PHYSICAL REVIEW D **88,** 124038 (2013)

# Finite-boost transformations in doubly special relativity

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(Received 8 October 2013; published 19 December 2013)

Doubly special relativity provides an intriguing scenario for studying possible modifications of special relativity. To the leading order of quantum corrections, the scenario is characterized simply by several free parameters in boost generators, dispersion relation and momentum addition law. In this scenario, we work out finite-boost transformations in $3+1$ dimensions. Constraints on parameters are obtained from compatibility analysis, in agreement with other studies. Combining two successive noncollinear boosts, we also obtain the Wigner rotation angle in this scenario. Our analysis is restricted mainly to the case of the commutative addition law.

DOI: 10.1103/PhysRevD.88.124038 PACS numbers: 04.60.Bc, 03.30.+p, 11.30.Cp

## I. INTRODUCTION

Einstein's special relativity (SR) is a cornerstone of modern physics. Embedded in one of its basic principles, it has a single fundamental constant, i.e., an observer-independent velocity scale $c$ (the speed of light). By introducing another fundamental constant, $L_p$, Amelino-Camelia explored a natural extension of SR one decade ago [1], now widely known as doubly special relativity (DSR). The constant $L_p$ is an observer-independent length scale, which may be regarded as the Planck length $L_p = \sqrt{\hbar G/c^3}$, where $\hbar$ is the Planck constant, and $G$ is the gravitational constant.

Although it is difficult to establish the full $L_p$ dependence of DSR hitherto, one can still parameterize DSR explicitly to the first order of $L_p$. This involves five model-dependent parameters in the generators of boost transformation, two parameters in the dispersion relation, and five in the addition law of energy-momentum. Such a reduced DSR scenario is sensible, because higher-order corrections are suppressed at sub-Planckian energies. As a concrete playground for studying possible modifications of SR, subcases of this scenario have been extensively studied in the literature [2,3], especially in $1+1$ dimensions [4–6]. However, its $3+1$ dimensional version has been relatively less explored until recently [7,8].

In this paper, we take a closer look at DSR in $3+1$ dimensions, focusing on transformations of energy and momentum under finite boosts. This is done with the exponential parameterization of boost, of which the idea will be recapped in Sec. II for Einstein's SR. Prepared with the deformed Poincaré algebra in Sec. III A, we derive in Sec. III B the transformation rule of energy and momentum between inertial reference frames. This result is cross-checked with a related method in Sec. III C and extended to a boost along any direction in Sec. III D. Using the obtained transformation rule, in Sec. IV, we prove that 12 parameters in the reduced DSR scenario are not independent: they are subject to nine constraints if the dispersion relation and the momentum addition law are observer-independent. The same constraints were previously found by Refs. [7,8] using different methods. In Sec. V, our result is applied to the Wigner rotation with two successive boosts in different directions. To our knowledge, this is the first time the Wigner rotation has been studied in DSR. We finally make some comments in Sec. VI. In Appendix A, we present the lengthy expression for energy and momentum after two perpendicular boosts. In Appendix B, the deformed Poincaré algebra is rewritten in terms of redefined energy-momentum variables. Then, in Appendix C, we relate our investigation to the $\kappa$-Poincaré algebra at the linear order of (the inverse of) the deformed parameter.

In Ref. [9], requiring the classical additivity of energy, it was elucidated that there are two types of DSR deformations: classical Poincaré algebra in nonlinear disguise (which has a commutative composition law of energy-momentum) and the $\kappa$-Poincaré algebra (with a noncommutative composition law). The present paper focuses on the former case. In particular, we assume that in the two-body system each particle's energy-momentum transforms independently under a finite boost.

Words for strong readers: To make the paper self-contained and clear in notation, we will present some trivial details in Sec. II. Section IV is simply a complementary analysis compared with results obtained by other authors. Strong readers are strongly recommended to skip them and go directly to Secs. III and V, which are the main parts of this paper and contain our new results.

Throughout this paper, we will work in the natural units $\hbar = c = 1$ for simplicity. In the discussion of DSR, all equalities should be understood to hold at $\mathcal{O}(L_p)$, while $\mathcal{O}(L_p^2)$ and higher-order terms are neglected.

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## II. FINITE BOOSTS IN EINSTEIN'S SPECIAL RELATIVITY

To set up the convention of notations and to illustrate our method, in this section, we will briefly review how to perform a finite-boost transformation in SR.

Recall that in SR, the generators of rotations $R_i$ and boosts $B_i$ can be put in differential forms,

$$R_i = -i\sum_{j,k}\epsilon_{ijk}p_j\frac{\partial}{\partial p_k}, \tag{1}$$

$$B_i = ip_i\frac{\partial}{\partial E} + iE\frac{\partial}{\partial p_i}, \tag{2}$$

where the indices $i$, $j$, $k$ stand for the spatial directions $x$, $y$, $z$. As is well known, energy $E$ and momentum $p_i$ are generators of translations. Together with $R_i$ and $B_i$, they form the Poincaré algebra, from which we will use the following commutators:

$$[B_i, E] = ip_i, \qquad [B_i, p_j] = iE\delta_{ij}. \tag{3}$$

Now consider two inertial reference frames, $|0\rangle$ and $|\xi_x\rangle$, associated with two different observers. They are related by a finite boost along the $x$ direction with rapidity $\xi_x$:

$$|\xi_x\rangle = e^{i\xi_x B_x}|0\rangle. \tag{4}$$

Suppose a particle has energy $E(0)$ and momentum $\vec{p}(0)$ in the reference frame $|0\rangle$. In frame $|\xi_x\rangle$, the particle's energy and momentum become $E(\xi_x)$ and $\vec{p}(\xi_x)$, respectively. For brevity, we denote $\vec{p} = (p_x, p_y, p_z)$ and $\vec{p}^2 = p_x^2 + p_y^2 + p_z^2$. From relation (4) and

$$\begin{aligned} E(\xi_x)|\xi_x\rangle &= e^{i\xi_x B_x}E(0)|0\rangle,\\ \vec{p}(\xi_x)|\xi_x\rangle &= e^{i\xi_x B_x}\vec{p}(0)|0\rangle, \end{aligned} \tag{5}$$

it is straightforward to demonstrate

$$\begin{aligned} E(\xi_x) &= e^{i\xi_x B_x}E(0)e^{-i\xi_x B_x}\\ &= E(0) + i\xi_x[B_x, E(0)] + \frac{1}{2!}(i\xi_x)^2[B_x,[B_x,E(0)]]\\ &\quad + \frac{1}{3!}(i\xi_x)^3[B_x,[B_x,[B_x,E(0)]]] + \cdots\\ &= \left[1 - \frac{1}{2!}(i\xi_x)^2 + \frac{1}{4!}(i\xi_x)^4 + \cdots\right]E(0)\\ &\quad + \left[i\xi_x - \frac{1}{3!}(i\xi_x)^3 + \cdots\right]ip_x(0)\\ &= E(0)\cosh(\xi_x) - p_x(0)\sinh(\xi_x), \end{aligned} \tag{6}$$

$$\begin{aligned} p_x(\xi_x) &= e^{i\xi_x B_x}p_x(0)e^{-i\xi_x B_x}\\ &= p_x(0) + i\xi_x[B_x, p_x(0)] + \frac{1}{2!}(i\xi_x)^2[B_x,[B_x,p_x(0)]]\\ &\quad + \frac{1}{3!}(i\xi_x)^3[B_x,[B_x,[B_x,p_x(0)]]] + \cdots\\ &= \left[1 - \frac{1}{2!}(i\xi_x)^2 + \frac{1}{4!}(i\xi_x)^4 + \cdots\right]p_x(0)\\ &\quad + \left[i\xi_x - \frac{1}{3!}(i\xi_x)^3 + \cdots\right]iE(0)\\ &= p_x(0)\cosh(\xi_x) - E(0)\sinh(\xi_x). \end{aligned} \tag{7}$$

In the above, we have utilized

$$\begin{aligned} [B_x, E] &= ip_x,\\ [B_x,[B_x,E]] &= -E,\\ [B_x,[B_x,[B_x,E]]] &= -ip_x,\\ [B_x,[B_x,[B_x,[B_x,E]]]] &= E, \end{aligned} \tag{8}$$

$$\begin{aligned} [B_x, p_x] &= iE,\\ [B_x,[B_x,p_x]] &= -p_x,\\ [B_x,[B_x,[B_x,p_x]]] &= -iE,\\ [B_x,[B_x,[B_x,[B_x,p_x]]]] &= p_x, \end{aligned} \tag{9}$$

which come from the commutators in Eq. (3). Likewise, it is easy to check $[B_x, p_y] = 0$, $[B_x, p_z] = 0$, and consequently

$$p_y(\xi_x) = p_y(0), \qquad p_z(\xi_x) = p_z(0). \tag{10}$$

Hence, we get the transformation rule of energy and momentum under a finite boost.

One can plug these expressions into the dispersion relation to show

$$E^2(\xi_x) - \vec{p}^2(\xi_x) = E^2(0) - \vec{p}^2(0) = m^2. \tag{11}$$

This confirms that the dispersion relation is indeed observer independent.

## III. FINITE BOOSTS IN DOUBLY SPECIAL RELATIVITY

### A. Deformed Poincaré algebra

In DSR, the rotational invariance is commonly assumed to be intact, with its generator taking the conventional form in Eq. (1). To the leading order of $L_p$, the generator of boost is deformed as [7,8]

$$B_i = i(p_i + \lambda_1 L_p E p_i)\frac{\partial}{\partial E} + i(E + \lambda_2 L_p E^2 + \lambda_3 L_p \vec{p}^2)\frac{\partial}{\partial p_i} + i\lambda_4 L_p p_i \sum_j p_j \frac{\partial}{\partial p_j} + i\lambda_5 L_p E \sum_{j,k} \epsilon_{ijk} p_j \frac{\partial}{\partial p_k}. \quad (12)$$

Here the deformation terms are fixed by dimensional analysis, but the dimensionless parameters $\lambda_1$, $\lambda_2$, $\lambda_3$, $\lambda_4$, $\lambda_5$ are left unspecified.

In accordance with Eqs. (1) and (12), the Poincaré algebra is deformed by $\mathcal{O}(L_p)$ terms. This is apparent in commutators involving $B_i$:

$$\begin{aligned}[B_i, E] &= i(p_i + \lambda_1 L_p E p_i),\\ [B_i, p_j] &= i\delta_{ij}(E + \lambda_2 L_p E^2 + \lambda_3 L_p \vec{p}^2) + i\lambda_4 L_p p_i p_j - i\lambda_5 L_p E \sum_k \epsilon_{ijk} p_k,\end{aligned} \quad (13)$$

$$\begin{aligned}[B_i, B_j] &= -i[1 + (\lambda_1 + 2\lambda_2 + 2\lambda_3 - \lambda_4)L_p E]\sum_k \epsilon_{ijk} R_k - 2i\lambda_5 L_p E \sum_k \epsilon_{ijk} B_k,\\ [R_i, B_j] &= i\sum_k \epsilon_{ijk} B_k - i\lambda_5 L_p E \sum_k \epsilon_{ijk} R_k.\end{aligned} \quad (14)$$

Other commutators do not contain $B_i$, and thus remain the same as in SR:

$$[R_i, E] = 0, \qquad [R_i, p_j] = i\sum_k \epsilon_{ijk} p_k, \qquad [R_i, R_j] = i\sum_k \epsilon_{ijk} R_k. \quad (15)$$

The above commutators are calculated by brute force. They will be utilized in the present and later sections.

### B. Boost transformation

A key point in Sec. II is parameterizing the finite boost exponentially; see Eq. (4), for instance. The exponential parameterization is well established and widely accepted in SR, since the Poincaré algebra is Lie algebra. As will be clear in the next subsection, the exponential parameterization method is also valid for DSR. Following the method, we are able to obtain expressions of energy $E(\xi_x)$ and momentum $\vec{p}(\xi_x)$ in DSR as below.

Consider again a boost in the $x$ direction. Then, according to Eq. (13), we find, to $\mathcal{O}(L_p)$,

$$\begin{aligned}[B_x, E] &= ip_x + i\lambda_1 L_p E p_x,\\ [B_x,[B_x,E]] &= -E - (\lambda_1+\lambda_2)L_p E^2 - \lambda_3 L_p \vec{p}^2 - (\lambda_1+\lambda_4)L_p p_x^2,\\ [B_x,[B_x,[B_x,E]]] &= -[B_x,E] - 2i(2\lambda_1+\lambda_2+\lambda_3+\lambda_4)L_p E p_x,\\ [B_x,[B_x,[B_x,[B_x,E]]]] &= -[B_x,[B_x,E]] + 2(2\lambda_1+\lambda_2+\lambda_3+\lambda_4)L_p[E^2+p_x^2],\end{aligned} \quad (16)$$

$$\begin{aligned}[B_x, p_x] &= iE + i\lambda_2 L_p E^2 + i\lambda_3 L_p \vec{p}^2 + i\lambda_4 L_p p_x^2,\\ [B_x,[B_x,p_x]] &= -p_x - (\lambda_1+2\lambda_2+2\lambda_3+2\lambda_4)\times L_p E p_x,\\ [B_x,[B_x,[B_x,p_x]]] &= -[B_x,p_x] - i(\lambda_1+2\lambda_2+2\lambda_3+2\lambda_4)\times L_p[E^2+p_x^2],\\ [B_x,[B_x,[B_x,[B_x,p_x]]]] &= p_x + 5(\lambda_1+2\lambda_2+2\lambda_3+2\lambda_4)\times L_p E p_x.\end{aligned} \quad (17)$$

Unlike in Sec. II, here the results are not periodic. So we should make an effort to work out general terms for the two sequences of commutators. Fortunately, for longer commutators, we are able to demonstrate generally

$$\begin{aligned}&\underbrace{[B_x,[B_x\cdots,[B_x}_{4n+1\text{ brackets}},E]]\cdots] - [B_x,E] = \frac{2}{3}i(16^n-1)(2\lambda_1+\lambda_2+\lambda_3+\lambda_4)L_p E p_x,\\ &\underbrace{[B_x,[B_x\cdots,[B_x}_{4n+2\text{brackets}},E]]\cdots] - [B_x,[B_x,E]] = -\frac{2}{3}(16^n-1)(2\lambda_1+\lambda_2+\lambda_3+\lambda_4)L_p[E^2+p_x^2],\end{aligned} \quad (18)$$

$$\begin{aligned}&\underbrace{[B_x,[B_x\cdots,[B_x}_{4n+3\text{ brackets}},E]]\cdots] - [B_x,[B_x,[B_x,E]]] = -\frac{8}{3}i(16^n-1)(2\lambda_1+\lambda_2+\lambda_3+\lambda_4)L_p E p_x,\\ &\underbrace{[B_x,[B_x\cdots,[B_x}_{4n+4\text{ brackets}},E]]\cdots] - [B_x,[B_x,[B_x,[B_x,E]]]] = \frac{8}{3}(16^n-1)(2\lambda_1+\lambda_2+\lambda_3+\lambda_4)L_p[E^2+p_x^2],\end{aligned} \quad (19)$$

$$\underbrace{[B_x,[B_x\cdots,[B_x}_{4n+1\text{ brackets}},p_x]]\cdots]-[B_x,p_x]$$
$$=\frac{1}{3}i(16^n-1)(\lambda_1+2\lambda_2+2\lambda_3+2\lambda_4)L_p[E^2+p_x^2],$$
$$\underbrace{[B_x,[B_x\cdots,[B_x}_{4n+2\text{ brackets}},p_x]]\cdots]-[B_x,[B_x,p_x]]$$
$$=-\frac{4}{3}(16^n-1)(\lambda_1+2\lambda_2+2\lambda_3+2\lambda_4)L_pEp_x, \quad (20)$$

$$\underbrace{[B_x,[B_x\cdots,[B_x}_{4n+3\text{ brackets}},p_x]]\cdots]-[B_x,[B_x,[B_x,p_x]]]$$
$$=-\frac{4}{3}i(16^n-1)(\lambda_1+2\lambda_2+2\lambda_3+2\lambda_4)L_p[E^2+p_x^2],$$
$$\underbrace{[B_x,[B_x\cdots,[B_x}_{4n\text{ brackets}},p_x]]\cdots]-p_x$$
$$=\frac{1}{3}(16^n-1)(\lambda_1+2\lambda_2+2\lambda_3+2\lambda_4)L_pEp_x \quad (21)$$

with $n\geq 1$.

The other useful commutators are periodic:

$$\begin{aligned}[B_x,p_y]&=i\lambda_4L_pp_xp_y-i\lambda_5L_pEp_z,\\ [B_x,[B_x,p_y]]&=-\lambda_4L_pEp_y+\lambda_5L_pp_xp_z,\\ [B_x,[B_x,[B_x,p_y]]]&=-i\lambda_4L_pp_xp_y+i\lambda_5L_pEp_z,\\ [B_x,[B_x,[B_x,[B_x,p_y]]]]&=\lambda_4L_pEp_y-\lambda_5L_pp_xp_z,\end{aligned} \quad (22)$$

$$\begin{aligned}[B_x,p_z]&=i\lambda_4L_pp_xp_z+i\lambda_5L_pEp_y,\\ [B_x,[B_x,p_z]]&=-\lambda_4L_pEp_z-\lambda_5L_pp_xp_y,\\ [B_x,[B_x,[B_x,p_z]]]&=-i\lambda_4L_pp_xp_z-i\lambda_5L_pEp_y,\\ [B_x,[B_x,[B_x,[B_x,p_z]]]]&=\lambda_4L_pEp_z+\lambda_5L_pp_xp_y.\end{aligned} \quad (23)$$

With the above expressions in hand, after tedious but straightforward computation parallel to that in Sec. II, one can prove that

$$\begin{aligned}E(\xi_x)=&\,E(0)\cosh(\xi_x)-p_x(0)\sinh(\xi_x)-\lambda_1L_pE(0)p_x(0)\sinh(\xi_x)\\ &+[(\lambda_1+\lambda_2)L_pE^2(0)+\lambda_3L_p\vec{p}^2(0)+(\lambda_1+\lambda_4)L_pp_x^2(0)][\cosh(\xi_x)-1]\\ &+\frac{2}{3}(2\lambda_1+\lambda_2+\lambda_3+\lambda_4)L_pE(0)p_x(0)\sinh(\xi_x)[1-\cosh(\xi_x)]\\ &+\frac{1}{3}(2\lambda_1+\lambda_2+\lambda_3+\lambda_4)L_p[E^2(0)+p_x^2(0)][\cosh(\xi_x)-1]^2,\end{aligned} \quad (24)$$

$$\begin{aligned}p_x(\xi_x)=&\,p_x(0)\cosh(\xi_x)-E(0)\sinh(\xi_x)\\ &-[\lambda_2L_pE^2(0)+\lambda_3L_p\vec{p}^2(0)+\lambda_4L_pp_x^2(0)]\ \sinh(\xi_x)\\ &+\frac{1}{3}(\lambda_1+2\lambda_2+2\lambda_3+2\lambda_4)L_pE(0)p_x(0)[2\cosh(\xi_x)+1][\cosh(\xi_x)-1]\\ &+\frac{1}{3}(\lambda_1+2\lambda_2+2\lambda_3+2\lambda_4)L_p[E^2(0)+p_x^2(0)]\ \sinh(\xi_x)[1-\cosh(\xi_x)],\end{aligned} \quad (25)$$

$$\begin{aligned}p_y(\xi_x)=&\,p_y(0)-[\lambda_4L_pp_x(0)p_y(0)-\lambda_5L_pE(0)p_z(0)]\ \sinh(\xi_x)\\ &+[\lambda_4L_pE(0)p_y(0)-\lambda_5L_pp_x(0)p_z(0)][\cosh(\xi_x)-1],\end{aligned} \quad (26)$$

$$\begin{aligned}p_z(\xi_x)=&\,p_z(0)-[\lambda_4L_pp_x(0)p_z(0)+\lambda_5L_pE(0)p_y(0)]\ \sinh(\xi_x)\\ &+[\lambda_4L_pE(0)p_z(0)+\lambda_5L_pp_x(0)p_y(0)][\cosh(\xi_x)-1].\end{aligned} \quad (27)$$

This is the expression of energy and momentum after a finite boost in DSR. It is one of the central results of this paper. We get it by extending the exponential parameterization of boost from SR to DSR. In the coming subsection, the same expression will be rederived by solving a system of differential equations directly.

### C. Cross-check

Finite-boost transformations have previously been studied for other subcases of DSR in the literature. In Ref. [1], the finite-boost transformation was studied in $1+1$ dimensions at $\mathcal{O}(L_p)$. Based on the $\kappa$-Poincaré Hopf algebra, Ref. [10] investigated the finite-boost transformation in $3+1$ dimensions. Both of them solve a system of differential equations to gain the expression of finite boosts. Although their method is apparently different from the exponential parameterization method, the start point is exactly the same.

To see this, we take the differential equation of $E$ as an example. Let us differentiate $E(\xi_x) = e^{i\xi_x B_x} E(0) e^{-i\xi_x B_x}$ with respect to $\xi_x$; then it results in $dE/d\xi_x = i[B_x, E]$. The commutator $[B_x, E]$ is always given in a specific model. Hence, indeed the exponential parameterization naturally leads to differential equations of energy and momentum.

In this subsection, we adopt the method of Refs. [1,10] to double-check our result in Sec. III B. For this purpose, we write down a system of differential equations corresponding to the scenario we considered:

$$\begin{aligned}
\frac{dE}{d\xi_x} &= i[B_x, E] = -p_x - \lambda_1 L_p E p_x,\\
\frac{dp_x}{d\xi_x} &= i[B_x, p_x] = -E - \lambda_2 L_p E^2 - \lambda_3 L_p \vec{p}^2 - \lambda_4 L_p p_x^2,\\
\frac{dp_y}{d\xi_x} &= i[B_x, p_y] = -\lambda_4 L_p p_x p_y + \lambda_5 L_p E p_z,\\
\frac{dp_z}{d\xi_x} &= i[B_x, p_z] = -\lambda_4 L_p p_x p_z - \lambda_5 L_p E p_y.
\end{aligned} \tag{28}$$

It is trivial to plug Eqs. (23–27) into these equations and confirm that they are well satisfied. Alternatively, one may solve the equations honestly, recasting them as

$$\begin{aligned}
\frac{d^2 p_x}{d\xi_x^2} &= p_x - (\lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4) L_p p_x \frac{dp_x}{d\xi_x},\\
\frac{dE}{d\xi_x} &= -p_x - \lambda_1 L_p E p_x,\\
\frac{d}{d\xi_x}(p_y^2 + p_z^2) &= -2\lambda_4 L_p p_x (p_y^2 + p_z^2),\\
\frac{d}{d\xi_x}\left(\frac{p_y}{p_z}\right) &= \lambda_5 L_p E\left[1 + \left(\frac{p_y}{p_z}\right)^2\right].
\end{aligned} \tag{29}$$

This system of equations can be solved order by order. We expect that the solution reproduces Eqs. (6), (7), (10) in the zeroth order, i.e., in the limit $L_p \to 0$. In the first order of $L_p$, there are five integration constants. These constants can be determined by initial conditions of $E$, $\vec{p}$ and the second line of Eq. (28) at $\xi_x = 0$ [10]. We have solved the equations in this way, and arrived at a solution exactly the same as that in Eqs. (24–27).

### D. Boost in any direction

So far, we have assumed the boost is along the $x$ direction. This does not lose much generality. Since the direction of momentum $\vec{p}$ is not restricted to a coordinate axis, one is free to choose the $x$ axis as the direction of boost, meeting our assumption. From the obtained results, we can get the boost rule along the $y$ direction by alternating indices $(x, y, z) \to (y, z, x)$, and the boost along the $z$ direction by $(x, y, z) \to (z, x, y)$.

As a preparation for Sec. V, it is nevertheless useful to consider an arbitrary boost in the $x$-$y$ plane with rapidity $\xi$. In this plane, the boost direction makes a polar angle $\phi$ with respect to the $x$ axis, and then the two reference frames are related by

$$|\xi\rangle_\phi = e^{i\phi R_z(-\phi;\xi)} e^{i\xi B_x(-\phi)} e^{-i\phi R_z(0)} |0\rangle, \tag{30}$$

leading to the transformation rules of energy and momentum:

$$\begin{aligned}
E(-\phi;\xi;\phi) &= e^{i\phi R_z(-\phi;\xi)} e^{i\xi B_x(-\phi)} e^{-i\phi R_z(0)}\\
&\quad \times E(0) e^{i\phi R_z(0)} e^{-i\xi B_x(-\phi)} e^{-i\phi R_z(-\phi;\xi)},\\
\vec{p}(-\phi;\xi;\phi) &= e^{i\phi R_z(-\phi;\xi)} e^{i\xi B_x(-\phi)} e^{-i\phi R_z(0)}\\
&\quad \times \vec{p}(0) e^{i\phi R_z(0)} e^{-i\xi B_x(-\phi)} e^{-i\phi R_z(-\phi;\xi)}.
\end{aligned} \tag{31}$$

In the above exponents, the parameters highlighted in the brackets tell us what kind of operator each one acts on. For example, $B_x(-\phi)$ should act on $E(-\phi)$ and $\vec{p}(-\phi)$ rather than $E(0)$ or $\vec{p}(0)$, though $B_x(-\phi)$ is independent of $\phi$, namely $\partial_\phi B_x(-\phi) = 0$. More details will appear in Sec. V.

The transformation rule stated in Eq. (31) is enough for our later discussion. The rule can be further generalized to any direction beyond the $x$-$y$ plane similarly by rotating the arbitrary direction of the boost to the $x$ direction.

## IV. COMPATIBILITY ANALYSIS

### A. Dispersion relation

The dispersion relation plays an important role in SR. When generalized to DSR, it contains a pair of free parameters in the $\mathcal{O}(L_p)$ terms,

$$m^2 = E^2 - \vec{p}^2 + \alpha_1 L_p E^3 + \alpha_2 L_p E \vec{p}^2, \tag{32}$$

where the parameters $\alpha_1$, $\alpha_2$ can be constrained by data, confronting DSR directly with observations and experiments.

In the spirit of the relativity principle, the dispersion relation is expected to be invariant in different inertial reference frames. If the deformed boost [Eq. (12)] is compatible with dispersion relation [Eq. (32)], then $E(\xi_x)$ and $\vec{p}(\xi_x)$ should satisfy Eq. (31) for arbitrary values of $\xi_x$. That is to say,

$$E^2(\xi_x) - \vec{p}^2(\xi_x) + \alpha_1 L_p E^3(\xi_x) + \alpha_2 L_p E(\xi_x)\vec{p}^2(\xi_x) = E^2(0) - \vec{p}^2(0) + \alpha_1 L_p E^3(0) + \alpha_2 L_p E(0)\vec{p}^2(0). \quad (33)$$

Putting Eqs. (23–27) into this equation, we find that $\alpha_1$, $\alpha_2$ can be expressed by parameters in the boost generator,

$$\alpha_1 = -\frac{2}{3}(\lambda_1 - \lambda_2 + 2\lambda_3 + 2\lambda_4), \quad \alpha_2 = 2(\lambda_3 + \lambda_4). \quad (34)$$

The same constraints have been derived by Ref. [7] in a Hamiltonian formalism.

### B. Addition law of energy-momentum

As argued in a footnote of Ref. [11], the addition law of energy-momentum becomes nonlinear in DSR because there is a special energy-momentum scale, just as the nonlinear addition law of velocity stems from a special velocity scale $c$ in SR. The addition law of two energies/momenta in DSR has been studied by Refs. [6,8] up to $\mathcal{O}(L_p)$. In $3+1$ dimensions, it takes the nonlinear form

$$E_a \oplus E_b = E_a + E_b + \beta_1 L_p E_a E_b + \beta_2 L_p \vec{p}_a \cdot \vec{p}_b, \quad \vec{p}_a \oplus \vec{p}_b = \vec{p}_a + \vec{p}_b + \gamma_1 L_p E_a \vec{p}_b + \gamma_2 L_p E_b \vec{p}_a + \gamma_3 L_p \vec{p}_a \times \vec{p}_b. \quad (35)$$

Supposing the law is universal in all inertial reference frames, we have

$$(E_a \oplus E_b)(\xi_x) = E_a(\xi_x) \oplus E_b(\xi_x), \quad (\vec{p}_a \oplus \vec{p}_b)(\xi_x) = \vec{p}_a(\xi_x) \oplus \vec{p}_b(\xi_x), \quad (36)$$

where the notation

$$(E_a \oplus E_b)(\xi_x) = (E_a(0) \oplus E_b(0)) \cosh(\xi_x) - (p_{ax}(0) \oplus p_{bx}(0)) \sinh(\xi_x) + \mathcal{O}(L_p) \quad (37)$$

as a result of Eq. (24), and similarly for $(\vec{p}_a \oplus \vec{p}_b)(\xi_x)$.

Equation (36) provides a compatibility condition for the boost [Eq. (12)] and the energy-momentum addition law [Eq. (35)]. Ignoring $\mathcal{O}(L_p^2)$ and higher-order terms, they should hold for arbitrary values of $E(0)$ and $\vec{p}(0)$. Along this line, we find the following relations between parameters:

$$\beta_1 = 2(\lambda_1 + \lambda_2 + 2\lambda_3), \quad \gamma_1 = \gamma_2 = \lambda_4 = \lambda_1 + 2\lambda_2 + 2\lambda_3, \quad \beta_2 = -2\lambda_3, \quad \gamma_3 = 0, \quad \lambda_5 = 0. \quad (38)$$

This result is in agreement with Ref. [8]. Note that here we have implicitly assumed that each particle's energy-momentum transforms independently under a finite boost, obeying the same transformation rule of the total energy-momentum.

We can see that the compatibility not only fixes all parameters in the addition law of energy-momentum, but also places limits on $\lambda_4$ and $\lambda_5$ in the boost generator. In contrast, Ref. [8] gains constraints on $\lambda_4$ and $\lambda_5$ "in order to reproduce the Lorentz algebra"[1] and other relations through infinitesimal boost transformations in DSR. More comparisons between our result and Ref. [8] will be presented in Sec. VI.

Combining Eqs. (34) and (38) together, it is not hard to reobtain the golden rule proposed on a physical ground in Ref. [6]:

$$\alpha_1 + \alpha_2 + \beta_1 + \beta_2 - \gamma_1 - \gamma_2 = 0. \quad (39)$$

## V. WIGNER ROTATION

In SR, two parallel boosts are equivalent to one boost along the same direction. However, the composition of two boosts in different directions differs from a single boost. Instead, it is equivalent to the composition of a boost and a rotation [12–14]. Such a rotation is known as the Wigner rotation.

It is natural to ask if the Wigner rotation receives corrections in DSR. To our knowledge, there has been no prior work on the Wigner rotation in DSR. As a preliminary exploration, in this section we consider two mutually perpendicular boosts in the $x$-$y$ plane, first along the $x$ direction and second along the $y$ direction. We focus on the situation where $\lambda_4$ and $\lambda_5$ are restricted by

$$\lambda_4 = \lambda_1 + 2\lambda_2 + 2\lambda_3, \quad \lambda_5 = 0. \quad (40)$$

A full investigation of more general situations will be done in a different scheme and reported elsewhere.

[1] See the first paragraph of Sec. III C in Ref. [8].

After two successive boosts, the energy and momentum transform as

$$E(\xi_x;\xi_y) = e^{i\xi_y B_y(\xi_x)} e^{i\xi_x B_x(0)} E(0) e^{-i\xi_x B_x(0)} e^{-i\xi_y B_y(\xi_x)},$$
$$\vec{p}(\xi_x;\xi_y) = e^{i\xi_y B_y(\xi_x)} e^{i\xi_x B_x(0)} \vec{p}(0) e^{-i\xi_x B_x(0)} e^{-i\xi_y B_y(\xi_x)}. \quad (41)$$

Again, in the above exponents, the parameters highlighted in the brackets remind us what kind of operator each one acts on. In this convention,

$$\begin{aligned} E(\xi_x;\xi_y) &= e^{i\xi_y B_y(\xi_x)} E(\xi_x) e^{-i\xi_y B_y(\xi_x)} \\ &= E(\xi_x)\cosh(\xi_y) - p_y(\xi_x)\sinh(\xi_y) + \mathcal{O}(L_p) \\ &\neq E(\xi_y)\cosh(\xi_x) - p_x(\xi_y)\sinh(\xi_x) + \mathcal{O}(L_p). \end{aligned} \quad (42)$$

The inequality in the last line means $E(\xi_x;\xi_y) \neq E(\xi_y;\xi_x)$ in both SR and DSR. It is easy to check this by applying the transformation rule of a single boost along the $x$ and $y$ directions in turn.

As we are interested in Wigner rotation, let us pause to explain the transformation rule of energy and momentum under rotations with respect to the $z$ axis. From Eq. 15, one can read directly

$$\begin{aligned} &[R_z, p_x] = ip_y, \quad [R_z,[R_z,p_x]] = p_x, \\ &[R_z, p_y] = -ip_x, \quad [R_z,[R_z,p_y]] = p_y, \\ &[R_z, p_z] = 0, \quad [R_z, E] = 0. \end{aligned} \quad (43)$$

Since the rotational invariance is unchanged, DSR does not lead to any corrections to the commutators above. As a result, after a finite rotation of angle $\theta$ with respect to the $z$ axis, the energy and momentum transform as follows:

$$\begin{aligned} E(\theta) &= e^{i\theta R_z} E(0) e^{-i\theta R_z} = E(0), \\ p_x(\theta) &= e^{i\theta R_z} p_x(0) e^{-i\theta R_z} = p_x(0)\cos(\theta) - p_y(0)\sin(\theta), \\ p_y(\theta) &= e^{i\theta R_z} p_y(0) e^{-i\theta R_z} = p_y(0)\cos(\theta) + p_x(0)\sin(\theta), \\ p_z(\theta) &= e^{i\theta R_z} p_z(0) e^{-i\theta R_z} = p_z(0). \end{aligned} \quad (44)$$

We proceed to study the Wigner rotation arising from two perpendicular boosts in DSR. It is tedious to write down the lengthy expressions of $E(\xi_x;\xi_y)$ and $\vec{p}(\xi_x;\xi_y)$, which are relegated to Appendix A. However, with a few lines of computer programming, we find to $\mathcal{O}(L_p)$ that

$$\begin{aligned} e^{i\theta R_z} E(\xi_x;\xi_y) e^{-i\theta R_z} &= E(-\phi;\xi;\phi), \\ e^{i\theta R_z} \vec{p}(\xi_x;\xi_y) e^{-i\theta R_z} &= \vec{p}(-\phi;\xi;\phi), \end{aligned} \quad (45)$$

as long as the equalities in Eq. (40) hold.[2] Here $E(-\phi;\xi;\phi)$, $\vec{p}(-\phi;\xi;\phi)$ and $E(\xi_x;\xi_y)$, $\vec{p}(\xi_x;\xi_y)$ are defined by Eqs. (31) and (41), with $\xi$, $\phi$, $\theta$ given by

$$\begin{aligned} &\cosh(\xi) = \cosh(\xi_x)\cosh(\xi_y), \\ &\sin(\phi) = \frac{\sinh(\xi_y)}{\sinh(\xi)}, \qquad \tan(\phi) = \frac{\tanh(\xi_y)}{\sinh(\xi_x)}, \\ &\sin(\theta) = -\frac{\sinh(\xi_x)\sinh(\xi_y)}{1+\cosh(\xi)}, \\ &\cos(\theta) = \frac{\cosh(\xi_x)+\cosh(\xi_y)}{1+\cosh(\xi)}. \end{aligned} \quad (46)$$

Recall that the right-hand side of Eq. (45) is the energy-momentum boosted along the direction at an angle $\theta$ with the $x$ axis in the $x$-$y$ plane. On the left-hand side, the energy-momentum is transformed under two perpendicular boosts and then a rotation. Hence, we have succeeded in proving the Wigner rotation in DSR to the first order of $L_p$. The angle $\theta$ is nothing else but the Wigner rotation angle. One can compare this result with Ref. [12] to see that DSR does not introduce corrections to the Wigner rotation, at least in the situation we considered.

The equalities in Eq. (45) have a simple interpretation: two successive perpendicular boosts are not equivalent to a single boost in DSR, but to a boost combined with a rotation. Reversing the logic, any boost in the $x$-$y$ plane can be decomposed into two mutually perpendicular boosts followed by a rotation.

Here we would like to highlight Eq. (40). In the previous section, we derived them as a part of the compatibility condition [Eq. (38)] for the transformation rule and the addition law of energy-momentum. In the current section, we encounter them as a necessary condition for recovering Wigner rotation in DSR. In the next section, we will make more comments on them.

## VI. COMMENTS

It is interesting to go back and look at the commutators in Eq. (14), whose $\mathcal{O}(L_p)$ terms disappear once Eq. (40) applies. In this perspective, Eq. (40) can be regarded as a restriction on deformations of the Poincaré algebra. If the deformed Poincaré algebra were not restricted in

[2]One may suspect that Eq. (45) continues to hold without imposing the conditions of Eq. (40) by appropriately tuning the relations in Eq. (46). To explore this, we have calculated $e^{i\theta R_z} p_z(\xi_x;\xi_y) e^{-i\theta R_z} - p_z(-\phi;\xi;\phi)$. Its full expression is too lengthy to print out, but we find it is of $\mathcal{O}(L_p)$ and, besides other terms, the coefficient of $p_y^2(0)$ in this expression is $\lambda_5 L_p \cos\phi \sin\phi(1-\cosh\xi)$. It indicates that no matter how we modify Eq. (46), Eq. (45) cannot hold unless the conditions in Eq. (40) are imposed.

this way, the boost transformation would be incompatible with the addition law of energy-momentum in Sec. IV B, and we would fail to recover the Wigner rotation in Sec. V. To be exact, in Ref. [8], Eq. (40) is not derived from the compatibility condition, but put forward as a condition to reproduce the Lorentz algebra from Eq. (14).

Still, it is unclear how to prove Eq. (45) directly from Eq. (14) after switching off the $\mathcal{O}(L_p)$ terms. To this end, we introduce new variables $\mathcal{E}$, $\vec{\mathcal{P}}$ through

$$E = \mathcal{E} + \delta_1 L_p \mathcal{E}^2 + \delta_2 L_p \vec{\mathcal{P}}^2, \quad \vec{P} = \vec{\mathcal{P}} + \delta_3 L_p \mathcal{E}\vec{\mathcal{P}}. \tag{47}$$

Substituting them into Eqs. (13–15), we find that $\mathcal{E}$, $\mathcal{P}_i$, $R_i$, $B_i$ can form the conventional Poincaré algebra at $\mathcal{O}(L_p)$ if

$$\delta_1 = \lambda_1 + \lambda_2 + 2\lambda_3, \qquad \delta_2 = -\lambda_3, \qquad \delta_3 = \lambda_4, \tag{48}$$

and Eq. (40) hold. Some details are given in Appendix B. That is to say, restricted by Eq. (40), the deformed Poincaré algebra in Sec. III A is equivalent to the undeformed Poincaré algebra in terms of new variables. In this form, the variables $\mathcal{E}$, $\vec{\mathcal{P}}$ behave like the standard SR energy-momenta. Given that the Wigner rotation is well established in SR, it is thus not hard to understand that Eq. (40) is a condition to recreate the Wigner rotation in DSR.

When imposing Eq. (36) in Sec. IV B, just like Ref. [15], we have implicitly assumed that under a finite boost, the set of momenta of different particles transform independently. Naturally, this leads to $\gamma_1 - \gamma_2 = \gamma_3 = 0$ in the compatibility condition [Eq. (38)], a special case called a commutative composition of momenta in Ref. [8]. It is worthwhile for future research to abandon this implicit assumption and extend our results to the noncommutative case.

## ACKNOWLEDGMENTS

The authors are grateful to the referee for enlightening comments. This work is supported by the National Natural Science Foundation of China (Grant No. 11105053) and partially by the Open Research Foundation of Shanghai Key Laboratory of Particle Physics and Cosmology (Grant No. 11DZ2230700).

## APPENDIX A: ENERGY-MOMENTUM AFTER TWO PERPENDICULAR BOOSTS

In this Appendix, we will show the transformation rule of energy and momentum under two mutually perpendicular boosts in DSR. For concreteness, in this paper we consider boosts in the $x$-$y$ plane, first along the $x$ direction and then along the $y$ direction.

In Sec. III B, we have presented the transformation rule of energy and momentum under a finite boost in the $x$ direction; see expressions (23–27). From them, the boost rule along the $y$ direction can be obtained by alternating indices $(x, y, z) \to (y, z, x)$. Then, in accordance with the definition [Eq. (41)], straightforward computations give the expression of energy and momentum after two boosts:

$$\begin{aligned}
E(\xi_x;\xi_y) = {} & E(0)\cosh\xi_x\cosh\xi_y - p_x(0)\cosh\xi_y\sinh\xi_x - p_y(0)\sinh\xi_y \\
& + L_p\bigg\{\cosh\xi_y\bigg[(E^2(0)(\lambda_1+\lambda_2) + \vec{p}^2(0)\lambda_3 + p_x^2(0)(\lambda_1+\lambda_4))(\cosh\xi_x - 1) \\
& + \frac{1}{3}(E^2(0)+p_x^2(0))(2\lambda_1+\lambda_2+\lambda_3+\lambda_4)(\cosh\xi_x-1)^2 - E(0)p_x(0)\lambda_1\sinh\xi_x \\
& - \frac{2}{3}E(0)p_x(0)(2\lambda_1+\lambda_2+\lambda_3+\lambda_4)(\cosh\xi_x-1)\sinh\xi_x\bigg] \\
& + \frac{1}{3}(2\lambda_1+\lambda_2+\lambda_3+\lambda_4)(\cosh\xi_y-1)^2(p_y^2(0) + (E(0)\cosh\xi_x - p_x(0)\sinh\xi_x)^2) \\
& + (\cosh\xi_y - 1)[p_y^2(0)(\lambda_1+\lambda_4) + (\lambda_1+\lambda_2)(E(0)\cosh\xi_x - p_x(0)\sinh\xi_x)^2 \\
& + \lambda_3(p_y^2(0) + p_z^2(0) + (p_x(0)\cosh\xi_x - E(0)\sinh\xi_x)^2)] \\
& + (p_x(0)p_z(0)\lambda_5 - E(0)p_y(0)\lambda_4)(\cosh\xi_x - 1)\sinh\xi_y + p_x(0)p_y(0)\lambda_4\sinh\xi_x\sinh\xi_y \\
& - E(0)p_z(0)\lambda_5\sinh\xi_x\sinh\xi_y - p_y(0)\lambda_1(E(0)\cosh\xi_x - p_x(0)\sinh\xi_x)\sinh\xi_y \\
& - \frac{2}{3}p_y(0)(2\lambda_1+\lambda_2+\lambda_3+\lambda_4)(\cosh\xi_y-1)(E(0)\cosh\xi_x - p_x(0)\sinh\xi_x)\sinh\xi_y\bigg\},
\end{aligned}$$

$$
\begin{aligned}
p_x(\xi_x;\xi_y) = {}& p_x(0)\cosh\xi_x - E(0)\sinh\xi_x \\
& + L_p\bigg\{\frac{1}{3}E(0)p_x(0)[\lambda_1+2(\lambda_2+\lambda_3+\lambda_4)](\cosh\xi_x-1)(1+2\cosh\xi_x) \\
& + p_y(0)p_z(0)\lambda_5(\cosh\xi_y-1) - E^2(0)\lambda_2\sinh\xi_x - \vec{p}^2(0)\lambda_3\sinh\xi_x - p_x^2(0)\lambda_4\sinh\xi_x \\
& - \frac{1}{3}(E^2(0)+p_x^2(0))[\lambda_1+2(\lambda_2+\lambda_3+\lambda_4)](\cosh\xi_x-1)\sinh\xi_x \\
& + \lambda_4(\cosh\xi_y-1)(p_x(0)\cosh\xi_x-E(0)\sinh\xi_x)(E(0)\cosh\xi_x-p_x(0)\sinh\xi_x) \\
& - p_y(0)\lambda_4(p_x(0)\cosh\xi_x-E(0)\sinh\xi_x)\sinh\xi_y \\
& - p_z(0)\lambda_5(E(0)\cosh\xi_x-p_x(0)\sinh\xi_x)\sinh\xi_y\bigg\},
\end{aligned}
$$

$$
\begin{aligned}
p_y(\xi_x;\xi_y) = {}& p_y(0)\cosh\xi_y + (-E(0)\cosh\xi_x + p_x(0)\sinh\xi_x)\sinh\xi_y \\
& + L_p\bigg\{(p_x(0)p_z(0)\lambda_5 - E(0)p_y(0)\lambda_4)(1-\cosh\xi_x)\cosh\xi_y \\
& - p_x(0)p_y(0)\lambda_4\cosh\xi_y\sinh\xi_x + E(0)p_z(0)\lambda_5\cosh\xi_y\sinh\xi_x \\
& + \frac{1}{3}p_y(0)[\lambda_1+2(\lambda_2+\lambda_3+\lambda_4)](\cosh\xi_y-1)(1+2\cosh\xi_y) \\
& \times(E(0)\cosh\xi_x-p_x(0)\sinh\xi_x) \\
& + \bigg[-(E^2(0)(\lambda_1+\lambda_2)+\vec{p}^2(0)\lambda_3+p_x^2(0)(\lambda_1+\lambda_4))(\cosh\xi_x-1) \\
& - \frac{1}{3}(E^2(0)+p_x^2(0))(2\lambda_1+\lambda_2+\lambda_3+\lambda_4)(\cosh\xi_x-1)^2 + E(0)p_x(0)\lambda_1\sinh\xi_x \\
& + \frac{2}{3}E(0)p_x(0)(2\lambda_1+\lambda_2+\lambda_3+\lambda_4)(\cosh\xi_x-1)\sinh\xi_x\bigg]\sinh\xi_y \\
& + \frac{1}{3}[\lambda_1+2(\lambda_2+\lambda_3+\lambda_4)](1-\cosh\xi_y)(p_y^2(0)+(E(0)\cosh\xi_x-p_x(0)\sinh\xi_x)^2) \\
& \times\sinh\xi_y + [-p_y^2(0)\lambda_4 - \lambda_2(E(0)\cosh\xi_x-p_x(0)\sinh\xi_x)^2 \\
& - \lambda_3(p_y^2(0)+p_z^2(0)+(p_x(0)\cosh\xi_x-E(0)\sinh\xi_x)^2)]\sinh\xi_y\bigg\},
\end{aligned}
$$

$$
\begin{aligned}
p_z(\xi_x;\xi_y) = {}& p_z(0) + L_p\{-E(0)p_z(0)\lambda_4 - p_x(0)p_y(0)\lambda_5 \\
& + (-2E(0)p_y(0)\lambda_5 + (E(0)p_y(0)\lambda_5 - p_x(0)p_z(0)\lambda_4)\cosh\xi_y)\sinh\xi_x \\
& - p_y(0)p_z(0)\lambda_4\sinh\xi_y + E(0)p_x(0)\lambda_5(\cosh^2\xi_x+\sinh^2\xi_x)\sinh\xi_y \\
& - \cosh\xi_x[(p_x(0)p_y(0)\lambda_5 - E(0)p_z(0)\lambda_4)\cosh\xi_y \\
& + \lambda_5(-2p_x(0)p_y(0) + (E^2(0)+p_x^2(0))\sinh\xi_x\sinh\xi_y)]\}.
\end{aligned}
$$

In principle, these results may also be gained by applying the Baker-Campbell-Hausdorff (BCH) formula to Eq. (41). But our method in this paper is more convenient to be implemented in computer programs. We hope the BCH formula will be useful for an all-order calculation.

In Sec. V, we further rotate the above results with respect to the $z$ axis and get the Wigner rotation at $\mathcal{O}(L_p)$ when certain conditions are satisfied. Switching off the $\mathcal{O}(L_p)$ terms, we can recover the familiar SR results.

## APPENDIX B: DEFORMED POINCARÉ ALGEBRA IN AUXILIARY VARIABLES

In terms of $\mathcal{E}$, $\mathcal{P}_i$ introduced by Eq. (47), we can put the deformed Poincaré algebra [Eqs. (13–15)] in the form

$$\begin{aligned}(1+2\delta_1 L_p\mathcal{E})[B_i,\mathcal{E}]+\sum_j 2\delta_2 L_p\mathcal{P}_j[B_i,\mathcal{P}_j]&=i\mathcal{P}_i[1+(\lambda_1+\delta_3)L_p\mathcal{E}],\\ (1+\delta_3 L_p\mathcal{E})[B_i,\mathcal{P}_j]+\delta_3 L_p\mathcal{P}_j[B_i,\mathcal{E}]&=i\delta_{ij}[\mathcal{E}+(\lambda_2+\delta_1)L_p\mathcal{E}^2+(\lambda_3+\delta_2)L_p\vec{\mathcal{P}}^2]\\ &\quad+i\lambda_4 L_p\mathcal{P}_i\mathcal{P}_j-i\lambda_5 L_p\mathcal{E}\sum_k\epsilon_{ijk}\mathcal{P}_k,\end{aligned} \tag{B1}$$

$$\begin{aligned}[B_i,B_j]&=-i[1+(\lambda_1+2\lambda_2+2\lambda_3-\lambda_4)L_p\mathcal{E}]\sum_k\epsilon_{ijk}R_k-2i\lambda_5 L_p\mathcal{E}\sum_k\epsilon_{ijk}B_k,\\ [R_i,B_j]&=i\sum_k\epsilon_{ijk}B_k-i\lambda_5 L_p\mathcal{E}\sum_k\epsilon_{ijk}R_k,\end{aligned} \tag{B2}$$

$$\begin{aligned}(1+2\delta_1 L_p\mathcal{E})[R_i,\mathcal{E}]+\sum_j 2\delta_2 L_p\mathcal{P}_j[R_i,\mathcal{P}_j]&=0,\\ (1+\delta_3 L_p\mathcal{E})[R_i,\mathcal{P}_j]+\delta_3 L_p\mathcal{P}_j[R_i,\mathcal{E}]&=i\sum_k\epsilon_{ijk}\mathcal{P}_k(1+\delta_3 L_p\mathcal{E}),\\ [R_i,R_j]&=i\sum_k\epsilon_{ijk}R_k.\end{aligned} \tag{B3}$$

If one requires that $\mathcal{E}$, $\mathcal{P}_i$, $R_i$, $B_i$ form the conventional Poincaré algebra, then the above relations at $\mathcal{O}(L_p)$ become a series of restrictions [Eqs. (40) and (48)]. In this case, the deformation can be interpreted with classical Poincaré symmetries in nonlinear disguise, as pointed out very early by Ref. [9].

## APPENDIX C: CONNECTION TO $\kappa$-DEFORMATION OF POINCARÉ ALGEBRA

In Ref. [9], it was shown that under quite plausible assumptions there are only two choices for the addition law of energy-momentum: either one chooses a classical one (symmetric) or the one provided by $\kappa$-deformation. The former case corresponds to the commutative addition law with $\gamma_1-\gamma_2=\gamma_3=0$ in Eq. (35). The latter refers to the quantum deformation of Poincaré algebra with a fundamental mass parameter $\kappa$, related to the noncommutative composition law.

The $\kappa$-deformed Poincaré algebra was introduced in 1991 in the so-called standard basis [16–18]; further, this deformation was rewritten in the bicrossproduct basis [19–21]. In the bicrossproduct basis, other commutation relations of Poincaré algebra remain undeformed except for [9,10,21]

$$[B_i,p_j]=\frac{i}{2}\delta_{ij}\left[\kappa\left(1-e^{-\frac{2E}{\kappa}}\right)+\frac{\vec{p}^2}{\kappa}\right]-\frac{i}{\kappa}p_ip_j. \tag{C1}$$

Relation (C1) is highly nonlinear in parameter $\kappa$. To connect it with our investigation in this paper, for $E/\kappa\ll 1$ we keep the deformation terms linear in $1/\kappa$,

$$[B_i,p_j]=i\delta_{ij}\left(E-\frac{E^2}{\kappa}+\frac{\vec{p}^2}{2\kappa}\right)-\frac{i}{\kappa}p_ip_j. \tag{C2}$$

Since our attention is paid to deformations of $\mathcal{O}(L_p)$, it is enough in this brief discussion to study Eq. (C2).

Identifying Eqs. (13–15) with the $\kappa$-deformed Poincaré algebra to the first order of $1/\kappa$, we get the condition

$$\lambda_1=0,\qquad \lambda_2=\lambda_4=-\frac{1}{L_p\kappa},\qquad \lambda_3=\frac{1}{2L_p\kappa} \tag{C3}$$

together with Eq. (40). It is easy to check that the condition is self-consistent. This indicates that Eq. (40) is also a necessary (but not sufficient) condition for reproducing $\kappa$-deformed Poincaré algebra at the linear order of $1/\kappa$.

Another way to see this point is through transformation (47). If $\mathcal{E}$, $\mathcal{P}_i$, $R_i$, $B_i$ satisfy the $\kappa$-deformed Poincaré algebra at the linear order of $1/\kappa$, then relations (B1–B3) at $\mathcal{O}(L_p)$ yield

$$\delta_1 = \lambda_1 + \lambda_2 + 2\lambda_3, \qquad \delta_2 = \frac{1}{2L_p\kappa} - \lambda_3,$$
$$\delta_3 = \frac{1}{L_p\kappa} + \lambda_4, \tag{C4}$$

and again Eq. (40).

Our results in this section imply that to the first order of deformation parameters ($L_p$ and $1/\kappa$), the two types of DSR deformations mentioned in Ref. [9] cannot be distinguished in the one-particle system. We note that condition (40) has been given in Ref. [8] for both the commutative and noncommutative cases.


[1] G. Amelino-Camelia, Int. J. Mod. Phys. D **11**, 35 (2002).
[2] G. Amelino-Camelia, D. Benedetti, F. D'Andrea, and A. Procaccini, Classical Quantum Gravity **20**, 5353 (2003).
[3] G. Amelino-Camelia, Symmetry **2**, 230 (2010).
[4] G. Amelino-Camelia, Phys. Lett. B **510**, 255 (2001).
[5] J. Magueijo and L. Smolin, Phys. Rev. Lett. **88**, 190403 (2002).
[6] G. Amelino-Camelia, Phys. Rev. D **85**, 084034 (2012).
[7] G. Amelino-Camelia, M. Matassa, F. Mercati, and G. Rosati, Phys. Rev. Lett. **106**, 071301 (2011).
[8] J. M. Carmona, J. L. Cortes, and F. Mercati, Phys. Rev. D **86**, 084032 (2012).
[9] J. Lukierski and A. Nowicki, Int. J. Mod. Phys. A **18**, 7 (2003).
[10] N. R. Bruno, G. Amelino-Camelia, and J. Kowalski-Glikman, Phys. Lett. B **522**, 133 (2001).
[11] G. Amelino-Camelia, J. Kowalski-Glikman, G. Mandanici, and A. Procaccini, Int. J. Mod. Phys. A **20**, 6007 (2005).
[12] R. Ferraro and M. Thibeault, Eur. J. Phys. **20**, 143 (1999).
[13] S. Baskal and Y. S. Kim, J. Phys. A **38**, 6545 (2005).
[14] K. O'Donnell and M. Visser, Eur. J. Phys. **32**, 1033 (2011).
[15] J. M. Carmona, J. L. Cortes, D. Mazon, and F. Mercati, Phys. Rev. D **84**, 085010 (2011).
[16] J. Lukierski, H. Ruegg, A. Nowicki, and V. N. Tolstoi, Phys. Lett. B **264**, 331 (1991).
[17] S. Giller, P. Kosinski, M. Majewski, P. Maslanka, and J. Kunz, Phys. Lett. B **286**, 57 (1992).
[18] J. Lukierski, A. Nowicki, and H. Ruegg, Phys. Lett. B **293**, 344 (1992).
[19] J. Lukierski and H. Ruegg, Phys. Lett. B **329**, 189 (1994).
[20] J. Lukierski, H. Ruegg, and W. J. Zakrzewski, Ann. Phys. (N.Y.) **243**, 90 (1995).
[21] S. Majid and H. Ruegg, Phys. Lett. B **334**, 348 (1994).