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Bound orbits and gravitational theory

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It can be easily shown that bound orbits around a static source can only exist in four dimensions for any force driven by the Laplace equation. This is true not only for Maxwell's electromagnetism and Newton's gravity, but for Einstein's theory of gravitation as well. In contrast to Maxwell's electrodynamics and Newton's gravity, general relativity has a natural and remarkable generalization in *higher dimensions* in Lovelock gravity. However, it is not Laplace driven and hence admits bound orbits around a static black hole in all even $D = 2N + 2$ dimensions, where N is the degree of the Lovelock polynomial action. This is as general a result as Bertrand's theorem of classical mechanics, in which the existence of closed orbits uniquely singles out the inverse square law for a long-range central force.

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The existence of bound orbits requires a repulsive centrifugal force that is able to counterbalance the attractive central force at two radii—giving rise to turning points of the orbit—whatever the source may be (electricity or gravity). This means that the effective potential—which is the sum of the repulsive and attractive parts—should have a minimum. It also indicates the existence of a stable circular orbit; that is, the existence of bound orbits is equivalent to the existence of a stable circular orbit. We know that the centrifugal potential always goes as $1/r^2$, while the attractive force obeying Gauss's law of flux conservation (or equivalently the Laplace equation) goes as $1/r^{n+1}$, where $n = d - 2$ with d the dimension of space. It then readily follows that the condition for a minimum of the effective potential is $n < 2$. Bound orbits could thus occur only in three space dimensions. This is true for any long-range Laplace-driven force, including both the electric and gravitational forces. In spatial dimensions >3, no bound orbits can occur, implying the total absence of structures, such as atoms and planetary systems. This means that there would be no life as well. Is this why the Universe we live in is four-dimensional? [1]

The conditions for a minimum of the effective potential $V(r)$ are $V'(r) = 0$, $V''(r) > 0$, where a prime denotes a derivative with respect to r. For a central attractive force, the effective potential is given by

$$V = -\frac{M}{r^n} + \frac{l^2}{r^2}. \tag{1}$$

It would be at a minimum whenever

$$n(2 - n) > 0, \tag{2}$$

which clearly demands $n = d - 2 < 2$, that is, the spatial dimension cannot be anything other than three. This would be true for any force that obeys Gauss's law or the Laplace equation. Thus bound orbits can only exist in three-dimensional space.

Unlike Maxwell's electrodynamics, Newtonian gravity has a relativistic generalization in Einstein's general relativity (GR). It turns out that the question of the existence of bound orbits does not depend on whether Newtonian or Einstein gravity is used, i.e., no bound orbits exist in GR for any spacetime dimension $D > 4$. This is because the potential of a static source in Einstein's theory is also Laplace driven.

For Einstein's theory, we write the effective potential as

$$V^2 = f(r)\left(\frac{l^2}{r^2} + 1\right) \tag{3}$$

for a radially symmetric static metric,

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega_{D-2}^2, \tag{4}$$

where $d\Omega_{D-2}^2$ is the metric on a $(D - 2)$-sphere, given by

$$(d\Omega_{D-2})^2 = d\theta_1^2 + \sin^2(\theta_1)d\theta_2^2 + \sin^2(\theta_1)\sin^2(\theta_2)d\theta_3^2 + \dot{c} + \left[\prod_{j=1}^{D-2} \sin^2(\theta_j)d\theta_{D-1}^2\right]. \tag{5}$$

This form of the metric is dictated by the requirement that a radially falling photon experiences no acceleration—the velocity of light remains constant in vacuum [2]. Now the conditions for a minimum of the effective potential give

$$\frac{l^2}{r^2} = \frac{rf'}{2f - rf'} \tag{6}$$

and

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$$rff'' - 2rf'^2 + 3ff' > 0. \tag{7}$$

For a D-dimensional solution of the Einstein vacuum equation describing static black hole, $R_{ab} = 0$ [3], we have

$$f(r) = 1 - M/r^n, \tag{8}$$

where $n = D - 3$ so that the potential satisfies the Laplace equation. Then we have

$$\frac{l^2}{r^2} = \frac{nM}{2r^n - (n+2)M}, \tag{9}$$

which gives the existence threshold for a circular orbit,

$$r_{\rm ex} > r_{\rm ph} = \left(\frac{n+2}{2}M\right)^{1/n}, \tag{10}$$

where $r_{\rm ph}$ is the radius of the photon's circular orbit. This marks the existence threshold as no circular orbit can exist inside the photon's circular orbit. The stability threshold is given by

$$r_{st} = \left(\frac{n+2}{2-n}M\right)^{1/n}. \tag{11}$$

Clearly, $n = D - 3 < 2$, that is, the spacetime dimension can only be four. Thus for GR as well bound orbits can only exist in four dimensions. So far as the existence of bound orbits is concerned both Newtonian gravity and GR are on the same footing because they are both Laplace driven. However, in the latter orbits have a minimum radius for an inner turning point, which is defined by the radius of the photon's circular orbit.

Further, GR has a natural generalization in higher dimensions known as Lovelock gravity (introduced by David Lovelock [4]), which includes GR in the linear order. The Lovelock action in $D(\geq 4)$-dimensional spacetime is given by

$$I = \frac{1}{2\kappa_D^2}\int d^Dx\sqrt{-g}\sum_{p=0}^{[D/2]}\alpha_{(p)}\mathcal{L}_{(p)},$$
$$\mathcal{L}_{(p)} := \frac{1}{2^p}\delta^{\mu_1\cdots\mu_p\nu_1\cdots\nu_p}_{\rho_1\cdots\rho_p\sigma_1\cdots\sigma_p}R_{\mu_1\nu_1}{}^{\rho_1\sigma_1}\cdots R_{\mu_p\nu_p}{}^{\rho_p\sigma_p}, \tag{12}$$

where $N = [(D-1)/2]$ and $\kappa_D := \sqrt{8\pi G_D}$. We assume $\kappa_D^2 > 0$ without any loss of generality, $\alpha_{(p)}$ is an arbitrary constant with dimension $(\text{length})^{2(p-1)}$, and $\mathcal{L}_{(p)}$ is the Euler density of a $2p$-dimensional manifold. The δ symbol denotes a totally antisymmetric product of Kronecker deltas, normalized to take the values 0 and ± 1 [4], defined by

$$\delta^{\mu_1\cdots\mu_p}_{\rho_1\cdots\rho_p} := p!\,\delta^{\mu_1}_{[\rho_1}\cdots\delta^{\mu_p}_{\rho_p]}. \tag{13}$$

$\alpha_{(0)}$ is related to the cosmological constant Λ by $\alpha_{(0)} = -2\Lambda$. Lovelock theories are distinct among the larger class of general higher-curvature theories, as their action is a homogeneous polynomial in the Riemann curvature, yet they have the remarkable unique property that the equation of motion for the field remains second order. If the polynomial degree is N, $N = 1$ corresponds to GR, $N = 2$ to Gauss-Bonnet, and so on. This means that Lovelock gravity is a general class that includes GR in the first order. For a given N, $D \geq 2N + 1$, and hence it is truly a higher-dimensional generalization. In GR, gravity is kinematic in three dimensions and it becomes dynamic in four dimensions. Could this be a general gravitational property in higher dimensions, i.e., kinematic in odd and dynamic in even dimensions? This is precisely what has recently been established for pure Lovelock gravity [5]. By pure we mean that the action contains only one Nth-order polynomial in the action (12), i.e., there is no summation over p as in case of Einstein-Lovelock gravity. Therefore, it is free of all terms $p < N$, including the Einstein-Hilbert linear term R. Further, it is possible to define an Nth-order analogue of the Riemann curvature with the property that the trace of its Bianchi derivative vanishes identically and that it yields a second-rank symmetric divergence-free tensor that is the same as the one that comes from the variation of the corresponding Nth-order Lovelock action [6]. Now gravity is kinematic in all odd ($D = 2N + 1$) dimensions because $R^{(N)}_{ab} = 0$ implies $R^{(N)}_{abcd} = 0$. That is, the Lovelock vacuum is Lovelock flat in $2N + 1$ dimensions; however, it will not be Riemann flat [5]. Thus pure Lovelock gravity is always kinematic in odd ($D = 2N + 1$) dimensions and becomes dynamic in even ($D = 2N + 2$) dimensions. This is a universal gravitational feature which was established for the first time in Ref. [5].

It has been proposed and strongly articulated [7] that the proper gravitational equation in higher dimensions is the pure Lovelock equation,

$$G^{(N)}_{ab} = -\Lambda g_{ab} + \kappa_N T_{ab}, \tag{14}$$

where $G^{(N)}_{ab}$ is defined as [6]

$$G^{(N)}_{ab} = N(R^{(N)}_{ab} - 1/2R^{(N)}g_{ab}). \tag{15}$$

Here $R^{(N)}_{ab}$ and $R^{(N)}$ are defined through the Lovelock curvature polynomial $R^{(N)}_{abcd}$ as

$$R^{(N)}_{abcd} = F^{(N)}_{abcd} - \frac{N-1}{N(D-1)(D-2)}F^{(N)}(g_{ac}g_{bd} - g_{ad}g_{bc}),$$
$$F^{(N)}_{abcd} = Q_{ab}{}^{mn}R_{cdmn}, \tag{16}$$

where

$$Q^{ab}{}_{cd} = \delta^{aba_1b_1\ldots a_Nb_N}_{cdc_1d_1\ldots c_Nd_N}R_{a_1b_1}{}^{c_1d_1}\ldots R_{a_{N-1}b_{N-1}}{}^{c_{N-1}d_{N-1}},$$
$$Q^{abcd}{}_{;d} = 0, \tag{17}$$

and

$$R^{(N)} = \frac{D-2N}{N(D-2)}F^{(N)}. \tag{18}$$

Note that $R^{(N)} = R^{(N)}_{ab}g^{ab} = 0$ in $2N$ dimensions for an arbitrary metric g_{ab}.

Note that the Einstein-Lovelock vacuum equation would be

$$\sum_N \alpha_N G_{ab}^{(N)} = -\Lambda g_{ab}, \tag{19}$$

which was solved for $N = 2$, describing an Einstein-Gauss-Bonnet static black hole [8]. This was followed by a general Einstein-Lovelock solution for any N [9]. The ultimate equation to be solved is an algebraic Nth-degree polynomial, which cannot in general be solved for $N > 4$. However, for pure Lovelock gravity there is no such difficulty [10] and we have the general solution for $D = 2N + 2$ given by

$$f(r) = 1 - \left(\frac{M}{r}\right)^{1/N}. \tag{20}$$

Here $n = 1/N$, which would always satisfy the required condition $n < 2$ for the existence of bound orbits. Note that here the governing equation for the potential is not the Laplace equation. Thus bound orbits would always exist (as shown in Fig. 1) for all even $2N + 2$ dimensions in pure Lovelock gravity. This question is however not pertinent for odd $2N + 1$ dimensions, as gravity is kinematic there. We have set $\Lambda = 0$ as it has no relevance in the region where bound orbits are being sought. This is indeed a very significant distinguishing feature of pure Lovelock gravity that is shared by neither Einstein and even Einstein-Lovelock gravity, nor by any other known modifications of GR.

For the pure Lovelock case, let us for transparency and clarity give the relevant expressions explicitly: they read

$$V^2 = \left[1 - \left(\frac{M}{r}\right)^{1/N}\right]\left(\frac{l^2}{r^2} + 1\right), \tag{21}$$

and

$$\frac{l^2}{r^2} = \frac{M^{1/N}}{(2Nr^{1/N} - (2N+1)M^{1/N})}, \tag{22}$$

which gives the existence threshold

$$r_{ex} > r_{ph} = \left(\frac{2N+1}{2N}\right)^N M. \tag{23}$$

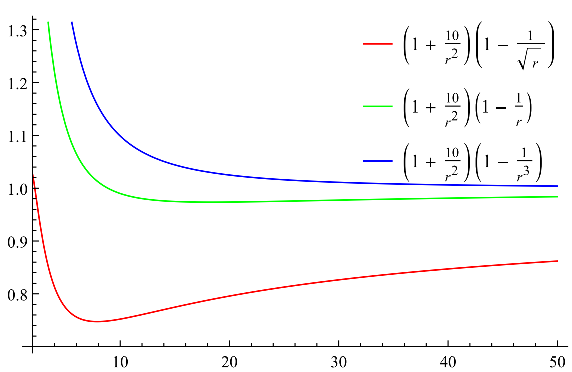

FIG. 1 (color online). Shape of the potential in Einstein and Gauss-Bonnet theories.

For the stability threshold we have

$$(V^2)'' = \frac{2M^{1/N}((2N-1)r^{1/N} - (2N+1)M^{1/N})}{Nr^{(2N+1)/N}(2Nr^{1/N} - (2N+1)M^{1/N})}, \tag{24}$$

and it is given by

$$r_{st} = \left(\frac{2N+1}{2N-1}\right)^N M. \tag{25}$$

For $N = 1$ we have the familiar limits $r_{\rm ph} = 3/2M$ and $r_{\rm st} = 3M$.

Let us first note that Lovelock gravity is a purely higher-dimensional generalization of GR as it necessarily requires $D \geq 2N + 1$, while all other modifications generally refer to four dimensions. If we consider flux conservation as an abiding physical principle for any classical theory, Gauss's law or the Laplace equation are natural consequences. Since pure Lovelock gravity is not Laplace driven, it cannot obey the conventional Gauss's law. Gauss's law is taken over to GR through the Komar integral, which defines the mass of a static black hole. Then the pertinent question that arises is, could we define the Komar integral with appropriate modifications for a pure Lovelock black hole? It turns out that it is possible to naturally extend the Komar integral to pure Lovelock gravity [11]. It could also be defined for an Einstein-Lovelock system, but this requires the addition of appropriate boundary terms, while for pure Lovelock gravity no such additional terms are required. This is yet another distinguishing feature of pure Lovelock gravity.

Finally, we have shown in all generality that so long as a force is Laplace driven, bound orbits can only exist only in four dimensions. However, they do always exist for pure Lovelock gravity in all even ($D = 2N + 2$) dimensions, as the force in this case not Laplace driven. This is an interesting distinction between Einstein gravity (with all its known generalizations) and pure Lovelock gravity. Lovelock gravity stands out from all other GR generalizations because it has the remarkable unique feature that it preserves the second-order character of the equation of motion. It is therefore not an effective generalization like other higher-derivative theories. It is a valid and true fundamental higher-dimensional theory. Also, pure Lovelock gravity provides another universal gravitational property: it dynamically distinguishes odd and even dimensions. It is therefore clear that so long as the equation of motion retains its second-order character, the requirement of the existence of bound orbits uniquely singles out pure Lovelock gravity, which however includes Einstein gravity for $N = 1$. This is in fact as general a result as Bertrand's theorem of classical mechanics, which by demanding the existence of closed orbits uniquely singles out the inverse square law for a long-range central force. We could thus say the following.

Unlike many other gravitational theories, pure Lovelock gravity has a second-order equation of motion and admits bound orbits in all even ($D = 2N + 2$) dimensions. Since pure Lovelock gravity is dynamic only in even dimensions, bound orbits only exist in even dimensions.

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