Two-point function for the Maxwell field in flat Robertson-Walker spacetimes

E. Huguet[1,*] and J. Renaud[2,†]

[1]*Université Paris Diderot-Paris 7, APC-Astroparticule et Cosmologie (UMR-CNRS 7164), Batiment Condorcet, 10 rue Alice Domon et Léonie Duquet, F-75205 Paris Cedex 13, France*

[2]*Université Paris-Est, APC-Astroparticule et Cosmologie (UMR-CNRS 7164), Batiment Condorcet, 10 rue Alice Domon et Léonie Duquet, F-75205 Paris Cedex 13, France*

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We obtain an explicit two-point function for the Maxwell field in flat Robertson-Walker spacetimes, thanks to a new gauge condition which takes the scale factor into account and assumes a simple form. The two-point function is found to have the short distance Hadamard behavior.

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I. INTRODUCTION

Despite numerous works on quantum field in curved spacetimes [1,2] and the importance of the flat Robertson-Walker (RW) spacetimes in current cosmology, it seems that, until recently, the search for a two-point function for the electromagnetic field in such spacetimes has been overlooked. In this paper, we make a proposal to fill this gap. Precisely, we derive a two-point function [whose explicit expression is given by Eq. (13)] for the electromagnetic field in flat RW spacetimes. To this end we use the Gupta-Bleuler (GB) quantization procedure and explain why it applies in this context. The quantization is performed in a covariant gauge [whose explicit expression is given by Eq. (9)] which reduces to the Lorenz gauge condition in a Minkowski space.

Two-point functions are of central importance in quantum field theory. In curved spacetimes, explicit expressions are known in a number of cases. For maximally symmetric spaces, general expressions have been given for both the scalar and the vector field [3]. The propagator of the graviton is the subject of continuous works especially in de Sitter and anti–de Sitter spaces (see for instance [4–6] for recent works). Nevertheless, an explicit expression for the two-point function of the (quantum) Maxwell field (the ‘‘photon’s propagator’’) in RW spaces is seemingly missing. A recent proposal in conformally flat spacetimes has been made [7] in which the electromagnetic field appears as a part of a six-dimensional $SO_0(2,4)$-invariant field in a conformally invariant gauge. A quantization using the Dirac's procedure for constrained systems has also been recently proposed in flat RW spacetimes [8]. But no explicit four-dimensional two-point function appears in these two works. In the second one, the choice of Dirac's method is mainly motivated by the alleged inapplicability of the GB condition in cosmological spacetimes. This is related to the fact that this condition makes explicit use of the annihilation operators, while it is well known that there is no preferred vacuum state in a general curved space. This has been the starting point of a discussion about the possible contribution of scalar photons to the dark energy [9]. However, following Parker [10] the situation of a flat RW space appears as an exception: since the Maxwell equations are conformally invariant, the choice of a preferred vacuum state, the so-called conformal vacuum, is possible in such conformally flat spacetimes. We will see that the GB condition makes sense in this context.

The quantization method introduced by Gupta and Bleuler was designed for the electromagnetic field in the Lorenz gauge in the Minkowski space. In this original application, the classical Maxwell equations are replaced by other equations ($\partial^2 A_\mu = 0$) which provide the modes used in this explicit canonical quantization. Unfortunately, this process, using the Lorenz gauge in more general spacetimes, yields equations which are often intractable in practice. This is precisely the situation in flat RW spacetimes. Fortunately, the GB process can be adapted to gauges other than the Lorenz one. The difficulty can then be circumvented thanks to the conformal relation between the flat RW space and its global Minkowskian chart, that is, the chart in which the metric is conformal to the Minkowskian metric $\eta = \mathrm{diag}(+,-,-,-)$. The point is that the conformal map allows us to choose a new gauge condition, in place of the Lorenz gauge on the flat RW manifold, which is conformally mapped to the Lorenz gauge in the Minkowskian chart. Then, thanks to the conformal invariance of the Maxwell equations, the problem in the Minkowskian chart is just that of the historical GB method.

II. A VIEW OF THE GB QUANTIZATION METHOD

In this paragraph we give a concise and practical view of the usual GB quantization method focusing on the electromagnetic field. This quantization, historically associated with the Lorenz gauge, is actually more general and can be, in particular, applied in other gauges. Essentially, the GB method can be viewed as an algorithm with the following steps (commented hereafter):

*huguet@apc.univ-paris7.fr
†jacques.renaud@univ-mlv.fr

(1) Define a scalar product on the space of the solutions of the field equations. This product is degenerate if gauge freedom is present.
(2) Extend the space of solutions (by considering new field equations) in order to eliminate the degeneracy of the scalar product.
(3) Apply canonical quantization to the field which is a solution of the extended equations.
(4) Select the subspace of physical states, which correspond to the initial (not extended) field equations using the so-called GB condition.

Note that only the second and the fourth points are specific to the GB scheme. The others belong to the standard canonical quantization, which can be presented in a number of equivalent ways.

In the first step, we consider a general bosonic field A on a globally hyperbolic curved spacetime. The field is assumed to satisfy the Euler-Lagrange equations coming from some quadratic Lagrangian L. Then, a natural Hermitian sesquilinear form on the space of solutions of the Euler-Lagrange equations results from the Lagrangian (see for instance Appendix B of [11] for a proof). For two solutions A and B, it reads

$$\langle A, B\rangle = -i\int_\Sigma \sigma_\mu \mathcal{J}^\mu(A, B^*), \tag{1}$$

where Σ is a Cauchy surface, and $\mathcal{J}^\mu$ is the divergenceless current corresponding to L. For the free Maxwell field, it reads

$$\mathcal{J}^\mu(A, B) := A_\nu \frac{\partial L}{\partial \nabla_\mu B_\nu} - \frac{\partial L}{\partial \nabla_\mu A_\nu} B_\nu. \tag{2}$$

Note that the general expression (1) is for the free scalar field nothing but the Klein-Gordon scalar product. In the case of the Maxwell field, the gauge invariance makes the above scalar product degenerate. This is due to the gauge solutions which are orthogonal to all solutions, including themselves. With no additional consideration, this property prohibits the canonical quantization.

The second step is the first part specific to the GB procedure. It amounts to finding equations $\mathcal{E}A = 0$ which are the Euler-Lagrange equations derived from a quadratic Lagrangian L' and which satisfy the following two conditions. First, the space of solutions of $\mathcal{E}A = 0$ can be equipped with a nondegenerate Hermitian product. Second, these equations, together with some constraint $\mathcal{C}A = 0$, are equivalent to the Maxwell equations $\mathcal{M}A = 0$, together with a (most frequently covariant) gauge condition $\mathcal{G}A = 0$. Thus, the spaces of solutions of $\mathcal{E}A = 0$ and $\mathcal{M}A = 0$ coincide on the subsets defined by their respective constraint:

$$(\mathcal{E}A = 0 \quad \text{and} \quad \mathcal{C}A = 0) \Leftrightarrow (\mathcal{M}A = 0 \quad \text{and} \quad \mathcal{G}A = 0). \tag{3}$$

In the historical case, for instance, one has $\mathcal{E}A_\mu = \partial^2 A_\mu$, $\mathcal{C}A = \mathcal{G}A = \partial \cdot A$, and $\mathcal{M}A_\mu = \partial^2 A_\mu - \partial_\mu(\partial \cdot A)$.

Then, the third step consists in quantizing the field defined through the equation $\mathcal{E}A = 0$. This part follows the canonical quantization scheme and is not specific to the GB method. In curved spacetimes it thus shares all the well-known difficulties of the quantization. In practice, it is performed (when possible) through the following steps (see for instance [2]): first find a basis of modes solution of the equation $\mathcal{E}A = 0$ with respect to the (nondegenerate) scalar product, and then determine (choose) the so-called positive frequency modes. From them, one obtains the annihilation (and creation) operators, the vacuum, the quantum field, the two-point functions, etc.

Finally, the last step of the GB scheme is to determine the physical states. These are the states which correspond to the classical positive frequency solutions satisfying the constraint $\mathcal{C}A = 0$. They can be determined thanks to the so-called GB condition. It reads in general

$$\mathcal{C}\hat{A}^{(+)}|\Psi_A\rangle = 0, \tag{4}$$

(see for instance Appendix B of [12] for this straightforward generalization of the historical condition $\nabla\hat{A}^{(+)}|\Psi_A\rangle = 0$). In this expression, the operator $\hat{A}^{(+)}$ is the annihilator part of the quantum field $\hat{A}$ and $|\Psi_A\rangle$ the state corresponding to the classical solution A [that is, $\langle 0|\hat{A}(x)|\Psi_A\rangle = A(x)$]. Finally, the quantum field $\hat{A}$ fulfills (3) in the mean on physical states:

$$\langle\Psi_{A_1}|\mathcal{M}\hat{A}|\Psi_{A_2}\rangle = 0 \quad \text{and} \quad \langle\Psi_{A_1}|\mathcal{G}\hat{A}|\Psi_{A_2}\rangle = 0, \tag{5}$$

as soon as $|\Psi_{A_1}\rangle$, $|\Psi_{A_2}\rangle$ are physical states.

It is worth noting that the well-known ambiguity in the definition of a vacuum in curved spacetimes affects the GB method both in the quantization of the field and in the determination of the physical states. Nevertheless, we will argue in the following that in the particular case of flat RW spacetime, for which the choice of the so-called conformal vacuum is possible, the GB procedure is still applicable.

III. LESSON FROM LORENZ GAUGE QUANTIZATION

First, let us attempt to apply the procedure just described to quantize the Maxwell field in the Lorenz gauge $\nabla \cdot A = 0$ in a conformally flat RW space. We first enlarge the space of solutions of the Maxwell equations to that of the equations

$$\Box A_\mu + R^\nu{}_\mu A_\nu = 0. \tag{6}$$

These equations can be obtained as usual by adding the gauge term $\frac{1}{2}(\nabla \cdot A)^2$ to the Lagrangian of electromagnetism $\frac{1}{4}F^2$, where F is the Faraday field strength tensor. In the Lorenz gauge they coincide with the Maxwell equations. We then have to determine a basis of modes solution of (6). One may attempt to take advantage of the conformal flatness of the spacetime. To this end, let us call Minkowskian coordinates the global coordinates system in which the RW metric element assumes the form

$$ds^2 = a^2(\tau)(d\tau^2 - d\boldsymbol{x}^2). \tag{7}$$

In that system of coordinates, Eq. (6) reads

$$\partial^2 A_\mu - W_\mu \partial \cdot A + (\partial_\mu - W_\mu) W \cdot A = 0, \tag{8}$$

where $W := \mathrm{d} \ln a^2$. Unfortunately, we did not succeed in finding a family of modes solution of Eq. (8). Consequently, we cannot explicitly complete the quantization process, although quantization is still theoretically possible [13].

Let us remark that the one-form W, which appears in Eq. (8), is generally defined by $W := \mathrm{d} \ln \Omega^2$, for a real positive conformal factor $\Omega(x)$. In the system of coordinates used in (7), one has $\Omega(x) = a(\tau)$ and the only non-vanishing component of W is $W_\tau = 2\mathcal{H}$, two times the comoving Hubble factor. Throughout this paper, we nevertheless keep the general notation W, since the derivations of all the expressions in which W appears do not make use of the exclusive dependence in τ, and they are still valid for $a := a(x)$.

Now, the above unsuccessful attempt leads us to a practicable route. Indeed, a look at Eq. (8) makes obvious that the conformal flatness does not lead to much simplification here. Let us consider more closely the equations (6) and (8) by themselves. Equation (8) is simply (6) written in the Minkowskian system of coordinates. Now, it is straightforward to show that Eq. (8) can also be obtained by adding to the usual Lagrangian of electromagnetism in Minkowski space the gauge-fixing term $\frac{1}{2}(\partial \cdot A + W \cdot A)^2$. In short, writing the Maxwell equations in Lorenz gauge in the RW space is equivalent to writing the Minkowskian Maxwell equations in the gauge $\partial \cdot A + W \cdot A = 0$.

The point is that writing equations over the RW manifold in the global Minkowskian chart, in which the Robertson-Walker metric element is (7), is equivalent to performing a conformal transformation (see for instance [2]) between the RW space and a Minkowski space, that is, a Weyl rescaling between the metric manifolds $(\mathbb{R}^4, g)$ and $(\mathbb{R}^4, \eta)$, where g and η are, respectively, the RW and the Minkowskian metric diag$(+,-,-,-)$. Under such a rescaling, the Maxwell equations are invariant and a form field solution A is mapped to itself since its so-called conformal weight is zero. In fact, the rescaling map induces a transport of all mathematical objects (fields, operators, etc.) between structures defined on the spacetimes. In particular, even the equations which are not conformally invariant can be moved between spaces. From this point of view, quantizing the Maxwell equations in Lorenz gauge in RW spacetime is equivalent to quantizing the Maxwell equations (since they are conformally invariant) in the Weyl rescaled gauge $\partial \cdot A + W \cdot A = 0$ in the Minkowski space.

Finally, the lesson from the unsuccessful Lorenz gauge quantization is that if one wishes to obtain a two-point function, one may recognize that the Lorenz gauge condition in the RW space is not the best choice. Since the Maxwell equations in Lorenz gauge in Minkowski space are well known, it is far more convenient to start from a gauge condition in RW space which reads as the Lorenz gauge in the Minkowskian coordinates (or equivalently which is conformally mapped to the Lorenz gauge in the Minkowski space). We take this approach in the sequel.

IV. QUANTIZATION IN THE *W* GAUGE

The Lorenz gauge condition in Minkowski space can be conformally lifted to the RW space where it reads

$$\nabla \cdot A - W \cdot A = 0, \tag{9}$$

or, specializing to the case $a = a(\tau)$,

$$\nabla \cdot A = 2\mathcal{H} A_\tau.$$

As explained in the previous section, we quantize the Maxwell equations on RW spaces in the above W gauge, because in the Minkowskian system it reduces to the historical GB quantization in Minkowski space in Lorenz gauge. From the point of view of conformal transformations, this amounts to pulling back in RW space the whole structure (enlarged space of solutions, basis of modes, two-point function, etc.) involved in the quantization process. Let us consider this construction step by step.

Because of the gauge invariance, the scalar product obtained from the Lagrangian over the space of solutions of the Maxwell equations is degenerate. Following the GB method, one first enlarges the space of solutions. This is done by adding to the Lagrangian of electromagnetism the gauge term $\frac{1}{2}(\nabla \cdot A - W \cdot A)^2$. The Euler equations then read

$$(MA)_\mu + (\nabla_\mu + W_\mu)(\nabla - W) \cdot A = 0, \tag{10}$$

where we have set $(MA)_\mu := \Box A_\mu - \nabla_\mu \nabla \cdot A + R^\nu{}_\mu A_\nu$. The space of solutions of (10) is endowed with the scalar product (1) derived from the gauge-fixed Lagrangian. Inspection of (10) makes obvious that the subset of solutions of (10) defined through the gauge condition (9) are solutions of the Maxwell equations.

The next step in the quantization procedure is to find a basis of modes for Eq. (10). To this end, they are expressed in the Minkowskian chart, which is equivalent to performing a Weyl rescaling, and they become $\partial^2 A_\mu = 0$, as expected from the consideration of the previous section. Then the modes are, in the Minkowskian coordinates, the familiar exponentials; they read

$$\Phi^{(\lambda)}_{k,\mu} := \epsilon^{(\lambda)}_\mu(k) \frac{1}{(2\pi)^3 \sqrt{2\omega_{\boldsymbol{k}}}} \exp\{-i(\omega_{\boldsymbol{k}} \tau - \boldsymbol{k} \cdot \boldsymbol{x})\}, \tag{11}$$

with $k^0 = \|\boldsymbol{k}\| =: \omega_{\boldsymbol{k}}$, and the forms $\{\epsilon^{(\lambda)}(k)\}$ being the polarization basis.

It is crucial to remark that, although (10) is not conformally invariant, it is the conformal lift of the Minkowskian equation $\partial^2 A_\mu = 0$. Consequently, the above functions are modes of both equations $\partial^2 A_\mu = 0$ and (10). In addition, these modes are of positive frequency with respect to the timelike Killing vector field of Minkowski space ∂_τ. Since the RW space is conformally flat, ∂_τ is also a timelike conformal Killing vector of the RW spacetime. The modes (11) are thus positive frequency with respect to the conformal time, which means that the vacuum they define is the so-called conformal vacuum.

It is known that the $\{\Phi^{(\lambda)}_{k,\mu}\}$ form a basis of the space of solutions of $\partial^2 A_\mu = 0$ endowed with the indefinite scalar product derived through (1) from the Minkowskian Lagrangian $L^M = \frac{1}{4}F^2 + \frac{1}{2}(\partial \cdot A)^2$. They also form a basis of the space of solutions of Eq. (10) endowed with the indefinite scalar product derived from the RW Lagrangian $L^{\mathrm{RW}} = \frac{1}{4}F^2 + \frac{1}{2}(\nabla \cdot A - W \cdot A)^2$. Indeed, these two spaces of solutions are identical. This can be seen as follows. First, since the conformal weight of the electromagnetic field is zero, these spaces of solutions contain the same functions. Then, the scalar products defined on them through (1) are equal. In fact, the conformal relation between spacetimes implies that $L^{\mathrm{RW}} = a^{-4}L^M$. Consequently, using the definition (2) of $\mathcal{J}^\mu$ and again the fact that the electromagnetic field is of null conformal weight, one obtains $\mathcal{J}^\mu_{\mathrm{RW}}(A,B) = a^{-4}\mathcal{J}^\mu_M(A,B)$. Since the surface form σ_μ in (1) scales as $\sqrt{g}$, one has $\sigma^{\mathrm{RW}}_\mu = a^4\sigma^M_\mu$. Finally, taking into account that a Cauchy surface Σ defined in the RW spacetime is also a Cauchy surface in the Minkowski chart, one obtains, through the definition (1), that $\langle A, B\rangle_{\mathrm{RW}} = \langle A, B\rangle_M$.

Once a basis of modes solution is known, the Wightman two-point function can be obtained straightforwardly. If one chooses, as usual, a polarization basis such that $\eta^{\mu\nu}\epsilon^{(\lambda)}_\mu\epsilon^{(\rho)}_\nu = \eta^{\lambda\rho}$, the two-point function takes the familiar Minkowskian form

$$D_{\mu\nu}(x,x') = -\eta_{\mu\nu}D^{(s)}_M(x,x'). \quad (12)$$

In this expression, x and x' denote two points of the RW spacetime whose Minkowskian Cartesian coordinates are (τ, x^i) and (τ', x'^i), and $D^{(s)}_M(x,x')$ is the two-point function for the conformal scalar field in Minkowski space. Using the Weyl rescaling and taking into account that the conformal weight of the conformal scalar is -1, this expression reads $D_{\mu\nu}(x,x') = -\eta_{\mu\nu}a(x)D^{(s)}(x,x')a(x')$, where $D^{(s)}(x,x') = a^{-1}(x)D^{(s)}_M(x,x')a^{-1}(x')$ is the two-point function for the conformal scalar field in RW space. Finally, taking into account the manifest symmetry of $D_{\mu\nu}(x,x')$ in (12), one obtains a more intrinsic expression for this two-point function in RW spacetime, namely,

$$D_{\mu\nu}(x,x') = -\frac{1}{2}\left(\frac{g_{\mu\nu}}{a^2} + \frac{g'_{\mu\nu}}{a'^2}\right)aa'D^{(s)}(x,x'), \quad (13)$$

where for brevity we have denoted by a prime the quantities which have to be evaluated at the point x'. The above expression is our central result.

It is worth noting that this two-point function has the Hadamard behavior. Inspection of (13) shows that the short distance behavior is that of $D^{(s)}(x,x')$. Then since $D^{(s)}(x,x') = a^{-1}(x)D^{(s)}_M(x,x')a^{-1}(x')$, the result follows from the fact that $\sigma_{\mathrm{RW}}(x,x') \simeq a^2(x)\sigma_M(x,x')$ for x close to x', the quantities σ_{RW} and σ_M being half of the square of geodesic length between two points x and x' in their respective spacetimes (see for instance [14]).

Finally, the basis of modes (11) also allows us to define the quantum field $\hat{A}$ in the usual way through the expansion over the modes. Then the Fock space is built using the standard procedure. The important point here is that the conformal flatness of the RW spacetime allows us, thanks to the existence of a timelike Killing vector field ∂_τ in Minkowski space, to unambiguously define positive frequency modes. These modes allow us in turn to define unambiguously annihilation (and creation) operators and consequently a preferred vacuum state: the conformal vacuum. This is a well-known result of Parker [1,10].

As a consequence, the GB condition (4) which defines the subspace of physical states is also meaningful. In the W gauge (9), it reads

$$(\nabla - W)\cdot A^{(+)}|\Psi_A\rangle = 0,$$

where $A^{(+)}$ is the annihilator part of the quantum field. This field fulfills the Maxwell equations together with the W-gauge condition, in the mean on physical states (5). This is not in agreement with the starting point of [8,9]. However, let us emphasize that the GB condition applies in flat RW spacetimes due to the existence of a conformal vacuum. On the contrary, this condition can lose its meaning in general curved spacetimes where the definition of the vacuum is ambiguous.

[1] L. Parker and D. Toms, *Quantum Field Theory in Curved Spacetime* (Cambridge University Press, Cambridge, England, 2009).

[2] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).

[3] B. Allen and T. Jacobson, Commun. Math. Phys. **103**, 669 (1986).
[4] P. J. Mora, N. C. Tsamis, and R. P. Woodard, J. Math. Phys. (N.Y.) **53**, 122502 (2012).
[5] M. Faizal and A. Higuchi, Phys. Rev. D **85**, 124021 (2012).
[6] M. Faizal, Classical Quantum Gravity **29**, 035007 (2012).
[7] S. Faci, Phys. Rev. D **87**, 064040 (2013).
[8] D. N. Vollick, Phys. Rev. D **86**, 084057 (2012).
[9] J. Beltran Jimenez and A. L. Maroto, Mod. Phys. Lett. A **26**, 3025 (2011).
[10] L. Parker, Phys. Rev. **183**, 1057 (1969).
[11] S. Faci, E. Huguet, and J. Renaud, Phys. Rev. D **84**, 124050 (2011).
[12] S. Faci, E. Huguet, J. Queva, and J. Renaud, Phys. Rev. D **80**, 124005 (2009).
[13] F. Finster and A. Strohmaier, arXiv:1307.1632.
[14] J. L. Synge, *Relativity: The General Theory* (North Holland Publishing Company, Amsterdam, 1960).