

Hawking radiation with dispersion versus breakdown of the WKB approximationR. Schützhold^{1,*} and W. G. Unruh^{2,†}¹*Fakultät für Physik, Universität Duisburg-Essen, Lotharstrasse 1, 47057 Duisburg, Germany*²*Department of Physics, CIAR Cosmology and Gravity Program, University of B.C., Vancouver, Canada V6T 1Z1*
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Inspired by the condensed matter analogues of black holes (a.k.a. dumb holes), we study Hawking radiation in the presence of a modified dispersion relation which becomes superluminal at large wave numbers. In the usual stationary coordinates (t, x) , one can describe the asymptotic evolution of the wave packets in WKB, but this WKB approximation breaks down in the vicinity of the horizon, thereby allowing for a mixing between initial and final creation and annihilation operators. Thus, one might be tempted to identify this point where WKB breaks down with the moment of particle creation. However, using different coordinates (τ, U) , we find that one can evolve the waves so that WKB in these coordinates is valid throughout this transition region, which contradicts the above identification of the breakdown of WKB as the cause of the radiation. Instead, our analysis suggests that the tearing apart of the waves into two different asymptotic regions (inside and outside the horizon) is the major ingredient of Hawking radiation.

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I. INTRODUCTION

Despite decades of research, the precise nature of Hawking [1] radiation—one of the most fundamental predictions of quantum field theory [2] in curved spacetimes—is not fully understood yet. E.g., open questions are:

- (i) Why does Hawking radiation give a thermal spectrum, i.e., why do black holes seem to behave as thermal [3] objects with an energy (mass), temperature, and because the temperature depends on energy, an entropy, etc.?
- (ii) What are the essential ingredients of Hawking radiation and how robust is it? Is the thermal spectrum and density matrix robust, or is it possible to encode some information into the seemingly thermal radiation given off?

The last point is especially relevant for the black hole information “paradox,” i.e., the question of whether (and if yes, how) the process of a gravitational collapse to a black hole and its subsequent evaporation can be described by a globally unitary (i.e., information preserving) process [4]. Because the field equations are local the system is always locally unitary. The question is essentially whether there are places (e.g., singularities, baby universes, etc.) into which information could be lost.

Sometimes it is argued that the information (possibly stored near the singularity or in the vicinity of the horizon) might perhaps leak out the black hole (during its evaporation process) hidden in correlations between the emitted Hawking quanta—this would require some sort of imprinting procedure during which the information is stored

(remembered) and later transferred to the emitted particles. These motivate a better understanding of the robustness and origin of Hawking radiation. In the following, we study this question for a setup including a modified dispersion relation at large wave numbers—an approach which is partly motivated by the condensed matter analogues [5] of black holes (a.k.a. dumb holes).

II. CONVENTIONAL DERIVATION

Let us start with a brief sketch of the conventional derivation of Hawking radiation in the presence of a modified dispersion relation [6]. Usually this is done in terms of the Painlevé-Gullstrand-Lemaître or the Eddington-Finkelstein coordinates [7]. Here, we employ the former since they are more closely related to the condensed matter analogues of black holes. In terms of these coordinates, the metric reads ($\hbar = c = 1$)

$$ds^2 = [1 - v^2(x)]dt^2 - 2v(x)dt dx - dx^2 = dt^2 - [dx + v(x)dt]^2, \quad (1)$$

where we restrict our consideration to 1 + 1 dimensions for simplicity. Here $v(x)$ corresponds to the fluid velocity of the sonic black hole analogues and we have a horizon where $v(x_{\text{horizon}}) = c = 1$. If we choose $v(x) = \sqrt{2M/x}$ this metric is exactly the Schwarzschild metric in Painlevé-Gullstrand-Lemaître coordinates with the angular coordinates suppressed.

In order to include a modified dispersion relation, the standard wave equation $\square\phi = 0$ is replaced by

$$\square\phi = F(\partial_x)\phi, \quad (2)$$

where F contains higher order spatial derivatives. Here we use $F = -\gamma^2\partial_x^4$, which corresponds to the dispersion

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relation of phonons in atomic Bose-Einstein condensates [8], for example, but our results can be generalized to other functions F .

The scale γ sets the cutoff wave number $k_{\text{cutoff}} = 1/\gamma$ where the dispersion relation starts to deviate significantly from a linear behavior. For atomic Bose-Einstein condensates, it is related to the healing length—whereas for real gravity, one might speculate that it is related to the Planck scale. Then Eq. (2) becomes

$$([\partial_t - \partial_x v][\partial_t - v \partial_x] - \partial_x^2 + \gamma^2 \partial_x^4) \phi = 0. \quad (3)$$

Since this wave equation is stationary and linear, we make the separation ansatz $\phi_\omega(x) e^{-i\omega t}$. However, due to the x -dependence of the velocity $v(x)$, an analogous (plane-wave) ansatz for $\phi_\omega(x) = e^{-ikx}$ is not an exact solution. Nevertheless, assuming that $v(x)$ varies slowly, we may use the WKB method to obtain an approximate solution $\phi_\omega(x) \approx \exp\{-iS_\omega(x)\}$. In order to ensure the applicability of the WKB approximation, the local wave number $k(x) = S'_\omega(x)$ should vary slowly compared to its magnitude $dk/dx = S''_\omega \ll k^2 = (S'_\omega)^2$. This WKB approximation yields the dispersion relation

$$(\omega - vk)^2 = k^2 + \gamma^2 k^4 \rightsquigarrow \omega = vk \pm k\sqrt{1 + \gamma^2 k^2}, \quad (4)$$

where the term with the square root on the right-hand side is the dispersion relation $\omega_{\text{Min}}(k) = k\sqrt{1 + \gamma^2 k^2}$ in Minkowski space-time (where $v = 0$). The plus sign in front of the square root refers to the left-moving solutions whereas the minus sign applies to the right-moving branch. Since the latter branch contains the relevant modes such as the outgoing Hawking particles, we focus on the minus sign in the following. Without loss of generality, we set $x_{\text{horizon}} = 0$ such that the velocity $v(x)$ can be approximated in the vicinity of the horizon by

$$v(x) = 1 - \kappa x + \mathcal{O}(x^2), \quad (5)$$

where κ is the surface gravity. Assuming that the frequency ω is of the order of the surface gravity κ and small compared to the dispersion scale $1/\gamma$ (i.e., $\gamma\omega \ll 1$), we may find an intermediate regime for x

$$1 \gg |\kappa x|^3 \gg \gamma^2 \omega^2 = \mathcal{O}(\gamma^2 \kappa^2). \quad (6)$$

This range is chosen such that the linearization (5) applies due to $|\kappa x|^3 \ll 1$ and, at the same time, we find two WKB solutions with large and nearly opposite wave numbers $k_\pm \approx \pm\sqrt{2|\kappa x|/\gamma^2}$ for $x < 0$, i.e., inside the black hole. In principle, these values k_\pm can be derived by solving the quartic equation (4) for k , but, to a very good approximation, we may Taylor expand the square root in the dispersion relation (4). Inserting the velocity profile (5), we get $\omega \approx -k(\kappa x + \gamma^2 k^2/2)$. Finally, using $|\kappa x|^3 \gg \gamma^2 \omega^2$ from the above condition (6), we find that $|\kappa x| \gg \omega/k_\pm$, which gives $k_\pm \approx \pm\sqrt{2|\kappa x|/\gamma^2}$. After integrating $k(x) = S'_\omega(x)$, the associated WKB solutions of the wave equation

read $\exp\{-i\omega t \pm \sqrt{8\kappa|x^3|/(9\gamma^2)}\}$. There are also other solutions of the dispersion relation (4), but they have much smaller k -values $k \approx \omega/(v-1)$ and are thus well separated from the k_\pm solutions.

However, when κx becomes too small (i.e., very close to the horizon) such that $|\kappa x|^3 \sim \gamma^2 \kappa^2$, this WKB approximation breaks down. One way to see this is to take the spatial derivative of the dispersion relation (4) giving

$$\frac{dk}{dx} = -\frac{dv}{dx} \frac{k}{v(x) - v_{\text{group}}(k)}, \quad (7)$$

where $v_{\text{group}} = d\omega_{\text{Min}}/dk$ is the group velocity in Minkowski space-time (where $v = 0$). When approaching the horizon (where $v = 1$), dk/dx diverges and thus the WKB approximation breaks down. To make this statement more precise, let us approximate the group velocity by $v_{\text{group}}(k_\pm) \approx 1 + 3\gamma^2 k_\pm^2/2 \approx 1 + 3|\kappa x|$, where we have used the same arguments as below Eq. (6). Then, inserting Eq. (5), the WKB condition $|dk_\pm/dx| \ll k_\pm^2$ translates into $|\kappa x|^3 \gg \gamma^2 \kappa^2$, consistent with Eq. (6).

At the WKB turning point near the horizon where $v(x)$ equals $v_{\text{group}}(k)$ and thus dk/dx diverges, the k_\pm -solutions are no longer well separated from the other modes and two real k -solutions ($k \approx \omega/(v-1)$ and k_+) of the dispersion relation (4) merge and become complex—i.e., the modes do not stay separated and mix. Thus, one might be tempted to identify this point with the place of particle creation.

III. LINEAR PROFILE

Since the critical point where the WKB approximation described above breaks down is very close to the horizon $|\kappa x|^3 \sim \gamma^2 \kappa^2 \ll 1$, we focus on this region and employ the near-horizon approximation by setting

$$v(x) = 1 - \kappa x. \quad (8)$$

Introducing the usual Kruskal light-cone coordinate U , the metric (1) can then be cast into the form

$$U = -x \exp\{-\kappa t\} \rightsquigarrow ds^2 = 2e^{\kappa t} dt dU - e^{2\kappa t} dU^2. \quad (9)$$

With $\sqrt{-g} = e^{\kappa t}$, we get the d'Alembert operator

$$\square \phi = e^{-\kappa t} (\partial_t e^{\kappa t} \partial_t + 2\partial_U \partial_t) \phi, \quad (10)$$

and the wave equation (2) with a modified dispersion relation $F = -\gamma^2 \partial_x^4$ reads (note that $\partial_x = -e^{-\kappa t} \partial_U$)

$$(\partial_t e^{\kappa t} \partial_t + 2\partial_U \partial_t) \phi = -\gamma^2 e^{-3\kappa t} \partial_U^4 \phi. \quad (11)$$

In terms of the new coordinates, the metric (9) becomes time-dependent but independent of U . Hence, ∂_t is not a Killing vector anymore but ∂_U is, which allows us to make the separation ansatz $\phi(t, U) = \phi_K(t) e^{iKU}$

$$\ddot{\phi}_K + \kappa \dot{\phi}_K + 2iK e^{-\kappa t} \dot{\phi}_K = -\gamma^2 K^4 e^{-4\kappa t} \phi_K. \quad (12)$$

For late times $t \uparrow \infty$, the exponential factors $e^{-\kappa t}$ and $e^{-4\kappa t}$ can be neglected and thus the above equation corresponds

to the motion of a particle with damping. Hence we see that the solution freezes $\phi(t \uparrow \infty, U) = \phi_K^\infty e^{iKU}$ at late times $t \uparrow \infty$. This motivates the introduction of a new time coordinate via

$$\tau = -\frac{1}{\kappa} e^{-\kappa t} \rightsquigarrow d\tau = e^{-\kappa t} dt, \quad (13)$$

such that the wave equations simplify to

$$(\partial_\tau^2 + 2iK\partial_\tau)\phi_K = -\gamma^2 K^4 (\kappa\tau)^2 \phi_K. \quad (14)$$

With the redefinition $\varphi_K(\tau) = e^{iK\tau} \phi_K(\tau)$, we may eliminate the $2iK\partial_\tau$ term and obtain the simple form

$$(\partial_\tau^2 + \gamma^2 K^4 (\kappa\tau)^2 + K^2)\varphi_K = 0. \quad (15)$$

Equations of this type are well known and can be solved in terms of Whittaker or Kummer functions [9]. Incidentally, the same equation occurs for the Sauter-Schwinger effect, i.e., electron-positron pair creation out of the QED vacuum due to a strong electric field [10].

IV. WKB ANALYSIS—LINEAR PROFILE

Even though the wave equation (15) can be solved exactly, let us find an approximate solution via the WKB method. The associated dispersion relation reads

$$\Omega_K(\tau) = \pm K \sqrt{1 + \gamma^2 \kappa^2 (K\tau)^2}. \quad (16)$$

If we now check the applicability of the WKB approximation in analogy to Eq. (7), we find that

$$\gamma\kappa \ll 1 \rightsquigarrow \left| \frac{1}{\Omega_K^2} \frac{d\Omega_K}{d\tau} \right| \ll 1, \quad (17)$$

and thus the WKB method applies for all τ , i.e., all the way from $t \downarrow -\infty$ down to $t \uparrow +\infty$. The two branches of solutions of (16) with positive and negative $\Omega_K(\tau)$ stay well separated throughout the evolution and there is negligible mode mixing between them.

Let us study the wave-packet trajectories for the two branches. First, we consider the case without dispersion $\gamma = 0$ for comparison. For the branch with $\Omega_K = +K$, the solution behaves as e^{iKU} and thus the trajectories have $U = \text{const}$, i.e., $x \propto e^{\kappa t}$. These are the modes peeling off the black hole horizon at $U = 0$ (i.e., $x = 0$), for example the Hawking modes. For the other branch with $\Omega_K = -K$, the solution behaves as $e^{iKU - 2iK\tau}$ and thus the trajectories have $U = 2\tau + \text{const}$, which translates into $x = 2/\kappa + e^{\kappa t} \times \text{const}$. This is a consequence of the linear velocity profile in (8) which yields another horizon (where $v = -1$) at $x = 2/\kappa$. Thus, the modes corresponding to the other branch peel off this additional horizon at $x = 2/\kappa$ but they propagate freely across the black hole horizon at $x = 0$.

As one would expect, this behavior changes in the presence of a modified dispersion relation (16). In this case, the modes which later form the Hawking radiation do not originate from the vicinity of the horizon at $U = 0$

but approach the horizon from the inside in the past. In analogy to (6), we use the smallness of $\gamma\kappa \ll 1$ to find an intermediate regime for τ where

$$\frac{1}{\gamma^2 \kappa^2} \gg (K\tau)^2 \gg 1. \quad (18)$$

In this regime, we may Taylor expand the square root in (16) and thus the final Hawking modes initially behave as $\varphi_K \approx \exp\{iS_K(\tau)\}$ with $dS_K/d\tau = \Omega_K(\tau)$ which gives $\varphi_K \approx \exp\{iK\tau + i\gamma^2 \kappa^2 K^3 \tau^3/6\}$. The complete field solution then reads $\phi(U, \tau) \approx \exp\{iKU + i\gamma^2 \kappa^2 K^3 \tau^3/6\}$. Using the stationary phase (or saddle point) approximation, we find the WKB trajectories as $U = U_0 - \gamma^2 \kappa^2 K^2 \tau^3/2$. As expected, the modes approach the horizon from the inside where $U > 0$ and $x < 0$ (note that $\tau < 0$).

V. PARTICLE CREATION

Now, after having studied the solutions of the wave equation, let us derive the consequences for Hawking radiation. To this end, we use the same trick as in [11] and consider the following function

$$\phi_\omega(U) = \begin{cases} e^{+\pi\omega/(2\kappa)} |\kappa U|^{i\omega/\kappa} & \text{for } U > 0 \\ e^{-\pi\omega/(2\kappa)} |\kappa U|^{i\omega/\kappa} & \text{for } U < 0. \end{cases} \quad (19)$$

After analytic continuation, this function is holomorphic in the entire upper half of the complex U plane $\Im(U) > 0$ while it has a branch cut in the lower half. As a result, the Fourier decomposition of this function $\phi_\omega(U)$ consists of modes e^{iKU} with positive $K > 0$ only. (For negative $K < 0$, the integral determining the overlap with $\phi_\omega(U)$ can be closed in the upper half of the complex plane and thus vanishes.)

As we have seen in the previous section, the modes which finally behave as $\exp\{iKU\}$ are initially of the form $\exp\{iKU + i\gamma^2 \kappa^2 K^3 \tau^3/6\}$. In terms of the original x -coordinate, we get $\exp\{iKx\kappa\tau + i\gamma^2 \kappa^2 K^3 \tau^3/6\}$. Now, calculating the time-derivative of this exponent (i.e., the eikonal function $S = Kx\kappa\tau + \gamma^2 \kappa^2 K^3 \tau^3/6$) for fixed x , we find $\partial_t S(t, x) = -\kappa K(U + \gamma^2 \kappa^2 K^2 \tau^3/2)$. Evaluating this expression along the WKB trajectories discussed above $U = U_0 - \gamma^2 \kappa^2 K^2 \tau^3/2$, we get a constant $\partial_t S = -\kappa K U_0$. This should be no surprise since our original problem is stationary in terms of the (t, x) coordinates, i.e., $\partial_t S = -\kappa K U_0$ is related to the conserved frequency $\omega \approx \kappa K U_0$. Thus, for Hawking radiation, mainly modes with $K U_0 = \mathcal{O}(1)$ will be important. Note, however, that the concept of well-localized WKB trajectories becomes somewhat inaccurate for $K U_0 = \mathcal{O}(1)$ since one cannot localize a wave packet more precisely than $\Delta U = \mathcal{O}(1/K)$. Hence, it is impossible to decide whether such a wave packet is located inside or outside the horizon at $U = 0$. This is precisely the reason for the tearing apart of the waves at the horizon and the resulting particle creation. Wave packets with large $|K U_0| \gg 1$ will

basically not be torn apart and thus there will be negligible particle creation for these modes.

As the next step, let us study how the initial mode of the form $\exp\{iKx\kappa\tau + i\gamma^2\kappa^2K^3\tau^3/6\}$ is experienced by a freely falling observer with the near-horizon trajectory $x(t) \approx x_0 - t$. Since (as we have seen above), the temporal variance $\partial_t S(t, x)$ is slow compared to the spatial variation $\partial_x S(t, x)$, the freely falling observer will see a rapid oscillation with a frequency of $\omega_{\text{in}} \approx K\kappa\tau$ initially, i.e., for $(K\tau)^2 \gg 1$. Thus, modes with positive K will have a large negative frequency ω_{in} due to $\tau < 0$ and vice versa.

As the final ingredient, let us convert the above function $\phi_\omega(U)$ into t, x coordinates, where we get

$$\phi_\omega(t, x) = \begin{cases} e^{+\pi\omega/(2\kappa)} |\kappa x|^{i\omega/\kappa} e^{-i\omega t} & \text{for } x < 0 \\ e^{-\pi\omega/(2\kappa)} |\kappa x|^{i\omega/\kappa} e^{-i\omega t} & \text{for } x > 0. \end{cases} \quad (20)$$

We see that we obtain the wave functions of the outgoing Hawking radiation (with positive pseudonorm) for $x > 0$ and its infalling partner particles (with negative pseudonorm) for $x < 0$ (where both have the same conserved frequency ω). Therefore, if we consider the evolution of a Gaussian wave packet peaked at $K_0 \gg \Delta K > 0$

$$\phi_{\Delta K}^{U_0, K_0}(\tau, U) \propto \int \phi_K(\tau) e^{iK(U-U_0) - (K-K_0)^2/\Delta K^2} dK, \quad (21)$$

we find the following picture: Initially, i.e., for values of τ satisfying (18), this wave packet approaches the horizon from the inside according to the trajectory $U = U_0 - \gamma^2\kappa^2K_0^2\tau^3/2$. For a freely falling observer, it has a large negative frequency $\omega_{\text{in}} \approx K_0\kappa\tau$. During the approach to the horizon, the wave packet is constantly redshifted according to $\omega_{\text{in}} \approx -K_0e^{-\kappa t}$. This reflects the time-translation invariance of our system: Modes with larger K_0 evolve in the same way as those with smaller K_0 , but at later times t . Finally, this wave packet is ripped apart at the horizon into an outgoing part and its infalling partner (unless K_0U_0 is too large, see above). Nevertheless, the WKB method in Sec. IV remains applicable during the whole process. Note that the above condition (18), together with $K_0U_0 = \mathcal{O}(1)$, also ensures that $\kappa x = U/\tau = U_0/\tau - \gamma^2\kappa^2K^2\tau^2/2$ is small and thus consistent with our linear approximation (8).

For a given final frequency ω , the amplitudes of these two parts directly yield the Bogoliubov coefficients α_ω and β_ω and from (20), we may read off

$$\left| \frac{\beta_\omega}{\alpha_\omega} \right| = \exp\left\{-\pi \frac{\omega}{\kappa}\right\}. \quad (22)$$

Together with the normalization $|\alpha_\omega|^2 - |\beta_\omega|^2 = 1$, this gives a thermal spectrum

$$\langle \hat{n}_\omega^{\text{out}} \rangle_{\text{in}} = |\beta_\omega|^2 = \frac{1}{\exp\{2\pi\omega/\kappa\} - 1}, \quad (23)$$

with the Hawking temperature ($\hbar = c = k_B = 1$)

$$T_{\text{Hawking}} = \frac{\kappa}{2\pi}. \quad (24)$$

VI. GENERAL PROFILE

For a more general velocity profile $v(x)$, the light cones lie at $dt = \pm[dx + v(x)dt]$ and thus the Kruskal light-cone variable U reads

$$U = -\frac{1}{\kappa} \exp\left\{-\kappa t - \kappa \int \frac{dx}{v(x)-1}\right\}, \quad (25)$$

where $\kappa = |v'(x_{\text{horizon}})|$ is again the surface gravity. Inserting the coordinate differential

$$dU = -\kappa U \left(dt + \frac{dx}{v(x)-1} \right), \quad (26)$$

the metric becomes

$$ds^2 = 2 \frac{v-1}{\kappa U} dU dt - \left(\frac{v-1}{\kappa U} \right)^2 dU^2. \quad (27)$$

Note that the factor

$$\chi = \sqrt{-g} = \frac{v-1}{\kappa U} \quad (28)$$

is regular at the horizon and behaves as $\chi = e^{\kappa t} f(x)$ with a regular function $f(x)$ satisfying $f(x_{\text{horizon}}) = 1$. Using the same modification $F(\partial_x)$ of the dispersion relation as before, the wave equation becomes

$$\square\phi = \frac{1}{\chi} (\partial_t \chi \partial_t + 2\partial_U \partial_t) \phi = F(\partial_x) \phi = -\gamma^2 \partial_x^4 \phi, \quad (29)$$

where $\partial_x = -\chi^{-1} \partial_U$.

VII. TANH-PROFILE

In order to deal with a smooth profile with well-defined asymptotics, let us consider the following example

$$v(x) = 1 - \tanh(\kappa x). \quad (30)$$

In this case, the Kruskal variable U simply becomes

$$U = -\frac{\sinh(\kappa x)}{\kappa} \exp\{-\kappa t\} = \tau \sinh(\kappa x), \quad (31)$$

and the factor χ in $ds^2 = 2\chi dU dt - \chi^2 dU^2$ reads

$$\chi = \frac{e^{\kappa t}}{\cosh(\kappa x)} = \frac{1}{\sqrt{e^{-2\kappa t} + \kappa^2 U^2}} = \frac{\kappa^{-1}}{\sqrt{\tau^2 + U^2}}. \quad (32)$$

The wave equation (29) assumes the form

$$(\chi \partial_t \chi \partial_t + 2\chi \partial_t \partial_U + \gamma^2 \chi^2 [\chi^{-1} \partial_U]^4) \phi = 0, \quad (33)$$

and, after transforming to the time coordinate τ , we get

$$(\xi \partial_\tau \xi \partial_\tau + 2\xi \partial_\tau \partial_U + \gamma^2 \chi^2 [\chi^{-1} \partial_U]^4) \phi = 0, \quad (34)$$

with the purely x -dependent factor

$$\xi = \frac{1}{\cosh(\kappa x)} = \frac{1}{\sqrt{1 + U^2/\tau^2}}. \quad (35)$$

Now let us consider the two limiting cases. For $\tau^2 \gg U^2$ (i.e., small x), we have $\xi \rightarrow 1$ and thus the wave equation simplifies and we get the same result as with the linear profile (as expected)

$$(\partial_\tau^2 + 2\partial_\tau\partial_U + \gamma^2\kappa^2\tau^2\partial_U^4)\phi = 0. \quad (36)$$

In the other limiting case $\tau^2 \ll U^2$ (i.e., large x), we have $\chi \rightarrow 1/|\kappa U|$ and thus the wave equation becomes approximately independent of t

$$(\partial_t^2 + 2|\kappa U|\partial_t\partial_U + \gamma^2[\kappa U\partial_U]^4)\phi = 0. \quad (37)$$

This equation has two branches of solutions

$$\phi_\theta^\pm \propto |\kappa U|^{i\theta} \exp\left[-i\kappa\theta t(1 \pm \sqrt{1 + \gamma^2\kappa^2\theta^2})\right], \quad (38)$$

labeled by $\theta = \text{const}$. Without dispersion, i.e., for $\gamma = 0$, we may interpret the two branches in the same way as in Sec. IV: one $\phi^+ \propto |\kappa U|^{i\theta} e^{-2i\kappa\theta t}$ is propagating into the black hole while the other $\phi^- \propto |\kappa U|^{i\theta}$ is trying to escape. However, most interesting is the case with dispersion, where we get

$$\phi_\theta^- \sim \exp\left\{i\kappa\theta(|x| - 2t + t\sqrt{1 + \gamma^2\kappa^2\theta^2})\right\}. \quad (39)$$

We see that the modes with $\gamma^2\kappa^2\theta^2 < 3$ are propagating away from the horizon while the others with $\gamma^2\kappa^2\theta^2 > 3$ are moving toward it. A final Hawking particle with frequency $\omega > 0$ stems from initial modes with $\theta < 0$ and

$$\gamma^2\kappa^2\theta^2 = 3 + \sqrt{\frac{16\gamma^2\omega^2}{3}}, \quad (40)$$

which propagate very slowly toward the horizon. Again, for a freely falling observer, these modes with $\theta < 0$ and $|\theta| \approx \sqrt{3}/(\gamma^2\kappa^2) \gg 1$ have a very large negative frequency.

VIII. WKB ANALYSIS—TANH-PROFILE

Since we can solve the wave equation (33) exactly in the two limiting cases $\tau^2 \gg U^2$ and $\tau^2 \ll U^2$, the remaining critical regime is where U and τ are roughly of the same order, i.e., where $\kappa x = \mathcal{O}(1)$. In this regime, we may obtain Ω_K and K via the transformation from the original t, x coordinates to the τ, U coordinates

$$\begin{pmatrix} \partial_t \\ \partial_x \end{pmatrix} = -\kappa \begin{pmatrix} \tau & U \\ 0 & \sqrt{U^2 + \tau^2} \end{pmatrix} \cdot \begin{pmatrix} \partial_\tau \\ \partial_U \end{pmatrix}. \quad (41)$$

Identifying $\partial_t \rightarrow \Omega$ and $\partial_x \rightarrow k$, as well as, $\partial_\tau \rightarrow \Omega_K$ and $\partial_U \rightarrow K$, we get the transformation

$$K = -\frac{k}{\sqrt{\kappa^2 U^2 + \kappa^2 \tau^2}} = -k\chi. \quad (42)$$

Furthermore, we find the exact conservation law

$$\tau\Omega_K + UK = -\frac{\omega}{\kappa} = \text{const}, \quad (43)$$

which reflects the fact that ∂_t is a Killing vector in the original t, x coordinates and thus the solution behaves as $e^{-i\omega t - iS_\omega(x)}$ with an eikonal function $S_\omega(x)$ depending on $\kappa x = \text{arcsinh}(U/\tau)$. Finally, using that $k \gg \omega$ in the regime of interest, we find

$$\Omega_K \approx -\frac{U}{\tau}K = \frac{k}{\sqrt{\kappa^2 U^2 + \kappa^2 \tau^2}} \frac{U}{\tau}. \quad (44)$$

Now, if U and τ are roughly of the same order, i.e., for $\kappa x = \mathcal{O}(1)$, we know that k is large $k \gg \kappa$ and slowly varying (while ω is exactly constant). Thus we find that K and Ω_K are also slowly varying, for example

$$\frac{1}{K^2} \frac{\partial K}{\partial U} \approx \frac{\kappa}{k} \frac{U}{\sqrt{U^2 + \tau^2}} \leq \frac{\kappa}{k} \ll 1, \quad (45)$$

and similarly for the other terms. Consequently, we again find that the WKB approximation does not break down when approaching the horizon.

IX. CONCLUSIONS

In summary, we studied the origin of Hawking radiation in the presence of a modified dispersion relation at large k which provides an effective UV cutoff. We find that the transformation from (t, x) coordinates to (τ, U) coordinates offers several advantages: First, if the UV cutoff scale $1/\gamma$ is much larger than the surface gravity κ (i.e., Hawking temperature), then the WKB approximation remains valid throughout the evolution. Second, the derivation of Hawking radiation becomes much simpler—for example, we avoid dealing with nontrivial complex contour integrals involving branch cuts and saddle points, see, e.g., [6].

Note that—in contrast to the Schwarzschild coordinates, for example—both coordinate systems, (t, x) and (U, τ) , are regular across the horizon (i.e., “horizon-penetrating”). However, the U coordinate is better adapted to follow the gravitational redshift (i.e., the tearing apart) of the waves near the horizon, which explains their different behavior in terms of the WKB method.

The persistent applicability of the WKB method in the (τ, U) coordinates could also shed light on the slightly puzzling observation that there is so little mixing between left and right-moving modes. At low wavelengths, slow spatial variation of the field, the equations are approximately conformally invariant, which naturally separates the left and right moving branches of the solutions, at higher spatial frequencies this conformal invariance fails, leading to the expectation that there should be mixing between the left and right moving modes. The absence of any evidence for such mixing in the numerical studies has long been a puzzle. In terms of the (t, x) coordinates, one

might expect that all these branches mix when the WKB approximation breaks down, but Eq. (15) shows that the (right-moving) branch of modes with $\Omega_K/K > 0$ containing the Hawking particles stays well separated from the other (left-moving) branch with $\Omega_K/K < 0$.

As a result of the validity of the WKB approximation throughout the evolution, we find that the genesis of Hawking radiation is an extremely robust process—as long as the (yet unknown) theory of quantum gravity incorporates some sort of general covariance, which allows us to go from (t, x) coordinates to (τ, U) coordinates. In view of the black hole information “paradox” mentioned in the Introduction, this robustness shows that it is not obvious how to encode information in the outgoing Hawking photons.

It is important to note that the thermal properties of the outgoing radiation do not come about because of any interaction with other degrees of freedom, a la Planck’s explanation of black body radiation in terms of the degrees

of freedom in the walls of the cavity [12]. They are a direct consequence of the free field evolution. The particles have no obvious sources except for the tidal disruption of the evolution by the stretching of the wavelengths. The energy is locally conserved by the equations of motion, and does arise due to emission or absorption of that radiation.

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