PHYSICAL REVIEW D **88,** 124004 (2013)

Quasilocal conserved charges in the presence of a gravitational Chern-Simons term

Wontae Kim,[1,2,*] Shailesh Kulkarni,[1,3,†] and Sang-Heon Yi[1,‡]

[1] *Center for Quantum Spacetime (CQUeST), Sogang University, Seoul 121-742, Korea*
[2] *Department of Physics, Sogang University, Seoul 121-742, Korea*
[3] *Department of Physics, University of Pune, Ganeshkhind, Pune 411007, India*

(Received 15 October 2013; published 3 December 2013)

We extend our recent work on the quasilocal formulation of conserved charges to a theory of gravity containing a gravitational Chern-Simons term. As an application of our formulation, we compute the off-shell potential and quasilocal conserved charges of some black holes in three-dimensional topologically massive gravity. Our formulation for conserved charges reproduces very effectively the well-known expressions on conserved charges and the entropy expression of black holes in the topologically massive gravity.

DOI: 10.1103/PhysRevD.88.124004 PACS numbers: 04.20.Fy, 04.50.−h

I. INTRODUCTION

Conserved charges in general relativity is a very important and rather subtle concept. The main obstacle is related to the construction of the completely generally covariant energy-momentum tensor for gravitational field. Many people, including Einstein himself, have unsuccessfully tried to find such a tensor or to construct its alternatives, for instance, an energy momentum pseudo-tensor [1]. Now there is a general consensus that such a construction does not exist, since the local conservation law for general relativity turns out to be meaningless. However, several approaches have been suggested to construct total conserved charges in general relativity, one of which is the formalism accomplished by Arnowit-Deser-Misner (ADM) [2]. This approach uses a linearization of metric around the asymptotically flat spacetime and becomes cumbersome for the gravity actions which contain higher curvature or higher derivative terms. An extension of ADM formalism to higher curvature theories of gravity—known as the Abbott-Deser-Tekin (ADT) formalism—was provided in [3,4]. Unlike the ADM formalism, the ADT method is covariant and also applicable to the asymptotically AdS geometry.

There also exist other approaches to conserved charges, which are based on quasilocal concepts (for review, see [5]). One of such formulations is the Brown-York formalism [6] which needs to be improved for asymptotic AdS space by introducing the appropriate counterterms [7]. This formulation has been especially useful in the context of the AdS/CFT correspondence. Another such formulation is known as the Komar integrals [8] which are not known to be completely consistent with the results in the existing literatures. For instance, the mass and angular momentum calculated via Komar integrals contain the well-known factor two discrepancy when compared to the ADM formalism. In the covariant phase space approach initiated by Wald, the conserved charges were computed by using the Noether potential [9–12]. Wald's formulation has a distinct advantage in that it holds for any generally covariant theory of gravity and captures the entropy of black holes which can be regarded as the natural extension of Beckestein-Hawking area law [13]. Furthermore, this method established the first law of black hole thermodynamics in any covariant theory of gravity.

There exists an interesting connection between the *on-shell* ADT potential and the linearized on-shell Noether potential. Indeed, it was observed that at the asymptotic boundary, the linearized Noether potential around the on-shell background (which solves the Einstein equations), when combined with the surface term, produces the known expression for the ADT potential [14–17]. This relation, although very interesting, is indirect and shown to hold in Einstein gravity only. In our recent work [18], we have provided a nontrivial generalization of the above connection to any covariant theory of gravity. This was achieved by elevating the ADT potential to the *off-shell* level. Then, by using the corresponding off-shell expression for the linearized Noether potential supplemented with the surface term, we were able to show the desired connection directly. Integrating the resultant expression for the ADT potential along the one parameter path in the solution space, we finally obtained the expression for the quasilocal conserved charges which is identical with the one given by Wald's covariant phase space approach. At the on-shell level, this result establishes that our construction through the quasilocal extension of the ADT formalism is completely equivalent to the covariant phase space formalism which encompasses the black hole entropy.

There are certain aspects of our formalism which we would like to highlight at this stage. We may recall that in the conventional analysis of ADT potentials/charges one has to use the equations of motion for the background. As a result, the procedures become highly complicated when

*wtkim@sogang.ac.kr
†skulkarnig@gmail.com
‡shyi@sogang.ac.kr

the higher curvature or higher derivative terms are present in the Lagrangian. On the other hand, our formalism uses the off-shell (or background independent) expression for ADT and Noether potentials which are shown to be related in a one to one fashion. One can exploit this correspondence to obtain the conserved ADT charges for any covariant theory of gravity in a more efficient way. The off-shell Noether potential has already been used in the literature in somewhat different ways. For instance, the entropy for black holes was computed from the off-shell Noether potential [19–21]. In another work [22], we have used the off-shell Noether potential to compute the entropy for rotating extremal as well as nonextremal BTZ black holes in new massive gravity coupled to a scalar field. We have also computed the angular momentum of hairy AdS black holes and shown its invariance along the radial direction. This fact was used to verify the holographic c-theorem for hairy AdS black holes.

Most of the studies on constructing the Noether potential and the corresponding conserved charges have been limited to covariant theories of gravity. There are some attempts to generalize the Wald's formalism to the apparently noncovariant Lagrangians which often include gravitational Chern-Simons terms. Gravitational Chern-Simons terms are closely related to anomaly and appear frequently in the string theory context. Moreover, it has important implications in the AdS/CFT correspondence. One such theory with a gravitational Chern-Simons term is the three-dimensional topologically massive gravity (TMG) proposed by Deser *et al.* [23]. The extension of Wald's procedure to the TMG, especially for the black hole entropy, was provided by Tachikawa in [24]. The entropy computed from this approach matches exactly with the one obtained by indirect ways [25–30]. This on-shell approach was extended in conjunction with the covariance of black hole entropy [31]. On the other hand, the mass and angular momentum for the noncovariant theories like TMG have been obtained independently of the entropy, for instance, by using the ADT formalism [32,33], by the canonical method [34], or by using the direct codified computer implementation [35] of the formalism given in [14]. Another interesting aspect of TMG is the existence of warped AdS black hole solutions. These solutions with central charge expressions were a starting point for the warped AdS/CFT correspondence [36] and the Kerr/CFT correspondence [37], which may be extended to the dS/CFT case [38–40].

Interestingly, the exact relationship among the ADT charges and the Noether potential is still missing for TMG. At first glance, the Noether potential introduced in our previous paper [18] becomes noncovariant for an apparently noncovariant Lagrangian like TMG. On the contrary, the ADT potential is completely covariant since its construction is based on the covariant equations of motion. Therefore, it is not clear that the formalism given in our previous paper can be extended to this case. In the present work we would expedite this apparently noncovariant case and show that the formalism works well even in this case. As a specific example, we will take the topologically massive gravity to elaborate our formalism.

This paper is organized as follows. In the next section we propose a general framework for calculating quasilocal conserved potentials and charges for a theory of gravity of the apparently noncovariant Lagrangian. We then implement this procedure to TMG and show that our formalism matches completely with the covariant phase space approach to TMG. The quasilocal mass and angular momentum for the rotating Banados-Teitelboim-Zanelli (BTZ) [41] and warped AdS black holes [42] are computed in Sec. III. Finally, we summarize our findings in Sec. IV.

II. CONSERVED CURRENTS AND POTENTIALS

A. Formalism

In this section we extend our formulation of quasilocal conserved charges developed in [18]. First, we obtain a generic expression for the off-shell Noether current. This current, apart from the usual covariant terms, involves a noncovariant piece. This means that the off-shell Noether current itself is not a covariant quantity and so does not have a good physical interpretation just like Levi-Civita connection. However, it turns out that its linearized expression is related to the off-shell covariant ADT potential and so it can be used for the computation of conserved charges through the one-parameter path on the on-shell solution space.

Let us consider an action which contains the apparently noncovariant term. The variation of the action with respect to $g^{\mu\nu}$ will be taken by

$$\delta I = \frac{1}{16\pi G}\int d^D x\, \delta(\sqrt{-g}L) = \frac{1}{16\pi G}\int d^D x[\sqrt{-g}\mathcal{E}_{\mu\nu}\delta g^{\mu\nu} + \partial_\mu \Theta^\mu(\delta g)], \quad (1)$$

where $\mathcal{E}^{\mu\nu} = 0$ denotes the equations of motion (EOM) for the metric and Θ denotes the surface term. Note that the surface term Θ becomes noncovariant since we are considering the apparently noncovariant Lagrangian, though $\mathcal{E}^{\mu\nu}$ is a covariant expression. Under the diffeomorphism denoted by the parameter ζ, the Lagrangian transforms as

$$\delta_\zeta(L\sqrt{-g}) = \partial_\mu(\sqrt{-g}\zeta^\mu L) + \partial_\mu \Sigma^\mu(\zeta), \quad (2)$$

where Σ^μ term denotes an additional noncovariant term when the Lagrangian contains a noncovariant term like the gravitational Chern-Simons term.

In this case, the identically conserved current can be introduced as

$$J^\mu \equiv \partial_\nu K^{\mu\nu} = \sqrt{-g}\mathcal{E}^{\mu\nu}\zeta_\nu + \sqrt{-g}\zeta^\mu L + \Sigma^\mu(\zeta) - \Theta^\mu(\zeta). \quad (3)$$

Unlike the covariant Lagrangian case, this off-shell Noether current J^μ, and the potential $K^{\mu\nu}$ are not warranted to be covariant. This is naturally expected, since the Lagrangian L, the surface term Θ, and the boundary term Σ, all take the noncovariant forms. Just as in the covariant case, there are some ambiguities in the form of the Noether potential K, which turn out not to affect the final expression for quasilocal conserved charges.

In contrast, the ADT potential [3,4,43–45] is introduced in a completely covariant way. The on-shell ADT current is introduced for a Killing vector ξ^μ as $\mathcal{J}^\mu = \delta\mathcal{E}^{\mu\nu}\xi_\nu$, which can be shown to be conserved by using EOM, the Bianchi identity, and the Killing property of ξ. Then, the ADT potential Q is introduced by $\mathcal{J}^\mu = \nabla_\nu Q^{\mu\nu}$. Since these on-shell current and potential, which use the EOM, are highly involved for a higher curvature or derivative theory of gravity, the background-independent ADT current and potential were used for TMG [33] and new massive gravity [46]. In Ref. [18], we have realized the importance of the identically conserved ADT current and extended its use to a generic case. Explicitly, the off-shell ADT current and its potential for a Killing ξ can be defined by

$$\begin{aligned}\mathcal{J}^\mu_{\rm ADT} &\equiv \nabla_\nu Q^{\mu\nu}_{\rm ADT}\\ &= \delta\mathcal{E}^{\mu\nu}\xi_\nu + \frac{1}{2}g^{\alpha\beta}\delta g_{\alpha\beta}\mathcal{E}^{\mu\nu}\xi_\nu + \mathcal{E}^{\mu\nu}\delta g_{\nu\rho}\xi^\rho\\ &\quad - \frac{1}{2}\xi^\mu\mathcal{E}^{\alpha\beta}\delta g_{\alpha\beta},\end{aligned} \tag{4}$$

which can be shown to be conserved identically by using the Bianchi identity and the Killing property of ξ without using EOM. Since this off-shell ADT potential is based on the covariant EOM, it takes the covariant form even for the apparently noncovariant Lagrangian. This covariant nature of the ADT potential may lead to some worries about the inapplicability of our formalism to the apparently noncovariant case. However, as we shall see below, the formalism can be extended successfully even to such a case.

For matching the linearized off-shell Noether potential and the off-shell ADT potential, the diffeomorphism parameter ζ will be taken as a Killing vector ξ in the following. To extend the formalism in Ref. [18] to this case, let us introduce the formal Lie derivative for the noncovariant Θ term as

$$\mathcal{L}_\zeta\Theta^\mu = \zeta^\nu\partial_\nu\Theta^\mu - \Theta^\nu\partial_\nu\zeta^\mu + \partial_\nu\zeta^\nu\,\Theta^\mu. \tag{5}$$

Note that this Lie derivative satisfies the Leibniz rule. This Lie derivative of a noncovariant quantity is different from its diffeomorphism variation, which is not the case for a covariant one. Let us denote the difference between the Lie derivative and the diffeomorphism variation of (noncovariant) Θ-term as

$$\delta_\zeta\Theta^\mu(g;\zeta) = \mathcal{L}_\zeta\Theta^\mu(g;\delta g) + A^\mu(g;\delta g,\zeta). \tag{6}$$

By using the property of the Θ-term [11,47] for a Killing vector ξ

$$\delta_\xi\Theta^\mu(g;\delta g) - \delta\Theta^\mu(g;\xi) = 0, \tag{7}$$

and introducing $\Xi^{\mu\nu}$ as

$$\partial_\nu\Xi^{\mu\nu}(g;\delta g,\xi) \equiv A^\mu(g;\delta g,\xi) - \delta\Sigma^\mu(g;\xi), \tag{8}$$

one can see that

$$\begin{aligned}2\sqrt{-g}Q^{\mu\nu}_{\rm ADT} &= \delta K^{\mu\nu}(\xi) - 2\xi^{[\mu}\Theta^{\nu]}(g;\delta g)\\ &\quad + \Xi^{\mu\nu}(g;\delta g,\xi).\end{aligned} \tag{9}$$

This is the extension of the quasilocal formula for conserved charges in the covariant Lagrangian case to the apparently noncovariant one. The left-hand side of the above equation is covariant by construction [see Eq. (4)], while each term in the right-hand side is not warranted generically to be covariant. One may note that the additional term $\Xi^{\mu\nu}$ is responsible for the covariantization of the right-hand side. We would like to emphasize again that this quasilocal ADT potential is defined only up to the total derivative of a certain antisymmetric tensor $U^{\mu\nu\rho}$ just in the covariant case. This ambiguity does not affect the final expression for the conserved charges since it is a total derivative under the integral over a closed subspace.

By using the above quasilocal ADT potential and using the one-parameter path in the solution space, just like the covariant case, one can introduce conserved charges for the Killing vector ξ as

$$Q(\xi) = \frac{1}{8\pi G}\int_0^1 ds\int d^{D-2}x_{\mu\nu}\sqrt{-g}Q^{\mu\nu}_{\rm ADT}(g|s). \tag{10}$$

We would like to emphasize that the background and the variation are on-shell in the end, since we have taken the path in the solution space. The on-shell conservation of $Q^{\mu\nu}_{\rm ADT}$ has been used for construction of conserved charges. Using Eq. (9), the conserved charge $Q(\xi)$ can be obtained through the Noether potential and surface terms as

$$\begin{aligned}Q(\xi) &= \frac{1}{16\pi G}\int d^{D-2}x_{\mu\nu}\\ &\quad\times\left(\Delta K^{\mu\nu}(\xi) - 2\xi^{[\mu}\int_0^1 ds\Theta^{\nu]} + \int_0^1 ds\,\Xi^{\mu\nu}(\xi)\right),\end{aligned} \tag{11}$$

where ΔK denotes the finite difference of K-values between two endpoints of the one-parameter path in the solution space. The right-hand side in Eq. (11) can be regarded as the extension of the covariant phase space expression to the apparently noncovariant Lagrangian case, which was done at the on-shell level in [24]. On the bifurcate Killing horizon H, the second term in the right-hand side would vanish and the final expression gives us the well-known Wald's entropy as $(\kappa/2\pi)\mathcal{S} = Q_H$. Our construction shows that the conventional ADT charges should agree exactly with those from the covariant phase space approach. Conversely speaking, the quasilocal extension of the ADT formalism can reproduce the Wald's entropy for black holes. One of the lessons in this formulation is that the ADT charges and Wald's entropy do not need to be computed independently. Rather, they are directly related in our formulation and should be consistent with the first law of black hole thermodynamics by construction, as was shown by Wald [9,11].

B. Gravitational Chern-Simons term

In this section we apply our formulation of quasilocal conserved charges to a specific example: TMG in three dimensions. It turns out that the ADT potential can be obtained in a very concise form and is consistent with the previously known results.

Let us take the action for TMG in three dimensions [23] as

$$I[g_{\mu\nu}] = \frac{1}{16\pi G}\int d^3x\sqrt{-g}\left[R + \frac{2}{L^2} + \frac{1}{\mu}L_{\rm CS}\right]. \quad (12)$$

The last term for the gravitational Chern-Simons term is given by

$$L_{\rm CS} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho}\left(\Gamma^{\alpha}_{\mu\beta}\partial_\nu\Gamma^{\beta}_{\rho\alpha} + \frac{2}{3}\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\gamma}\Gamma^{\gamma}_{\rho\alpha}\right), \quad (13)$$

where the ϵ-tensor is defined such that $\sqrt{-g}\epsilon^{012} = 1$. Our convention for the curvature tensor is taken as $[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu}V^\sigma$ and the mostly plus metric signature is employed.

The equations of motion for TMG are given by

$$G_{\mu\nu} - \frac{1}{L^2}g_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} = 0, \quad (14)$$

where $G_{\mu\nu}$ denotes the Einstein tensor and $C_{\mu\nu}$ denotes the Cotton tensor defined by

$$C^{\mu\nu} = \epsilon^{\alpha\beta\mu}\nabla_\alpha\left(R^\nu_\beta - \frac{1}{4}\delta^\nu_\beta R\right). \quad (15)$$

The above Cotton tensor is traceless, symmetric, and divergence-free, which is the three-dimensional analog of the Weyl tensor. One may note that it can also be written as $C^{\mu\nu} = \epsilon^{\alpha\beta(\mu}\nabla_\alpha R^{\nu)}_\beta$. In the following, we use $h_{\mu\nu}$ for the linearized metric interchangeably with $\delta g_{\mu\nu}$ and all the indices are raised and lowered by the background metric g.

The quasilocal ADT potential for the Ricci scalar part has been known to be given by

$$Q^{\mu\nu}_R(\xi) = \frac{1}{2}h\nabla^{[\mu}\xi^{\nu]} - h^{\alpha[\mu}\nabla_\alpha\xi^{\nu]} - \xi^{[\mu}\nabla_\alpha h^{\nu]\alpha} + \xi_\alpha\nabla^{[\mu}h^{\nu]\alpha} + \xi^{[\mu}\nabla^{\nu]}h, \quad (16)$$

which can also be derived from the quasilocal ADT formalism given in [18]. Since the construction has already been done for the covariant terms, let us focus on the gravitational Chern-Simons term in the following. The surface term for the gravitational Chern-Simons term under a generic variation turns out to be

$$\Theta^\mu(\delta g) = \frac{1}{2}\sqrt{-g}[\epsilon^{\mu\nu\rho}\Gamma^{\alpha}_{\rho\beta}\delta\Gamma^{\beta}_{\nu\alpha} + \epsilon^{\alpha\nu\rho}R_{\nu\rho}{}^{\mu\beta}\delta g_{\alpha\beta}]. \quad (17)$$

Note that the surface Θ-term is noncovariant though the EOM is covariant. Under a diffeomorphism with a parameter ζ, the Christoffel symbol transforms as

$$\delta_\zeta\Gamma^\rho_{\mu\nu} = \mathcal{L}_\zeta\Gamma^\rho_{\mu\nu} + \partial_\mu\partial_\nu\zeta^\rho, \quad (18)$$

where $\mathcal{L}_\zeta$ denotes the Lie derivative defined in the same way with the Θ-term as

$$\mathcal{L}_\zeta\Gamma^\rho_{\mu\nu} = \zeta^\sigma\partial_\sigma\Gamma^\rho_{\mu\nu} - \Gamma^\sigma_{\mu\nu}\partial_\sigma\zeta^\rho + 2\Gamma^\rho_{\sigma(\mu}\partial_{\nu)}\zeta^\sigma.$$

Then, one can see that $L_{\rm CS}$ transforms under diffeomorphism as

$$\delta_\zeta L_{\rm CS} = \partial_\mu(\sqrt{-g}\zeta^\mu L_{\rm CS}) + \partial_\mu\Sigma^\mu_{\rm CS}, \quad (19)$$

where the additional boundary term $\Sigma_{\rm CS}$ is given by

$$\Sigma^\mu_{\rm CS} = \frac{1}{2}\sqrt{-g}\epsilon^{\mu\nu\rho}\partial_\nu\Gamma^\beta_{\rho\alpha}\partial_\beta\zeta^\alpha. \quad (20)$$

The surface term for this diffeomorphism is given by

$$\Theta^\mu_{\rm CS}(\zeta) = \sqrt{-g}\left[\frac{1}{2}\epsilon^{\mu\nu\rho}\Gamma^\alpha_{\rho\beta}\delta_\zeta\Gamma^\beta_{\nu\alpha} + 2\epsilon^{\mu\nu(\alpha}R^{\beta)}_\nu\nabla_\alpha\zeta_\beta\right]. \quad (21)$$

One may note that the above Σ-term and the Θ-term have some ambiguities. Nevertheless, those do not affect our essential steps and so the above explicit expressions are taken for definiteness.

According to the generic formulation given in Eq. (3), the off-shell current and Noether potential for a gravitational Chern-Simons term are introduced by

$$J^\mu_{\rm CS} \equiv \partial_\nu K^{\mu\nu}_{\rm CS} = 2\sqrt{-g}C^{\mu\nu}\zeta_\nu + \sqrt{-g}\zeta^\mu L_{\rm CS} + \Sigma^\mu_{\rm CS}(\zeta) - \Theta^\mu_{\rm CS}(\zeta). \quad (22)$$

By using three-dimensional identities given in the appendix, one can obtain the off-shell Noether potential in the form of[1]

$$K^{\mu\nu}_{\rm CS} = 2\sqrt{-g}\epsilon^{\mu\nu\rho}\left[\left(R^\sigma_\rho - \frac{1}{4}R\delta^\sigma_\rho\right)\zeta_\sigma - \frac{1}{4}\Gamma^\beta_{\rho\alpha}\nabla_\beta\zeta^\alpha\right]. \quad (23)$$

The additional term $\Xi^{\mu\nu}_{\rm CS}$ can be shown to be given by

[1] See [48] for the on-shell Noether potential for a gravitational Chern-Simons term.

$$\Xi^{\mu\nu}_{\rm CS}(g;\delta g,\zeta)=-\frac{1}{2}\sqrt{-g}\epsilon^{\mu\nu\rho}\delta\Gamma^{\beta}_{\rho\alpha}\partial_{\beta}\zeta^{\alpha}$$
$$=-\frac{1}{2}\sqrt{-g}\epsilon^{\mu\nu\rho}\delta\Gamma^{\beta}_{\rho\alpha}\nabla_{\beta}\zeta^{\alpha}$$
$$+\frac{1}{2}\sqrt{-g}\epsilon^{\mu\nu\rho}\Gamma^{\alpha}_{\beta\sigma}\delta\Gamma^{\beta}_{\rho\alpha}\zeta^{\sigma}. \quad (24)$$

Collecting the above results, one can obtain the contribution of the gravitational Chern-Simons term to conserved charges and the entropy of black holes. First, let us consider the contribution of the Chern-Simons term to the entropy of black holes. By using Eq. (9) with the on-shell background metric and taking ζ as the Killing vector ξ_H for the Killing horizon H, one can show that the contribution of the Chern-Simons term is given by

$$\frac{\kappa}{2\pi}\delta\mathcal{S}_{\rm CS}=-\frac{1}{16\pi G}\int_H d^{D-2}x_{\mu\nu}\sqrt{-g}\epsilon^{\mu\nu\rho}\delta\Gamma^{\beta}_{\rho\alpha}\nabla_{\beta}\xi^{\alpha}_{H}, \quad (25)$$

where we have used the property of the bifurcate Killing horizon such that ξ vanishes on H. This expression is completely covariant and can be integrated into a finite form which is consistent with the one obtained in the covariant phase space approach [24,31].

Now, by using our relation given in Eq. (9), one can obtain the quasilocal ADT potential for the three-dimensional gravitational Chern-Simons term as[2]

$$Q^{\mu\nu}_{\rm CS}=\epsilon^{\mu\nu\rho}\xi^{\sigma}\delta\left(R_{\rho\sigma}-\frac{1}{4}Rg_{\rho\sigma}\right)-\frac{1}{2}\epsilon^{\mu\nu\rho}\delta\Gamma^{\beta}_{\rho\alpha}\nabla_{\beta}\xi^{\alpha}$$
$$-\xi^{[\mu}\epsilon^{\nu]\rho(\alpha}R^{\beta)}_{\rho}\delta g_{\alpha\beta}. \quad (26)$$

Note that this expression is completely covariant as was shown generically to be the case in the previous section. We would like to compare our results to the previously known expressions of the ADT potential for the gravitational Chern-Simons term. To achieve this goal, let us introduce the totally antisymmetric tensor $U^{\mu\nu\rho}$ as

$$U^{\mu\nu\rho}\equiv\frac{1}{2}(\epsilon^{\rho\alpha\beta}\nabla_{\beta}\xi^{[\mu}h^{\nu]}_{\alpha}+\epsilon^{\alpha\beta[\mu}\nabla_{\beta}\xi^{\nu]}h^{\rho}_{\alpha}$$
$$+h^{[\mu}_{\alpha}\epsilon^{\nu]\alpha\beta}\nabla_{\beta}\xi^{\rho}). \quad (27)$$

Using the Killing property of ξ and the identities given in the appendix, one can show that $U^{\mu\nu\rho}$ satisfies

$$\nabla_{\rho}U^{\mu\nu\rho}=\frac{1}{2}hR\epsilon^{\mu\nu\rho}\xi_{\rho}+hR^{[\mu}_{\alpha}\epsilon^{\nu]\alpha\beta}\xi_{\beta}+Rh^{[\mu}_{\alpha}\epsilon^{\nu]\alpha\beta}\xi_{\beta}$$
$$-\frac{1}{2}\epsilon^{\mu\nu\rho}\xi_{\rho}h^{\alpha\beta}R_{\alpha\beta}+\epsilon^{\alpha\beta\rho}h^{[\mu}_{\alpha}R^{\nu]}_{\beta}\xi_{\rho}$$
$$-h_{\alpha\beta}R^{\alpha[\mu}\epsilon^{\nu]\beta\rho}\xi_{\rho}-R_{\alpha\beta}h^{\alpha[\mu}\epsilon^{\nu]\beta\rho}\xi_{\rho}.$$

[2]We have been informed from B. Tekin that essentially the same expression was obtained in [49].

As a result, one can verify that the above expression of the ADT potential for the gravitational Chern-Simons term $Q^{\mu\nu}_{\rm CS}$ can be rewritten as[3]

$$Q^{\mu\nu}_{\rm CS}=\nabla_{\rho}U^{\mu\nu\rho}+Q^{\mu\nu}_{R}(\eta)+\epsilon^{\mu\nu\rho}\left[\delta G^{\lambda}_{\rho}\xi_{\lambda}-\frac{1}{2}\delta G\xi_{\rho}\right.$$
$$\left.+\frac{1}{2}\xi_{\rho}h^{\alpha\beta}G_{\alpha\beta}+\frac{1}{4}h\left(\xi_{\sigma}G^{\sigma}_{\rho}+\frac{1}{2}\xi_{\rho}R\right)\right], \quad (28)$$

where η is defined by $\eta^{\mu}\equiv\frac{1}{2}\epsilon^{\mu\alpha\beta}\nabla_{\alpha}\xi_{\beta}$. This computation shows us explicitly the equivalence of our expression of the background independent or off-shell ADT potential to the one given in [33]. Though our expression of the off-shell ADT potential is more succinct and illuminating, we would like to emphasize that we can use Eq. (11) for the computation of conserved charges instead of the explicit expression of the ADT potential. Using Eq. (11), one can also obtain the entropy of black holes in TMG at one stroke.

In order to apply Eq. (11) to black hole solutions in TMG in the next section, let us summarize what we have computed. In TMG, the off-shell Noether potential, Θ-term, and Ξ-term are given by

$$K^{\mu\nu}_{\rm TMG}(g;\zeta)=\sqrt{-g}\left[2\nabla^{[\mu}\zeta^{\nu]}+\frac{2}{\mu}\epsilon^{\mu\nu\rho}\right.$$
$$\left.\times\left\{\left(R^{\sigma}_{\rho}-\frac{1}{4}R\delta^{\sigma}_{\rho}\right)\zeta_{\sigma}-\frac{1}{4}\Gamma^{\beta}_{\rho\alpha}\nabla_{\beta}\zeta^{\alpha}\right\}\right],$$
$$\Theta^{\mu}_{\rm TMG}(g,\delta g)=\sqrt{-g}\left[\nabla^{\mu}(g_{\alpha\beta}\delta g^{\alpha\beta})-\nabla_{\nu}\delta g^{\mu\nu}\right.$$
$$\left.+\frac{1}{\mu}\left\{\frac{1}{2}\epsilon^{\mu\nu\rho}\Gamma^{\alpha}_{\rho\beta}\delta\Gamma^{\beta}_{\nu\alpha}+\epsilon^{\mu\nu(\alpha}R^{\beta)}_{\nu}\delta g_{\alpha\beta}\right\}\right],$$
$$\Xi^{\mu\nu}_{\rm TMG}(g;\delta g,\zeta)=-\frac{1}{2\mu}\sqrt{-g}\epsilon^{\mu\nu\rho}\delta\Gamma^{\beta}_{\rho\alpha}\partial_{\beta}\zeta^{\alpha}. \quad (29)$$

It is interesting to note that each of the above expressions are noncovariant, as expected.

III. BLACK HOLES AND THEIR CHARGES

In this section, we compute the mass and angular momentum of some black holes in TMG as the simplest example of our formulation. Since our formulation was shown to give us the background independent ADT potential which is equivalent to the previously known expression in [33], the mass and angular momentum for black holes[4] in TMG are assured to be given by the same expression. However, it is illuminating and fruitful to reproduce these results by using the expression of conserved charges given in Eq. (11). In all the given examples, upper index components of relevant Killing vectors are taken to be constant and so the Ξ-term contribution vanishes.

[3]See appendix for some details of this computation.

[4]Since the computation of the entropy of these black holes is identical to the covariant phase space approach and gives nothing new, we omit these parts.

In our convention, the metric of the BTZ black hole [41] is taken in the following form of

$$ds^2 = L^2\left[-\frac{(r^2-r_+^2)(r^2-r_-^2)}{r^2}dt^2 + \frac{r^2}{(r^2-r_+^2)(r^2-r_-^2)}dr^2 + r^2\left(d\theta - \frac{r_+r_-}{r^2}dt\right)^2\right]. \quad (30)$$

The Killing vectors for the time-translational and rotational symmetry will be chosen as $\xi = \frac{\partial}{L\partial t}, \frac{\partial}{\partial\theta}$, respectively. To utilize the formula given in Eq. (11), take an infinitesimal parametrization of a one-parameter path in the solution space as follows

$$r_+ \to r_+ + dr_+, \qquad r_+ \to r_- + dr_-.$$

By expanding the above BTZ metric in terms of $dr_\pm$ and keeping terms up to the relevant order, one can obtain the infinitesimal expression of the Θ-term. And then, one can integrate this expression to obtain conserved charges. Let us consider the quasilocal angular momentum of the BTZ black hole at first. After a bit of computation, one can see that, just like the covariant case, the quasilocal angular momentum for the rotational Killing vector $\xi_R = \frac{\partial}{\partial\theta}$ comes entirely from the ΔK-term, of which the relevant component is

$$\Delta K^{rt}_{\mathrm{TMG}}(\xi_R) = -2Lr_+r_- + \frac{1}{\mu}(r_+^2 + r_-^2).$$

As a result, the angular momentum of the BTZ black hole is given by

$$J = \frac{1}{16\pi G}\int_0^{2\pi} d\theta \Delta K^{rt}_{\mathrm{TMG}} = -\frac{Lr_+r_-}{4G} + \frac{r_+^2+r_-^2}{8G\mu}. \quad (31)$$

By noting that the nonvanishing components of the infinitesimal Θ-term and the ΔK-term for a Killing vector $\xi_T = \frac{1}{L}\frac{\partial}{\partial t}$ are

$$\Theta^r = Ld(r_+^2 + r_-^2), \qquad \Delta K^{rt}(\xi_T) = -2\frac{r_+r_-}{L\mu},$$

one can show that the mass of the BTZ black hole in TMG is given in the form of

$$M = \frac{1}{16\pi G}\int_0^{2\pi} d\theta\left(\Delta K^{rt}_{\mathrm{TMG}}(\xi_T) + \xi_T^t \int \Theta^r\right) = \frac{r_+^2 + r_-^2}{8G} - \frac{r_+r_-}{4GL\mu}. \quad (32)$$

These expressions match completely with the known results. Note that our convention is such that the first law of black hole thermodynamics holds in the form of $dM = T_H d\mathcal{S}_{\mathrm{BH}} - \Omega dJ$.

Now, let us consider the warped AdS black hole in TMG, of which expressions for the mass and angular momentum are rather involved. The metric of the warped AdS black hole may be taken as [42]

$$ds^2 = -\beta^2\frac{\rho^2-\rho_0^2}{Z^2}dt^2 + \frac{d\rho^2}{\zeta^2\beta^2(\rho^2-\rho_0^2)} + Z^2\left(d\theta - \frac{\rho + (1-\beta^2)\omega}{Z^2}dt\right)^2, \quad (33)$$

where $Z^2 \equiv \rho^2 + 2\omega\rho + (1-\beta^2)\omega^2 + \beta^2\rho_0^2/(1-\beta^2)$. Two of the four parameters in the above metric, β and ζ, are related to the Lagrangian parameter $1/L^2$ and $1/\mu$ as follows

$$\beta^2 \equiv \frac{1}{4}\left(1 + \frac{27}{\mu^2L^2}\right), \qquad \zeta = \frac{2}{3}\mu. \quad (34)$$

The other two parameters ω and ρ_0 are related to the mass and angular momentum of this black hole. In this case one can choose the infinitesimal one-parameter path in the solution space as[5]

$$\omega \to \omega + d\omega, \qquad \rho_0^2 \to \rho_0^2 + d\rho_0^2.$$

As in the case of the BTZ black hole, it is sufficient to keep various terms up to linear parts of the variations. Then, the quasilocal conserved angular momentum for the rotational Killing vector $\xi_R = \frac{\partial}{\partial\theta}$ can be shown to come entirely from the $\Delta K^{\mu\nu}$-term, while the quasilocal mass for the timelike Killing vector $\xi_T = \frac{\partial}{\partial t}$ has another contribution from the Θ-term.

Let us consider the angular momentum of the warped AdS black hole at first. By using the relevant component of the ΔK-term for the rotational Killing vector ξ_R

$$\Delta K^{\rho t}_{\mathrm{TMG}}(\xi_R) = -\frac{2}{3}\zeta\beta^2\left[(1-\beta^2)\omega^2 - \frac{1+\beta^2}{1-\beta^2}\rho_0^2\right], \quad (35)$$

one can obtain the quasilocal angular momentum of the warped AdS_3 black hole as

$$J = \frac{1}{16\pi G}\int_0^{2\pi} d\theta \Delta K^{\rho t}_{\mathrm{TMG}}(\xi_R) = -\frac{\zeta\beta^2}{12G}\left[(1-\beta^2)\omega^2 - \frac{1+\beta^2}{1-\beta^2}\rho_0^2\right]. \quad (36)$$

Now, let us turn to the mass of the black hole for the timelike Killing vector ξ_T. In this case the nonvanishing component of the infinitesimal Θ-term for the above chosen path turns out to be

$$\Theta^\rho_{\mathrm{TMG}} = \frac{2}{3}\zeta\beta^2(1-\beta^2)d\omega. \quad (37)$$

By combining this with the ΔK contribution

[5] Note that we do not need to expand in terms of $d\rho_0$ since the form of ρ_0^2 only appears in the metric.

$$\Delta K^{\rho t}_{\rm TMG}(\xi_T) = \frac{2}{3}\zeta\beta^2(1-\beta^2)\omega, \tag{38}$$

one can see that mass is given by

$$M = \frac{1}{16\pi G}\int_0^{2\pi} d\theta\Big(\Delta K^{\rho t}_{\rm TMG}(\xi_T) + \xi^t_T\int\Theta^\rho\Big) = \frac{\zeta\beta^2}{6G}(1-\beta^2)\omega. \tag{39}$$

Note that the above expressions for the mass and angular momentum match completely with those given in [33] up to sign convention for angular momentum. (See [36,50,51] for a dual CFT interpretation for these black holes.)

IV. CONCLUSION

In this paper we have extended our previous formalism for quasilocal conserved charges to a theory of gravity with a gravitational Chern-Simons term. This formulation turns out to be very effective in obtaining the ADT potential and quasilocal charges. In fact, we have shown that this quasilocal extension of the ADT method even to an apparently noncovariant Lagrangian is completely equivalent to the covariant phase space approach. We have explicitly verified that this formulation reproduces the known background independent ADT potential for TMG up to the irrelevant total derivative of a totally antisymmetric tensor. Furthermore, quasilocal conserved charges for the BTZ black holes and the warped AdS black holes are reproduced, which are completely consistent with the previously known results.

It would be very interesting to develop this formulation further to encompass the asymptotic Killing vectors, which is relevant to the construction of the asymptotic Virasoro algebra in the context of the AdS/CFT, Kerr/CFT, and dS/CFT correspondence. This would allow us to extract the information of the central charge and eventually the black hole entropy. Another interesting direction would be to use the off-shell Noether potential, $K_{\mu\nu}$ [see Eq. (29)] in the stretched horizon approach developed by Carlip [52]. This will lead to the near horizon Virasoro algebra and the entropy of black holes.

ACKNOWLEDGMENTS

We would like to thank B. Tekin for a useful correspondence. This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIP) through the CQUeST of Sogang University with Grant No. 2005-0049409. W. K. was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MOE) (2010-0008359). S.-H. Y. was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MOE) (No. 2012R1A1A2004410). S. Kulkarni was also supported by the INSPIRE faculty scheme (IFA-13 PH-56) by the Department of Science and Technology (DST), India.

APPENDIX

Here we shall give some identities and formulas which are useful in the text, especially in Sec. II B. In three dimensions we have the following identities

$$\epsilon^{\mu\nu\rho}V^\sigma = g^{\mu\sigma}\epsilon^{\nu\rho\alpha}V_\alpha + g^{\nu\sigma}\epsilon^{\rho\mu\alpha}V_\alpha + g^{\rho\sigma}\epsilon^{\mu\nu\alpha}V_\alpha,$$
$$R_{\mu\nu\rho\sigma} = R_{\mu\rho}g_{\nu\sigma} + R_{\nu\sigma}g_{\mu\rho} - R_{\mu\sigma}g_{\nu\rho} - R_{\nu\rho}g_{\mu\sigma} - \frac{1}{2}R(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma}). \tag{A1}$$

We have used the following convention for ϵ tensor and the integration measure

$$\sqrt{-g}\epsilon^{tr\theta} = 1, \qquad dx_{\mu\nu} \equiv dx^\rho\epsilon_{\mu\nu\rho}\frac{1}{2\sqrt{-g}}. \tag{A2}$$

A Killing vector ξ satisfies

$$\nabla_{(\mu}\xi_{\nu)} = 0, \qquad \nabla_\mu\nabla_\nu\xi_\rho = \xi^\sigma R_{\sigma\mu\nu\rho}. \tag{A3}$$

For a Killing vector ξ, let us introduce another vector field η formed by contracting the covariant derivative of ξ with the ϵ-tensor

$$\eta^\mu \equiv \frac{1}{2}\epsilon^{\mu\alpha\beta}\nabla_\alpha\xi_\beta. \tag{A4}$$

Such a vector field η obeys

$$\nabla^{[\mu}\eta^{\nu]} = \frac{1}{2}R\epsilon^{\mu\nu\rho}\xi_\rho + R^{[\mu}_\alpha\epsilon^{\nu]\alpha\beta}\xi_\beta, \qquad -h^{\alpha[\mu}\nabla_\alpha\eta^{\nu]} = \frac{1}{2}Rh^{[\mu}_\alpha\epsilon^{\nu]\alpha\beta}\xi_\beta + \epsilon^{\alpha\beta\rho}h^{[\mu}_\alpha R^{\nu]}_\beta\xi_\rho,$$
$$\eta^{[\mu}\nabla_\alpha h^{\nu]\alpha} = \frac{1}{2}\nabla_\rho\xi_\sigma\epsilon^{\rho\sigma[\mu}\nabla_\alpha h^{\nu]\alpha}, \qquad \eta^{[\mu}\nabla^{\nu]}h = \frac{1}{2}\nabla_\rho\xi_\sigma\epsilon^{\rho\sigma[\mu}\nabla^{\nu]}h,$$
$$-\frac{1}{2}\epsilon^{\mu\nu\rho}\delta\Gamma^\beta_{\rho\alpha}\nabla_\beta\xi^\alpha = \eta_\alpha\nabla^{[\mu}h^{\nu]\alpha} - \eta^{[\mu}\nabla_\alpha h^{\nu]\alpha} + \eta^{[\mu}\nabla^{\nu]}h,$$
$$\eta_\alpha\nabla^{[\mu}h^{\nu]\alpha} = \frac{1}{2}\nabla_\rho\xi_\sigma\epsilon^{\rho\sigma\alpha}\nabla^{[\mu}h^{\nu]}_\alpha = \epsilon^{\alpha\rho\beta}\nabla_\rho\xi^{[\mu}\nabla_\beta h^{\nu]}_\alpha$$
$$= 2h\nabla^{[\mu}\eta^{\nu]} + \frac{5}{2}Rh^{[\mu}_\alpha\epsilon^{\nu]\alpha\rho}\xi_\rho - \epsilon^{\mu\nu\rho}\xi_\rho h^{\alpha\beta}R_{\alpha\beta} - 2h^\beta_\alpha R^{[\mu}_\beta\epsilon^{\nu]\alpha\rho}\xi_\rho - 3R^\rho_\alpha h^{[\mu}_\rho\epsilon^{\nu]\alpha\rho}\xi_\rho + 2\epsilon^{\alpha\beta\rho}h^{[\mu}_\alpha R^{\nu]}_\beta\xi_\rho - \nabla_\beta(\epsilon^{\alpha\rho[\mu}\nabla_\rho\xi^{\nu]}h^\beta_\alpha + h^{[\mu}_\alpha\epsilon^{\nu]\alpha\rho}\nabla_\rho\xi^\beta). \tag{A5}$$

Here, $h_{\mu\nu}$ and h represents the linearized metric and its trace, respectively. Another useful identity for the $\nabla_\rho U^{\mu\nu\rho}$ computation is

$$\nabla_\beta(\epsilon^{\alpha\rho\beta}\nabla_\rho \xi^{[\mu} h_\alpha^{\nu]}) = \epsilon^{\alpha\rho\beta}\nabla_\rho \xi^{[\mu}\nabla_\beta h_\alpha^{\nu]} + \frac{1}{2} R h_\alpha^{[\mu}\epsilon^{\nu]\alpha\rho}\xi_\rho + R_\alpha^{[\mu} h_\beta^{\nu]}\epsilon^{\alpha\beta\rho}\xi_\rho + \xi_\sigma R_\rho^\sigma \epsilon^{\alpha\rho[\mu} h_\alpha^{\nu]}.$$

[1] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon Press, Oxford, 1975).
[2] R. L. Arnowitt, S. Deser, and C. W. Misner, Gen. Relativ. Gravit. **40**, 1997 (2008).
[3] L. F. Abbott and S. Deser, Nucl. Phys. **B195**, 76 (1982).
[4] S. Deser and B. Tekin, Phys. Rev. D **67**, 084009 (2003).
[5] L. B. Szabados, Living Rev. Relativity **7**, 4 (2004).
[6] J. D. Brown and J. W. York, Phys. Rev. D **47**, 1407 (1993).
[7] V. Balasubramanian and P. Kraus, Commun. Math. Phys. **208**, 413 (1999).
[8] A. Komar, Phys. Rev. **113**, 934 (1959).
[9] R. M. Wald, Phys. Rev. D **48**, R3427 (1993).
[10] T. Jacobson, G. Kang, and R. C. Myers, Phys. Rev. D **49**, 6587 (1994).
[11] V. Iyer and R. M. Wald, Phys. Rev. D **50**, 846 (1994).
[12] R. M. Wald and A. Zoupas, Phys. Rev. D **61**, 084027 (2000).
[13] J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973).
[14] G. Barnich and F. Brandt, Nucl. Phys. **B633**, 3 (2002).
[15] G. Barnich, Classical Quantum Gravity **20**, 3685 (2003).
[16] G. Barnich and G. Compere, Phys. Rev. D **71**, 044016 (2005); **73**, 029904(E) (2006).
[17] G. Barnich and G. Compere, J. Math. Phys. (N.Y.) **49**, 042901 (2008).
[18] W. Kim, S. Kulkarni, and S.-H. Yi, Phys. Rev. Lett. **111**, 081101 (2013).
[19] B. R. Majhi and T. Padmanabhan, Phys. Rev. D **85**, 084040 (2012).
[20] B. R. Majhi and T. Padmanabhan, Phys. Rev. D **86**, 101501 (2012).
[21] B. R. Majhi, Adv. High Energy Phys. **2013**, 386342 (2013).
[22] W. Kim, S. Kulkarni, and S.-H. Yi, J. High Energy Phys. 05 (2013) 041.
[23] S. Deser, R. Jackiw, and S. Templeton, Ann. Phys. (N.Y.) **140**, 372 (1982); **185**, 406(E) (1988); **185**, 406 (1988); **281**, 409 (2000).
[24] Y. Tachikawa, Classical Quantum Gravity **24**, 737 (2007).
[25] J.-H. Cho, J. Korean Phys. Soc. **44**, 1355 (2004).
[26] P. Kraus and F. Larsen, J. High Energy Phys. 09 (2005) 034.
[27] P. Kraus and F. Larsen, J. High Energy Phys. 01 (2006) 022.
[28] S. N. Solodukhin, Phys. Rev. D **74**, 024015 (2006).
[29] B. Sahoo and A. Sen, J. High Energy Phys. 07 (2006) 008.
[30] M.-I. Park, Phys. Rev. D **77**, 026011 (2008).
[31] L. Bonora, M. Cvitan, P. D. Prester, S. Pallua, and I. Smolic, J. High Energy Phys. 07 (2011) 085; Classical Quantum Gravity **28**, 195009 (2011); J. High Energy Phys. 10 (2012) 077; Phys. Rev. D **87**, 024047 (2013).
[32] S. Deser and B. Tekin, Classical Quantum Gravity **20**, L259 (2003).
[33] A. Bouchareb and G. Clement, Classical Quantum Gravity **24**, 5581 (2007).
[34] K. Hotta, Y. Hyakutake, T. Kubota, and H. Tanida, J. High Energy Phys. 07 (2008) 066.
[35] B. Chen, J.-j. Zhang, J.-d. Zhang, and D.-l. Zhong, J. High Energy Phys. 04 (2013) 055.
[36] D. Anninos, W. Li, M. Padi, W. Song, and A. Strominger, J. High Energy Phys. 03 (2009) 130.
[37] M. Guica, T. Hartman, W. Song, and A. Strominger, Phys. Rev. D **80**, 124008 (2009).
[38] A. Strominger, J. High Energy Phys. 10 (2001) 034.
[39] D. Anninos, J. High Energy Phys. 02 (2010) 046.
[40] S. de Buyl, S. Detournay, G. Giribet, and G. S. Ng, arXiv:1308.5569.
[41] M. Banados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. **69**, 1849 (1992).
[42] K. A. Moussa, G. Clement, and C. Leygnac, Classical Quantum Gravity **20**, L277 (2003).
[43] S. Deser and B. Tekin, Phys. Rev. Lett. **89**, 101101 (2002).
[44] C. Senturk, T. C. Sisman, and B. Tekin, Phys. Rev. D **86**, 124030 (2012).
[45] A. J. Amsel and D. Gorbonos, Phys. Rev. D **87**, 024032 (2013).
[46] S. Nam, J.-D. Park, and S.-H. Yi, Phys. Rev. D **82**, 124049 (2010).
[47] J. Lee and R. M. Wald, J. Math. Phys. (N.Y.) **31**, 725 (1990).
[48] A. Borowiec, M. Ferraris, and M. Francaviglia, J. Phys. A **31**, 8823 (1998); **36**, 2589 (2003); A. Borowiec, L. Fatibene, M. Ferraris, and M. Francaviglia, Int. J. Geom. Methods Mod. Phys. **03**, 755 (2006); **04**, 277 (2007).
[49] C. Nazaroglu, Y. Nutku, and B. Tekin, Phys. Rev. D **83**, 124039 (2011).
[50] G. Compere and S. Detournay, Classical Quantum Gravity **26**, 012001 (2009); **26**, 139801(E) (2009).
[51] A. Ghodsi and D. M. Yekta, J. High Energy Phys. 06 (2012) 131.
[52] S. Carlip, Classical Quantum Gravity **16**, 3327 (1999).