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Radion stability and induced, on-brane geometries in an effective scalar-tensor theory of gravity

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About a decade ago, using a specific expansion scheme, effective, on-brane scalar tensor theories of gravity were proposed by Kanno and Soda [Phys. Rev. D **66**, 083506 (2002)] in the context of the warped two-brane model of Randall–Sundrum. The inter-related effective theories on both the branes were derived with the space-time dependent radion field playing a crucial role. Taking another look at this effective theory, we find cosmological and spherically symmetric, static solutions sourced by a radion-induced, effective stress energy, as well as additional, on-brane matter. The distance between the branes (governed by the time or space dependent radion) is shown to be stable and asymptotically nonzero, thereby setting aside any possibility of brane collisions. It turns out that the inclusion of on-brane matter plays a decisive role in stabilising the radion—a fact which we demonstrate through our solutions.

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I. INTRODUCTION

The possible existence of extra spatial dimensions is now a well-known theoretical assumption where our four-dimensional world is considered to be a 3-brane embedded in a higher-dimensional spacetime. Such a description emerges naturally in the backdrop of various string-inspired models [1]. Moreover, extradimensional models were developed as a nonsupersymmetric, alternative approach in tackling the well-known fine-tuning/gauge hierarchy problem in the regime of the Standard Model of particle physics. It became more and more evident that gravity may become an integral part to address issues on physics beyond the Standard Model.

The extradimensional models can broadly be classified into those having large compact radii [2] or having small compact radii [3]. Regarding their geometry, these models are generally compactified under various topological setups. The uncompactified, four-dimensional spacetime then emerges as a low energy effective theory which contains signatures of the higher-dimensional theory.

However, among all models proposed so far, we will confine ourselves to the Randall–Sundrum (RS) model [3], which has two 3-branes, with equal and opposite brane tensions, embedded in a five-dimensional spacetime. This model was initially developed to combat the unnatural fine-tuning involved in determining the mass of the Higgs boson. While determining the theoretically predicted mass of the Higgs boson (100–125 GeV) from higher order self-energy calculations, this boson gets quantum corrections typically of the order of the Planck energy scale. As a result, an extreme fine-tuning needs to be carried out at every order of perturbation theory to obtain the theoretically predicted value. This fine-tuning is often known as the Higgs mass hierarchy problem or naturalness problem in particle physics. Without introducing any intermediate scale in the theory, the RS model successfully resolved the fine-tuning problem by exponentially suppressing all mass scales on one of the 3-branes, known as the visible brane. Thus, the entire low energy theory is reproduced on the negative tension visible brane at TeV scale. By far, this is one of the most successful approaches for addressing the naturalness problem for a constant interbrane separation.

However, the RS model suffered from the stabilization problem. In the absence of any stabilization scheme, the two-brane system can collapse under the influence of equal and opposite brane tensions. Therefore, a reasonably generic method for stabilizing the brane separation distance r_c or the modulus field was proposed by Goldberger and Wise in Ref. [4], in which a stabilizing potential for the modulus field is generated by a five-dimensional bulk scalar field with the appropriate value at the boundary. The minimum of the modulus potential corresponds to the vacuum expectation value (vev) of the modulus field (kr_c). From this condition the vev of the modulus field can be set as $kr_c \simeq 11.5$ (to resolve the naturalness problem) without any fine-tuning of the four-dimensional parameters. In other words, the stabilization is achieved without sacrificing the conditions necessary to solve the gauge hierarchy problem.

Besides offering explanations to the problems beyond the Standard Model of particle physics, the RS model has attracted the attention of cosmologists due to its unique interpretation of the cosmological constant fine-tuning problem. Therefore, over the last decade, various cosmological

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and astrophysical issues like galaxy formation, the existence of anisotropies in the cosmic microwave background, dark energy and dark matter, and black hole formation have been extensively studied in the context of the RS two-brane model (see Ref. [5] and references therein).

In the present paper, we consider the effective, on-brane, scalar-tensor theories formulated by Kanno and Soda [6] where the radion field, which measures the interbrane separation between the visible brane and the Planck brane is not a constant quantity. In fact, while studying the cosmological solution on the visible or the Planck brane, the radion is taken as a time dependent field. Similarly, for spherically symmetric, static on-brane geometries, the radion field depends on the radial coordinate. The spatial or temporal dependence of the radion therefore leads to the requirement that it must be nonzero everywhere in order to avoid brane collisions. We are able to demonstrate that by assuming the existence of on-brane matter, a stable nonzero distance between the branes is possible.

In the next section, we provide an overview of the effective scalar-tensor theories proposed by Kanno and Soda [6]. Subsequently in Sec. III we deal with cosmological solutions, and in Sec. IV we look at spherically symmetric solutions. In the last section, we provide our summary and conclusions.

II. GRADIENT EXPANSION SCHEME AND THE KANNO–SODA EFFECTIVE THEORY

Let us now briefly discuss the low energy effective theory on a 3-brane developed by Kanno and Soda [6] in the context of the two-brane model developed by Randall and Sundrum. The two 3-branes being Z_2 symmetric are located at orbifold fixed points $y = 0$ and $y = l$ such that the geometry under consideration in this model is $M^{1,3} \times S^1/Z_2$. Our Universe is assumed to be on the visible 3-brane, which is a hypersurface embedded in a five-dimensional anti-de Sitter bulk filled with only a five-dimensional bulk cosmological constant. The bulk curvature scale is l. Typically, in the RS model, the Einstein equations are determined by keeping the interbrane distance fixed and considering a flat 3-brane. However, the scenario drastically changes once the interbrane separation distance or the proper length becomes a function of the spacetime coordinates and the on-brane geometry is curved. These generalizations are incorporated while deriving the effective equations of motion on a 3-brane [6]. Beginning with Ref. [7], there has been a lot of work on the effective Einstein equations on the brane under various assumptions [8]. In fact, the effective equations for the two-brane system as obtained in Ref. [6] were also rederived in a different approach in Ref. [9]. An interesting recent work on slanted warped extra dimensions and its phenomenological consequences appeared in Ref. [10].

To determine the effective theory, we assume the following five-dimensional action and a five-dimensional metric with a spacetime varying proper distance between the two 3-branes. The action functional is given as

$$S = \frac{1}{2\kappa^2}\int d^5x\sqrt{-g}\left(R + \frac{12}{l^2}\right) - \Sigma_{i=a,b}\sigma_i \int d^4x\sqrt{-g^{i\mathrm{brane}}} + \Sigma_{i=a,b}\int d^4x\sqrt{-g^{i\mathrm{brane}}}L^i{}_{\mathrm{matter}}, \tag{1}$$

where the tensions on the Planck brane and visible brane are, respectively, given by $\sigma_a = \frac{6}{\kappa^2 l}$ and $\sigma_b = -\frac{6}{\kappa^2 l}$. Let us consider the most general five-dimensional line element,

$$ds^2 = e^{2\phi(x)}dy^2 + g_{\mu\nu}(y, x^\mu)dx^\mu dx^\nu, \tag{2}$$

where κ^2 is a five-dimensional gravitational coupling constant. Since both cosmological and astrophysical solutions that we consider in the present case occur at energy scales much lower than that of the Planck scale, therefore in the effective theory approach, the brane curvature radius L is much larger compared to bulk curvature l. As a result, perturbation theory can be used with a dimensionless perturbation parameter ϵ such that $\epsilon = (\frac{l}{L})^2 \ll 1$. This method, called the gradient approximation scheme, is a metric-based iterative method in which the bulk metric and extrinsic curvature are expanded with increasing order of ϵ in perturbation theory. The effective Einstein equations on a brane are determined with the solutions of these quantities and the junction conditions. In this method, the RS fine-tuning condition is reproduced at the zeroth order when the interbrane separation is constant and the two 3-branes are characterized by opposite brane tensions. The effective Einstein equations are then obtained at the first order incorporating nonzero contributions of the radion field and brane matter. Using the gradient expansion scheme, the effective Einstein equations on the visible brane are as follows: [6]

$$G_{\mu\nu} = \frac{\kappa^2}{l\Phi}T^b_{\mu\nu} + \frac{\kappa^2(1+\Phi)}{l\Phi}T^a_{\mu\nu} + \frac{1}{\Phi}(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\Phi - f_{\mu\nu}\tilde{\nabla}^\alpha\tilde{\nabla}_\alpha\Phi) - \frac{3}{2\Phi(1+\Phi)}\left(\tilde{\nabla}_\mu\Phi\tilde{\nabla}_\nu\Phi - \frac{1}{2}f_{\mu\nu}\tilde{\nabla}^\alpha\Phi\tilde{\nabla}_\alpha\Phi\right), \tag{3}$$

where $\Phi = e^{2\frac{d}{l}} - 1$ and d is the proper distance between the branes, which in general is a spacetime dependent quantity. κ^2 is the five-dimensional gravitational coupling constant. $T^a_{\mu\nu}$, $T^b_{\mu\nu}$ are the matter on the Planck brane and the visible brane, respectively. All covariant derivatives in the above expression are defined with respect to the metric on the visible brane (denoted by the superscript b) given by $f_{\mu\nu}$.

The proper distance, a spacetime dependent function, between the two 3-branes in the interval $y = 0$ and $y = l$ is defined as

$$d(x) = \int_0^l e^{\phi(x)} dy, \tag{4}$$

and the corresponding equation of motion of the scalar field on the negative tension brane is given by

$$\tilde{\nabla}^\alpha \tilde{\nabla}_\alpha \Phi = \frac{\kappa^2}{l} \frac{T^a + T^b}{2\omega + 3} - \frac{1}{2\omega + 3} \frac{d\omega}{d\Phi} (\tilde{\nabla}^\alpha \Phi)(\tilde{\nabla}_\alpha \Phi). \tag{5}$$

Here, T^a and T^b are traces of energy momentum tensors on Planck brane and visible brane, respectively. The coupling function $\omega(\Phi)$ in terms of Φ can be expressed as

$$\omega(\Phi) = -\frac{3\Phi}{2(1+\Phi)}. \tag{6}$$

It is, however, known that the gravity on both the branes is not independent. The dynamics on the Planck brane situated at $y = 0$ is related to that of the visible brane by the following transformation [6]:

$$\Phi(x) = \frac{\Psi}{1 - \Psi}, \tag{7}$$

where Ψ is the radion field defined on Planck brane. Now, the induced metric on the visible brane can be expressed in terms of Ψ as

$$g_{\mu\nu}^{b\text{-brane}} = (1 - \Psi)[h_{\mu\nu} + g_{\mu\nu}^{(1)}(h_{\mu\nu}, \Psi, T^a_{\mu\nu}, T^b_{\mu\nu}, y = l)], \tag{8}$$

where $g_{\mu\nu}^{(1)}$ is the first order correction term.

It is to be noted that in the subsequent calculations, we will assume that the on-brane stress energy is present only on the "b"-brane, i.e., on the visible brane.

An important feature of the effective equations given above is that, unlike the ones derived in Ref. [7], there is no nonlocal contribution (bulk-Weyl dependent $\mathcal{E}_{\mu\nu}$ [7]) from bulk geometry.

III. COSMOLOGICAL SOLUTIONS

To study the cosmological solution on the negative tension, visible brane, we assume the radion field to be time dependent. Therefore, the proper distance between the orbifold fixed points, i.e., $y = 0$ to $y = l$, is given by

$$d(t) = \int_{y=0}^{y=l} e^{\phi(t)} dy = l e^{\phi(t)}. \tag{9}$$

The Friedmann–Robertson–Walker (FRW) solutions of the Einstein equations can be obtained for three different types of spatial curvature, $k = -1, 0, 1$. In this section, we study the solutions corresponding to each of these values of k separately. The FRW metric with a nonzero spatial curvature is given by

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin\theta^2 d\phi^2)\right], \tag{10}$$

where r, θ, ϕ are the radial coordinates and $a(t)$ is the scale factor to be determined. Substituting the above metric in Eq. (3), the Einstein equations with spatial curvature k are obtained as

$$3\left(\frac{\dot{a}}{a}\right)^2 + 3\frac{\dot{a}}{a}\frac{\dot{\Phi}}{\Phi} + \frac{3k}{a^2} + \frac{3\dot{\Phi}^2}{4\Phi(1+\Phi)} = \frac{\kappa^2}{l\Phi}\rho \tag{11}$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} - \frac{\dot{a}}{a}\frac{\dot{\Phi}}{\Phi} - \frac{\dot{\Phi}^2}{4\Phi(1+\Phi)} = \frac{\kappa^2}{3l}(-\rho + 3p) - \frac{\kappa^2}{3l\Phi}\rho, \tag{12}$$

and the scalar field equation is given by

$$\ddot{\Phi} + 3\frac{\dot{a}}{a}\dot{\Phi} = \frac{\kappa^2(\rho - 3p)(1+\Phi)}{3l} + \frac{\dot{\Phi}^2}{2(1+\Phi)}, \tag{13}$$

where an overdot represents the derivative with respect to time t. It is to be noted that Eq. (12) is obtained by substituting Eq. (13) in ii-th component of the Einstein's equations. The scalar field equation is found to be independent of spatial curvature k, and hence the equation remains the same for any value of k. However, the scalar field profile is different for different k values due to the different functional forms of $a(t)$.

Let us now consider each value of k separately and study the cosmological solution in the presence of a radion field with a time dependence.

A. Spatially flat solution ($k = 0$)

To construct a spatially flat FRW Universe on the visible brane in the presence of a time dependent radion field, we consider the line element given by Eq. (10), which for $k = 0$ reduces to

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j. \tag{14}$$

We initially assume that both the 3-branes are devoid of brane energy densities and pressures. Therefore, when $\rho = 0 = p$, Eq. (13) can be reexpressed in terms of first integral of the Φ equation. The scalar field equation reduces to

$$\dot{\Phi}^2 = \frac{C_1^2}{a^6}(1+\Phi). \tag{15}$$

After substituting $k = 0$ and $\rho = 0 = p$ in Eqs. (11) and (12) and then adding the two, we get

$$\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = 0. \tag{16}$$

Integrating Eq. (16), we get

$$a(t) = \sqrt{2\tilde{C}_1 t - C_2}. \tag{17}$$

Now, we can choose the dimensionful factor $\tilde{C}_1 = 1$ by a scaling choice so that the solution of scale factor is rewritten as

$$a(t) = (2t - C_2)^{\frac{1}{2}}, \tag{18}$$

where C_2 is a constant of integration. Substituting Eq. (15) and the scale factor into Eq. (11) (with $k = 0$) and then integrating it gives the solution for the time dependent scalar field as

$$\Phi(t) = \frac{C_1^2}{4(2t - C_2)} + \frac{C_1}{(2t - C_2)^{\frac{1}{2}}}, \tag{19}$$

where C_1 is a nonzero constant with dimensions of $L^{\frac{1}{2}}$. The constant C_2 may be set to zero by time translation so that $a(0) = 0$. However, C_1 must be strictly nonzero so that the scalar field $\Phi(t)$ remains nonzero as well. From the above solution of $\Phi(t)$, we can construct the proper distance $d(t)$ as given below:

$$d(t) = \frac{l}{2} \ln\left[1 + \frac{C_1^2}{8t} + \frac{C_1}{\sqrt{2t}}\right]. \tag{20}$$

The above solution indicates that the scale factor has a decelerating (but expanding) nature and the scalar field approaches zero in the later time, whereas it is large in the early Universe. The obtained solution is similar to that of the FRW radiation-dominated Universe. However, $d(t)$, which measures the interbrane distance, tends to zero in the limit $t \to \infty$, thereby indicating an instability.

Let us now consider a perfect fluid but with the equation of state $p = \frac{\rho}{3}$ and then construct the solutions. The traceless property of the energy momentum tensor for a perfect fluid with $p = \frac{\rho}{3}$ offers some simplifications. With the above-mentioned equation of state, the addition of Eqs. (11) and (12) for $k = 0$ produces the same differential equation for the scale factor $a(t)$ as before and hence the same solution, which is

$$a(t) = \sqrt{2t}, \tag{21}$$

where we have set the constant $C_2 = 0$. Using the scale factor derived above in Eq. (15), the solution of the scalar field can now be written as

$$\Phi(t) = \frac{C_1^2}{8t} \pm \frac{C_1 A}{2\sqrt{2t}} + \frac{A^2 - 4}{4}, \tag{22}$$

where we now have an extra parameter A. Now using the solutions of $a(t)$ and $\Phi(t)$, the energy density on the visible brane is given by

$$\rho(t) = \frac{l}{\kappa^2} \frac{3(A^2 - 4)}{16t^2}. \tag{23}$$

We note that when $A = 2$, Eq. (22) exactly reduces to the solution of $\Phi(t)$ given in Eq. (19) (with $C_2 = 0$), which is the scalar field solution in the absence of the brane matter on both the 3-branes. The nature of the variation of $\Phi(t)$ vs t is shown in Fig. 1, where $A = 2$, $C_1 = 2\sqrt{2}$ (red curve), and $A = 3$, $C_1 = 2\sqrt{2}$ (green curve). The horizontal line (blue) shows the nonzero asymptotic value of $\Phi(t)$ when brane matter is present. Now in the presence of matter, the proper distance between the branes using Eq. (22) is found to be

$$d(t) = \frac{l}{2} \ln\left[1 + \frac{C_1^2}{8t} \pm \frac{C_1 A}{2\sqrt{2t}} + \frac{A^2 - 4}{4}\right]. \tag{24}$$

As $t \to \infty$, $d(t)$ is always nonzero and tends to a constant value for all $A > 2$.

Hence, the proper distance never vanishes, and therefore no instability exists. Thus, the perfect fluid matter on the brane with equation of state $p = \frac{\rho}{3}$ stabilizes the distance between the branes. It is to be noted that such an equation of state corresponds to a perfect fluid comprising of relativistic particles.

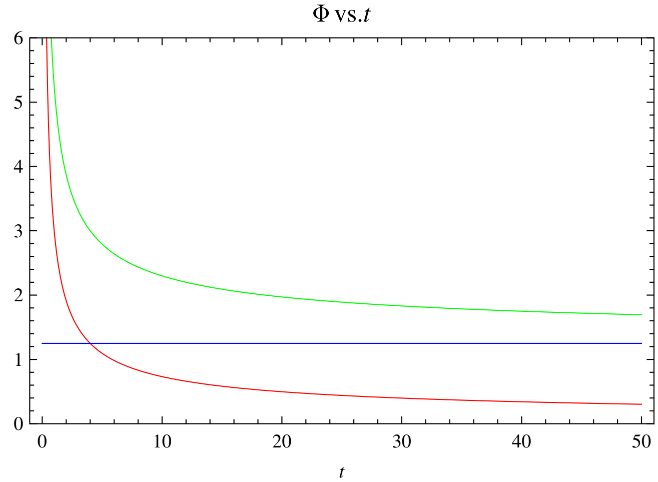

FIG. 1 (color online). Plot of $\Phi(t)$ vs t in the absence of visible brane matter [$A = 2$ and $C_1 = 2\sqrt{2}$ (red curve)], in the presence of visible brane matter [$A = 3$ and $C_1 = 2\sqrt{2}$ (green curve)], and the nonzero asymptotic value of $\Phi(t)$ when brane matter is present (horizontal blue line).

B. Spatially curved solutions ($k = -1, +1$)

Let us now construct the FRW solution on the visible brane with nonzero spatial curvature. The Einstein equations given by Eqs. (11) and (12) lead to an interesting observation: for $p = \frac{\rho}{3}$ the known Friedmann solutions for $k = +1$ and $k = -1$ survive. The scalar field equation remains unchanged, but the scalar field profile is obviously different due to the different functional forms of $a(t)$ for $k = +1$ and $k = -1$.

When $k = +1$, the addition of Eqs. (11) and (12) (with $p = \frac{\rho}{3}$) yields

$$\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{1}{a^2} = 0, \tag{25}$$

which on solving gives

$$a(t) = \sqrt{A_1^2 - (t - A_1)^2}. \tag{26}$$

This is the well-known Friedmann scale factor where the Universe begins at $t = 0$ and there is a big crunch at $t = 2A_1$. Similarly for $k = -1$, the addition of Eqs. (11) and (12) results in

$$\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} - \frac{1}{a^2} = 0, \tag{27}$$

and the scale factor becomes

$$a(t) = \sqrt{((t + A_1)^2 - A_1^2)}. \tag{28}$$

With appropriate time translation ($t + A_1 \rightarrow t$), the solution of the scale factor may, in general, be written as

$$a(t) = \sqrt{t^2 + K}, \tag{29}$$

where K is a real integration constant. In our form of the solution, we have chosen $K = -A_1^2 < 0$ and $a(0) = 0$. Here, the Universe is eternally expanding, though with deceleration.

If we now write the scalar field as $1 + \Phi(t) = e^{\frac{2d(t)}{l}}$, then using this and Eq. (15), we can express the proper distance $d(t)$ in terms of integral of the scale factor as

$$e^{\frac{d(t)}{l}} = \pm \frac{C_1}{2} \int \frac{dt}{a^3(t)} + B_1, \tag{30}$$

where B_1 is a constant of integration. Thus, given the scale factor for any spatial curvature $k = 0, -1, 1$, Eq. (30) is the most general expression that determines the proper distance between the two 3-branes. To verify whether a given scale factor always admits a nonzero $d(t)$, we need to verify that the lhs of the Eq. (30) is never be equal to 1.

Substituting the solution of the scale factor for $k = -1$, i.e., Eq. (28) in Eq. (30), we get

$$\sqrt{1 + \Phi(t)} = e^{\frac{d(t)}{l}} = \mp \frac{C_1}{2A_1^2} \frac{A_1 + t}{\sqrt{t(2A_1 + t)}} + B. \tag{31}$$

Similarly, for $k = +1$ using Eq. (26) in Eq. (30), we get

$$\sqrt{1 + \Phi(t)} = e^{\frac{d(t)}{l}} = \pm \frac{C_1}{2A_1^2} \frac{-A_1 + t}{\sqrt{t(t - 2A_1)}} + D, \tag{32}$$

where B and D are integration constants.

Let us now try to see if $\Phi(t)$ can become zero for any t. This will be possible for some t if the squares of the rhs of Eqs. (31) and (32) become equal to 1.

For $k = -1$, setting the square of the rhs of Eq. (31) equal to 1, we obtain the following roots for t:

$$t_{\pm} = -1 \mp \sqrt{\frac{1}{1 - \frac{C_1^2}{4B'}}}, \tag{33}$$

where we have set $A_1 = 1$ (without any loss of generality) and $B' = (\pm 1 - B) \geq 0$. Thus, if $C_1 > 4B'$, the roots are complex conjugates, and hence $\Phi(t)$ is never zero. For $C_1^2 < 4B'$, there is a positive root for which $\Phi(t)$ can become zero. However, choosing $B = 1$ and the upper sign in B', one may eliminate this possibility, too.

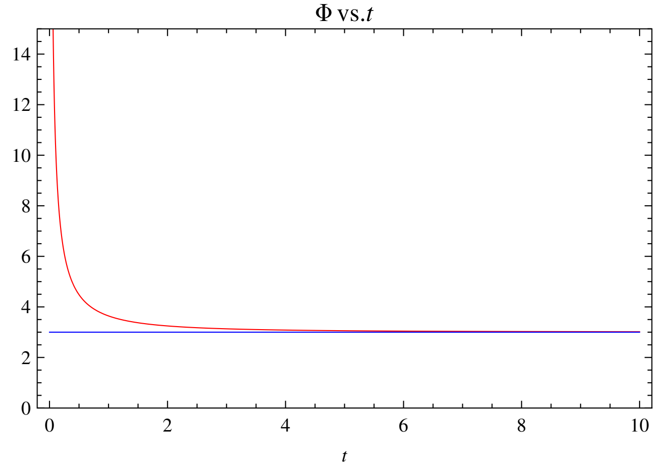

FIG. 2 (color online). $\Phi(t)$ vs t for $k = -1$; $C_1 = 2$, $A_1 = 1$, $B = 1$. The blue line shows the asymptotic value.

Similarly, for $k = 1$, we can obtain the roots for t when $\Phi(t)$ may become zero. These turn out to be (with $A_1 = 1$ and $D' = (\pm 1 - D)^2$)

$$t_{\pm} = 1 \mp \sqrt{\frac{1}{1 + \frac{C_1^2}{4D'}}}. \tag{34}$$

Here, it is clear that both roots lie within the domain of t. which is $0 \leq t \leq 2$. If $D' = 0$ (i.e., $D = 1$, with the upper sign in the expression for D'), then there ia a single root at $t = 1$. The variation of radion field $\Phi(t)$ with time for both $k = -1, 1$ are shown in Figs. 2 and 3 respectively, and they confirm the above discussion. It is clear that in the $k = +1$ case, an instability (brane collision) arises during the evolution of the Universe.

The condition under which $d(t)$ can be never equal to 1 for the spatially flat case has already been shown earlier.

IV. SPHERICALLY SYMMETRIC, STATIC SOLUTIONS

Let us now look at spherically symmetric static solutions of the effective Einstein equations on the visible brane. In constructing such a solution, it is legitimate to assume a radial coordinate, i.e., r dependent radion field $\Phi(r)$. We begin with a line element of the Majumdar–Papapetrou [11] form, which uses isotropic coordinates,

$$ds^2 = -\frac{1}{U^2(r)} dt^2 + U^2(r)[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2], \tag{35}$$

where $U(r)$ is the unknown function to be determined by solving Einstein's equations. First, let us assume that the branes are empty, i.e., $T^a{}_{\mu\nu} = T^b{}_{\mu\nu} = 0$. Substituting the metric ansatz given by Eq. (35) in Eq. (3), we arrive at the following field equations:

$$-2\frac{U''}{U}+\left(\frac{U'}{U}\right)^2-4\frac{U'}{Ur}=-\frac{\Phi'^2}{4\Phi(1+\Phi)}+\frac{U'}{U}\frac{\Phi'}{\Phi} \quad (36)$$

$$-\left(\frac{U'}{U}\right)^2=-\frac{3\Phi'^2}{4\Phi(1+\Phi)}-\frac{U'}{U}\frac{\Phi'}{\Phi}-\frac{2\Phi'}{\Phi r} \quad (37)$$

$$\left(\frac{U'}{U}\right)^2=\frac{\Phi'^2}{4\Phi(1+\Phi)}+\frac{U'}{U}\frac{\Phi'}{\Phi}+\frac{\Phi'}{\Phi r}. \quad (38)$$

Here, a prime denotes a derivative with respect to r. Adding Eqs. (37) and (38), one obtains

$$\frac{\Phi'}{\Phi}\left(\frac{\Phi'}{2(1+\Phi)}+\frac{1}{r}\right)=0. \quad (39)$$

Since $\Phi'(r)\neq 0$ one can consider the term in brackets in the above equation as a condition on Φ and its derivative. However, the scalar field equation for $\Phi(r)$ given by

$$\Phi''+2\frac{\Phi'}{r}=\frac{\Phi'^2}{2(1+\Phi)} \quad (40)$$

can be readily integrated once to get

$$\frac{\Phi'}{\sqrt{1+\Phi}}=\frac{2C_1}{r^2}, \quad (41)$$

where C_1 is a positive, nonzero integration constant. Consistency of Eq. (39) (i.e., the equation $\frac{\Phi'}{2(1+\Phi)}+\frac{1}{r}=0$) and Eq. (41) for $\Phi'(r)$ leads to a unique form of $\Phi(r)$ given by

$$\Phi(r)=\frac{C_1^2}{r^2}-1. \quad (42)$$

Further, we can use the condition in Eq. (39) to rewrite the Einstein equations in the following form:

$$-2\frac{U''}{U}+\left(\frac{U'}{U}\right)^2-4\frac{U'}{Ur}=\frac{\Phi'}{2\Phi r}+\frac{U'}{U}\frac{\Phi'}{\Phi} \quad (43)$$

$$-\left(\frac{U'}{U}\right)^2=-\frac{\Phi'}{2\Phi r}-\frac{U'}{U}\frac{\Phi'}{\Phi} \quad (44)$$

$$\left(\frac{U'}{U}\right)^2=\frac{\Phi'}{2\Phi r}+\frac{U'}{U}\frac{\Phi'}{\Phi}. \quad (45)$$

We note that the rhs of the above field equations lead to the tracelessness requirement on the lhs. Therefore, $U(r)$ must satisfy the following differential equation:

$$U''+2\frac{U'}{r}=0, \quad (46)$$

which is the Laplace equation $\nabla^2 U=0$ expressed in spherical polar coordinates (this result is the same as what follows in Einstein–Maxwell theory for Majumdar–Papapetrou-type solutions [11]). The solution for $U(r)$ is therefore straightforward and is given by

$$U(r)=C_2-\frac{C_3}{r}, \quad (47)$$

where C_2 and C_3 are two positive, nonzero constants. Substituting the solutions obtained for $U(r)$, $\Phi(r)$ and their derivatives in either of the two Einstein equations, i.e., Eq. (43) or (44), we find a single condition between the nonzero constants given as

$$C_1^2C_2^2=C_3^2. \quad (48)$$

Hence, the final solutions for $U(r)$ and $\Phi(r)$ in terms of C_1, C_2, and C_3 become

$$\Phi(r)=\frac{C_3^2}{C_2^2r^2}-1 \quad (49)$$

$$U(r)=C_2-\frac{C_3}{r}. \quad (50)$$

At $r=C_1=\frac{C_3}{C_2}$, $U(r)=0$, which implies the existence of a black hole horizon. Now for the same value of r, the radion field $\Phi(r)$ or the interbrane distance vanishes, suggesting an instability which needs to be removed. To keep $\Phi(r)$ always nonzero, we apply the method adopted in the case of cosmology (see the earlier section of this article). We add traceless matter on the visible brane. Therefore, using Eq. (35) in Eq. (3) once again (but with the presence of matter on the visible brane), we now obtain the following Einstein equations on the visible brane:

$$-2\frac{U''}{U}+\left(\frac{U'}{U}\right)^2-4\frac{U'}{Ur}=-\frac{\Phi'^2}{4\Phi(1+\Phi)}+\frac{U'}{U}\frac{\Phi'}{\Phi}+\frac{\kappa^2}{l\Phi}\rho \quad (51)$$

$$-\left(\frac{U'}{U}\right)^2=-\frac{3\Phi'^2}{4\Phi(1+\Phi)}-\frac{U'}{U}\frac{\Phi'}{\Phi}-\frac{2\Phi'}{\Phi r}+\frac{\kappa^2}{l\Phi}\tau \quad (52)$$

$$\left(\frac{U'}{U}\right)^2=\frac{\Phi'^2}{4\Phi(1+\Phi)}+\frac{U'}{U}\frac{\Phi'}{\Phi}+\frac{\Phi'}{\Phi r}+\frac{\kappa^2}{l\Phi}p, \quad (53)$$

where $\rho(r)$, $\tau(r)$, and $p(r)$ are the diagonal components (in the frame basis) of the energy momentum tensor on the visible brane. As long as this additional brane matter is traceless, i.e.,

$$-\rho+\tau+2p=0, \quad (54)$$

there is no change in the scalar field differential equation. The general solution of the scalar field equation, however, needs to be taken as

$$\Phi(r)=\left(\frac{C_1}{r}+\frac{C_4}{2}\right)^2-1, \quad (55)$$

where C_4 is a positive constant which is responsible for generating the brane matter. Even though with $C_4=0$ the r dependent $\Phi(r)$ produces a nonflat on-brane metric, it involves an unstable radion and also corresponds to the case when the visible brane is empty. We can easily see that as long as $C_4 > 2$, $\Phi(r)$ never vanishes, and by having traceless matter on the visible brane, the instability disappears for this particular spherically symmetric solution with a r dependent interbrane distance $\Phi(r)$. The plot of $\Phi(r)$ vs r is shown in Fig. 4. It is to be noted further that the solution for $U(r)$ remains unaltered under the tracelessness condition on brane matter. However, it is now possible to choose $\frac{C_3}{C_2}$ to be different from C_1. We assume

$$U(r) = 1 - \frac{C_5}{r}. \tag{56}$$

From the above expressions for $U(r)$ and $\Phi(r)$, the visible brane matter energy momentum, i.e., ρ, τ and p turn out to be

$$\rho = \frac{l}{\kappa^2}\frac{1}{(1-\frac{C_5}{r})^2}\frac{1}{r^4}\left(C_5C_1C_4 + C_1^2 - C_5^2 + \frac{C_5^2C_4^2}{4}\right) \tag{57}$$

$$\tau = \frac{l}{\kappa^2}\frac{1}{(1-\frac{C_5}{r})^2} \times \left(\frac{3C_1C_4C_5 - C_1^2 + C_5^2 - \frac{C_5^2C_4^2}{4}}{r^4} - \frac{2C_1C_4}{r^3} - \frac{2C_1C_4C_5^2}{r^5}\right) \tag{58}$$

$$p = \frac{l}{\kappa^2}\frac{1}{(1-\frac{C_5}{r})^2} \times \left(\frac{-C_1C_4C_5 + C_1^2 - C_5^2 + \frac{C_5^2C_4^2}{4}}{r^4} + \frac{C_1C_4}{r^3} + \frac{C_1C_4C_5^2}{r^5}\right). \tag{59}$$

We note that C_5 as well as C_1 cannot be zero in order to ensure a nonconstant $U(r)$ and $\Phi(r)$. At the same time, $C_4 = 0$ is also not desirable because it would lead to an instability [i.e., $\Phi(r)$ becoming zero at some r]. Further, all three constants must satisfy $C_1 > 0$, $C_4 > 0$ and $C_5 > 0$. It is possible to have both C_1 and C_4 negative, but this does not affect the functional forms of ρ, τ, and p or $\Phi(r)$. However, if one chooses $C_5 < 0$, the solution leads to a naked singularity. It is also clear that we cannot have $\tau = p$ because this condition leads to a quadratic equation for r which implies specific r values as its solutions. The only allowed condition is the one for traceless matter, i.e., $\rho = \tau + 2p$. In addition, the weak energy condition (WEC) or null energy condition (NEC) will be violated. In particular,

$$\rho + \tau = -\frac{l}{\kappa^2}\frac{2C_1C_4}{r^4}. \tag{60}$$

Since we must have $C_1, C_4 > 0$ for stability, $\rho + \tau < 0$, but one can satisfy $\rho > 0$ and $\rho + p > 0$ by choosing the constants appropriately. Even though the ρ, τ, and p violate WEC and NEC, the "effective matter," which is equal to the *total* expressions in the rhs of Eqs. (51)–(53) does satisfy

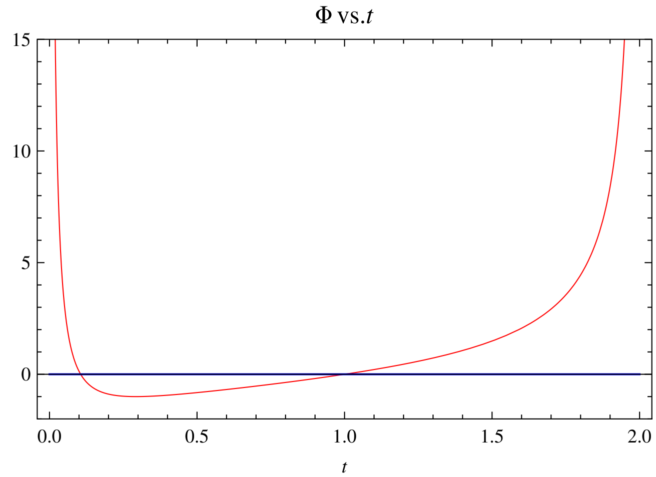

FIG. 3 (color online). $\Phi(t)$ vs t for $k = +1$; $C_1 = 2$, $A_1 = 1$, $D = 1$. $\Phi(t)$ equals zero at $t = 1$.

the WEC or NEC. One can easily check this by renaming the quantities on the rhs of Eqs. (51)–(53) as $\rho_{\rm eff}$, $\tau_{\rm eff}$, $p_{\rm eff}$ and verifying the validity of $\rho_{\rm eff} > 0$, $\rho_{\rm eff} + \tau_{\rm eff} = 0$, and $\rho_{\rm eff} + p_{\rm eff} > 0$. The functional forms of ρ, τ, and p are shown in Fig. 5 for a specific choice of the parameters, with $C_1 = C_5$. We have also checked (not shown here) that the profiles of ρ, τ, and p are similar when $C_1 \neq C_5$. It is now easy to convert the metric solution (and the scalar field solution) into the extremal Reissner–Nordström black hole form by the following identifications:

$$r = r' - M; \qquad C_1 = M. \tag{61}$$

This leads to the extremal Reissner–Nordström black hole metric given as

$$ds^2 = -\left(1 - \frac{M}{r'}\right)^2 dt^2 + \frac{dr'^2}{(1-\frac{M}{r'})^2} + r'^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{62}$$

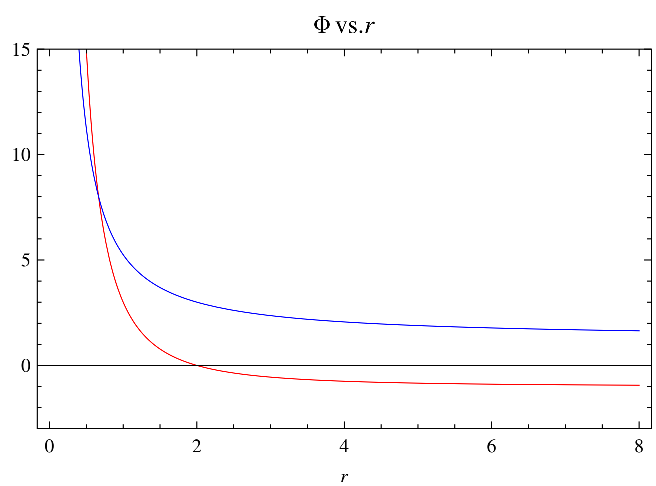

FIG. 4 (color online). Plot of $\Phi(r)$ vs r when $C_1 = 2$, $C_4 = 0$ (red curve) and $C_1 = 2$, $C_4 = 3$ (blue curve).

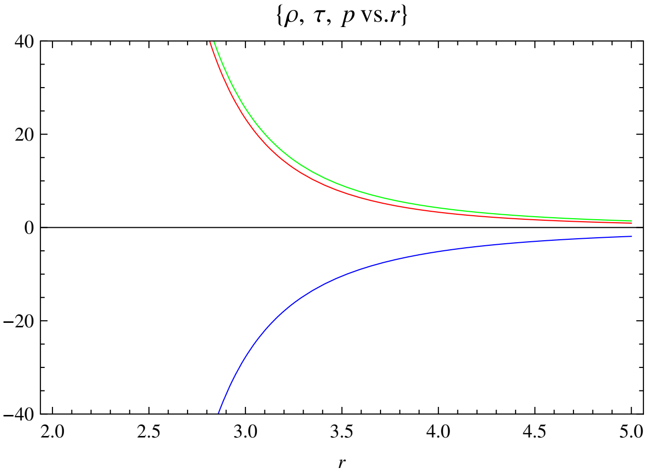

FIG. 5 (color online). Plot of $\frac{\kappa^2}{3l}\rho(r)$, $\frac{\kappa^2}{3l}\tau(r)$, $\frac{\kappa^2}{3l}p(r)$ vs r in red, blue, and green, respectively. Here $C_1 = 2$ and $C_4 = 3$, and the y axis values are scaled by a factor of 10.

We note that $r' = M$ is the location of horizon as well as the spacetime singularity.

For such spherically symmetric solutions, we can also obtain the $\Psi(r)$ by exploiting the relation between $\Phi(r)$ and $\Psi(r)$ given in Ref. [6]. For example, in the simple case (without visible brane matter), we have

$$\Psi(r) = 1 - \frac{r^2}{C_1^2} \tag{63}$$

$$h_{ij} = \frac{C_1^2}{r^2} f_{ij}, \tag{64}$$

where the h_{ij} is the metric on the Planck brane and the visible brane metric functions, f_{ij}, are given in terms of the $U(r)$ obtained above.

V. CONCLUSION

In summary, we have shown the following:

(i) In the cosmological case, for traceless matter ($p = \frac{\rho}{3}$) on the visible brane, we find analytic solutions for the scale factor and the radion field. In the spatially flat Universe, the scale factor is that of the radiation dominated FRW case, while the radion is stable and never zero. Instability arises when there is no on-brane matter. In a spatially curved Universe with traceless, radiative matter, the results are similar for the case of negative spatial curvature. With positive spatial curvature, instabilities arise even with on-brane matter.

(ii) In the spherically symmetric, static case, in isotropic coordinates, we find that the solution obtained is nothing but the extremal Reissner–Norström solution. However, there is no physical charge or mass here (like in the Einstein–Maxwell theory), and the radion field parameters play the role of an equivalent charge or mass.

For the case when the matter on the brane is not necessarily traceless, we are unable to find analytical solutions. Numerical work (not discussed here) suggests that the nature of the solutions for, say, $p = 0$ or $p = -\rho$ are different from the solutions for $p = \frac{\rho}{3}$ discussed here.

It is noteworthy that our analytic solutions are all obtained using traceless, on-brane matter. However, we also note that the stability of the radion may not necessarily have any connection with the tracelessness of on-brane matter, though the need for some on-brane matter to achieve stability has been demonstrated in our examples. A hint about what kind of matter can achieve stability of the radion can be obtained by setting $C_4 = 0$ in the expressions for ρ, τ, and p. Notice [from Eqs. (57)–(59)] that for $C_1^2 > C_5^2$, the NEC and WEC will be satisfied. Does this indicate that a stable radion requires energy condition violating on-brane matter? A general statement is unlikely here, though one may surely try to explore the exact link between the nature of on-brane matter and radion stability in future investigations.

Finally, the fact that we have rediscovered known solutions (i.e., the FRW scale factors in cosmology and the extremal Reissner–Nordström in the static, spherically-symmetric case) in the context of a theory different from general relativity is certainly welcome. This feature was also noticed in the first analytic solution in the Shiromizu–Maeda–Sasaki on-brane, effective theory [7] where the Reissner–Nordström solution was rediscovered as an exact solution [12]. There, the interpretation of a charge or mass was entirely geometric and largely dependent on the presence of the extra dimensions. Here, too, it is the presence of extra dimensions, through the space or time dependent radion, which is responsible for the nature of the solutions, though on-brane matter seems to be crucial in maintaining stability.

[1] S. B. Giddings, S. Kachru, and J. Polchinski, Phys. Rev. D **66**, 106006 (2002).

[2] N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. B **429**, 263 (1998); I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, *ibid.* **436**, 257 (1998).

[3] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999); **83**, 4690 (1999).

[4] W. D. Goldberger and M. B. Wise, Phys. Rev. Lett. **83**, 4922 (1999).

[5] R. Maartens and K. Koyama, Living Rev. Relativity **13**, 5 (2010).

[6] S. Kanno and J. Soda, Phys. Rev. D **66**, 083506 (2002).

[7] T. Shiromizu, K. Maeda, and M. Sasaki, Phys. Rev. D **62**, 024012 (2000).

[8] S. Kanno and J. Soda, Phys. Rev. D **66**, 043526 (2002); T. Shiromizu, K. Koyama, S. Onda, and T. Torii, Phys. Rev. D **68**, 063506 (2003); S. Kanno and J. Soda, Gen. Relativ. Gravit. **36**, 689 (2004); T. Kobayashi, T. Shiromizu, and N. Deruelle, Phys. Rev. D **74**, 104031 (2006); L. Cotta-Ramusino and D. Wands, Phys. Rev. D **75**, 104001 (2007); S. Fujii, T. Kobayashi, and T. Shiromizu, Phys. Rev. D **76**, 104052 (2007); F. Arroja, arXiv:0812.1431.

[9] T. Shiromizu and K. Koyama, Phys. Rev. D **67**, 084022 (2003); T. Shiromizu, K. Koyama, and K. Takahashi, Phys. Rev. D **67**, 104011 (2003).

[10] D. Hernandez and M. Sher, Phys. Lett. B **698**, 403 (2011).

[11] S. D. Majumdar, Phys. Rev. **72**, 390 (1947); A. Papapetrou, *Proceedings of the Royal Irish Academy Section A: Mathematical and Physical Sciences* (Royal Irish Academy, Dublin, 1945–1948), Vol. 51, pp. 191–204.

[12] N. Dadhich, R. Maartens, P. Papadopoulos, and V. Rezania, Phys. Lett. B **487**, 1 (2000).