

Thermal field theory with nonuniform chemical potentialMasato Arai,^{1,*} Yoshishige Kobayashi,^{2,†} and Shin Sasaki^{3,‡}¹*Fukushima National College of Technology Iwaki, Fukushima 970-8034, Japan and Institute of Experimental and Applied Physics, Czech Technical University in Prague, Horská 3a/22, 128 00 Prague 2, Czech Republic*²*Department of Physics, Tokyo Institute of Technology Tokyo 152-8551, Japan*³*Department of Physics, Kitasato University Sagami-hara 252-0373, Japan*

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We investigate thermal one-loop effective potentials in multiflavor models with chemical potentials. We study four-dimensional models in which each flavor has different global $U(1)$ charges. Accordingly they have different chemical potentials. We call these “nonuniform chemical potentials,” which are organized into a diagonal matrix $\hat{\mu}$. The mass matrix at a vacuum does not commute with $\hat{\mu}$. We find that the effective potential is divided into three parts. The first part is the Coleman-Weinberg potential. The UV divergence resides only in this part. The second is the correction to the Coleman-Weinberg potential that is independent of temperature, and the third depends on both temperature and $\hat{\mu}$. Our result is a generalization of the thermal potentials in previous studies for models with single and multiflavors with (uniform) chemical potentials, and it reproduces all the known results correctly.

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I. INTRODUCTION

Thermal field theories with chemical potentials play an important role in understanding many issues in physics, such as the quark-gluon plasma [1,2], the color superconductivity in QCD [3], cosmology, and astrophysics of neutron stars [4]. One of the characteristic features of these fields is phase transitions, which are possibly caused by quantum corrections, finite temperature, and density effects.

When the theory admits perturbative approximations, the one-loop effective potential is a powerful tool for finding the structure of vacua. Thermal one-loop effective potentials of single-flavor models with a chemical potential have been intensively studied (see [5–7] and references therein). On the other hand, chemical potentials in multiflavor models have been introduced in a flavor-independent way in the literature [8–10]. In general, each flavor can have a different value of chemical potential, which we call “nonuniform chemical potentials.” The nonuniform chemical potentials become important when one studies multiflavor models with different $U(1)$ global charges, such as generalized O’Raifeartaigh models [11] and so on.

The generalized O’Raifeartaigh model has phenomenological interests since it yields a spontaneous supersymmetry breaking with $U(1)$ R-symmetry breaking, allowing gauginos to be massive. The key issue to realize the $U(1)$ R-symmetry breaking is that the model should include multiflavors with a peculiar choice of different $U(1)$ R charges [11]. The generalized O’Raifeartaigh models with finite temperatures have been studied to investigate thermal history of supersymmetry breaking vacua in the early Universe [12–19]. Effects of finite temperature and a

chemical potential to the $U(1)$ R-symmetry breaking have been studied for a supersymmetric model with a single flavor [20],¹ where the chemical potential breaks the $U(1)$ R-symmetry even at high temperatures. However, in this model a spontaneous supersymmetry breaking is not considered. In order to study the thermal history of the generalized O’Raifeartaigh model, we need to understand how to calculate the effective potential with finite temperature and nonuniform chemical potential.

One reason that people have not paid attention to theories with nonuniform chemical potentials might be a rather technical issue. In a Lagrangian \mathcal{L} , the nonuniform chemical potentials are organized into a diagonal matrix $\hat{\mu}$, which generically does not commute with a mass matrix \hat{m} in \mathcal{L} . As we will see in this paper, the straightforward generalization of formulas for single-flavor models into those for models with the nonuniform chemical potential is not valid due to the noncommutative nature of the matrices $\hat{\mu}$ and \hat{m} .

The purpose of this paper is to establish the precise calculational scheme of the thermal one-loop effective potential of the multiflavor models with nonuniform chemical potentials. The calculations are quite delicate and need special attention on the UV regularization due to the noncommutativity of the matrices $\hat{\mu}$ and \hat{m} . We find that the one-loop effective potential is the sum of the Coleman-Weinberg potential [24], which is independent of the temperature and $\hat{\mu}$, and the terms that depend on them. A nontrivial finding in this result is that the terms that depend on temperature and $\hat{\mu}$ are UV finite. Since the main purpose of this paper is the calculations themselves, we

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¹Effects of nonuniform chemical potentials for lepton numbers are considered in the MSSM framework [21–23]. However, there are no mass matrices that do not commute with the chemical potentials. This setting is essentially different from ours.

will show the detailed treatment of terms step by step in the calculations.

The organization of this paper is as follows. In the next section, we review the calculation of the effective potential in the single-flavor models. In Sec. III, we generalize the calculation to the multiflavor models. This procedure involves various nontrivial aspects in the calculation. In Sec. IV, the result in Sec. III is applied to a two-flavor model as a simple example. Section V is devoted to conclusion and discussions. In Appendix A, we prove the positive definiteness of the matrix $\sqrt{p^2 \mathbf{1} + \hat{m}^2} \pm \hat{\mu}$. In Appendix B, we show the asymptotic behavior of the determinant quantities defined in Sec. III.

II. EFFECTIVE POTENTIAL: SINGLE-FLAVOR MODEL

We begin with the model including a single complex scalar field ϕ . The Lagrangian is

$$\mathcal{L} = \partial_m \phi \partial^m \phi^\dagger - V(\phi, \phi^\dagger), \quad (2.1)$$

where $m = 0, 1, 2, 3$ is the space-time vector index. We use the mostly minus convention of the metric $\eta_{mn} = \text{diag}(1, -1, -1, -1)$. The scalar potential $V(\phi, \phi^\dagger)$ has at least one extremum where the scalar field ϕ develops its vacuum expectation value (VEV) ϕ_{cl} . The model exhibits a $U(1)$ global symmetry $\phi' = e^{iq} \phi$, where q is the $U(1)$ charge of ϕ . The one-loop effective potential of the model at finite temperature with a chemical potential is calculated through the partition function. The temperature T is introduced by letting $ix_0 = \tau$ and by imposing the periodic boundary condition $\phi(\tau, \vec{x}) = \phi(\tau + \beta, \vec{x})$, where $\beta = 1/T$ and $\vec{x} = (x_1, x_2, x_3)$. The chemical potential μ is introduced by gauging the $U(1)$ global symmetry in the Lagrangian (2.1) [5,6,25]. The space-time derivative ∂_m is replaced by the gauge covariant derivative $D_m = \partial_m + iqA_m$. The nondynamical gauge field A_m is introduced as a VEV only in the zeroth component $\langle A_m \rangle = (i\mu, \mathbf{0})$. Then the partition function is given as

$$\begin{aligned} Z &= \text{Tr} e^{-\beta(H - \mu \mathcal{N})} \\ &= C \int_{\phi(\tau) = \phi(\tau + \beta)} \mathcal{D}\phi \mathcal{D}\phi^\dagger e^{-\int_0^\beta d\tau \int d^3x (D_0 \phi D_0 \phi^\dagger + \vec{\nabla} \phi \cdot \vec{\nabla} \phi^\dagger + V)}, \end{aligned} \quad (2.2)$$

where H is the Hamiltonian associated with the Lagrangian (2.1) and \mathcal{N} is the Noether current of the $U(1)$ symmetry. The constant C is a normalization factor and $D_0 = \frac{\partial}{\partial \tau} - \mu$. Here we set $q = 1$ for simplicity. $\vec{\nabla}$ is the differentiation with respect to \vec{x} . The generating function is derived from the partition function (2.2) from which we obtain the Feynman rules, including the temperature and the chemical potential. With the use of the Feynman rules, the effective potential is obtained as [5,6]

$$\begin{aligned} V_B^{\beta, \mu}(\phi_{\text{cl}}) &= V_B^{(0)\beta, \mu}(\phi_{\text{cl}}) + V_B^{(1)\beta, \mu}(\phi_{\text{cl}}) \\ &\quad + (\text{higher-loop corrections}). \end{aligned} \quad (2.3)$$

The first term $V_B^{(0)\beta, \mu}(\phi_{\text{cl}}) = V(\phi_{\text{cl}}, \phi_{\text{cl}}^\dagger)$ is the potential at tree level. The second term $V_B^{(1)\beta, \mu}$ is the one-loop part of the effective potential which is given by

$$\begin{aligned} V_B^{(1)\beta, \mu}(\phi_{\text{cl}}) &= -\frac{1}{2\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^3 p}{(2\pi)^3} \log(\omega_n^2 + \omega_p^2), \\ \omega_p^2 &\equiv p^2 + m_B^2, \quad \omega_n \equiv 2\pi\beta^{-1}n - i\mu, \quad n \in \mathbb{Z}, \end{aligned} \quad (2.4)$$

where m_B^2 is the mass squared of ϕ at the vacuum $\phi = \phi_{\text{cl}}$,

$$m_B^2 = \left. \frac{\partial^2 V}{\partial \phi^\dagger \partial \phi} \right|_{\phi = \phi_{\text{cl}}}. \quad (2.5)$$

The subscript “ B ” stands for quantities associated with boson fields. In general, the summation over the discrete momentum modes n diverges. In order to regularize the infinity, we rewrite the summation over n to an auxiliary integration over $a^2 \in \mathbb{R}$ [26,27]:

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \log(\omega_n^2 + \omega_p^2) &= \int_{1/\beta^2}^{\omega_p^2} da^2 \sum_{n=-\infty}^{\infty} \frac{1}{\omega_n^2 + a^2} \\ &\quad + \sum_{n=-\infty}^{\infty} \log(\omega_n^2 + 1/\beta^2). \end{aligned} \quad (2.6)$$

Next, we evaluate the sum $\sum_n \frac{1}{\omega_n^2 + a^2}$ in (2.6). Since the function $\frac{\beta}{2} \cot\left(\frac{\beta\omega}{2}\right)$ has poles at $\omega = 2\pi\beta^{-1}n$ with residue 1, the summation is rewritten as [27]

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{1}{(2\pi\beta^{-1}n - i\mu)^2 + a^2} &= \sum_{\omega \in 2\pi\beta^{-1}\mathbb{Z}} \frac{\beta}{2} \text{Res} \left[\cot\left(\frac{\beta\omega}{2}\right) \frac{1}{(\omega - i\mu)^2 + a^2} \right] \\ &= - \sum_{\omega \notin 2\pi\beta^{-1}\mathbb{Z}} \frac{\beta}{2} \text{Res} \left[\cot\left(\frac{\beta\omega}{2}\right) \frac{1}{(\omega - i\mu)^2 + a^2} \right], \end{aligned} \quad (2.7)$$

where the sums in the right-hand sides are taken over poles of the function in the square bracket. Here we used the fact that the following contour integral vanishes for a sufficiently large circle C_R with the radius R :

$$\lim_{R \rightarrow \infty} \oint_{C_R} d\omega \frac{\beta}{2} \cot\left(\frac{\beta\omega}{2}\right) \frac{1}{(\omega - i\mu)^2 + a^2} = 0. \quad (2.8)$$

The poles of the function $[(\omega - i\mu)^2 + a^2]^{-1}$ are found to be $\omega = i(\mu \pm \sqrt{a^2})$. Then the summation in the last expression in (2.7) is easily performed:

$$\begin{aligned}
& \sum_{\omega \notin 2\pi\beta^{-1}\mathbb{Z}} \text{Res} \left[\cot \left(\frac{\beta\omega}{2} \right) \frac{1}{(\omega - i\mu)^2 + a^2} \right] \\
&= -\frac{1}{2\sqrt{a^2}} \left[\coth \left(\frac{\beta(\mu + \sqrt{a^2})}{2} \right) \right. \\
&\quad \left. + \coth \left(\frac{\beta(-\mu + \sqrt{a^2})}{2} \right) \right]. \quad (2.9)
\end{aligned}$$

The above result is integrated by a^2 and we obtain

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \log(\omega_n^2 + \omega_p^2) &= \log \left[\sinh \left(\frac{\beta}{2} (\omega_p + \mu) \right) \right] \\
&\quad + \log \left[\sinh \left(\frac{\beta}{2} (\omega_p - \mu) \right) \right] \\
&\quad + C(T, \mu), \quad (2.10)
\end{aligned}$$

where $C(T, \mu)$ is a p -independent “constant,” which may or may not depend on T and/or μ . The quantity $C(T, \mu)$ comes from the lower limit of the integration and the infinite sum in (2.6), namely,

$$\begin{aligned}
C(T, \mu) &= -\log \left[\sinh \left(\frac{1}{2} (1 + \beta\mu) \right) \right] \\
&\quad - \log \left[\sinh \left(\frac{1}{2} (1 - \beta\mu) \right) \right] \\
&\quad + \sum_{n=-\infty}^{\infty} \log(\omega_n^2 + 1/\beta^2). \quad (2.11)
\end{aligned}$$

Actually $C(T, \mu)$ is a constant which is independent of both T and μ . To see this fact, we differentiate $C(T, \mu)$ with respect to μ :

$$\begin{aligned}
\frac{\partial C(T, \mu)}{\partial \mu} &= \frac{\beta}{2} \left[-\coth \left(\frac{1}{2} (1 + \beta\mu) \right) \right. \\
&\quad \left. + \coth \left(\frac{1}{2} (1 - \beta\mu) \right) \right] - \sum_{n=-\infty}^{\infty} \frac{2i\omega_n}{\omega_n^2 + 1/\beta^2}. \quad (2.12)
\end{aligned}$$

The convergence of the infinite sum in the above equation is not obvious; however, the sum can be deformed by using the symmetry $n \rightarrow -n$ of the sum,

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \frac{2i\omega_n}{\omega_n^2 + 1/\beta^2} &= \sum_{n=-\infty}^{\infty} \left[\frac{1}{\beta^{-1} - i\omega_n} - \frac{1}{\beta^{-1} + i\omega_n} \right] \\
&= \sum_{n=-\infty}^{\infty} \left[\frac{1}{2} \left(\frac{1}{\beta^{-1} - i\omega_n} + \frac{1}{\beta^{-1} - i\omega_{-n}} \right) - \frac{1}{2} \left(\frac{1}{\beta^{-1} + i\omega_n} + \frac{1}{\beta^{-1} + i\omega_{-n}} \right) \right] \\
&= \sum_{n=-\infty}^{\infty} \left[\frac{\beta^{-1} - \mu}{(\beta^{-1} - \mu)^2 + (2\pi\beta^{-1}n)^2} - \frac{\beta^{-1} + \mu}{(\beta^{-1} + \mu)^2 + (2\pi\beta^{-1}n)^2} \right]. \quad (2.13)
\end{aligned}$$

The infinite sum in the last line is convergent obviously. The infinite sum can be calculated as shown previously, which gives $\frac{\beta}{2} \coth(\frac{1}{2}(1 - \beta\mu))$ and $-\frac{\beta}{2} \coth(\frac{1}{2}(1 + \beta\mu))$. Then we find

$$\frac{\partial C(T, \mu)}{\partial \mu} = 0. \quad (2.14)$$

Furthermore, in the case of $\mu = 0$, $C(T, \mu)$ is independent of T [27]. Thus $C(T, \mu)$ depends on neither T nor μ .

Expanding the hyperbolic sine function and factoring out $e^{\frac{\beta}{2}\omega_p}$ in (2.12), the one-loop part of the effective potential is given by

$$\begin{aligned}
V_B^{(1)\beta, \mu}(\phi_{\text{cl}}) &= - \int \frac{d^3 p}{(2\pi)^3} \left[\frac{\omega_p}{2} + \frac{1}{2\beta} \log(1 - e^{-\beta(\omega_p + \mu)}) \right. \\
&\quad \left. + \frac{1}{2\beta} \log(1 - e^{-\beta(\omega_p - \mu)}) \right] \\
&\quad + \text{const}, \quad (2.15)
\end{aligned}$$

where const. is just a numerical constant and does not contain the VEV, T and μ . The explicit value of the constant depends on models. The first term in the integrand is the temperature-independent zero-point energy which

turns out to be the ordinary Coleman-Weinberg potential [24]. The second and third terms are contributions coming from finite temperature and chemical potential. We note that the potential becomes complex when $|\mu| > |m|$. Thus, the chemical potential has the upper bound $|\mu| \leq |m|$.

It is straightforward to incorporate the fermionic contributions to the effective potential. We consider the model with the following Lagrangian:

$$\begin{aligned}
\mathcal{L} &= \partial_m \phi \partial^m \phi^\dagger - V(\phi, \phi^\dagger) + i \bar{\psi} \gamma^m \partial_m \psi \\
&\quad - \bar{\psi} G(\phi, \phi^\dagger) \psi, \quad (2.16)
\end{aligned}$$

where ψ is a Dirac fermion field and G is a linear function of ϕ, ϕ^\dagger . The fermion transforms as $\psi' = e^{i\tilde{q}} \psi$ by the $U(1)$ group, where \tilde{q} is the $U(1)$ charge of the fermion. The function G is chosen such that the Lagrangian (2.16) is invariant under the $U(1)$ transformation and gives the mass for the fermion $m_F = G|_{\phi=\phi_{\text{cl}}}$. Now we promote the global $U(1)$ symmetry to gauge symmetry. The chemical potential is introduced as the VEV of the zeroth component of the $U(1)$ gauge field, $\langle A_m \rangle = (i\mu, \mathbf{0})$. We calculate the one-loop contributions to the effective potential from the fermion ψ . Since the fermion satisfies the antiperiodic

boundary condition along the τ direction, the summation over the discrete momentum in (2.7) is changed. For the fermion field, the modes n are shifted by $1/2$, namely, $\omega_n = 2\pi\beta^{-1}(n + 1/2) - i\mu$. Again we rewrite the summation $\sum_n \frac{1}{\omega_n^2 + a^2}$ into the contour integral by using the function $-\tan(\beta\omega/2)$ instead of $\cot(\beta\omega/2)$. The calculation is performed in parallel with the bosonic case. The result is

$$V_F^{(1)\beta,\mu}(\phi_{\text{cl}}) = 2 \int \frac{d^3p}{(2\pi)^3} \left[\frac{\omega_p}{2} + \frac{1}{2\beta} \log(1 + e^{-\beta(\omega_p + \mu)}) + \frac{1}{2\beta} \log(1 + e^{-\beta(\omega_p - \mu)}) \right] + \text{const}, \quad (2.17)$$

where const is independent of the VEV, T and μ . The subscript “ F ” stands for quantities associated with fermion fields. The factor 2 in front of the above integral appears because the fermion ψ has twice the degrees of freedom of a scalar field. We also note that the sign in front of the exponential factor in the logarithmic function is plus. Therefore, in contrast to the bosonic case, the argument of the logarithmic function does not become negative and there are no upper bounds for the chemical potential $|\mu|$ in the contribution from the fermionic field.

III. MULTIFLAVOR GENERALIZATION

In this section, we consider the multiflavor models, including N complex scalar fields ϕ^i ($i = 1, \dots, N$). The Lagrangian is given by

$$\mathcal{L} = \partial_m \phi^i \partial^m \phi_i^\dagger - V(\phi^i, \phi_i^\dagger). \quad (3.1)$$

The model has a $U(1)$ global symmetry $\phi^i \rightarrow e^{iq^i} \phi^i$, where q^i are $U(1)$ charges of the fields ϕ^i . In the multiflavor case, one repeats steps similar to those discussed in the previous section to obtain the effective potential. In the single-flavor model, it is easy to perform the summation over the modes n through the a^2 integral as in (2.8). However, we need to elaborate on the calculation in some points in the multiflavor models. In the following subsections, we show how to perform the summation of the modes n and discuss the regularization with respect to the momentum integral.

A. Effective potential

The effective potential is a function of the VEVs of the scalar fields whose dependence is attributed to the mass matrix of the fields. The $N \times N$ mass matrix at a vacuum is given by

$$(\hat{m}_B^2)_{ij} = \left. \frac{\partial^2 V}{\partial \phi_i^\dagger \partial \phi_j} \right|_{\phi^i = \phi_{\text{cl}}^i}. \quad (3.2)$$

We assume that the matrix $(\hat{m}_B^2)^T = \hat{m}_B^2$ is real and symmetric.² In addition, we assume that the mass matrix \hat{m}_B^2 is positive definite to avoid tachyonic modes at the vacuum. As we explained in the previous section, the chemical potentials are introduced through the gauging of the $U(1)$ symmetry. We denote the chemical potential of ϕ^i as $\mu^i = q^i \mu$, where μ is a real parameter. We define the $N \times N$ chemical potential matrix, which is real and diagonal, as

$$\hat{\mu} \equiv \text{diag}(\mu^1, \dots, \mu^N). \quad (3.3)$$

As in the single-flavor case, the one-loop part of the effective potential of the model is found to be

$$V_B^{(1)\beta,\mu}(\phi_{\text{cl}}^i) = -\frac{1}{2\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3} \text{Tr} \log(\omega_n^2 + \omega_p^2), \quad (3.4)$$

where ϕ_{cl}^i is the VEV of ϕ^i and

$$\omega_p^2 \equiv p^2 \mathbf{1} + \hat{m}_B^2, \quad \omega_n \equiv 2\pi\beta^{-1}n \mathbf{1} - i\hat{\mu}. \quad (3.5)$$

Here $\mathbf{1}$ is the $N \times N$ unit matrix. Now we calculate the summation over the discrete momentum modes n . We define

$$\hat{v}(\mu) \equiv \sum_{n=-\infty}^{\infty} \text{Tr} \log[p^2 \mathbf{1} + \hat{m}_B^2 + \omega_n^2]. \quad (3.6)$$

Again the summation over n diverges. We generalize the single-flavor result (2.7) to the multiflavor models. In the multiflavor case, both the mass parameter and the chemical potential become matrices. Following the single-flavor case, we rewrite the sum over n by the $a^2 \in \mathbb{R}$ integral. Although ω_p and ω_n are matrices, we can rewrite $\hat{v}(\mu)$ as

$$\hat{v}(\mu) = \int_0^{p^2} da^2 \sum_{n=-\infty}^{\infty} \text{Tr} \{ (a^2 \mathbf{1} + \hat{m}_B^2 + \omega_n^2)^{-1} \} + \text{const}, \quad (3.7)$$

where the power -1 in the trace stands for the inverse matrix. In the above equation, the last term const. is not a constant itself, because it may depend on T and $\hat{\mu}$. However, we can show that the sum of the last term and the value of the lower limit of the integration in the first term is completely independent of both T and $\hat{\mu}$, as shown for $C(T, \mu)$ in (2.10). So hereafter we keep only the upper limit of the integration. The summation over n is written as

²In general, \hat{m}_B^2 is a Hermitian matrix and the off-diagonal blocks $\frac{\partial^2 V}{\partial \phi^i \partial \phi^j}, \frac{\partial^2 V}{\partial \phi_i^\dagger \partial \phi_j^\dagger}$ appear in the mass matrix. In this case, in order to make the mass matrix be real and symmetric one may have to divide a complex field into two real fields, and rewrite the mass matrix in the bases of the real fields. Although the apparent size of the mass matrix is doubled, the subsequent discussion is essentially the same.

$$\begin{aligned}
& \sum_{n=-\infty}^{\infty} \text{Tr}\{(a^2 \mathbf{1} + \hat{m}_B^2 + \omega_n^2)^{-1}\} \\
&= - \sum_{\omega \notin 2\pi\beta^{-1}\mathbb{Z}} \text{Res} \left[\frac{\beta}{2} \cot\left(\frac{\beta\omega}{2}\right) \text{Tr}\{(a^2 \mathbf{1} + \hat{m}_B^2 \right. \\
&\quad \left. + (\omega \mathbf{1} - i\hat{\mu})^2)^{-1}\} \right]. \tag{3.8}
\end{aligned}$$

$$\text{Tr}\{(\hat{m}_B^2 + a^2 \mathbf{1} + \omega^2 \mathbf{1})^{-1}\} = \text{Tr} \left[\text{diag} \left(\frac{1}{a^2 + m_1^2 + \omega^2}, \frac{1}{a^2 + m_2^2 + \omega^2}, \dots, \frac{1}{a^2 + m_N^2 + \omega^2} \right) \right], \tag{3.9}$$

where we have used the fact that the trace of the inverse matrix is written as the sum of the inverse of the corresponding eigenvalues. Here m_i^2 are the eigenvalues of the mass matrix \hat{m}_B^2 . Since the singularities of the integrand are located at $a^2 + m_i^2 + \omega^2 = 0$, we then find

$$\begin{aligned}
& \sum_{\omega \notin 2\pi\beta^{-1}\mathbb{Z}} \text{Res} \left[\frac{\beta}{2} \cot\left(\frac{\beta\omega}{2}\right) \text{Tr}\{(a^2 \mathbf{1} + \hat{m}_B^2 + \omega^2 \mathbf{1})^{-1}\} \right] \\
&= - \sum_{i=1}^N \frac{\coth\left(\frac{\beta}{2} \sqrt{a^2 + m_i^2}\right)}{\sqrt{a^2 + m_i^2}}. \tag{3.10}
\end{aligned}$$

Finally, we perform the integration by a^2 . The result in the case $\hat{\mu} = 0$ is therefore

$$\begin{aligned}
\hat{v}(0) &= \int_0^{p^2} da^2 \sum_{i=1}^N \frac{\coth\left(\frac{\beta}{2} \sqrt{a^2 + m_i^2}\right)}{\sqrt{a^2 + m_i^2}} \\
&= \frac{4}{\beta} \sum_{i=1}^N \log \left[\sinh\left(\frac{\beta}{2} \sqrt{p^2 + m_i^2}\right) \right] + \text{const.} \tag{3.11}
\end{aligned}$$

This precisely recovers the thermal effective potential for the multiflavor models [6].

Now we consider the case $\hat{\mu} \neq 0$. We need to evaluate the summation over the residues

$$\sum_{\omega \notin 2\pi\beta^{-1}\mathbb{Z}} \text{Res} \left[\cot\left(\frac{\beta\omega}{2}\right) \text{Tr}\{(a^2 \mathbf{1} + \hat{m}_B^2 + (\omega \mathbf{1} - i\hat{\mu})^2)^{-1}\} \right]. \tag{3.12}$$

Before going to the situation where $\hat{\mu} \neq 0$, we first consider the $\hat{\mu} = 0$ case. Generally speaking, finding the trace of the inverse matrix is cumbersome; however, when $\hat{\mu} = 0$, we can perform the calculation by diagonalizing the matrix \hat{m}_B^2 . When $\hat{\mu} = 0$, the trace of the matrix is calculated as

When the matrices \hat{m}_B and $\hat{\mu}$ can be simultaneously diagonalized, the two matrices commute with each other $[\hat{m}_B, \hat{\mu}] = 0$ and the calculation is the same in the previous section. This is because, by using new diagonal matrices \hat{m}_B' and $\hat{\mu}'$ which are obtained from \hat{m}_B and $\hat{\mu}$, the effective potential is the simple summation of that of the N -independent single-flavor (complex) fields. In the following, we consider the general case where $[\hat{m}_B, \hat{\mu}] \neq 0$.

In order to evaluate the residues in (3.12), we need to find singularities of the function $\text{Tr}(a^2 \mathbf{1} + \hat{m}_B^2 + (\omega \mathbf{1} - i\hat{\mu})^2)^{-1}$. In general, the inverse of a matrix M is given by

$$M^{-1} = (\det M)^{-1} \tilde{M}, \tag{3.13}$$

where \tilde{M} is the cofactor matrix of M . Since all the elements of the matrix $M = a^2 \mathbf{1} + \hat{m}_B^2 + (\omega \mathbf{1} - i\hat{\mu})^2$ are polynomials of ω , the elements of the cofactor matrix \tilde{M} do not have any poles in ω . Therefore, all the singularities in $\text{Tr} M^{-1}$ come from the zeros of $\det M$; namely, the singularities in the ω plane satisfy the following equation:

$$\det M = \det(a^2 \mathbf{1} + \hat{m}_B^2 + (\omega \mathbf{1} - i\hat{\mu})^2) = 0. \tag{3.14}$$

The determinant of M is written as a polynomial of ω , namely,

$$\det M = \prod_{i=1}^{2N} (\omega - \chi_i), \tag{3.15}$$

where χ_i are the solutions to Eq. (3.14). Then we have

$$\begin{aligned}
& \sum_{\omega \notin 2\pi\beta^{-1}\mathbb{Z}} \text{Res} \left[\cot\left(\frac{\beta\omega}{2}\right) \text{Tr}\{(a^2 \mathbf{1} + \hat{m}_B^2 + (\omega \mathbf{1} - i\hat{\mu})^2)^{-1}\} \right] \\
&= \sum_{\{\omega | \det M=0\}} \text{Res} \left[\cot\left(\frac{\beta\omega}{2}\right) \frac{1}{(\omega - \chi_1)(\omega - \chi_2) \cdots (\omega - \chi_{2N})} \text{Tr} \tilde{M} \right] \\
&= \sum_{i=1}^{2N} \frac{\cot\left(\frac{\beta\chi_i}{2}\right)}{(\chi_i - \chi_1) \cdots (\chi_i - \chi_{i-1})(\chi_i - \chi_{i+1}) \cdots (\chi_i - \chi_{2N})} \text{Tr} \tilde{M}|_{\omega=\chi_i}. \tag{3.16}
\end{aligned}$$

For simplicity, we assume that the solutions χ_i are general and not degenerate. The next task is to perform the integration of (3.16) by a^2 . In order to find the a^2 dependence of the solutions χ_i , we show the following facts. We consider $\det M$ as a function of a^2 and ω , $g(a^2, \omega) \equiv \det M$. Then $\omega = \chi_i$ are solutions to the equation $g(a^2, \omega(a^2)) = 0$. From the implicit function theorem, we have the following relation:

$$\frac{\partial \chi_i}{\partial a^2} = - \frac{\frac{\partial \det M}{\partial a^2}}{\frac{\partial \det M}{\partial \omega}} \bigg|_{\omega=\chi_i}. \quad (3.17)$$

The denominator in (3.17) is evaluated as

$$\begin{aligned} \frac{\partial \det M}{\partial \omega} \bigg|_{\omega=\chi_i} &= (\chi_i - \chi_1) \cdots (\chi_i - \chi_{i-1}) \\ &\quad \times (\chi_i - \chi_{i+1}) \cdots (\chi_i - \chi_{2N}). \end{aligned} \quad (3.18)$$

On the other hand, since a^2 enters into each element in the matrix M as $M = a^2 \mathbf{1} + \cdots$, where \cdots are terms that are independent of a^2 , the numerator in (3.17) is evaluated as

$$\begin{aligned} \frac{\partial \det M}{\partial a^2} &= \det \begin{pmatrix} 1 & 0 & \cdots & 0 \\ M_{21} & M_{22} & \cdots & M_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ M_{N1} & \cdots & \cdots & M_{NN} \end{pmatrix} \\ &\quad + \det \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1N} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ M_{N1} & \cdots & \cdots & M_{NN} \end{pmatrix} \\ &\quad + \cdots + \det \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1N} \\ \vdots & \ddots & \vdots & \vdots \\ M_{(N-1)1} & M_{(N-1)2} & \cdots & M_{(N-1)N} \\ 0 & 0 & \cdots & 1 \end{pmatrix}. \end{aligned} \quad (3.19)$$

This is nothing but the trace part of the cofactor matrix. Then, we obtain

$$\frac{\partial \det M}{\partial a^2} = \text{Tr} \tilde{M}. \quad (3.20)$$

In short, we find the following formula:

$$\begin{aligned} \frac{d\chi_i}{da^2} &= - \frac{1}{(\chi_i - \chi_1) \cdots (\chi_i - \chi_{i-1})(\chi_i - \chi_{i+1}) \cdots (\chi_i - \chi_{2N})} \\ &\quad \times \text{Tr} \tilde{M} \big|_{\omega=\chi_i}. \end{aligned} \quad (3.21)$$

Using this formula, we can perform the integration over a^2 . The result is

$$\begin{aligned} \int da^2 \sum_{n=-\infty}^{\infty} \text{Tr} \{ (a^2 \mathbf{1} + \hat{m}_B^2 + \omega_n^2)^{-1} \} \\ = \frac{2}{\beta} \sum_{i=1}^{2N} \log \left(\sin \left(\frac{\beta \chi_i}{2} \right) \right) + \text{const.} \end{aligned} \quad (3.22)$$

Therefore, we obtain the following expression of the effective potential:

$$\begin{aligned} V_B^{(1)\beta, \mu}(\phi_{\text{cl}}) &= - \frac{1}{2\beta} \int \frac{d^3 p}{(2\pi)^3} \sum_{i=1}^{2N} \log \left(\sin \left(\frac{\beta \chi_i}{2} \right) \right) \\ &\quad + \text{const.} \end{aligned} \quad (3.23)$$

Now we consider the model that includes N Dirac fermionic fields ψ^i ($i = 1, \dots, N$),

$$\begin{aligned} \mathcal{L} &= \partial_m \phi^i \partial^m \phi_i^\dagger - V(\phi^i, \phi_i^\dagger) + i \bar{\psi}_i \gamma^m \partial_m \psi^i \\ &\quad - \bar{\psi}_i \hat{M}^i_j(\phi, \phi^\dagger) \psi^j. \end{aligned} \quad (3.24)$$

The model has $U(1)$ global symmetry $\phi^{i'} = e^{iq^i} \phi^i$ and $\psi^{i'} = e^{i\bar{q}^i} \psi^i$. The same calculations are applied to the fermionic sector where the mass \hat{m}_B for the bosonic fields is replaced by that for the fermionic fields $\hat{m}_F = \hat{M}|_{\phi=\phi_{\text{cl}}}$ and the summation is separated into the parts $n = 1, 3, \dots$ and $n = 2, 4, \dots$ [7]. We just replace the function $(\beta/2) \times \cot(\beta\omega/2)$ with $-(\beta/2) \tan(\beta\omega/2)$ in the calculation. Then the contributions to the one-loop effective potential from the Dirac fermions ψ^i are found to be

$$V_F^{(1)\beta, \mu}(\phi_{\text{cl}}) = \frac{1}{2\beta} \int \frac{d^3 p}{(2\pi)^3} \sum_{i=1}^{2N} \log \left(\cos \left(\frac{\beta \chi_i}{2} \right) \right) + \text{const.} \quad (3.25)$$

The expressions (3.23) and (3.25) are one of the main results in this work. A few comments are in order. The constant parts in (3.23) and (3.25) are set to be zero which is determined in the limit $\hat{\mu} \rightarrow 0$. The expression (3.23) is the natural generalization of the single-flavor formula [5]. We note that in order to find the explicit expression of the effective potential, one needs to find all the solutions χ_i to the equation $\det M = 0$. The detail of the solutions χ_i depends on the respective models and we never look for the explicit solutions in this paper. Although we have performed the summation over the modes n and regularized the divergences coming from the sum, the expressions (3.23) and (3.25) still contain a divergent part stemming from the momentum integration. In the following section, we study the properties of the solutions χ_i that do not depend on the details of models. We will show that the divergent part in the integration by the momentum p is isolated and regularized.

B. Properties of solutions

We have obtained the one-loop effective potentials (3.23) and (3.25). In order to find the explicit expressions of the potentials, one needs the solutions to the equation for ω :

$$\det(p^2\mathbf{1} + \hat{m}^2 + (\omega\mathbf{1} - i\hat{\mu})^2) = 0. \quad (3.26)$$

Here we do not distinguish the bosonic and fermionic masses. In the following, we study properties of the solutions to Eq. (3.26). Hereafter we assume that $\hat{m}^2 - \hat{\mu}^2$ is a positive definite matrix. This condition is a generalization of the bound of the chemical potential $|\mu| \leq m$ for a single bosonic field. The violation of this condition implies the existence of tachyonic modes in the vacuum for the scalar fields.

1. Pure imaginary nature of the solution

Let ω be a solution to Eq. (3.26) and $|0\rangle$ the eigenvector of the matrix $p^2\mathbf{1} + \hat{m}^2 + (\omega\mathbf{1} - i\hat{\mu})^2$ associated with the zero eigenvalue. Then we find

$$\begin{aligned} \langle 0|(p^2\mathbf{1} + \hat{m}^2 + (\omega\mathbf{1} - i\hat{\mu})^2)|0\rangle \\ = \omega^2 + p^2 - 2i\langle 0|\hat{\mu}|0\rangle\omega + \langle 0|(\hat{m}^2 - \hat{\mu}^2)|0\rangle = 0, \end{aligned} \quad (3.27)$$

where $\langle 0|$ is the Hermitian conjugate of $|0\rangle$, which is normalized as $\langle 0|0\rangle = 1$. Therefore, we have

$$\omega = i(\langle 0|\hat{\mu}|0\rangle \pm \sqrt{p^2 + \langle 0|(\hat{m}^2 - \hat{\mu}^2)|0\rangle + \langle 0|\hat{\mu}|0\rangle^2}). \quad (3.28)$$

Because (3.27) is a necessary condition for the given solution ω and the associated eigenvector $|0\rangle$, (3.28) does not completely determine ω , but only shows that ω is equal to one of the two quantities of the right-hand side. The quantity $\langle 0|(\hat{m}^2 - \hat{\mu}^2)|0\rangle$ is positive by the assumption, and $\langle 0|\hat{\mu}|0\rangle$ is real because $\hat{\mu}$ is a real and diagonal matrix. Therefore, ω is always pure imaginary for any p . In the following discussions, we denote the imaginary solutions with the positive sign by χ_i^+ , and with the negative sign by χ_i^- .

2. Solutions and relation to their reference values

For later use, we clarify the relation between the solutions to Eq. (3.26) and the eigenvalues of the matrices $i(\hat{\mu} \pm \sqrt{p^2\mathbf{1} + \hat{m}^2})$. We define the following function:

$$\begin{aligned} f(\eta, \omega) &= \det \left[\left(\omega\mathbf{1} - i \left(\hat{\mu} - \sqrt{p^2\mathbf{1} + \hat{m}^2} \right) \right) \right. \\ &\quad \times \left(\omega\mathbf{1} - i \left(\hat{\mu} + \sqrt{p^2\mathbf{1} + \hat{m}^2} \right) \right) \\ &\quad \left. + \eta \left[\hat{\mu}, \sqrt{p^2\mathbf{1} + \hat{m}^2} \right] \right] \\ &= \det \left[p^2\mathbf{1} + \hat{m}^2 - \hat{\mu}^2 - 2i\hat{\mu}\omega \right. \\ &\quad \left. + \omega^2\mathbf{1} + (\eta - 1) \left[\hat{\mu}, \sqrt{p^2\mathbf{1} + \hat{m}^2} \right] \right], \end{aligned} \quad (3.29)$$

where $0 \leq \eta \leq 1$ is a parameter. The matrix $\sqrt{p^2\mathbf{1} + \hat{m}^2}$ should be chosen to be positive definite.³ Equation (3.26) is equivalent to $f(1, \omega) = 0$. We first solve $f(0, \omega) = 0$ and then discuss the properties of the solutions to (3.26). When $\eta = 0$, the equation $f(0, \omega) = 0$ becomes

$$\begin{aligned} f(0, \omega) &= \det \left(\omega\mathbf{1} - i \left(\hat{\mu} - \sqrt{p^2\mathbf{1} + \hat{m}^2} \right) \right) \\ &\quad \times \det \left(\omega\mathbf{1} - i \left(\hat{\mu} + \sqrt{p^2\mathbf{1} + \hat{m}^2} \right) \right) \\ &= 0. \end{aligned} \quad (3.30)$$

Equation (3.30) is equivalent to

$$\det \left(\omega\mathbf{1} - i \left(\hat{\mu} + \sqrt{p^2\mathbf{1} + \hat{m}^2} \right) \right) = 0, \quad (3.31)$$

or

$$\det \left(\omega\mathbf{1} - i \left(\hat{\mu} - \sqrt{p^2\mathbf{1} + \hat{m}^2} \right) \right) = 0. \quad (3.32)$$

The solutions to these equations are given by the eigenvalues of the matrices $M_{\pm} \equiv i(\hat{\mu} \pm \sqrt{p^2\mathbf{1} + \hat{m}^2})$, respectively. Note that they are the exact solutions to the equation $f(\eta, \omega) = 0$ as long as $[\hat{m}, \hat{\mu}] = 0$. Since $\sqrt{p^2\mathbf{1} + \hat{m}^2} \pm \hat{\mu}$ are positive definite matrices (see Appendix A), the eigenvalues of M_+ , which we call κ_i^+ , are pure imaginary with the positive sign. On the other hand, the eigenvalues of M_- , which we call κ_i^- , are pure imaginary with the negative sign.

We find that the solutions to the equation $f(\eta, \omega) = 0$ are pure imaginary in the whole range of $0 \leq \eta \leq 1$. To see it, we rewrite the matrix in (3.29) as

³In general, there are 2^N square roots of an $N \times N$ matrix. We have chosen the branch where all the eigenvalues of the matrix $\sqrt{p^2\mathbf{1} + \hat{m}^2}$ are positive. This is always possible when \hat{m}^2 is positive definite.

$$\begin{aligned}
& p^2 \mathbf{1} + \hat{m}^2 - \hat{\mu}^2 - 2i\hat{\mu}\omega + \omega^2 \mathbf{1} \\
& + (\eta - 1) \left[\hat{\mu}, \sqrt{p^2 \mathbf{1} + \hat{m}^2} \right] \\
& = \eta(p^2 \mathbf{1} + \hat{m}^2 - \hat{\mu}^2) + (1 - \eta) \left(\sqrt{p^2 \mathbf{1} + \hat{m}^2} - \hat{\mu} \right) \\
& \quad \times \left(\sqrt{p^2 \mathbf{1} + \hat{m}^2} + \hat{\mu} \right) - 2i\hat{\mu}\omega + \omega^2 \mathbf{1}. \quad (3.33)
\end{aligned}$$

Now we consider the eigenvector $|0\rangle \neq 0$, which satisfies the following relation:

$$\begin{aligned}
& \left[\eta(p^2 \mathbf{1} + \hat{m}^2 - \hat{\mu}^2) + (1 - \eta) \left(\sqrt{p^2 \mathbf{1} + \hat{m}^2} - \hat{\mu} \right) \right. \\
& \quad \times \left. \left(\sqrt{p^2 \mathbf{1} + \hat{m}^2} + \hat{\mu} \right) - 2i\hat{\mu}\omega + \omega^2 \mathbf{1} \right] |0\rangle = 0. \quad (3.34)
\end{aligned}$$

The solutions to $f(\eta, \omega) = 0$ are given by

$$\omega = i\langle 0 | \hat{\mu} | 0 \rangle \pm i \sqrt{\eta(p^2 + \langle 0 | (\hat{m}^2 - \hat{\mu}^2) | 0 \rangle) + (1 - \eta) \langle 0 | \left(\sqrt{p^2 \mathbf{1} + \hat{m}^2} - \hat{\mu} \right) \left(\sqrt{p^2 \mathbf{1} + \hat{m}^2} + \hat{\mu} \right) | 0 \rangle + \langle 0 | \hat{\mu} | 0 \rangle^2}. \quad (3.35)$$

The matrix $(\sqrt{p^2 \mathbf{1} + \hat{m}^2} - \hat{\mu})(\sqrt{p^2 \mathbf{1} + \hat{m}^2} + \hat{\mu})$ is positive definite since both $\sqrt{p^2 \mathbf{1} + \hat{m}^2} - \hat{\mu}$ and $\sqrt{p^2 \mathbf{1} + \hat{m}^2} + \hat{\mu}$ are real symmetric and positive definite matrices. Therefore, the quantity inside the square root in (3.35) is positive. Then the solutions ω to the equation $f(\eta, \omega) = 0$ are always pure imaginary and nonzero. We note that the solutions ω change continuously when η runs from 0 to 1. However, as we have seen in the above discussion, the solutions ω in (3.35) are always pure imaginary and nonzero. Then κ_i^+ and κ_i^- change continuously into χ_i^+ and χ_i^- , respectively, without flipping their signs. Therefore, we conclude that the solutions to Eq. (3.26) are pure imaginary, half of which have positive signs and the other half have negative signs. All those have the one-to-one correspondence to κ_i^\pm .

3. Large- p behavior

In order to find the final expression of the effective potentials (3.23) and (3.25), we need to perform the momentum integration. The integral diverges at the large momentum p . In the following, we examine the large- p behavior of the solutions χ_i^\pm .

Equation (3.26) generally has $2N$ solutions. In the large- p limit, the solutions can be written in the following form:

$$\begin{aligned}
\chi_i^\pm &= i \left(\mu_i \pm \sqrt{p^2 + m_{ii}^2} + \mathcal{O}(1/p) \right) \\
&= i(\pm p + \mu_i + \mathcal{O}(1/p)), \quad (3.36)
\end{aligned}$$

where μ^i is the i th-diagonal component of $\hat{\mu}$ and m_{ii}^2 is a diagonal (i, i) component of \hat{m}^2 . Although all ω are linearly divergent at large p , the solutions are distinguishable from one another at order $\mathcal{O}(p^0)$ as long as the chemical potentials μ_i are general values. The eigenvalues κ_i^\pm have the same asymptotic form as in (3.36). In this sense, κ_i^\pm are the approximate solutions to Eq. (3.26), at least in the large- p regime. Since all the solutions χ_i^\pm to Eq. (3.26) [$f(1, \omega) = 0$] have the one-to-one correspondence to the solutions κ_i^\pm to $f(0, \omega) = 0$, each χ_i^+ (χ_i^-) approaches the

corresponding one among the set $\{\kappa_i^+\}$ ($\{\kappa_i^-\}$). In the following, we estimate how χ_i^\pm approach their counterparts.

The determinant of (3.26) is rewritten as

$$\begin{aligned}
& \det(p^2 \mathbf{1} + \hat{m}^2 + (\omega \mathbf{1} - i\hat{\mu})^2) \\
&= \prod_{i=1}^N (\omega - \chi_i^+) (\omega - \chi_i^-) = \prod_{i=1}^N \Delta_i^+ \Delta_i^-, \quad (3.37)
\end{aligned}$$

where $\Delta_i^\pm \equiv \omega - \chi_i^\pm$. Differentiating both sides of (3.37) with respect to ω , we obtain

$$\begin{aligned}
& \frac{d}{d\omega} \det(p^2 \mathbf{1} + \hat{m}^2 + (\omega \mathbf{1} - i\hat{\mu})^2) \\
&= (\Delta_2^+ \cdots \Delta_N^+ \Delta_1^- \cdots \Delta_N^-) \\
&+ (\Delta_1^+ \Delta_3^+ \cdots \Delta_N^+ \Delta_1^- \cdots \Delta_N^-) \\
&+ \cdots + (\Delta_1^+ \cdots \Delta_N^+ \Delta_1^- \cdots \Delta_{N-2}^- \Delta_N^-) \\
&+ (\Delta_1^+ \cdots \Delta_N^+ \Delta_1^- \cdots \Delta_{N-1}^-). \quad (3.38)
\end{aligned}$$

Now we choose one of $\{\kappa_i^\pm\}$, say κ_k^+ , and substitute ω with κ_k^+ . As shown in Appendix B, in the large- p limit, (3.37) and (3.38) behave in such a way that $\det(p^2 \mathbf{1} + \hat{m}^2 + (\kappa_k^+ \mathbf{1} - i\hat{\mu})^2) \sim \mathcal{O}(p^{N-4})$ and $\frac{d}{d\omega} \det(p^2 \mathbf{1} + \hat{m}^2 + (\kappa_k^+ \mathbf{1} - i\hat{\mu})^2) \sim \mathcal{O}(p^N)$. We introduce the parameters ε_i^\pm that represent the power of p in Δ_i^\pm in the large p ; namely, we have $\Delta_i^\pm \sim \mathcal{O}(p^{\varepsilon_i^\pm})$. The asymptotic behavior of (3.37) and (3.38) tells us that

$$S = N - 4 \quad \left(S \equiv \sum_{i=1}^N (\varepsilon_i^+ + \varepsilon_i^-) \right), \quad (3.39)$$

$$\max_{i=1, \dots, N} \{S - \varepsilon_i^+, S - \varepsilon_i^-\} = N. \quad (3.40)$$

On the other hand, because κ_k^+ has the positive sign, the large- p behavior of each Δ_i^\pm in the case that $\omega = \kappa_k^+$ is given by

$$\Delta_i^+ = \mathcal{O}(p^0), \quad \Delta_i^- = +2ip + \mathcal{O}(p^0) \quad (i = 1, 2, \dots, N), \quad (3.41)$$

from the asymptotic behavior (3.36). So one can impose the following conditions on ε_i^\pm :

$$\varepsilon_1^- = \varepsilon_2^- = \dots = \varepsilon_N^- = 1, \quad (3.42)$$

and

$$\varepsilon_i^+ \leq 0 \quad (i = 1, \dots, N). \quad (3.43)$$

The only possibility satisfying all the conditions (3.39), (3.40), (3.42), and (3.43) is

$$\varepsilon_i^+ = \begin{cases} -4 & (i = l) \\ 0 & (i \neq l) \end{cases} \text{ for some } l. \quad (3.44)$$

Among $2N$ components of Δ_i^\pm , only $\Delta_l^+ = \chi_l^+ - \kappa_k^+$ behaves as $\Delta_l^+ = \mathcal{O}(1/p^4)$. It is obvious that χ_l^+ is the corresponding solution to κ_k^+ . Though both χ_l^+ and κ_k^+ diverge in the large- p limit, the difference $\chi_l^+ - \kappa_k^+$ decreases at the inverse of the fourth power of p . The above discussion can be applied to arbitrary χ_i^\pm . Therefore, one can conclude that the difference between any χ_i^\pm and the corresponding κ_k^\pm drops as $\mathcal{O}(1/p^4)$ in $p \rightarrow \infty$.

C. Regularized effective potential

In this subsection, we study the regularization of the momentum integral in the effective potentials (3.23) and (3.25). Since the solutions χ_i^\pm are pure imaginary, these are represented as $\chi_i^\pm = \pm i|\chi_i^\pm|$. We rewrite the potential (3.23) in such a way that the logarithmic function is well defined in the large- p regime:

$$\begin{aligned} \log\left(\sin\frac{\beta}{2}\chi_i^\pm\right) &= \log\left(\pm\sin\frac{i\beta}{2}|\chi_i^\pm|\right) \\ &= \log\left(\pm\frac{i}{2}e^{\frac{\beta}{2}|\chi_i^\pm|}(1 - e^{-\beta|\chi_i^\pm|})\right) \\ &= \mp i\frac{\beta}{2}\chi_i^\pm + \log(1 - e^{-\beta|\chi_i^\pm|}) \\ &\quad + \log(\pm i/2). \end{aligned} \quad (3.45)$$

The last term is the constant that includes the phase factor of χ_i^\pm . Then the sum of all $2N$ solutions becomes

⁴Indeed, the sum over all the differences $\chi_i^\pm - \kappa_i^\pm$ drops more quickly,

$$\sum_{i=1}^N \chi_i^\pm = \sum_{i=1}^N \kappa_i^\pm + \mathcal{O}(1/p^5).$$

However, the $\mathcal{O}(p^{-4})$ behavior of Δ_i^\pm at large p is sufficient to discuss the convergence of the integration over the momentum \vec{p} .

$$\begin{aligned} &\sum_{i=1}^N \left[\log\left(\sin\left(\frac{\beta\chi_i^+}{2}\right)\right) + \log\left(\sin\left(\frac{\beta\chi_i^-}{2}\right)\right) \right] \\ &= -i\frac{\beta}{2} \sum_{i=1}^N (\chi_i^+ - \chi_i^-) + \sum_{i=1}^N [\log(1 - e^{-\beta|\chi_i^+|}) \\ &\quad + \log(1 - e^{-\beta|\chi_i^-|})]. \end{aligned} \quad (3.46)$$

The first term in the last line contains the Coleman-Weinberg potential. Here we drop an irrelevant constant in the last line. The sum of all the phase factors vanishes because the sum contains $\log i$ as many as $\log(-i)$.

In order to find the large- p behavior of the quantity (3.46), we define $\delta_i^\pm(p) \equiv \chi_i^\pm(p) - \kappa_i^\pm(p)$. Then the first term in (3.46) is split into two parts,

$$\begin{aligned} \sum_{i=1}^N (\chi_i^+ - \chi_i^-) &= \sum_{i=1}^N \{(\kappa_i^+ + \delta_i^+) - (\kappa_i^- + \delta_i^-)\} \\ &= 2i \sum_{i=1}^N \sqrt{p^2 + m_{ii}^2} + \sum_{i=1}^N (\delta_i^+ - \delta_i^-). \end{aligned} \quad (3.47)$$

Here we used the relation $\sum_{i=1}^N \kappa_i^\pm = i \text{Tr}(\hat{\mu} \pm \sqrt{p^2 \mathbf{1} + \hat{m}^2})$. Collecting all the terms together, we obtain the one-loop part of the effective potential as

$$\begin{aligned} V_B^{(1)\beta,\mu} &= -\frac{1}{\beta} \int \frac{d^3p}{(2\pi)^3} \sum_{i=1}^N \left[\beta \sqrt{p^2 + m_{Bii}^2} \right. \\ &\quad \left. - \frac{i\beta}{2} ((\chi_i^+ - \kappa_i^+) - (\chi_i^- - \kappa_i^-)) \right. \\ &\quad \left. + \log(1 - e^{-\beta|\chi_i^+|}) + \log(1 - e^{-\beta|\chi_i^-|}) \right]. \end{aligned} \quad (3.48)$$

The first term in the integrand is nothing but the ordinary Coleman-Weinberg potential term, of which the regularization scheme is well known [24]. The second term is corrections to the Coleman-Weinberg potential stemming from the nonuniform chemical potential. Because $\chi_i^\pm - \kappa_i^\pm$ decreases like p^{-4} at large p , this term stays finite after the integration over the momentum. The third and fourth terms are corrections from both chemical potentials and temperature. They are the familiar bosonic thermal potentials, where the arguments of the exponential functions are replaced by χ_i^\pm . The momentum integration of these terms converges since the exponential factors behave well in the large- p regime. Therefore, the divergent part comes only from the Coleman-Weinberg potential term. We stress that in the derivation of the first term in (3.48), the contributions coming from the chemical potential $\hat{\mu}$ are canceled out. We emphasize that all the divergent pieces in (3.48) are completely independent of the temperature and the chemical potential. This is the necessary condition for the renormalization of the quantum field theory.

D. Upper bound of $\hat{\mu}$ in fermionic contributions

As in the case of the single-flavor models, it is straightforward to derive the contribution from fermions to the one-loop effective potential in the multiflavor models. However, there is one significant difference on the upper bound of the chemical potentials. So far we have assumed that $\hat{m}^2 - \hat{\mu}^2$ is positive definite. For the chemical potentials of the scalar fields, the upper bound has clear physical meaning. Actually it is equivalent to the condition that the system has no tachyonic modes at tree level. However, the physical reason for the upper bound on the chemical potential is not obvious for fermionic fields. Indeed, the contributions to the one-loop thermal effective potential from the fermion in the single-flavor model (2.17) are well defined as both $\mu < m$ and $\mu \geq m$. For the multiflavor models, the positive definiteness of the matrix $\hat{m}^2 - \hat{\mu}^2$ leads to the property that the solutions to (3.26), ω , are pure imaginary and are divided into equal numbers of solutions χ_i^+ and χ_i^- . These properties are not guaranteed if we do not assume the positive definiteness of the matrix.

We take a closer look at the potential (3.25). When ω is pure imaginary, namely, $\omega = ir$ ($r \in \mathbb{R}$), the integrand of (3.25) can be decomposed as

$$\begin{aligned} \log\left(\cos\frac{\beta}{2}\omega\right) &= \log\left(\cos\frac{i\beta}{2}r\right) = \log\left(\frac{1}{2}e^{\frac{\beta}{2}r}(1 + e^{-\beta r})\right) \\ &= \frac{\beta}{2}\text{Im}(\omega) + \log(1 + e^{-\beta\text{Im}(\omega)}) - \log 2. \end{aligned} \quad (3.49)$$

In (3.49), the last term is real and there are no complex phase terms [see the last term in (3.45) for the scalar field contributions] and the point $\omega = 0$ is not singular. Therefore, for fermionic fields, unlike for scalar fields, the potential is well defined and physical as long as ω is pure imaginary, and ω can freely move on the imaginary axis, including zero. As we have shown, ω is pure imaginary when the following quantity is positive:

$$p^2 + \langle 0 | (\hat{m}^2 - \hat{\mu}^2) | 0 \rangle + \langle 0 | \hat{\mu} | 0 \rangle^2. \quad (3.50)$$

In the single-flavor models, (3.50) becomes $p^2 + m^2$, which is real and positive, and therefore ω is always pure imaginary. It is the reason that the potential is well defined as both $\mu < m$ and $\mu \geq m$. In the multiflavor models, it becomes a little more complicated. When $\hat{m}^2 - \hat{\mu}^2$ is positive definite, the solutions ω are pure imaginary. However, when $\hat{m}^2 - \hat{\mu}^2$ is not positive definite, it can be negative at small p . In that case, ω may have not only an imaginary part but a real part. Then complex quantities may come out from the logarithmic function in the effective potential, which cannot be absorbed into a

p -independent constant. As a result, the potential becomes a complex value, which may be recognized as the instability of the corresponding vacuum. This is a new phenomenon only for multiflavor models with nonuniform chemical potential. In the case that $\hat{m}^2 - \hat{\mu}^2$ is not positive definite, the small- p behaviors of the solutions ω depend on the details of the matrices \hat{m} and $\hat{\mu}$. We did not find any restriction that ensures the appropriate behaviors of ω at small p . Even if all ω are pure imaginary for the entire range of p , each ω does not have the definite sign any more, which may flip as p goes from zero to infinity. Therefore, in the cases where $\hat{m}^2 - \hat{\mu}^2$ is not positive definite, in order to extract the UV divergent piece as in (3.49), we need to extract the linear term in ω with the appropriate sign. This sign is determined by its large- p behavior which is necessary for the convergence of the momentum integral of the logarithmic function.

With the above discussion, the one-loop part of the effective potential from fermionic fields is given by

$$\begin{aligned} V_F^{(1)\beta,\mu} &= \frac{2}{\beta} \int \frac{d^3p}{(2\pi)^3} \sum_{i=1}^N \left[\beta \sqrt{p^2 + m_{Fii}^2} \right. \\ &\quad - \frac{i\beta}{2} ((\chi_i'^+ - \kappa_i'^+) - (\chi_i'^- - \kappa_i'^-)) \\ &\quad \left. + \log(1 + e^{-\beta|\chi_i'^+|}) + \log(1 + e^{-\beta|\chi_i'^-|}) \right], \end{aligned} \quad (3.51)$$

where we have denoted the solutions to $f(1, \omega) = 0$ and $f(0, \omega) = 0$ as $\chi_i'^{\pm}$ and $\kappa_i'^{\pm}$, respectively. Here the solutions $\chi_i'^+$ ($\chi_i'^-$) and $\kappa_i'^+$ ($\kappa_i'^-$) have positive (negative) sign at large p .

IV. SIMPLE EXAMPLE

In this section, we demonstrate an example of the explicit calculation of the thermal one-loop potential (3.48). We consider a two-flavor bosonic model as the most simple multiflavor model. The mass matrix and chemical potential of the model are

$$\hat{m}^2 = m^2 \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}, \quad \hat{\mu} = \mu \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (4.1)$$

where $m > \mu > 0$ are real parameters.

The matrices \hat{m} and $\hat{\mu}$ do not commute with each other and $\hat{m}^2 - \hat{\mu}^2$ is positive definite as long as $m > \mu$. The two eigenvalues of the mass matrix \hat{m}^2 are m^2 and $9m^2$. The square root of the matrix $p^2 \mathbf{1} + \hat{m}^2$ can be written in an explicit form by

$$\sqrt{p^2 \mathbf{1} + \hat{m}^2} = \frac{1}{2} \begin{pmatrix} \sqrt{p^2 + m^2} + \sqrt{p^2 + 9m^2} & \sqrt{p^2 + m^2} - \sqrt{p^2 + 9m^2} \\ \sqrt{p^2 + m^2} - \sqrt{p^2 + 9m^2} & \sqrt{p^2 + m^2} + \sqrt{p^2 + 9m^2} \end{pmatrix}. \quad (4.2)$$

The eigenvalues of $i(\hat{\mu} + \sqrt{p^2 \mathbf{1} + \hat{m}^2})$ and $i(\hat{\mu} - \sqrt{p^2 \mathbf{1} + \hat{m}^2})$ are imaginary with positive sign and negative sign, respectively. We assign those eigenvalues to κ_i^+ and κ_i^- as

$$\begin{aligned} \kappa_1^+ &= -\kappa_1^- \\ &= \frac{i}{2} \left(\sqrt{p^2 + m^2} + \sqrt{p^2 + 9m^2} \right. \\ &\quad \left. + \sqrt{\left(\sqrt{p^2 + m^2} - \sqrt{p^2 + 9m^2} \right)^2 + 4\mu^2} \right), \end{aligned} \quad (4.3)$$

$$\begin{aligned} \kappa_2^+ &= -\kappa_2^- \\ &= \frac{i}{2} \left(\sqrt{p^2 + m^2} + \sqrt{p^2 + 9m^2} \right. \\ &\quad \left. - \sqrt{\left(\sqrt{p^2 + m^2} - \sqrt{p^2 + 9m^2} \right)^2 + 4\mu^2} \right). \end{aligned} \quad (4.4)$$

Equation (3.26) is

$$\begin{aligned} \det(p^2 \mathbf{1} + \hat{m}^2 + (\omega \mathbf{1} - i\hat{\mu})^2) \\ = \omega^4 + 2(p^2 + 5m^2 + \mu^2)\omega^2 + p^4 \\ + (10m^2 - 2\mu^2)p^2 + 9m^4 - 10\mu^2 m^2 + \mu^4 \\ = 0. \end{aligned} \quad (4.5)$$

This equation has four solutions which can be found analytically,

$$\chi_1^+ = i\sqrt{p^2 + 5m^2 + \mu^2 + 2\sqrt{\mu^2 p^2 + 4m^4 + 5\mu^2 m^2}}, \quad (4.6)$$

$$\chi_2^+ = i\sqrt{p^2 + 5m^2 + \mu^2 - 2\sqrt{\mu^2 p^2 + 4m^4 + 5\mu^2 m^2}}, \quad (4.7)$$

$$\chi_1^- = -i\sqrt{p^2 + 5m^2 + \mu^2 + 2\sqrt{\mu^2 p^2 + 4m^4 + 5\mu^2 m^2}}, \quad (4.8)$$

$$\chi_2^- = -i\sqrt{p^2 + 5m^2 + \mu^2 - 2\sqrt{\mu^2 p^2 + 4m^4 + 5\mu^2 m^2}}. \quad (4.9)$$

The sum of κ_i^\pm is

$$\frac{i}{2} \sum_{i=1}^2 (\kappa_i^+ - \kappa_i^-) = \sqrt{p^2 + m^2} + \sqrt{p^2 + 9m^2}, \quad (4.10)$$

which gives the Coleman-Weinberg potential after applying an appropriate regularization. In the large- p limit, the difference between corresponding κ_i^\pm and χ_i^\pm is expanded in the series of $1/p$,

$$\begin{aligned} \chi_1^+ - \kappa_1^+ &= -(\chi_1^- - \kappa_1^-) \\ &= -2i \frac{\mu m^4}{p^4} + 2i \frac{\mu^2 m^4}{p^5} + \mathcal{O}(p^{-6}), \end{aligned} \quad (4.11)$$

$$\begin{aligned} \chi_2^+ - \kappa_2^+ &= -(\chi_2^- - \kappa_2^-) \\ &= 2i \frac{\mu m^4}{p^4} + 2i \frac{\mu^2 m^4}{p^5} + \mathcal{O}(p^{-6}), \end{aligned} \quad (4.12)$$

which decrease at the order of $\mathcal{O}(p^{-4})$ as proved in Sec. III B. The sum of them, on the other hand, behaves like the order $\mathcal{O}(p^{-5})$

$$\sum_{i=1}^2 (\chi_i^+ - \kappa_i^+) = 4i \frac{\mu^2 m^4}{p^5} + \mathcal{O}(p^{-6}), \quad (4.13)$$

as mentioned in the footnote in Sec. III B. Finally we obtain the thermal effective potential of this example,

$$\begin{aligned} V_B^{(1)\beta, \mu} &= - \int \frac{d^3 p}{(2\pi)^3} \left[\sqrt{p^2 + m^2} + \sqrt{p^2 + 9m^2} \right. \\ &\quad + \frac{i}{2} \sum_{i=1}^2 (\chi_i^+ - \chi_i^- - \kappa_i^+ + \kappa_i^-) \\ &\quad \left. + \frac{1}{\beta} \sum_{i=1}^2 (\log(1 - e^{i\beta \chi_i^+}) + \log(1 - e^{-i\beta \chi_i^-})) \right]. \end{aligned} \quad (4.14)$$

V. CONCLUSION AND DISCUSSIONS

In this paper, we studied the thermal one-loop effective potential of multiflavor models with the nonuniform chemical potential. The nonuniform chemical potential is necessary when the fields in the multiflavor models have different $U(1)$ charges.

Due to the noncommutativity between the matrices $\hat{\mu}$ and \hat{m} , the rigorous treatment of the nonuniform chemical potentials in the effective potential had been less understood in the literature. We wrote down the explicit expressions of the thermal effective potentials with the nonuniform chemical potential. We showed the detailed calculations in each step and obtained the expressions (3.23) and (3.25).

The effective potential is completely determined by the zeros of the determinant $\det(p^2\mathbf{1} + \hat{m}^2 + (\omega\mathbf{1} - i\hat{\mu})^2)$, which is the natural generalization of the single-flavor case. Although the expressions (3.23) and (3.25) contain terms with a UV divergent piece, a careful analysis of the large- p behavior of the solution to $\det(p^2\mathbf{1} + \hat{m}^2 + (\omega\mathbf{1} - i\hat{\mu})^2) = 0$ reveals that the divergent parts of the effective potential come only from the Coleman-Weinberg potential. At the same time, we found that terms that depend on temperature and the chemical potential give finite contributions to the effective potential. The generalization to the models with fermions is straightforward, except for subtleties of the signs of the solutions. We then obtained the final results (3.48) and (3.51). In supersymmetric models, the Coleman-Weinberg potential is exactly canceled out because the bosonic and fermionic mass matrices are coincident. In the case of models with nonuniform chemical potentials, however, there are finite corrections to the Coleman-Weinberg potential that depend on $[\hat{m}, \hat{\mu}]$, which are not canceled in general.

We again stress that the result (3.48) is not the simple generalization of the well-known single-flavor case. The noncommutativity of the chemical potential matrix $\hat{\mu}$ and the mass matrix \hat{m} makes the calculation nontrivial. We demonstrated the explicit calculations in a simple two-flavor model. The analytic solutions $\chi_{1,2}^\pm$ to the equation for the zeros of the determinant are found. The large momentum behavior of the solutions is consistent with the general discussions. The expression of the thermal effective potential is shown analytically. The effective potential is completely governed by the solutions to the equation $\det(p^2\mathbf{1} + \hat{m}^2 + (\omega\mathbf{1} - i\hat{\mu})^2) = 0$ but extracting the concrete expression of the solutions is cumbersome when the number of flavors N is large. This fact forces one to utilize the numerical analysis to find the physics from (3.48). It is interesting to study applications of our result (3.48) by focusing on a specific model. For example, phase transitions of supersymmetry breaking vacua in the early Universe is an interesting topic. Precise treatments of finite density effects in more realistic models are also interesting. Apart from the applications, generalizations of our calculational scheme to models with vector fields are important problems.

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APPENDIX A: POSITIVITY OF $\sqrt{p^2\mathbf{1} + \hat{m}^2} \pm \hat{\mu}$

In this appendix, we prove that the matrices $\sqrt{p^2\mathbf{1} + \hat{m}^2} \pm \hat{\mu}$ are positive definite when the matrices \hat{m} and $\hat{m}^2 - \hat{\mu}^2$ are positive definite. It is sufficient to prove this in the case for $p = 0$. Suppose that \hat{m} and $\hat{m}^2 - \hat{\mu}^2$ are positive definite, and assuming that $\hat{m} - \hat{\mu}$ has a negative eigenvalue, we have

$$(\hat{m} - \hat{\mu})|a\rangle = a|a\rangle, \quad a < 0, \quad (\text{A1})$$

where $|a\rangle$ is an eigenvector associated with the negative eigenvalue a . By adding Eq. (A1) multiplied by $\hat{\mu}$, to Eq. (A1) multiplied by \hat{m} , we have⁵

$$\langle a|(\hat{m}^2 - \hat{\mu}^2)|a\rangle = a\langle a|(\hat{m} + \hat{\mu})|a\rangle. \quad (\text{A2})$$

Since the left-hand side is positive by the assumption, and $a < 0$, we find

$$\langle a|(\hat{m} + \hat{\mu})|a\rangle < 0. \quad (\text{A3})$$

We now consider the following relation:

$$\langle a|(\hat{m} + \hat{\mu})|a\rangle + \langle a|(\hat{m} - \hat{\mu})|a\rangle = 2\langle a|\hat{m}|a\rangle. \quad (\text{A4})$$

The right-hand side is positive. On the other hand, the left-hand side is negative as we have shown in (A1) and (A3). This is a contradiction. Therefore, we conclude that the matrix $\hat{m} - \hat{\mu}$ does not have a negative eigenvalue.

If $\hat{m} - \hat{\mu}$ has zero eigenvalue ($a = 0$), we have

$$\langle a|(\hat{m}^2 - \hat{\mu}^2)|a\rangle = 0, \quad (\text{A5})$$

which contradicts the fact that $\hat{m}^2 - \hat{\mu}^2$ is positive definite. Consequently, it is proved that $\hat{m} - \hat{\mu}$ has neither zero nor negative eigenvalues. Similarly, one can prove that $\hat{m} + \hat{\mu}$ is positive definite.

APPENDIX B: ASYMPTOTIC BEHAVIORS OF DETERMINANTS

In this appendix, we show the large- p behavior of the following quantities:

$$\det(p^2\mathbf{1} + \hat{m}^2 + (\omega\mathbf{1} - i\hat{\mu})^2) \sim \mathcal{O}(p^{N-4}), \quad (\text{B1})$$

$$\frac{d}{dw} \det(p^2\mathbf{1} + \hat{m}^2 + (\omega\mathbf{1} - i\hat{\mu})) \sim \mathcal{O}(p^N). \quad (\text{B2})$$

⁵To derive the following equation, we use the relation $\langle a|[\hat{m}, \hat{\mu}]|a\rangle = 0$, which comes from the fact that $\langle a|A|a\rangle = 0$ for any antisymmetric matrix A and any real vector $|a\rangle$. Since the matrix $\hat{m} - \hat{\mu}$ is real and symmetric, every eigenvalue of $\hat{m} - \hat{\mu}$ is real and the corresponding eigenvector is a real vector except for an overall factor.

Here we assume that any two of the diagonal elements in $\hat{\mu}$ are not coincident, although one can reach the same conclusion without this assumption. First we show the relation (B1). We consider the function $f(\eta, \omega)$ in (3.29). We investigate the order in p by expanding $f(\eta, \omega)$ with

respect to η . If we assign an approximate solution $\kappa_k^+(k = 1, \dots, N)$ to ω in $f(\eta, \omega)$, the expansion around $\eta = 0$ must start with the order $\mathcal{O}(\eta^1)$ since the term at $\mathcal{O}(\eta^0)$ vanishes from $f(0, \kappa_i^\pm) = 0$.

The matrix in (3.29) has the following form:

$$\begin{pmatrix} d_1 & m_{12}^2 - \alpha_{12} + \eta\alpha_{12} & \cdots & m_{1N}^2 - \alpha_{1N} + \eta\alpha_{1N} \\ m_{12}^2 + \alpha_{12} - \eta\alpha_{12} & d_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ m_{1N}^2 + \alpha_{1N} - \eta\alpha_{1N} & \cdots & \cdots & d_N \end{pmatrix}, \quad (\text{B3})$$

where m_{ij}^2 and α_{ij} are (i, j) components of \hat{m}^2 and $[\hat{\mu}, \sqrt{p^2 \mathbf{1} + \hat{m}^2}]$, respectively. We have used the properties $m_{ij} = m_{ji}$ and $\alpha_{ij} = -\alpha_{ji}$. For $p \gg 1$, the diagonal component d_i is approximately given by

$$d_i \sim 2p(\mu_i - \mu_k) - \mu_i^2 - \mu_k^2 + 2\mu_i\mu_k + m_{ii}^2 - m_{kk}^2 + \mathcal{O}(1/p). \quad (\text{B4})$$

Thus, $d_i \sim \mathcal{O}(p)$ for $i \neq k$ and $d_k \sim \mathcal{O}(1/p)$. Since the off-diagonal components m_{ij}^2 are at order $\mathcal{O}(1)$, the highest order term in p includes as many diagonal components except d_k as possible. On the other hand, η appears only in off-diagonal components, and therefore nonvanishing terms contain at least one off-diagonal component. Thus, the terms that have the highest power in p should be the following form:

$$d_1 d_2 \cdots (m_{ik}^2 + \alpha_{ik} - \eta\alpha_{ik}) \cdots (m_{ik}^2 - \alpha_{ik} + \eta\alpha_{ik}) \cdots d_k \\ = \left(\prod_{j \neq i, k} d_j \right) ((m_{ik}^2)^2 - \alpha_{ik}^2 + 2\eta\alpha_{ik}^2 - \eta^2\alpha_{ik}^2). \quad (\text{B5})$$

For $p \gg 1$, we find

$$\alpha_{ij} = \left[\hat{\mu}, p + \frac{\hat{m}^2}{2p} - \frac{(\hat{m}^2)^2}{8p^3} + \cdots \right] \sim \frac{1}{2p} [\hat{\mu}, \hat{m}^2] \sim \mathcal{O}\left(\frac{1}{p}\right). \quad (\text{B6})$$

Substituting them back into (B5), we find that

$$\mathcal{O}(f(\eta, \omega)) = \mathcal{O}\left(\left(\prod_{j \neq i, k} d_j\right)(2\eta\alpha_{ik}^2 - \eta^2\alpha_{ik}^2)\right) \\ = \mathcal{O}(p^{N-4}). \quad (\text{B7})$$

Next we show the relation (B2). After differentiating $f(\eta, \omega)$ with respect to ω , the relation $\frac{\partial f(0, \omega)}{\partial \omega}|_{\omega=\kappa_k^\pm} = 0$ no longer holds and the $\mathcal{O}(\eta^0)$ term does not vanish. Since ω is only in diagonal components, the highest order term in p is obtained when d_k is differentiated. From this we find

$$\mathcal{O}\left(\frac{d}{d\omega} f(\eta, \omega)|_{\omega=\kappa_k^+}\right) \\ = \mathcal{O}(d_1 d_2 \cdots d_{k-1} (2(\omega - i\mu_k)) \\ \times d_{k+1} \cdots d_N|_{\omega=\kappa_k^+}) = \mathcal{O}(p^N). \quad (\text{B8})$$

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