}^B = -\sigma(AB)\sigma(A)\sigma(B)\bar{W}^B \bar{W}^A$ and $X_{AB} = \sigma(AB)X_{BA}$ rule out invariants such as $\bar{W}_1(1)\bar{W}_1 = 0$. Longer strings of coordinates either vanish due to the $[X\bar{X}]_{24} = 0$ constraint or can be reduced to the basic invariants using the $[XX]_{16} = 0$ constraint. For example, $(1\bar{2}1\bar{2})_A^B$ is proportional to $\langle 1\bar{2} \rangle$ times $(1\bar{2})_A^B$. Finally, by the scaling properties, all two-point functions reduce to powers of $\bar{W}_1(1\bar{2})W_2, \bar{W}_2(2\bar{1})W_1$ and $\langle 1\bar{2} \rangle, \langle 2\bar{1} \rangle$.

We find using these results that the two point function of $\Phi_1 \sim (j, \bar{j}, q, \bar{q})$ with another superfield Φ_2 is nonvanishing only for Φ_2 transforming in the representation (\bar{j}, j, \bar{q}, q) . This two-point function can be read off the term in $\langle \Phi_1 \Phi_2 \rangle$ proportional to $2j$ powers of W_{2A}, \bar{W}_1^B and $2\bar{j}$ powers of W_{1A}, \bar{W}_2^B . It is

$$\frac{[\bar{W}_1(1\bar{2})W_2]^{2j} [\bar{W}_2(2\bar{1})W_1]^{2\bar{j}}}{\langle 1\bar{2} \rangle^{q+j} \langle 2\bar{1} \rangle^{\bar{q}+\bar{j}}}. \quad (76)$$

For example, the two-point function of superfields $\Phi_{1A}(Z) \sim (\frac{1}{2}, 0, q, \bar{q})$ and $\Phi_2^A(Z) \sim (0, \frac{1}{2}, \bar{q}, q)$ has the form $\langle \Phi_{1A} \Phi_2^B \rangle = \langle 1\bar{2} \rangle^{-q-\frac{1}{2}} \langle 2\bar{1} \rangle^{-\bar{q}} (1\bar{2})_A^B$. Using

$$(1\bar{2})_{A=a}^{B=b} = -\frac{i}{4} X_1^+ \bar{X}_2^+ (y_{1\bar{2}} + 4i\theta_1 \bar{\theta}_2)_{ab} \quad (77)$$

($y_{\bar{1}2} \equiv \bar{y}_1 - y_2 = -y_{2\bar{1}}$) together with the rules given in Sec. III for projecting onto four-dimensional superfields then yields

$$\langle \phi_{1a}(x_1, \theta_1, \bar{\theta}_1) \phi_{2b}(x_2, \theta_2, \bar{\theta}_2) \rangle = \frac{(y_{\bar{1}2} + 4i\theta_1\bar{\theta}_2)_{ab}}{[(y_{2\bar{1}} + 2i\theta_1\sigma\bar{\theta}_2)^2]^{q+1/2}[(y_{\bar{1}2} + 2i\theta_2\sigma\bar{\theta}_1)^2]^q}. \quad (78)$$

Similarly the two-point function of the supercurrent $\mathcal{T}_{ab}(x, \theta, \bar{\theta}) \sim (\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2})$ is given by the embedding space expression

$$\langle [\bar{W}_1 \mathcal{T}(Z_1) W_1][\bar{W}_2 \mathcal{T}(Z_2) W_2] \rangle = \frac{c_{\mathcal{T}\mathcal{T}}}{16} \frac{[\bar{W}_1(1\bar{2})W_2][\bar{W}_2(2\bar{1})W_1]}{\langle 1\bar{2} \rangle^2 \langle 2\bar{1} \rangle^2}, \quad (79)$$

or in four-dimensional language

$$\langle T_{ab}(x_1, \theta_1, \bar{\theta}_1) T_{cd}(x_2, \theta_2, \bar{\theta}_2) \rangle = -\frac{c_{\mathcal{T}\mathcal{T}}}{16} \frac{(y_{\bar{1}2} + 4i\theta_1\bar{\theta}_2)_{ad}(y_{2\bar{1}} + 4i\theta_2\bar{\theta}_1)_{cb}}{[(y_{2\bar{1}} + 2i\theta_1\sigma\bar{\theta}_2)^2(y_{\bar{1}2} + 2i\theta_2\sigma\bar{\theta}_1)^2]^2}, \quad (80)$$

in agreement with Ref. [17].

B. Three-point functions

Supertraces constructed from strings of products of the three coordinates $(X, \bar{X})_{i=1,2,3}$ reduce to products of the bilinears $\langle i\bar{j} \rangle$, as can be seen for instance in a frame in which two points are fixed to the origin and infinity, respectively. The remaining symmetries in this frame are sufficient to fix the bosonic part of the third point but not to set its Grassmann part to zero. Thus, in contrast to non-supersymmetric theories, there is an invariant cross ratio for three points [18]. It can be taken to be

$$u = \frac{\langle 1\bar{2} \rangle \langle 2\bar{3} \rangle \langle 3\bar{1} \rangle}{\langle 2\bar{1} \rangle \langle 3\bar{2} \rangle \langle 1\bar{3} \rangle}. \quad (81)$$

Because superfield three-point correlators can have arbitrary dependence on u , it might seem at first that predictive power is completely lost. Fortunately, the functional dependence on u is fixed up to three numerical constants: In the frame with points $X_{1,2}$ fixed to the origin and infinity, respectively, there is residual $SL(2, \mathbb{C}) \times U(1)_R$ plus dilation symmetry that fixes the unique independent invariant to be $\theta_3 x_3 \cdot \sigma \bar{\theta}_3 / x_3^2$. This invariant is related to u by

$$z \equiv \frac{1-u}{1+u} = -\frac{2i\theta_3 x_3 \cdot \sigma \bar{\theta}_3}{x_3^2}. \quad (82)$$

Thus, in any frame, the most general function $f(u)$ is a quadratic polynomial in z so $SU(2, 2|1)$ symmetry does yield some predictions, in the form of relations between component field correlators. Many known 4D SCFTs have in addition global symmetries, for which the (possibly anomalous) Ward identities provide extra constraints.

Consider, for instance, the three-point function of conserved currents $L^I(x, \theta, \bar{\theta})$. By including suitable improvement terms if necessary, the correlator can be made symmetric in the exchange of operator labels. Thus, in terms of the structure constants f^{IJK} and anomaly tensor d^{IJK} of the global symmetry group G ,

$$\langle L^I_\varepsilon(Z_1) L^J_\varepsilon(Z_2) L^K_\varepsilon(Z_3) \rangle = \frac{d^{IJK}(\lambda_0 + \lambda_2 z^2) + \lambda_1 z f^{IJK}}{[\langle 1\bar{2} \rangle \langle 2\bar{1} \rangle \langle 1\bar{3} \rangle \langle 3\bar{1} \rangle \langle 2\bar{3} \rangle \langle 3\bar{2} \rangle]^{\frac{1}{2}}}. \quad (83)$$

We have used the property $z \rightarrow -z$ under interchange of coordinates to fix the dependence on u .

To fix the constants $\lambda_{0,1,2}$, we impose Ward identities that relate the coefficient of f^{IJK} to the two-point function and the coefficient of d^{IJK} to the $\text{Tr}G^3$ chiral anomaly. Because we lack an embedding space formulation of contact terms, we impose these Ward identities directly on the component fields. First, send $x_1 \rightarrow 0$ and $x_2 \rightarrow \infty$, and pick out the coefficient of $\theta_3 \bar{\theta}_3$ from Eq. (83),

$$\langle C^I(x_1 \rightarrow 0) C^J(x_2 \rightarrow \infty) j^K_\mu(x_3) \rangle = -\frac{8i\lambda_1 f^{IJK}}{x_2^4} \frac{\partial}{\partial x_3^\mu} (x_3^2)^{-1}, \quad (84)$$

where $C^I(x) = L^I(x, \theta = \bar{\theta} = 0)$. From the $\langle C^I C^J \rangle$ component of Eq. (70) and the G -symmetry Ward identity

$$\partial_3^\mu \langle T C^I(x_1) C^J(x_2) j^K_\mu(x_3) \rangle = -i f^{IJK} [\delta^4(x_1 - x_3) - \delta^4(x_2 - x_3)] \frac{\tau}{16\pi^4 x_{12}^4}, \quad (85)$$

we find, using $\square(x^2)^{-1} = 4\pi^2 i \delta^4(x)$, that $\lambda_1 = -i\tau/(512\pi^6)$. To fix λ_2 in terms of λ_0 , note that $\langle C^I C^J C^K \rangle$ is

$$\langle C^I(x_1 \rightarrow 0) C^J(x_2 \rightarrow \infty) C^K(x_3) \rangle = \frac{8}{x_2^4 x_3^2} d^{IJK} \lambda_0, \quad (86)$$

while the $\theta_3^2 \bar{\theta}_3^2$ component gives

$$\langle C^I(x_1 \rightarrow 0) C^J(x_2 \rightarrow \infty) \square C^K(x_3) \rangle = -\frac{32}{x_2^4 x_3^4} d^{IJK} (2\lambda_2 - \lambda_0), \quad (87)$$

where $L(x, \theta, \bar{\theta})|_{\theta^2 \bar{\theta}^2} = -\frac{1}{4} \square C(x)$. Up to contact terms, these two results are only consistent if $\lambda_2 = \frac{1}{2} \lambda_0$. Finally, λ_0 is fixed in terms of the chiral anomaly, which from Eq. (83) is given by

$$\langle j^I_{\mu_1}(x_1) j^J_{\mu_2}(x_2) j^K_{\mu_3}(x_3) \rangle|_d = \frac{8i(\lambda_0 + 6\lambda_2)}{x_{12}^4 x_{23}^4 x_{31}^4} d^{IJK} (\text{Tr}[\sigma \cdot x_{12} \bar{\sigma}_{\mu_2} \sigma \cdot x_{23} \bar{\sigma}_{\mu_3} \sigma \cdot x_{31} \bar{\sigma}_{\mu_1}] - \text{Tr}[\sigma \cdot x_{31} \bar{\sigma}_{\mu_3} \sigma \cdot x_{23} \bar{\sigma}_{\mu_2} \sigma \cdot x_{12} \bar{\sigma}_{\mu_1}]), \quad (88)$$

which in turn is equal to the one-loop anomaly of k free chiral fermions, provided we adjust $\lambda_0 = k/(1024\pi^6)$. Thus, the three-point function can be written as

$$\begin{aligned} & \langle L^I_\mathcal{E}(Z_1)L^J_\mathcal{E}(Z_2)L^K_\mathcal{E}(Z_3) \rangle \\ &= \frac{1}{1024\pi^6} \frac{-2i\tau f^{IJK}z + kd^{IJK}(1 + \frac{1}{2}z^2)}{[\langle 1\bar{2} \rangle \langle 2\bar{1} \rangle \langle 1\bar{3} \rangle \langle 3\bar{1} \rangle \langle 2\bar{3} \rangle \langle 3\bar{2} \rangle]^{\frac{1}{2}}}. \end{aligned} \quad (89)$$

The correlator $\langle \mathcal{T} L^I L^J \rangle$ is obtained by similar considerations. Imposing $SU(2, 2|1)$ invariance, symmetry under exchange $(1, I) \leftrightarrow (2, J)$, and reality conditions on the fields, the most general form is

$$\begin{aligned} & \langle L^I(Z_1)L^J(Z_2)[\bar{W}_3 \mathcal{T}(Z_3)W_3] \rangle \\ &= \frac{\delta^{IJ}}{\langle 3\bar{1} \rangle \langle 1\bar{3} \rangle \langle 3\bar{2} \rangle \langle 2\bar{3} \rangle} \left[\hat{\lambda}_0 \cdot \bar{W}_3 \left(\frac{3\bar{2}1\bar{3}}{\langle 1\bar{2} \rangle} + \frac{3\bar{1}2\bar{3}}{\langle 2\bar{1} \rangle} \right) W_3 \right. \\ & \quad \left. + z\hat{\lambda}_1 \cdot \bar{W}_3 \left(\frac{3\bar{2}1\bar{3}}{\langle 1\bar{2} \rangle} - \frac{3\bar{1}2\bar{3}}{\langle 2\bar{1} \rangle} \right) W_3 \right] \end{aligned} \quad (90)$$

for some constants $\hat{\lambda}_{0,1}$. (A possible term proportional to z^2 times the symmetric invariant in the first line of this equation vanishes, as can be seen in a frame with Z_3 at the origin and Z_2 at infinity). To fix the constants, we impose Ward identities. Taking $x_1 = 0$ and $x_2 \rightarrow \infty$, the $\theta_3\bar{\theta}_3$ component of this equation is

$$\langle T_{\mu\nu}(x_3)C^I(0)C^J(x_2 \rightarrow \infty) \rangle = \frac{2\delta^{IJ}}{x_2^4 x_3^4} \left[\hat{\lambda}_0 \eta_{\mu\nu} + 2\hat{\lambda}_1 \frac{x_{3\mu}x_{3\nu}}{x_3^2} \right]. \quad (91)$$

Comparing this to the energy-momentum Ward identity [31]

$$\begin{aligned} & \partial_3^\mu \langle TT_{\mu\nu}(x_3)C^I(0)C^J(x_2 \rightarrow \infty) \rangle \\ &= -i\partial_\mu^3 [\delta^4(x_3 - x_1) + \delta^4(x_3 - x_2)] \langle TC^I(x_1)C^J(x_2) \rangle \end{aligned} \quad (92)$$

requires that $\hat{\lambda}_1 = -2\hat{\lambda}_0$ and $\hat{\lambda}_0 = \tau/(2^6\pi^6)$. This yields

$$\begin{aligned} & \langle L^I(Z_1)L^J(Z_2)[\bar{W}_3 \mathcal{T}(Z_3)W_3] \rangle \\ &= \frac{\tau\delta^{IJ}}{64\pi^6 \langle 3\bar{1} \rangle \langle 1\bar{3} \rangle \langle 3\bar{2} \rangle \langle 2\bar{3} \rangle} \\ & \quad \times \bar{W}_3 \left[(1-2z) \frac{3\bar{2}1\bar{3}}{\langle 1\bar{2} \rangle} + (1+2z) \frac{3\bar{1}2\bar{3}}{\langle 2\bar{1} \rangle} \right] W_3. \end{aligned} \quad (93)$$

Because this correlator also contains the term $\langle j_{\mu_1}^R j_{\mu_2}^I j_{\mu_3}^J \rangle$, we recover the relation between the normalization of the global current two-point function and the mixed $\text{Tr}RG^2$ anomaly of the SCFT.

As our last application, we consider the correlator $\langle \mathcal{T} \mathcal{T} L \rangle$. The form of tensors that appear in this correlator is restricted by the gauge invariance of Eqs. (72) and (73) to be of the form

$$\begin{aligned} & [\bar{W}_1(1 \dots \bar{1})W_1] \cdot [\bar{W}_2(2 \dots \bar{2})W_2], \\ & [\bar{W}_1(1 \dots \bar{2})W_2] \cdot [\bar{W}_2(2 \dots \bar{1})W_1], \\ & [\bar{W}_1(1 \dots 2)\bar{W}_2] \cdot [W_1(\bar{1} \dots \bar{2})W_2] \end{aligned} \quad (94)$$

times powers of z and z^2 . The omitted expressions, denoted by (\dots) , are strings of products of supercoordinates $Z_{1,2,3}$,

which are constructed by imposing $SU(2, 2|1)$ invariance together with the covariant constraints, Eqs. (19) and (20). The $[X\bar{X}]_{24} = 0$ and $[X\bar{X}]_1 = 0$ constraints imply that such strings cannot contain a coordinate times its conjugate in adjacent positions. The $[XX]_{16} = 0$ constraint written as in Eq. (22) is useful for rearranging/shortening strings with repeated insertions of a given coordinate. As a specific example, it follows readily from Eq. (22) that

$$(\bar{1}\bar{2})_{AB} = \frac{1}{2} \langle \bar{1}\bar{2} \rangle X_{1AB}. \quad (95)$$

Using the constraints in this way to simplify possible tensors, the $\langle \mathcal{T} \mathcal{T} L \rangle$ correlator reduces to six structures with definite reality and permutation symmetry properties:

$$t_1 = \frac{[\bar{W}_1(\bar{1}\bar{2})W_2][\bar{W}_2(2\bar{1})W_1]}{\langle \bar{1}\bar{2} \rangle \langle 2\bar{1} \rangle}, \quad (96)$$

$$t_2 = \frac{[\bar{W}_1(1\bar{3}\bar{2})\bar{W}_2][W_1(\bar{1}3\bar{2})W_2]}{(\langle \bar{1}\bar{2} \rangle \langle 2\bar{1} \rangle \langle 1\bar{3} \rangle \langle 3\bar{1} \rangle \langle 2\bar{3} \rangle \langle 3\bar{2} \rangle)^{\frac{1}{2}}}, \quad (97)$$

$$\begin{aligned} t_3 &= \frac{[\bar{W}_1(1\bar{2}\bar{3}\bar{1})W_1][\bar{W}_2(2\bar{1}\bar{3}\bar{2})W_2]}{\langle \bar{1}\bar{2} \rangle \langle 2\bar{1} \rangle \langle 3\bar{1} \rangle \langle 3\bar{2} \rangle} \\ & \quad + \frac{[\bar{W}_1(1\bar{3}\bar{2}\bar{1})W_1][\bar{W}_2(2\bar{3}\bar{1}\bar{2})W_2]}{\langle \bar{1}\bar{2} \rangle \langle 2\bar{1} \rangle \langle 1\bar{3} \rangle \langle 2\bar{3} \rangle}, \end{aligned} \quad (98)$$

$$\begin{aligned} t_4 &= \frac{[\bar{W}_1(1\bar{2}\bar{3}\bar{1})W_1][\bar{W}_2(2\bar{1}\bar{3}\bar{2})W_2]}{\langle \bar{1}\bar{2} \rangle \langle 2\bar{1} \rangle \langle 3\bar{1} \rangle \langle 3\bar{2} \rangle} \\ & \quad - \frac{[\bar{W}_1(1\bar{3}\bar{2}\bar{1})W_1][\bar{W}_2(2\bar{3}\bar{1}\bar{2})W_2]}{\langle \bar{1}\bar{2} \rangle \langle 2\bar{1} \rangle \langle 1\bar{3} \rangle \langle 2\bar{3} \rangle}, \end{aligned} \quad (99)$$

$$\begin{aligned} t_5 &= \frac{[\bar{W}_1(1\bar{2}\bar{3}\bar{1})W_1][\bar{W}_2(2\bar{3}\bar{1}\bar{2})W_2]}{\langle \bar{1}\bar{2} \rangle^2 \langle 3\bar{1} \rangle \langle 2\bar{3} \rangle} \\ & \quad + \frac{[\bar{W}_1(1\bar{3}\bar{2}\bar{1})W_1][\bar{W}_2(2\bar{1}\bar{3}\bar{2})W_2]}{\langle 2\bar{1} \rangle^2 \langle 1\bar{3} \rangle \langle 3\bar{2} \rangle}, \end{aligned} \quad (100)$$

$$\begin{aligned} t_6 &= \frac{[\bar{W}_1(1\bar{2}\bar{3}\bar{1})W_1][\bar{W}_2(2\bar{3}\bar{1}\bar{2})W_2]}{\langle \bar{1}\bar{2} \rangle^2 \langle 3\bar{1} \rangle \langle 2\bar{3} \rangle} \\ & \quad - \frac{[\bar{W}_1(1\bar{3}\bar{2}\bar{1})W_1][\bar{W}_2(2\bar{1}\bar{3}\bar{2})W_2]}{\langle 2\bar{1} \rangle^2 \langle 1\bar{3} \rangle \langle 3\bar{2} \rangle}. \end{aligned} \quad (101)$$

Not all these objects are independent. By going to the special frame where $Z_{2,3}$ are fixed, and covariantizing the result, it is possible to establish certain relations between the structures given above. These relations are $zt_4 = 0$, $zt_6 = -z^2 t_3$, $t_3 + t_5 = -z^2 t_2$, and $z^2(t_3 - t_5) = z^2(t_1 + 2t_2)$. Thus, the most general form of the three-point function $\langle \mathcal{T} \mathcal{T} L \rangle$ can be taken to be of the form

$$\begin{aligned} & \langle [\bar{W}_1 \mathcal{T}(Z_1)W_1][\bar{W}_2 \mathcal{T}(Z_2)W_2]L(Z_3) \rangle \\ &= \frac{(\lambda_1 + \tilde{\lambda}_1 z^2)t_1 + \lambda_2 t_2 + \lambda_4 t_4 + (\lambda_5 + \tilde{\lambda}_5 z^2)t_5}{[\langle \bar{1}\bar{2} \rangle \langle 2\bar{1} \rangle \langle 1\bar{3} \rangle \langle 3\bar{1} \rangle \langle 2\bar{3} \rangle \langle 3\bar{2} \rangle]^{\frac{1}{2}}}. \end{aligned} \quad (102)$$

The coefficients in this expression are related by conservation of the supercurrent, $D^a T_{a\dot{a}} = \bar{D}^{\dot{a}} T_{a\dot{a}} = 0$. In the

special frame with $X_2 \rightarrow 0$, $X_3 \rightarrow \infty$, Eq. (102) contains the component correlation function

$$\langle j_\mu^R(x) j_\nu^R(0) C(\infty) \rangle = \frac{1}{x^4 x_3^4} \left[-\eta_{\mu\nu} \left(\lambda_1 - \frac{1}{2} \lambda_2 \right) + \frac{x_\mu x_\nu}{x^2} (2\lambda_1 - \lambda_5) \right]. \quad (103)$$

Conservation of the R current $\partial^\mu j_\mu^R = 0$ then fixes $\lambda_5 = 2(\lambda_2 - \lambda_1)$. In this frame, the correlator involving the energy-momentum tensor is

$$\langle T_{\mu\nu}(x) j_\rho^R(0) C(\infty) \rangle = \frac{1}{x^6 x_3^4} \left[\lambda_1 \epsilon_{\mu\nu\rho\alpha} x^\alpha - \frac{i}{2} \lambda_4 (x_\mu \eta_{\nu\rho} - x_\rho \eta_{\nu\mu}) \right], \quad (104)$$

so that symmetry $T_{\mu\nu} = T_{\nu\mu}$ requires $\lambda_1 = \lambda_4 = 0$. Finally, the $\theta^2 \bar{\theta}^2$ component of the superfield $\mathcal{T}_{a\dot{a}}(x, \theta, \bar{\theta})$ vanishes, which implies that $\tilde{\lambda}_1 = -\tilde{\lambda}_5 = -\lambda_2$. Combining together the constraints from different components, we obtain

$$\begin{aligned} & \langle [\bar{W}_1 \mathcal{T}(Z_1) W_1] [\bar{W}_2 \mathcal{T}(Z_2) W_2] L(Z_3) \rangle \\ &= \frac{c_{TTL}}{(\langle 1\bar{2} \rangle \langle 2\bar{1} \rangle \langle 1\bar{3} \rangle \langle 3\bar{1} \rangle \langle 2\bar{3} \rangle \langle 3\bar{2} \rangle)^{\frac{1}{2}}} \left[z^2 \frac{[\bar{W}_1(1\bar{2}) W_2] [\bar{W}_2(2\bar{1}) W_1]}{\langle 1\bar{2} \rangle \langle 2\bar{1} \rangle} - \frac{[\bar{W}_1(1\bar{3}2) \bar{W}_2] [W_1(\bar{1}3\bar{2}) W_2]}{(\langle 1\bar{2} \rangle \langle 2\bar{1} \rangle \langle 1\bar{3} \rangle \langle 3\bar{1} \rangle \langle 2\bar{3} \rangle \langle 3\bar{2} \rangle)^{\frac{1}{2}}} \right. \\ & \left. - (2 + z^2) \left(\frac{[\bar{W}_1(1\bar{2}3\bar{1}) W_1] [\bar{W}_2(2\bar{3}1\bar{2}) W_2]}{\langle 1\bar{2} \rangle^2 \langle 3\bar{1} \rangle \langle 2\bar{3} \rangle} + \frac{[\bar{W}_1(1\bar{3}2\bar{1}) W_1] [\bar{W}_2(2\bar{1}3\bar{2}) W_2]}{\langle 2\bar{1} \rangle^2 \langle 1\bar{3} \rangle \langle 3\bar{2} \rangle} \right) \right]. \quad (105) \end{aligned}$$

Therefore, superconformal invariance determines the $\langle \mathcal{T} \mathcal{T} L \rangle$ correlator up to the overall normalization, which is in agreement with Ref. [17].

V. CONCLUSIONS

In this paper, we have shown how superconformal multiplets in representations (j, \bar{j}, q, \bar{q}) fit into the superembedding framework introduced in Ref. [9]. Physically, the most important examples correspond to the real scalar multiplet that contains the global conserved current j_μ and the supercurrent multiplet $\mathcal{T}_{a\dot{b}}$ containing j_R^μ , the supercurrent S_a^μ , and the energy-momentum tensor T_ν^μ . Constructing the relevant correlators is reduced to the task of enumerating $SU(2, 2|1)$ invariants that appear in the products of several copies of the linear representations X_{AB} , \bar{X}^{AB} , W_A , \bar{W}^A . This index-free approach yields relatively compact expressions for the Green's functions. Although the examples presented are not new, the manifestly covariant forms we presented are, and we hope that the simplifications that come with working in the superembedding formalism will eventually lead to new results.

At present we have no way of representing contact terms in the embedding formalism. This would be necessary, for

instance, to deal with the anomaly structure of conserved current three-point functions in a covariant way (rather than imposing that the formalism satisfies the correct anomaly relations componentwise, as we did in this paper). To this end, a direct embedding space formulation of conservation laws such as $D^2 L = \bar{D}^2 L = 0$, etc., in terms of embedding space differential operators would also be required. Finally, besides the extension of the formalism discussed here to the case of extended superconformal invariance, another useful direction might be to see if the recent techniques developed in Ref. [8] for efficiently computing conformal blocks have a natural extension to the supersymmetric case.

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