

Compact (A)dS boson stars and shellsBetti Hartmann,^{1,*} Burkhard Kleihaus,^{2,†} Jutta Kunz,^{2,‡} and Isabell Schaffer^{2,§}¹*School of Engineering and Science, Jacobs University, Postfach 750 561 D-28725 Bremen, Germany*²*Institut für Physik, Universität Oldenburg, Postfach 2503, D-26111 Oldenburg, Germany*

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We present compact Q -balls in an (anti-)de Sitter background in D dimensions, obtained with a V-shaped potential of the scalar field. Beyond critical values of the cosmological constant $\hat{\Lambda}_{\text{cr}}(D)$ compact Q -shells arise. By including the gravitational backreaction, we obtain boson stars and boson shells with (anti-)de Sitter asymptotics. We analyze the physical properties of these solutions and determine their domain of existence. In four dimensions we address some astrophysical aspects.

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I. INTRODUCTION

Inspired by Wheeler's quest for the existence of geons [1], boson stars were introduced by Feinblum and McKinley [2], Kaup [3], and Ruffini and Bonazzola [4]. In these boson stars the electromagnetic vector field was replaced by some tentative scalar field. With the discovery of a Higgs-like boson last year at the LHC [5,6] the first fundamental scalar field has been found. But numerous scalar fields have been predicted to exist in high energy physics and cosmology.

Boson stars arise as stationary localized solutions of the coupled Einstein-Klein-Gordon equations [2–4,7]. The physical properties of boson stars depend strongly on the type of scalar field potential employed (see e.g. the review articles [8–13]). Miniboson stars arise, when only a mass term is present but no self-interaction. They can reach only relatively small masses. Larger boson stars are obtained when a repulsive quartic self-interaction is included [14]. For these two types of boson stars gravity is a necessary ingredient.

In contrast, in the presence of a sextic potential one obtains solitonic boson stars [8], which possess a flat space-time limit, where they correspond to nontopological solitons [15] (or Q -balls [16]). Moreover, these solitonic boson stars can reach even higher masses [8].

Here we consider boson stars, which are compact in the sense that the scalar field of these spherically symmetric configurations is finite inside a ball of radius r_0 , but vanishes identically outside this radius. In this respect the compact Q -balls resemble stars [17]. Obtained from a V-shaped self-interaction potential, these compact boson stars also possess a flat space-time limit, compact Q -balls [18–20]. These represent solutions of the signum-Gordon equation.

The study of scalar fields with a V-shaped self-interaction potential has revealed interesting physical phenomena. When coupled to electromagnetism, the balance of forces allows for shell-like configurations [19]. In these Q -shells the scalar field vanishes identically both inside a certain radius r_i and outside a certain radius r_o . The scalar field thus forms a shell of charged matter, $r_i < r < r_o$.

When coupling these shells to gravity the resulting boson shells possess an empty Minkowski space interior $r < r_i$. However, the shells need not be empty in their interior; they can harbor a black hole in there [21]. Thus one finds that uniqueness and no-hair theorems for black holes can be avoided in the presence of boson shells [21,22].

Whereas these previous studies considered only asymptotically flat solutions, we here include a cosmological constant. On the one hand, a positive cosmological constant is relevant from an observational point of view, since it can model the dark energy of the Universe. Since boson stars are very compact objects that can possess very high densities, they have been suggested as alternatives to supermassive black holes, e.g. in the center of galaxies [23]. Even if that would be excluded by observations (for a discussion see e.g. [24]) boson stars could still act as toy models for very compact objects, e.g. neutron stars. Such a model of a compact star in a space-time with a positive cosmological constant should be a more realistic description of compact stars in the Universe, since all observations seem to indicate the existence of a form of dark energy.

A negative cosmological constant, on the other hand, leads to solutions that can be interpreted within the AdS/CFT correspondence [25,26]. Recently, the study of boson stars in anti-de Sitter (AdS) space-time received increasing attention [27–34]. This is related to the fact that within the context of a holographic description of superconductors and superfluids [35–37] (for reviews see [38–40]) the formation of scalar hair on charged solitons in asymptotically AdS has been interpreted as an insulator/superconductor phase transition [41,42]. The limit of setting the electric charge of the scalar field e to infinity, which due to the scaling symmetries corresponds to setting Newton's

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constant G to zero, is called the “probe limit” in this context. We adapt this nomenclature here and refer to the case, where the matter field equation is solved in a fixed background, as “the probe limit.” In the opposite limit $e = 0$, the gauge symmetry becomes global and the resulting solutions are uncharged solitons in AdS. These are essentially uncharged boson stars and have been suggested to be the holographic description of glueball condensates [41].

Boson stars in asymptotic AdS are also of interest from another point of view. It has been suggested that the dynamical formation of a black hole in AdS is the dual description of thermalization in a strongly coupled quantum field theory. As such the stability of AdS space-time was studied with respect to perturbations, and it was conjectured that AdS is unstable under arbitrarily small scalar perturbations and that eventually a black hole would form due to the reflection of the perturbations on the AdS boundary [43]. However, in [44] it was shown that boson stars appear to be nonlinearly stable. If that were true, thermalization in the dual field theory would not occur. Hence, boson stars in AdS play an important role in the context of the nonlinear (in)stability of AdS space-time, and thus of its dual description.

Here we first consider the set of boson star solutions for various space-time dimensions $D \geq 3$ in the probe limit. Interestingly, the presence of a positive cosmological constant allows for the existence of boson shells without an electromagnetic field. We note that all objects constructed here are electrically neutral.

By solving the coupled set of the Einstein-signum-Gordon equations, we subsequently determine the domain of existence of the compact (A)dS boson stars and shells in $D \geq 3$ space-time dimensions. We analyze their physical properties and briefly address the stability of the boson stars from a catastrophe theory point of view [45–49]. We also address astrophysical aspects of boson stars and boson shells in four dimensions for the physical value of the cosmological constant.

The paper is organized as follows. In Sec. II we present the action, the Ansatz, the equations of motion together with the scaling property, the boundary conditions and the global charges. We present the solutions in the probe limit in Sec. III. The boson stars and shells obtained with the backreaction taken into account are discussed in Sec. IV. We end with our conclusions and an outlook in Sec. V.

II. MODEL

A. Action

We consider the action of a self-interacting complex scalar field Φ coupled to Einstein gravity in D dimensions

$$S = \int \left[\frac{1}{16\pi G} (R - 2\Lambda) - \frac{1}{2} (\partial_\mu \Phi)^* (\partial^\mu \Phi) - U(|\Phi|) \right] \times \sqrt{-g} d^D x, \quad (1)$$

with curvature scalar R , cosmological constant Λ , and Newton’s constant G , and the asterisk denotes complex conjugation. The scalar potential U is chosen as

$$U(|\Phi|) = \lambda |\Phi|^4. \quad (2)$$

Note that we set $c = \hbar \equiv 1$ here. To be able to compare with observations we would have to reinsert the physical values of these constants. For $D = 4$ the dimensionful action has been given in [17], and we refer the reader to this paper. In the present paper we will also give physical quantities of the boson stars in observationally relevant units for $\Lambda \geq 0$ and $D = 4$.

Variation of the action with respect to the metric and the matter fields leads, respectively, to the Einstein equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (R - 2\Lambda) = 8\pi G T_{\mu\nu} \quad (3)$$

with stress-energy tensor

$$T_{\mu\nu} = g_{\mu\nu} L_M - 2 \frac{\partial L_M}{\partial g^{\mu\nu}} \quad (4)$$

and the matter field equation

$$\nabla_\mu \nabla^\mu \Phi = -\lambda \frac{\Phi}{|\Phi|}, \quad (5)$$

where ∇_μ denotes the covariant derivative.

Invariance of the action under the global phase transformation

$$\Phi \rightarrow \Phi e^{i\chi} \quad (6)$$

leads to the conserved current

$$j^\mu = -i(\Phi^* \partial^\mu \Phi - \Phi \partial^\mu \Phi^*), \quad j^\mu{}_{;\mu} = 0, \quad (7)$$

and the associated conserved charge Q .

B. Ansatz

To construct spherically symmetric boson star solutions we employ Schwarzschild-like coordinates and adopt the spherically symmetric metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -A^2 N dt^2 + N^{-1} dr^2 + r^2 d\Omega_{D-2}^2, \quad (8)$$

where $d\Omega_{D-2}^2$ is the metric on the $D - 2$ dimensional unit sphere.

The associated Ansatz for the boson field takes the form

$$\Phi = \phi(r) e^{i\omega t} \quad (9)$$

with the frequency ω . The conserved scalar charge Q

$$Q = - \int j^t |g|^{1/2} d^{D-1} x \quad (10)$$

is then proportional to ω .

We next introduce dimensionless quantities by

$$r = \hat{r}/\omega, \quad \phi = \hat{\phi}\lambda/\omega^2, \quad \Lambda = \hat{\Lambda}\omega^2, \quad 8\pi G = \alpha\omega^4/\lambda^2. \quad (11)$$

Thus the coupling strength of gravity is expressed in terms of the coupling constant α . This yields the set of equations

$$\frac{1}{A}A' = \frac{2\alpha}{D-2}\hat{r}\left\{\frac{\hat{\phi}^2}{N^2} + (\hat{\phi}')^2\right\}, \quad (12)$$

$$\hat{r}N' = (D-3)(1-N) - 2\frac{\hat{r}^2\hat{\Lambda}}{D-2} - 2\frac{\alpha\hat{r}^2}{N(D-2)}(N^2(\hat{\phi}')^2 + 2N\hat{\phi} + \hat{\phi}^2/A^2), \quad (13)$$

$$\hat{\phi}'' = \frac{A^2N - \hat{\phi}}{A^2N^2} - \frac{(D-3) + N}{\hat{r}N}\hat{\phi}' + \frac{2\hat{r}\hat{\Lambda}}{(D-2)N}\hat{\phi}' - 4\frac{\hat{r}\alpha}{(D-2)N}\hat{\phi}\hat{\phi}'. \quad (14)$$

C. Boundary conditions

Let us now specify the boundary conditions for the metric and the boson field. For the metric function A we adopt

$$A(\hat{r}_o) = 1, \quad (15)$$

where \hat{r}_o is the outer radius of the boson star. Since it retains this value to infinity, this fixes the time coordinate. For the metric function $N(\hat{r})$ we require at the origin the regularity condition for globally regular ball-like boson star solutions

$$N(0) = 1, \quad (16)$$

and for globally regular shell-like solutions

$$N(\hat{r}_i) = 1 - \frac{2\hat{\Lambda}}{(D-2)(D-1)}\hat{r}_i^2, \quad (17)$$

where \hat{r}_i is the inner radius of the shell.

For boson stars we require for the boson field function one condition at the origin and two conditions at the outer radius \hat{r}_o

$$\hat{\phi}'(0) = 0, \quad \hat{\phi}(\hat{r}_o) = 0, \quad \hat{\phi}'(\hat{r}_o) = 0. \quad (18)$$

Since this is one condition too many, we introduce another auxiliary differential equation, $\hat{r}'_o = 0$, by treating \hat{r}_o as a function. Thus \hat{r}_o is constant, but the value of the constant is adjusted in the numerical scheme such that the boundary conditions Eqs. (18) are satisfied. This determines the outer radius of the star.

For boson shells, on the other hand, we require at the inner radius \hat{r}_i and at the outer radius \hat{r}_o the conditions

$$\hat{\phi}(\hat{r}_i) = 0, \quad \hat{\phi}'(\hat{r}_i) = 0, \quad \hat{\phi}(\hat{r}_o) = 0, \quad \hat{\phi}'(\hat{r}_o) = 0. \quad (19)$$

We now also make the ratio of inner and outer radii \hat{r}_i/\hat{r}_o an auxiliary (constant) variable.

For the numerical computation we introduce the scaled coordinate $x = (\hat{r} - \hat{r}_i)/(\hat{r}_o + \hat{r}_i)$, such that the inner radius is at $x = 0$ and the outer radius is at $x = 1$.

D. Outer solutions

We refer to the solution in the exterior region $\hat{r} \geq \hat{r}_o$ as the outer solution of the boson stars and boson shells. In the asymptotically flat case, the outer solution is given by the Schwarzschild solution. In the presence of a cosmological constant the Schwarzschild–de Sitter and Schwarzschild–anti–de Sitter solutions

$$\begin{aligned} \hat{\phi}(\hat{r}) &= 0, & A(\hat{r}) &= 1, \\ N(\hat{r}) &= 1 - \frac{2\mu}{\hat{r}^{D-3}} - \frac{2\hat{\Lambda}}{(D-2)(D-1)}\hat{r}^2, \\ \mu &= \text{const} \end{aligned} \quad (20)$$

are exact solutions of the ordinary differential equations (ODEs) in the exterior region.

Hence the mass parameter of the solutions is given by

$$\mu = \left(1 - N(\hat{r}_o) - \frac{2\hat{r}_o^2\hat{\Lambda}}{(D-2)(D-1)}\right)\frac{\hat{r}_o^{D-3}}{2}. \quad (21)$$

E. Inner solutions

Analogously, we refer to the solution in the interior region $\hat{r} \leq \hat{r}_i$ as the inner solution of the boson shells. In the asymptotically flat case, the regular inner solution corresponds to flat Minkowski space, whereas in the presence of a cosmological constant the regular inner solutions correspond to either de Sitter or anti–de Sitter space. Note, however, that in general $A(\hat{r}) = \text{const} = A_i \neq 1$. Thus a rescaling of the time coordinate, $t \rightarrow t/A_i$, is required to obtain the de Sitter or anti–de Sitter line element in the standard form.

In principle, we could replace these regular inner solutions by the appropriate black hole solutions, analogously to the asymptotically flat case [21,22]. Then these inner solutions would correspond to Schwarzschild–de Sitter or Schwarzschild–anti–de Sitter solutions.

III. PROBE LIMIT

Here we present the families of solutions in the so-called probe limit. Thus we obtain the solutions for vanishing coupling to gravity, i.e., $\alpha = 0$, in the respective background.

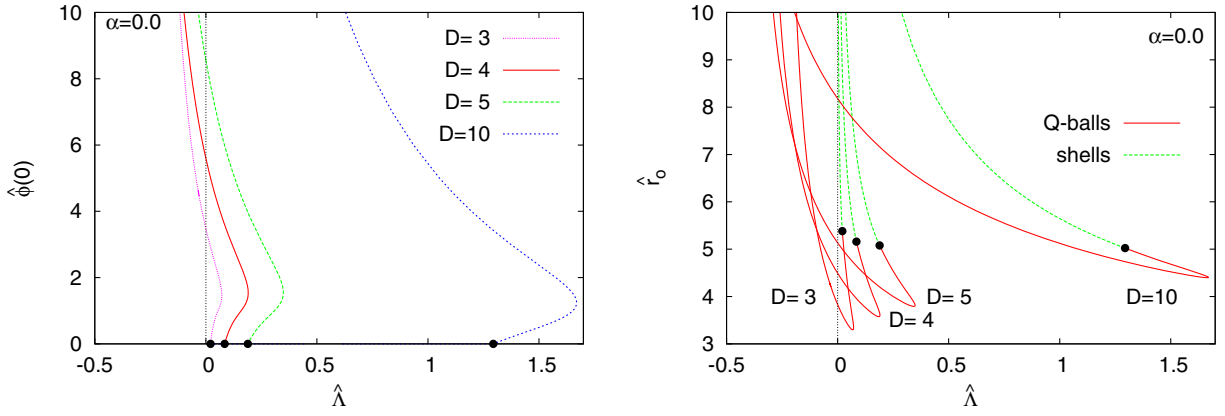


FIG. 1 (color online). Left: The value of the scalar field at the origin $\hat{\phi}(0)$ and Right: the value of the outer radius \hat{r}_o for Q -ball and Q -shell solutions in the (anti-)de Sitter background in $D = 3, 4, 5$ and 10 dimensions. The black dots label the transition points between Q -balls and Q -shells.

A. Q -ball solutions in the Minkowski background

For a vanishing cosmological constant the Q -ball solutions can be found analytically and expressed in terms of Bessel functions. The $D = 4$ solution was given in [18,19]

$$\hat{\phi}(\hat{r}) = \begin{cases} 1 - \frac{\hat{r}_o}{\hat{r}} \frac{\sin \hat{r}}{\sin \hat{r}_o} & \text{if } 0 \leq \hat{r} \leq \hat{r}_o \\ 0 & \text{if } \hat{r} \geq \hat{r}_o. \end{cases} \quad (22)$$

For $D \geq 3$ dimensions this generalizes according to

$$\hat{\phi}(\hat{r}) = \hat{r}^{-n} C_1 J_n(\hat{r}) + 1, \quad N(\hat{r}) = 1, \quad A(\hat{r}) = 1, \quad (23)$$

where $n = (D - 3)/2$. The constant C_1 and the outer radius \hat{r}_o are determined by the conditions $\hat{\phi}(\hat{r}_o) = 0$ and $\hat{\phi}'(\hat{r}_o) = 0$. The latter yields $J_{n+1}(\hat{r}_o) = 0$. Hence \hat{r}_o is the smallest nonvanishing zero of J_{n+1} . From the first condition it then follows that $C_1 = -\hat{r}_o^n / J_n(\hat{r}_o)$. Thus we find

$$\hat{\phi}(\hat{r}) = 1 - \left(\frac{\hat{r}_o}{\hat{r}}\right)^n \frac{J_n(\hat{r})}{J_n(\hat{r}_o)}, \quad \hat{r} \leq \hat{r}_o, \quad \hat{\phi}(\hat{r}) = 0, \quad \hat{r} > \hat{r}_o. \quad (24)$$

The properties of the unique solution in $D = 4$ dimensions was discussed in [17]. When going to higher dimensions, the properties of the respective solutions vary only slowly with D , and likewise when going to $D = 3$. This is seen in Fig. 1, when restricting to a vanishing cosmological constant, $\hat{\Lambda} = 0$.

B. Q -ball solutions in an (anti-)de Sitter background

Let us now consider the Q -ball solutions in an (anti-)de Sitter background in D dimensions. To obtain these solutions, we have solved the scalar field equation in the respective background numerically, employing a Newton-Raphson scheme.

When the scaled cosmological constant $\hat{\Lambda}$ is varied, the solutions change smoothly from the Minkowski background solutions. In Fig. 1 we exhibit the dependence of the

solutions on $\hat{\Lambda}$. Here we show the value of the scalar field at the origin $\hat{\phi}(0)$ together with the value of the outer radius \hat{r}_o of the solutions for $D = 3, 4, 5$ and 10 dimensions.

As the scaled cosmological constant $\hat{\Lambda}$ increases from zero, the value of the scalar field at the origin $\hat{\phi}(0)$ decreases along with the outer radius \hat{r}_o . Interestingly, there is a maximal value $\hat{\Lambda}_{\max}(D)$, which increases with the dimension D , for which these solutions exist. At $\hat{\Lambda}_{\max}(D)$ a second branch of solutions is encountered. Moving backwards along this second branch the value of the scalar field at the origin $\hat{\phi}(0)$ continues to decrease, until it reaches zero at a critical value $\hat{\Lambda}_{\text{cr}}(D)$.

At $\hat{\Lambda}_{\text{cr}}(D)$ the solutions change character and Q -shells arise. As $\hat{\Lambda}$ decreases further, the inner radius \hat{r}_i increases along with the outer radius \hat{r}_o . In the limit $\hat{\Lambda} \rightarrow 0$ the size of the shells diverges while the ratio \hat{r}_i/\hat{r}_o tends to one. Thus there are no Q -shells in a Minkowski or anti-de Sitter background.

Likewise, when the Q -ball solutions are continued to negative values of the cosmological constant, they change smoothly from the Minkowski background solutions, as seen in Fig. 1. As the scaled cosmological constant $\hat{\Lambda}$ decreases from zero, the value of the scalar field at the origin $\hat{\phi}(0)$ increases along with the outer radius \hat{r}_o until a minimal value $\hat{\Lambda}_{\min}(D)$ is encountered, beyond which no such solutions exist. Our data indicate that $\hat{\Lambda}_{\min}(D) = -(D - 2)/(2(D - 1))$. For a derivation of this limit in $D = 4$ dimensions see Appendix A.

IV. BACKREACTION

To study the backreaction of the Q -balls and Q -shells on the space-time, we have solved the coupled system of equations for the metric and the scalar field numerically. In the following we first discuss the case of $D = 4$ dimensions, and then turn to other dimensions.

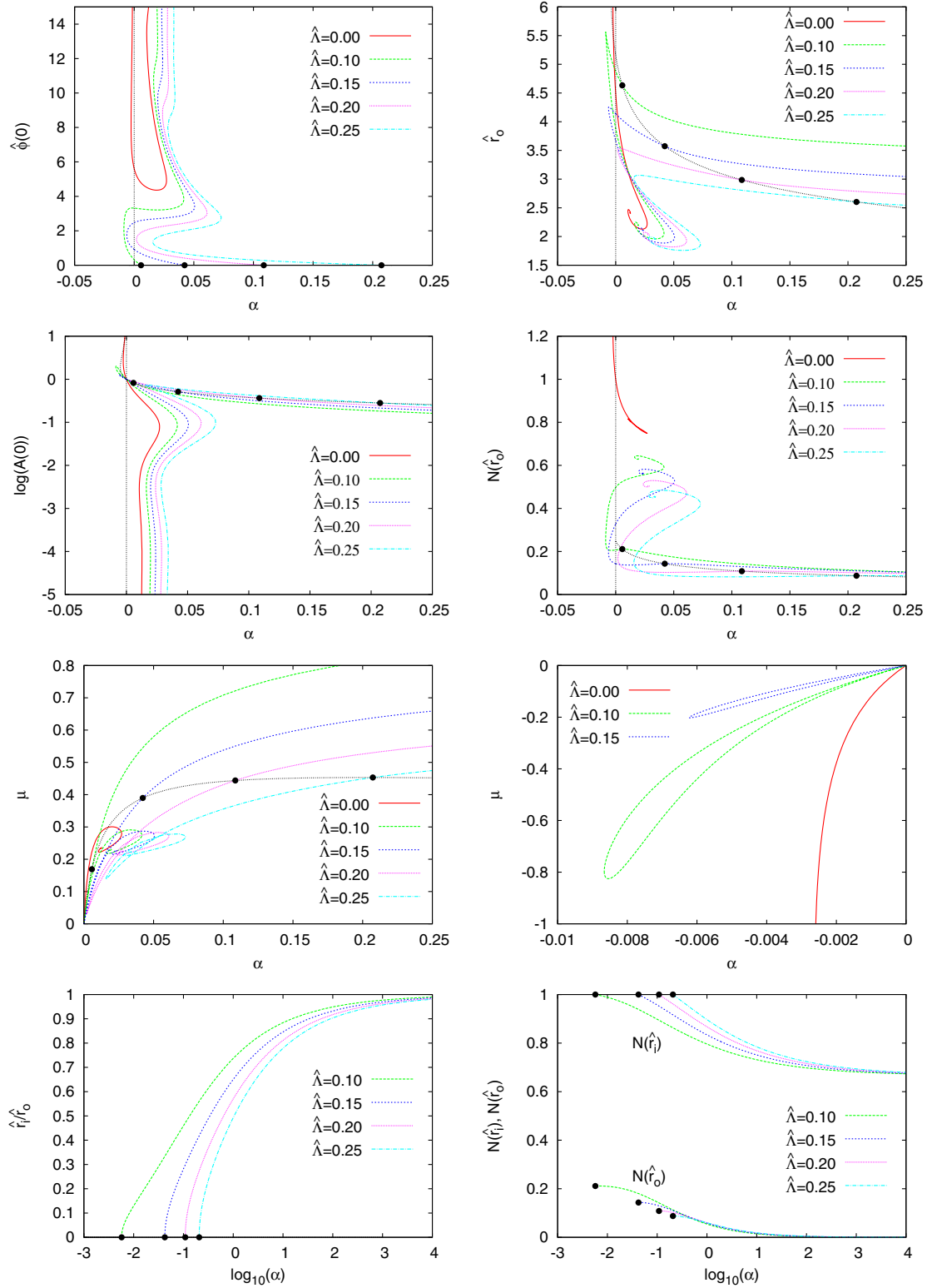


FIG. 2 (color online). Properties of the boson stars and boson shells in $D = 4$ are shown versus α for several values of $\hat{\Lambda} \geq 0$: (a) the scalar field $\hat{\phi}$ at the origin, $\hat{\phi}(0)$; (b) the outer radius \hat{r}_o ; (c) the metric function A at the origin, $A(0)$; (d) the metric function N at the outer radius, $N(\hat{r}_o)$; (e) the scaled mass μ for positive α ; (f) the scaled mass μ for negative α ; (g) the ratio of the inner and outer radii \hat{r}_i/\hat{r}_o for shells only; and (h) the values of the metric function at the inner and outer radii, $N(\hat{r}_i)$ and $N(\hat{r}_o)$, respectively. The black dots label the transition points between boson stars and boson shells. We label the subfigures from left to right and then top to bottom by the letters (a), (b), (c), (d), (e), (f), (g), and (h).

A. Boson stars and boson shells in $D = 4$

1. Asymptotically de Sitter boson stars and shells

Let us start by briefly recalling the properties of the single family of asymptotically flat compact boson star solutions found in [17]. At $\alpha = 0$ this family starts from the Q -ball solution in the Minkowski background. As seen in Fig. 2, with increasing α the value of the scalar field at the origin $\hat{\phi}(0)$ decreases, until it reaches a finite minimum, and then increases strongly, while α undergoes damped oscillations. In other physical quantities these damped oscillations with respect to α lead to a spiral-like pattern, as seen for the outer radius \hat{r}_o or the mass parameter μ . Such a behavior is typical for boson stars and neutron stars.

Here we have extended this family of compact boson star solutions to negative values of the coupling constant α , as seen in Fig. 2. The physical interpretation of the solutions with negative α is that they represent compact solutions made from phantom scalar fields. Thus the negative sign of α can be absorbed by the negative Lagrangian of the phantom field. Ever since the relevance of dark energy for cosmology became apparent, such phantom fields are found ubiquitously in the literature. Moreover, phantom fields also allow for the formation of various types of wormholes.

Let us now turn to compact de Sitter boson stars by increasing the value of $\hat{\Lambda}$ from zero. We demonstrate the effect of a positive cosmological constant on the compact solutions in Fig. 2 by exhibiting the physical properties of such families of solutions for several values of $\hat{\Lambda}$. We see that for finite $\hat{\Lambda}$ the minimum of $\hat{\phi}(0)$ reaches zero. This signals the occurrence of boson shells.

In particular, for a given value of $\hat{\Lambda}$, $\hat{\phi}(0)$ reaches zero at a critical value $\alpha_{\text{cr}}(\hat{\Lambda})$. Then for $\alpha > \alpha_{\text{cr}}(\hat{\Lambda})$ boson shells exist. $\alpha_{\text{cr}}(\hat{\Lambda})$ is exhibited in Fig. 3. For very small positive

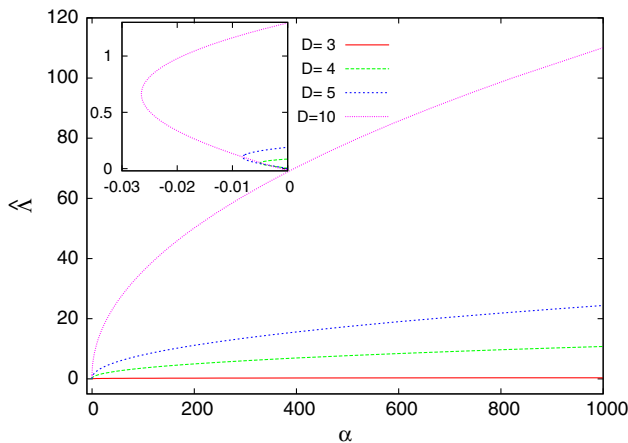


FIG. 3 (color online). The critical value α_{cr} , where the transition from boson stars to boson shells occurs, versus $\hat{\Lambda}$ for $D = 3, 4, 5$ and 10 dimensions.

$\hat{\Lambda}$, the critical value $\alpha_{\text{cr}}(\hat{\Lambda})$ is negative. Thus there exist phantom boson shells in this case, which turn into ordinary boson shells, as α increases beyond zero. Extrapolation to $\hat{\Lambda} = 0$ indicates that indeed no shells exist in this limit.

For all values of $\hat{\Lambda}$ the compact boson stars exhibit the characteristic spirals. In Fig. 2 these are seen for the value of the outer radius \hat{r}_o , the value of the metric function at the outer radius $N(\hat{r}_o)$ and the value of the scaled mass μ . Interestingly, phantom-type boson star solutions exist only for small values of $\hat{\Lambda}$. Since the mass μ has the same sign as α , branches with a negative mass exist only for small values of $\hat{\Lambda}$.

Whereas there is an upper bound α_{max} for the compact boson stars, there is no such bound for the boson shells. However, with increasing α their outer radius decreases and tends to a finite limiting value. At the same time, the ratio of the inner and outer radii \hat{r}_i/\hat{r}_o increases and tends to one. Thus the shells become smaller and thinner, while their scaled mass μ grows. On the other hand, as α is kept fixed while $\hat{\Lambda}$ is decreased, the shells grow in size, while the ratio \hat{r}_i/\hat{r}_o tends to one. Here in the limit $\alpha \rightarrow 0$ the shell size diverges.

2. Astrophysical considerations

In the above subsection we have constructed the domain of existence of compact boson stars and boson shells in terms of dimensionless quantities. We can obtain physical solutions with dimensional quantities by scaling these dimensionless solutions appropriately.

In the case of a vanishing cosmological constant we have considered compact stars [17]. In particular, we have shown that when the mass of these boson stars is on the order of the solar mass, then their radius is on the order of ten(s) of kilometers; thus they can correspond in mass and size to neutron stars. Moreover, spirals are also encountered for neutron stars, when they approach the black hole limit.

Concerning their stability, we employ arguments from catastrophe theory [45–49]. According to catastrophe theory, the stability changes only at turning points. Thus when starting from a stable configuration, the stability should change at the maximum of the mass. Therefore solutions inside the spiral should be unstable. For neutron stars or ordinary boson stars this has been confirmed by a mode analysis.

Since the value of the cosmological constant as obtained from cosmology is very small, $\Lambda \lesssim 10^{-52} \text{ m}^{-2}$ in metric units, its presence hardly affects the properties of boson stars that have masses on the order of the mass of the sun. Thus the results for boson stars correspond to those obtained before [17]. However, the presence of a positive cosmological constant, no matter how small, does allow for boson shells. But those boson shells possess cosmological mass and length scales. It would be interesting to see

whether such thin boson shells can be associated with voids, i.e. with vast regions of empty space surrounded by a shell of matter.

To see the effect of Λ on the compact boson stars let us now consider very big scales.

We fix $\Lambda = 3\Omega_\Lambda H_0^2/c^2$ at its physical value, with $\Omega_\Lambda = 0.7$ and $H_0 = 75 \text{ km}/(\text{Mpc s})$, as an example. This translates into the mass scale $M_0 = c^2/(G\sqrt{\Lambda}) \approx 5.7 \times 10^{22} M_\odot$. From the relation (11) we find $\omega = c\sqrt{\Lambda/\hat{\Lambda}} \approx 3.52 \times 10^{-18} \text{ s}^{-1}/\sqrt{\hat{\Lambda}}$ and the corresponding time scale $T = 2\pi/\omega \approx 56.6\sqrt{\hat{\Lambda}}$ gigayears.

In Fig. 4 we show our scaled results for the mass M in units of M_0 for boson stars and boson shells versus the outer radius \hat{r}_o in units of gigaparsec. Shown in addition are the black hole horizon and the cosmological horizon of the corresponding Schwarzschild–de Sitter space-times. Note that these two horizons coincide for the extremal configuration with the maximum value of the mass.

These Schwarzschild–de Sitter values form the boundary, within which all extended objects must remain. As seen in Fig. 4, the boson stars reside well within these bounds. Note that since we consider only positive values of the gravitational coupling, i.e., no phantom fields, the boson star curves associated with the smaller values of the parameter T consist of two disconnected parts.

The boson shells, in contrast, exist until they reach this cosmological bound. In particular, all boson shell curves extend precisely to the extremal value of the Schwarzschild–de Sitter curve, where the two horizons coincide. When approaching this limiting configuration, the inner radius of the shells approaches the outer radius. Thus the ratio \hat{r}_i/\hat{r}_o tends to the value one in this limit. More massive shells cannot exist.

Converting the values in Fig. 4 into numerical values we find that with $\Lambda = 10^{-52} \text{ m}^{-2}$ the mass of the boson stars is on the order of 10^{52} kg and their radius is on the order of Gigaparsec (Gpc). These are sizes that are beyond those of galaxies and galaxy clusters. If there were dark matter distributions on such large scales, our solutions would be able to model those. Choosing somewhat smaller values of T , on the other hand, we would find sizes relevant for galaxies or galaxy clusters, so that these solutions could be considered to model the dark matter halo of galaxies or the dark matter in galaxy clusters, respectively. Note that in the limit of vanishing mass the radius r_o becomes spurious since the nonscaled boson field vanishes identically.

3. Asymptotically anti–de Sitter boson stars

As expected, compact boson stars exist also for negative values of the cosmological constant. Indeed, the asymptotically Minkowski solutions can be smoothly extended to negative values of $\hat{\Lambda}$, thus yielding asymptotically AdS boson stars. In Fig. 5 we exhibit some of their physical

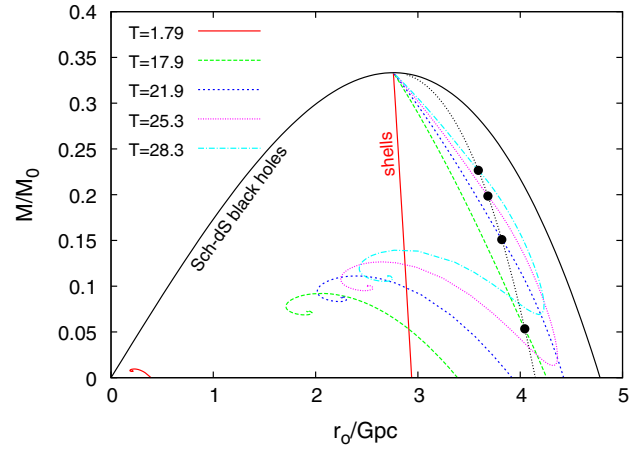


FIG. 4 (color online). The mass M in units of $M_0 = c^2/(G\sqrt{\Lambda})$ versus the outer radius \hat{r}_o in gigaparsec for several values of $T = 2\pi/\omega \propto \sqrt{\hat{\Lambda}}$. The black dots label the transition points between boson stars and boson shells. The black solid curve represents the black hole horizon and the cosmological horizon of the Schwarzschild–de Sitter space-times.

properties versus the coupling constant α , for several values of $\hat{\Lambda}$.

We note that the domain of existence of these compact AdS boson stars decreases with decreasing $\hat{\Lambda}$. This suggests that there is a limiting minimal value for $\hat{\Lambda}$ for compact boson stars. Analogous to the dS case, some of the physical properties of AdS boson stars exhibit damped oscillations with respect to α , whereas other properties exhibit spirals. Moreover, as in the dS case, there are phantom boson stars, associated with negative values of α .

However, we do not find AdS boson shells. In the AdS case, the scalar field $\hat{\phi}$ never reaches the value zero at the origin, which is necessary for boson shells to arise. We conclude, that the extra attraction associated with negative Λ inhibits the formation of AdS shells even stronger than in the asymptotically flat case, $\Lambda = 0$. The existence of shells needs repulsion that can be provided either by a positive cosmological constant, as seen in the previous subsection, or by the presence of electric charge [18,19,21,22].

B. Boson stars and boson shells in $D \neq 4$

Here we consider the domain of solutions and their properties for various space-time dimensions. We have made a complete study for dimensions $D = 3, 5$ and 10 . Since the dependence on D is mostly rather smooth, we exhibit only a number of selected cases. As an example of a solution, we show an asymptotically de Sitter boson shell solution in Fig. 6.

1. Asymptotically de Sitter boson stars and shells

We start our discussion by considering boson stars and boson shells in $D = 5$ dimensions. Some of their properties

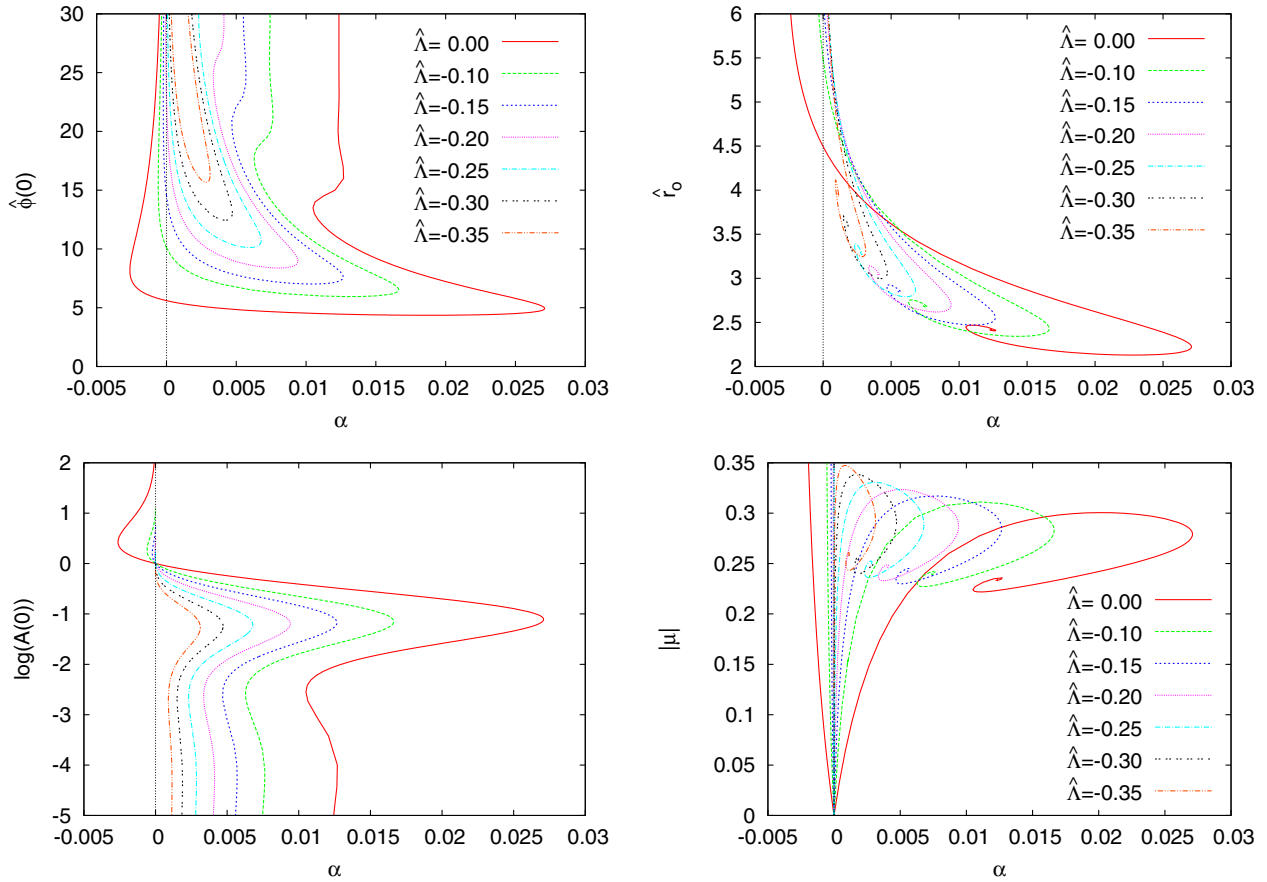


FIG. 5 (color online). Properties of the boson stars in $D = 4$ are shown versus α for several values of $\hat{\Lambda} \leq 0$: Top left: the scalar field $\hat{\phi}$ at the origin, $\hat{\phi}(0)$; Top right: the outer radius \hat{r}_o ; bottom left: the metric function A at the origin, $A(0)$; and bottom right: the absolute value of the scaled mass μ .

are exhibited in Fig. 7 and can be compared to those of Fig. 2.

A surprising feature is that in $D = 5$ boson shells seem to exist in the asymptotically flat case, $\hat{\Lambda} = 0$. This follows from the additional (almost vertical) line present in Fig. 7

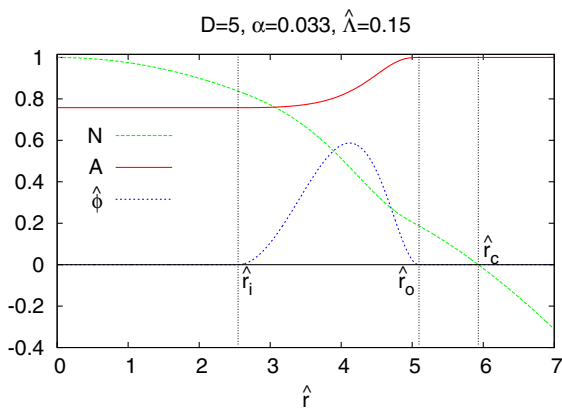


FIG. 6 (color online). The metric functions A , N and the scalar field function $\hat{\phi}$ are shown for the boson shell solution in $D = 5$ for parameters $\alpha = 0.33$ and $\hat{\Lambda} = 0.15$. r_c indicates the cosmological horizon.

for $\hat{\phi}(0)$. However, since this line is reaching zero for a very small negative value of α , those shells are phantom shells. The small branch of phantom shells is seen more clearly in the plot of $A(0)$ in the inset. As $\alpha \rightarrow 0$, the size of these phantom shells diverges.

For finite values of $\hat{\Lambda}$, however, we obtain also ordinary dS boson shells. The critical value $\alpha_{cr}(\hat{\Lambda})$ of the transition between the boson stars and shells is seen in Fig. 3. In particular, the resulting phantom dS boson shells continue to exist beyond $\alpha = 0$, where they smoothly turn into ordinary boson shells. As in $D = 4$, at fixed $\hat{\Lambda}$ with increasing α the outer radius \hat{r}_o of these boson shells decreases, tending to a finite limiting value, while their ratio \hat{r}_i/\hat{r}_o of inner and outer radii increases toward one.

The compact dS boson stars, on the other hand, exist only below a maximal value of α , which is given either by the onset of the spiral or by the transition to dS boson shells. For a given $\hat{\Lambda}$, the domain of existence of dS boson stars in $D = 5$ with respect to α is smaller than in $D = 4$. When going to higher dimensions, this trend continues. In contrast, the domain of existence of phantom dS boson stars increases with increasing D .

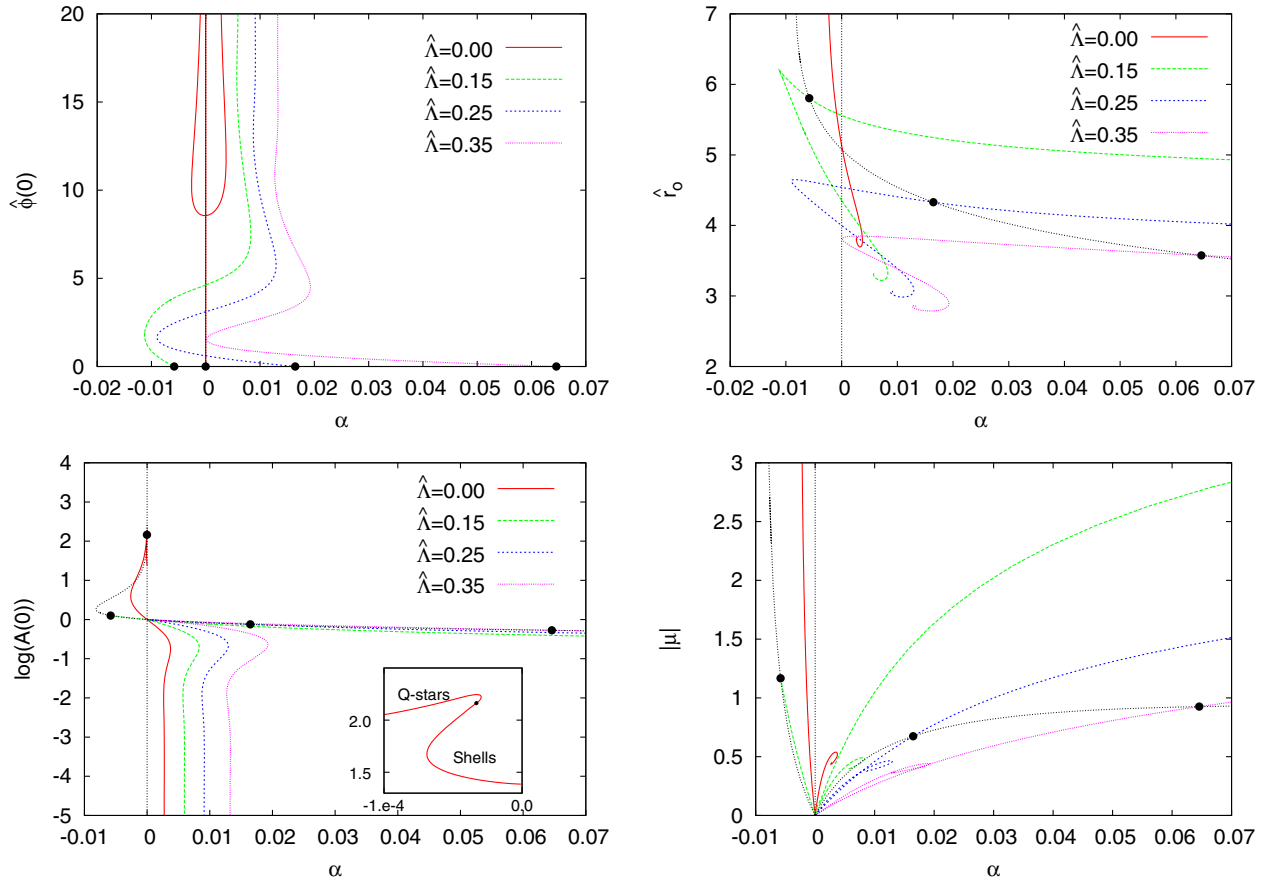


FIG. 7 (color online). Properties of the boson stars and boson shells in $D = 5$ are shown versus α for several values of $\hat{\Lambda} \geq 0$: Top left: the scalar field $\hat{\phi}$ at the origin, $\hat{\phi}(0)$; Top right: the outer radius \hat{r}_o ; bottom left: the metric function A at the origin, $A(0)$; and bottom right: the absolute value of the scaled mass μ . The black dots label the transition points between boson stars and boson shells.

In $D = 3$ dimensions gravity is nondynamic. Therefore we may expect that boson stars may exhibit a different behavior in $D = 3$ dimensions than in higher dimensions.

Let us first inspect the dependence of some of the properties of the compact three-dimensional objects on α for several values of $\hat{\Lambda}$. Figure 8 exhibits the scalar field $\hat{\phi}$ at the origin, $\hat{\phi}(0)$, the outer radius \hat{r}_o , and the scaled mass μ . Here we observe that indeed some properties of compact boson stars are very different in $D = 3$ dimensions. First of all, we notice a maximal value of $\hat{\phi}(0)$ that decreases with increasing $\hat{\Lambda}$. This maximal value is encountered for negative values of α , and thus these configurations correspond to phantom boson stars.

As this maximal value of $\hat{\phi}(0)$ two branches of solutions merge. Along one of these branches $\hat{\phi}(0)$ reaches zero. Thus boson shells emerge at the corresponding critical value α_{cr} . With increasing $\hat{\Lambda}$ the critical value α_{cr} increases (at least for positive α_{cr}), analogously to other dimensions.

The second branch of boson star solutions, however, exhibits a different behavior from the one observed before. Clearly, the damped oscillations of $\hat{\phi}(0)$ with α are not

present. Instead a monotonic decrease of $\hat{\phi}(0)$ with increasing α is observed, and no maximal value of α is encountered. We further observe that for small $\hat{\Lambda}$, $\hat{\phi}(0)$ and the outer radius \hat{r}_o depend only weakly on $\hat{\Lambda}$.

The absence of a maximal value of α suggests to consider the dependence of the compact boson stars on $\hat{\Lambda}$, choosing fixed large values of α . As can be seen from Figs. 8(e) and 8(f), for a given α boson star solutions exist only up to a maximal value of $\hat{\Lambda}$ that increases with α . Finally, we note that the solutions with $\hat{\Lambda} = 0$ are not asymptotically flat, since their mass parameter μ is finite. This holds for boson stars and boson shells alike.

2. Asymptotically anti-de Sitter boson stars

Let us turn finally to AdS boson stars in D dimensions. Some properties of asymptotically AdS boson stars in $D = 5$ are exhibited in Fig. 9. As in four dimensions, there are no ordinary AdS boson shells in other than four dimensions. However, there is a very small region of phantom AdS boson shells. This is seen by inspecting the critical curve $\alpha_{\text{cr}}(\hat{\Lambda})$ for negative values of α close to zero.

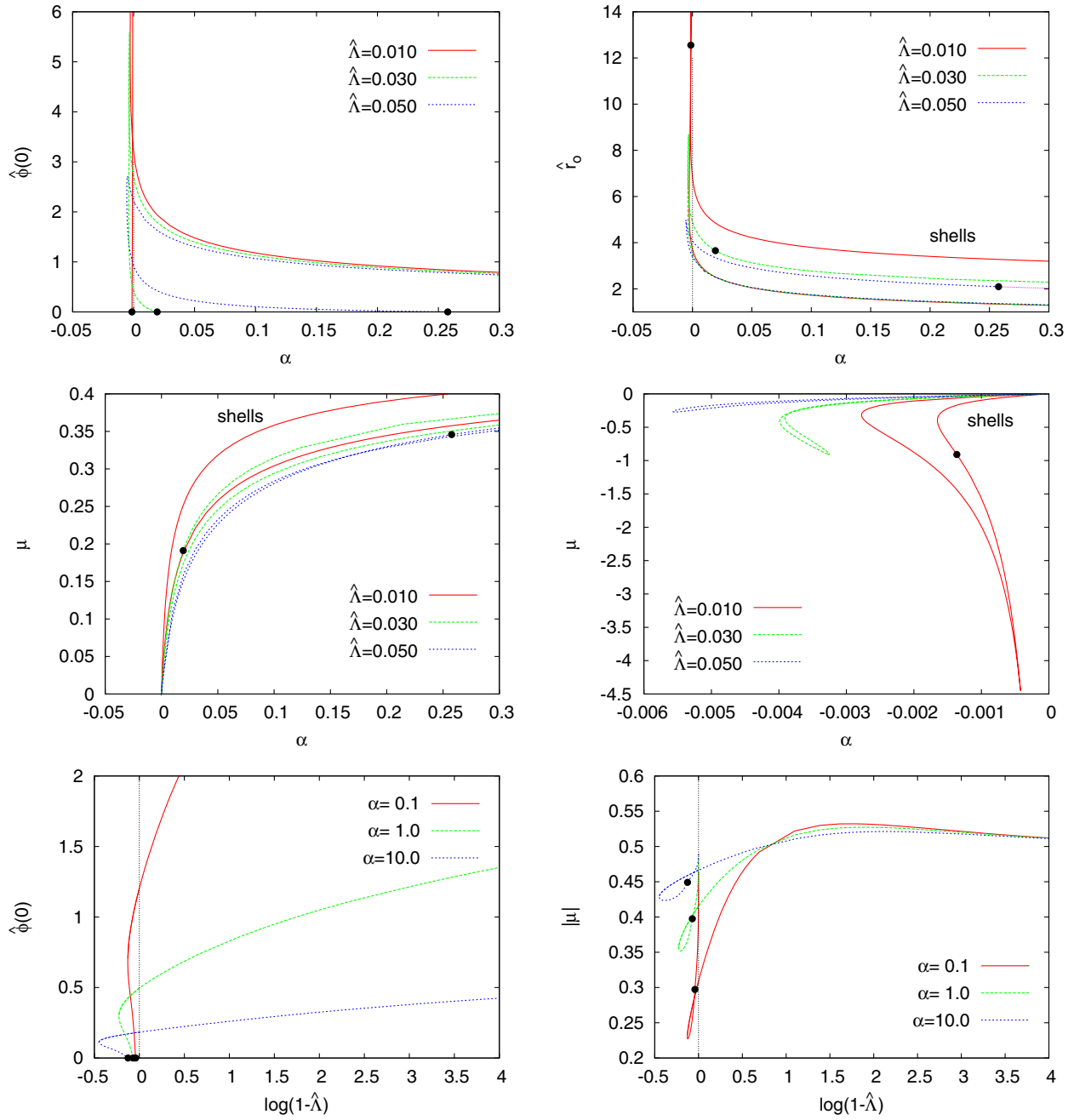


FIG. 8 (color online). Properties of the boson stars and boson shells in $D = 3$: (a) the scalar field $\hat{\phi}$ at the origin, $\hat{\phi}(0)$; (b) the outer radius \hat{r}_o ; (c) and (d) the scaled mass μ is shown versus α for several values of $\hat{\Lambda} > 0$; (e) the scalar field $\hat{\phi}$ at the origin, $\hat{\phi}(0)$; and (f) the absolute value of the scaled mass μ is shown versus $\log(1 - \hat{\Lambda})$ for several values of α . The black dots label the transition points between boson stars and boson shells. We label the subfigures from left to right and then top to bottom by the letters (a), (b), (c), (d), (e), and (f).

Here the critical curve passes negative values of $\hat{\Lambda}$ as well. Like the phantom shells for $\hat{\Lambda} = 0$ these phantom AdS shells diverge in size as $\alpha \rightarrow 0$.

Otherwise all the basic properties of these AdS solutions in $D = 5$ are similar to those of the $D = 4$ AdS solutions discussed above. Moreover, we observe only gradual changes with increasing D . In particular, the domain of existence of AdS boson stars with respect to α decreases

with D . Only the domain of phantom AdS shells increases slightly, as seen in Fig. 3.

The lower dimensional case $D = 3$ is special again, as can be seen in Figs. 8(e) and 8(f), where the dependence of $\hat{\phi}(0)$ and μ on $\hat{\Lambda}$ is shown for several values of α . Interestingly, for a given α there is no lower bound of $\hat{\Lambda}$ encountered. Moreover, the mass becomes practically independent of α as $\hat{\Lambda}$ becomes sufficiently small.

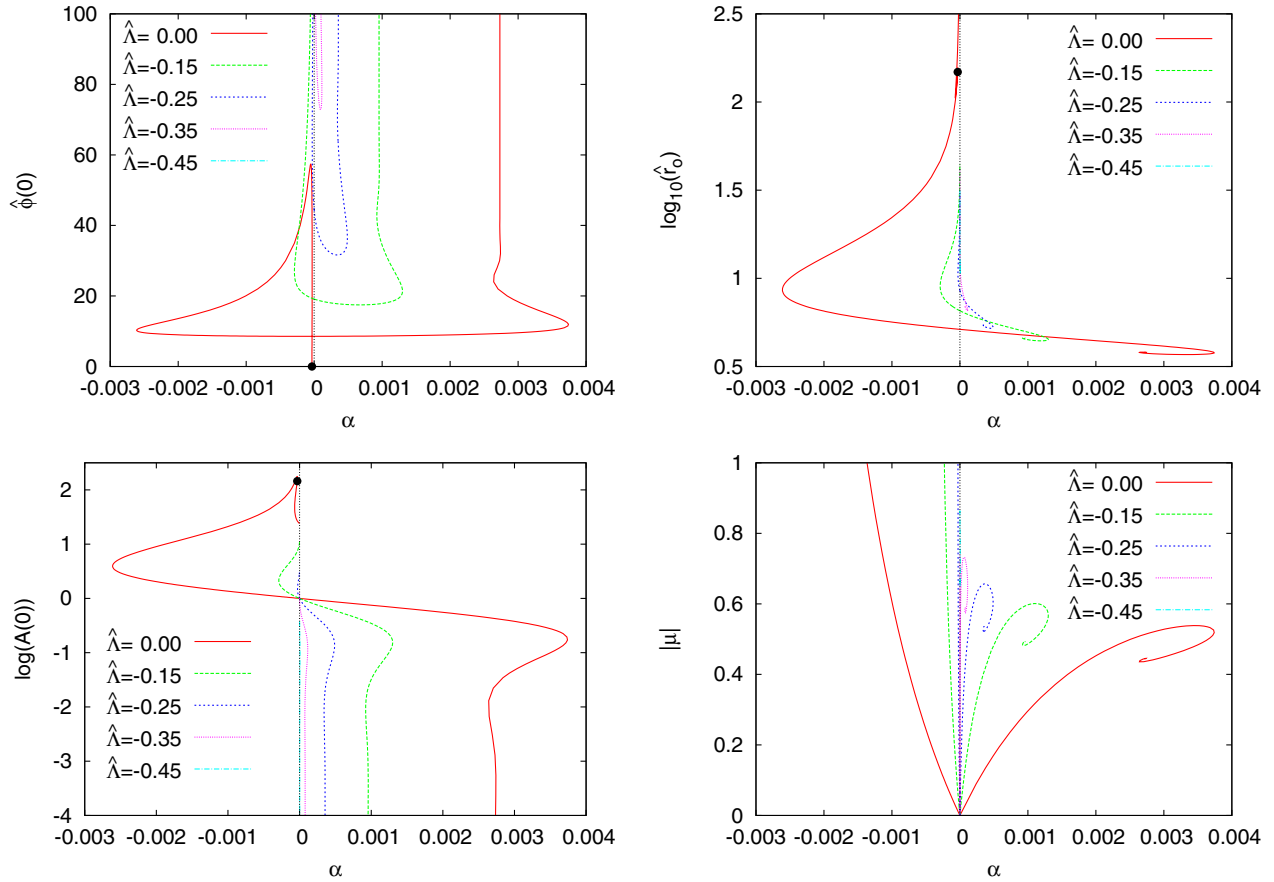


FIG. 9 (color online). Properties of the boson stars and boson shells in $D = 5$ are shown versus α for several values of $\hat{\Lambda} \leq 0$: Top left: the scalar field $\hat{\phi}$ at the origin, $\hat{\phi}(0)$; Top right: the outer radius \hat{r}_o ; bottom left: the metric function A at the origin, $A(0)$; and bottom right: the absolute value of the scaled mass μ .

When $\hat{\Lambda}$ is fixed instead, while α is varied, no spirals are encountered for the compact three-dimensional boson stars, whereas spirals are present in all higher dimensions. This was observed before for ordinary boson stars [28]. Restricting to positive α , there is a maximal value of $\hat{\phi}(0)$ for a given $\hat{\Lambda}$. However, when allowing for phantom fields, a minimal value of α is encountered, beyond which $\hat{\phi}(0)$ increases without bound, while the mass decreases.

V. CONCLUSIONS AND OUTLOOK

We have studied compact boson stars and shells obtained with a V-shaped interaction potential in $D \geq 3$ dimensions. The V-shaped potential confines the scalar field to a finite region, which can be ball-like or shell-like.

In the probe limit, we have given the general analytical solution for Q -balls in a Minkowski background. Here no Q -shells exist. Q -shells arise only beyond a critical value $\Lambda_{\text{cr}}(D) > 0$ of the cosmological constant, which increases with the number of dimensions D . Likewise, Q -balls exist only above a minimal value of the cosmological

constant, which seems to correspond to $\hat{\Lambda}_{\text{min}}(D) = -(D-2)/(2(D-1))$.

Subsequently, we have taken the backreaction into account. The resulting configurations correspond to compact boson stars and boson shells. By solving the coupled set of Einstein-scalar field equations, we have obtained the full set of solutions, subject to Minkowski, de Sitter and anti-de Sitter asymptotics for a number of space-time dimensions, ranging from 3 to 10.

For any dimension $D \geq 3$ there are compact boson stars with all three types of asymptotics. But concerning their properties, we see a distinct behavior in three dimensions that is different from the common behavior encountered in all higher dimensions. In four and higher dimensions, these boson stars exist in a finite interval $\alpha_{\text{min}}(D, \hat{\Lambda}) \leq \alpha \leq \alpha_{\text{max}}(D, \hat{\Lambda})$. In contrast, in three dimensions there is no upper bound on the value of α .

Also, all boson stars in four and higher dimensions exhibit a spiral-like dependence of the outer radius and the mass on the coupling constant α . At the same time, the scalar field value $\hat{\phi}(0)$ exhibits damped oscillations. In contrast, in three dimensions the respective boson star

properties do not exhibit such a spiral-like dependence or damped oscillations.

By exploring the parameter space, we also find boson stars for negative values of α . These boson stars with negative α correspond to phantom boson stars, since the negative sign of α can be reinterpreted as a negative sign associated with the scalar field in the Lagrangian, and thus with a phantom field. Ample motivation for the consideration of phantom fields is nowadays provided by cosmology.

In contrast to boson stars, boson shells do not exist for Minkowski asymptotics, if there is no additional force present, balancing the gravitational attraction. Consequently, there are no ordinary AdS boson shells. However, dS boson shells do exist. Here the positive cosmological constant provides the necessary repulsion. On the other hand, there exist small regions of phantom AdS shells in more than four dimensions.

In four dimensions we have also considered astrophysical aspects of the compact boson stars and boson shells. While we can always adjust the parameters of the solutions to describe compact astrophysical objects with masses and sizes of neutron stars as discussed in [17], the influence of the cosmological constant on these objects is negligible, when the physical value of Λ is taken. The new feature is, however, that in addition to boson stars there exist also boson shells for positive values of Λ .

We have then addressed the question, what the properties of such compact objects would be, if we set the scale by the physical value of Λ . Interestingly, in this case the resulting sets of boson stars reach huge masses and sizes that are more akin to structures on the largest scales of the Universe. The boson shells, on the other hand, can grow in mass until they reach the limit, set by the extremal Schwarzschild–de Sitter solution, for which the event horizon and the cosmological horizon merge.

For negative Λ one might be tempted to consider the AdS/CFT correspondence and try to interpret the solutions within this framework. However, all of our solutions are compact. The outer solutions are all given in terms of the Schwarzschild–AdS solutions. Thus the AdS boundary does not feel anything of the solutions except for their mass. Consequently, for such compact solutions the concept of holography does not work. Indeed, lots of different compact objects may sit in the bulk, and if they have the same mass, the boundary does not notice a difference.

As our next step we plan to include rotation [50]. Rotating boson stars are known for noncompact configurations in four dimensions [12,13,51–55]. Interestingly, their angular momentum J is quantized in terms of their particle number Q , $J = nQ$, where n is an integer. Rotating boson stars have also been constructed in odd dimensions (with equal magnitude angular momenta for $D \geq 5$) [56–59]. We expect that the rotation of compact boson stars will lead to interesting new features. Moreover, there may be rotating boson shells in the presence of a cosmological constant.

It should also be interesting to construct interacting compact Q -balls and Q -shells for a finite cosmological constant. These should arise in the presence of several complex scalar fields [55,60,61].

ACKNOWLEDGMENTS

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APPENDIX A: ADS SOLUTIONS IN THE PROBE LIMIT

Here we consider the AdS solutions in the probe limit. We introduce the scaled coordinate $\eta = \hat{r}/\ell$, where $\ell = \sqrt{-\frac{(D-2)(D-1)}{2\Lambda}}$, and the scaled scalar field $\psi = \hat{\phi}/\hat{\phi}(0)$. This yields for the (ODE) of the function ψ

$$(\eta^m N \psi')' + \frac{\ell^2 \eta^m}{N} \psi = \frac{\ell^2 \eta^m}{\hat{\phi}(0)}, \quad (\text{A1})$$

where $m = (D - 2)$, $N = 1 + \eta^2$, and prime denotes the derivative with respect to η .

We rewrite the ODE, Eq. (A1), as

$$\psi'' + p \psi' + q \psi = F, \quad \text{with } p = (\log(\eta^m N))',$$

$$q = \frac{\ell^2}{N^2}, \quad F = \frac{\ell^2}{N \hat{\phi}(0)}. \quad (\text{A2})$$

If y_1 is a solution of the homogeneous ODE, then a second independent solution of the homogeneous ODE can be found,

$$y_2(\eta) = c_0 y_1(\eta) \int_0^\eta \frac{d\tau}{\tau^m N(y_1(\tau))^2}, \quad (\text{A3})$$

where c_0 is a constant.

Now the general solution of the inhomogeneous ODE can be written as

$$\psi = c_1 y_1 + c_2 y_2 + y_{\text{inh}}, \quad (\text{A4})$$

where c_1 and c_2 are constants and

$$y_{\text{inh}}(\eta) = \frac{\ell^2}{c_0 \hat{\phi}(0)}$$

$$\times \left[y_2(\eta) \int_0^\eta \tau^m y_1(\tau) d\tau - y_1(\eta) \int_0^\eta \tau^m y_2(\tau) d\tau \right] \quad (\text{A5})$$

is a special solution of the inhomogeneous ODE.

Let us assume that y_1 is regular at $\eta = 0$ and $y_1(0) = 1$. As a consequence the integral in Eq. (A3) diverges and y_2 is singular at $\eta = 0$. However, the special solution of the inhomogeneous ODE is regular and vanishes at $\eta = 0$. Thus to obtain the regular solution with $\psi(0) = 1$ we set $c_1 = 1$ and $c_2 = 0$.

Next we consider the conditions for compact solutions, i.e. $\psi(\eta_o) = 0$ and $\psi'(\eta_o) = 0$ at some η_o .

$$\begin{aligned} \psi(\eta_o) &= y_1(\eta_o) + \frac{\ell^2}{\hat{\phi}(0)} \left[y_2(\eta_o) \int_0^{\eta_o} \tau^m y_1(\tau) d\tau \right. \\ &\quad \left. - y_1(\eta_o) \int_0^{\eta_o} \tau^m y_2(\tau) d\tau \right] \\ &= 0, \end{aligned} \quad (\text{A6})$$

$$\begin{aligned} \psi'(\eta_o) &= y_1'(\eta_o) + \frac{\ell^2}{\hat{\phi}(0)} \left[y_2'(\eta_o) \int_0^{\eta_o} \tau^m y_1(\tau) d\tau \right. \\ &\quad \left. - y_1'(\eta_o) \int_0^{\eta_o} \tau^m y_2(\tau) d\tau \right] \\ &= 0. \end{aligned} \quad (\text{A7})$$

The linear superpositions $y_1'(\eta_o)\psi(\eta_o) - y_1(\eta_o)\psi'(\eta_o) = 0$ and $y_2'(\eta_o)\psi(\eta_o) - y_2(\eta_o)\psi'(\eta_o) = 0$ yield

$$0 = [(y_1'(\eta_o)y_2(\eta_o) - y_2'(\eta_o)y_1(\eta_o))] \int_0^{\eta_o} \tau^m y_1(\tau) d\tau, \quad (\text{A8})$$

$$\begin{aligned} 0 &= [(y_1'(\eta_o)y_2(\eta_o) - y_2'(\eta_o)y_1(\eta_o))] \\ &\quad \times \left(1 - \frac{\ell^2}{\hat{\phi}(0)} \int_0^{\eta_o} \tau^m y_2(\tau) d\tau \right), \end{aligned} \quad (\text{A9})$$

respectively, where we set $c_0 = 1$.

Since y_1 and y_2 are solutions of the homogeneous ODE, it follows that $y_1'y_2 - y_2'y_1 = c_h/(\eta^m N)$ for some constant c_h . As a consequence $[(y_1'(\eta_o)y_2(\eta_o) - y_2'(\eta_o)y_1(\eta_o))] \neq 0$ and the conditions (A8) and (A9) reduce to

$$0 = \int_0^{\eta_o} \tau^m y_1(\tau) d\tau, \quad (\text{A10})$$

$$\hat{\phi}(0) = \ell^2 \int_0^{\eta_o} \tau^m y_2(\tau) d\tau. \quad (\text{A11})$$

The first equation determines the point η_o and the second the value of $\hat{\phi}(0)$.

Now we are left with the problem to determine the range of ℓ for which solutions of Eq. (A10) exist. Although the solutions of the homogeneous ODE can be expressed in terms of hypergeometric functions, we did not succeed to determine the minimal values of ℓ in the general case, i.e. in all dimensions.

However, in $D = 4$ dimensions the solution y_1 can be expressed in terms of trigonometric functions,

$$y_1 = \frac{1}{\ell \eta} [\eta \cos(\ell \arctan \eta) - \ell \sin(\ell \arctan \eta)], \quad (\text{A12})$$

which simplifies the problem considerably. Here we consider only $\ell \neq 1$. The case $\ell = 1$ needs special treatment.

We will show that Eq. (A10) has only a solution η_o if $\ell > 3$. Clearly, the integral in Eq. (A10) can vanish only if

the integrand possesses (at least) one zero. To analyze this condition we introduce a new coordinate $z = \arctan \eta$, with $0 \leq z < \pi/2$, and consider the function

$$\begin{aligned} &\eta \cos(\ell \arctan \eta) - \ell \sin(\ell \arctan \eta) \\ &= \tan z \cos(\ell z) - \ell \sin(\ell z) \\ &= (\sin z \cos(\ell z) - \ell \cos z \sin(\ell z)) / \cos z \\ &= F(z) / \cos z. \end{aligned} \quad (\text{A13})$$

Since $\cos z$ does not change sign on the interval $0 \leq z < \pi/2$, it is sufficient to consider the function

$$F = \sin z \cos(\ell z) - \ell \cos z \sin(\ell z). \quad (\text{A14})$$

Now we are left with the question for which values of ℓ the function F does not possess a zero.

Let us first restrict to $\ell > 1$. Expanding the function F for small values of z we find $F = z(1 - \ell^2) + \mathcal{O}(z^3) < 0$. On the other hand, evaluating F at $z = \pi/2$ yields $F(\pi/2) = \cos(\ell \pi/2) > 0$ for $3 < \ell < 5$. Thus, $F(z)$ possesses at least one zero for $3 < \ell < 5$. Consequently, we can restrict to $1 < \ell < 3$.

We rewrite Eq. (A14) as

$$\begin{aligned} F &= -\frac{1}{2} [(\ell - 1) \sin(z(\ell + 1)) + (\ell + 1) \sin(z(\ell - 1))] \\ &= -\frac{\ell + 1}{2} [b \cos a \sin a + \sin(ba)], \end{aligned} \quad (\text{A15})$$

where $b = 2\frac{\ell-1}{\ell+1}$, $a = z(\ell + 1)/2$ are restricted to $0 < b < 1$ and $0 < a < \frac{\pi}{2-b} < \pi$. It can be seen from Fig. 10 that $F(a, b)$ indeed does not possess a zero in the region $(0, \pi) \times (0, 1)$. Consequently there are no compact solutions with $1 < \ell < 3$.

Now we turn to the case $0 < \ell < 1$. With $\ell' = 1/\ell > 1$ and $z' = z/\ell'$ the function F reads

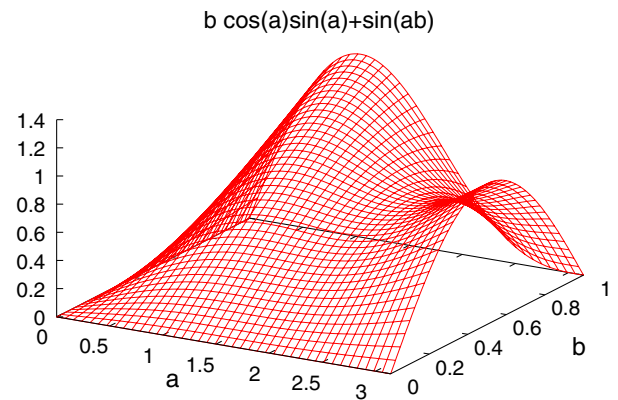


FIG. 10 (color online). The function $F(a, b)$ as given in (A15).

$$\begin{aligned}
F &= \sin(\ell' z') \cos(z') - \frac{1}{\ell'} \cos(\ell' z') \sin(z') \\
&= -\frac{1}{\ell'} [\sin(z') \cos(\ell' z') - \ell' \cos(z') \sin(\ell' z')] \\
&= \frac{1}{2\ell'} [(\ell' - 1) \sin(z'(\ell' + 1)) + (\ell' + 1) \sin(z'(\ell' - 1))].
\end{aligned} \tag{A16}$$

Note that now $0 < z' < \pi/2\ell' < \pi/2$. Therefore, $z'(\ell' + 1) < \frac{\pi}{2}(1 + 1/\ell') < \pi$ and $z'(\ell' - 1) < \frac{\pi}{2}(1 - 1/\ell') < \pi$ imply $\sin(z'(\ell' + 1)) > 0$ and $\sin(z'(\ell' - 1)) > 0$. This shows that the function F does not have a zero for $0 < \ell' < 1$.

To conclude, we have found a lower bound for ℓ , i.e. $\ell_{\min} = 3$, below which no compact solution can exist. However, this does not prove that solutions exist for $\ell \geq \ell_{\min}$, since the condition Eq. (A10) is more restrictive than $F = 0$ for some η . Indeed, for $\ell = 3$ Eq. (A10) yields an equation of the form $x - \tanh(x) = 0$, which has no solution for $x > 0$. However, consider the case when the function $F(z)$ possesses only one zero on the interval $(0, \pi/2)$ for some $\ell > 3$. Denote the zero by z_1 . The condition (A10) can then be written as

$$\begin{aligned}
0 &= \int_0^{z_o} F(z) \frac{\sin z}{\cos^4 z} dz \\
&= \int_0^{z_1} F(z) \frac{\sin z}{\cos^4 z} dz + \int_{z_1}^{z_o} F(z) \frac{\sin z}{\cos^4 z} dz.
\end{aligned} \tag{A17}$$

The first integral yields a finite negative contribution, since $F(z)$ is bounded and negative on $(0, z_1)$. The second integral, on the other hand, assumes any positive value between zero and infinity, as z_o ranges between z_1 and $\pi/2$, since $F(z)$ is bounded and positive on $(z_1, \pi/2)$. Consequently, there exists a z_o on $(z_1, \pi/2)$ for which both integrals in Eq. (A17) cancel, implying that compact solutions exist for some $\ell > \ell_{\min}$.

As an example we computed the exact solution for $\ell = 4$. We found

$$\begin{aligned}
\psi(\eta) &= \frac{1}{15(1 + \eta^2)^2} \left\{ 15 - 10\eta^2 - \eta^4 \right. \\
&\quad - \frac{1}{4\eta\phi_0} (4[(7\eta^4 + 100\eta^2 - 30)\eta \\
&\quad - 30(5\eta^2 - 1) \arctan \eta] - 15(\eta^4 \\
&\quad \left. + 10\eta^2 - 15)\eta \log(1 + \eta^2) \right\}.
\end{aligned} \tag{A18}$$

For this solution the values $\eta_o = 7.0657485298$ and $\phi_o = 20.286677195$ have been computed numerically. Remarkably, they coincide with the corresponding values of our numerically computed solution of the ODE up to eight digits.

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