Interacting viscous dark fluids

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We revise the conditions for the physical viability of a cosmological model in which dark matter has bulk viscosity and also interacts with dark energy. We have also included radiation and baryonic matter components; all matter components are represented by perfect fluids, except for the dark matter one that is modeled as an imperfect fluid. We impose upon the model the condition of a complete cosmological dynamics that results in an either null or negative bulk viscosity, but the latter also disagrees with the local second law of thermodynamics (LSLT). The model is also compared with cosmological observations at different redshifts: type Ia supernova, the acoustic peak of baryon acoustic oscillation, the Hubble parameter $H(z)$, and the angular scale of the cosmic microwave background encoded in the first peak. Taken together, observations consistently point to a negative value of the bulk viscous coefficient, that is in disagreement with the LSLT. From the different cases that we study, the best model that we find corresponds to the case of a dark matter with a *null* viscosity, interacting with a phantom dark energy. Also, overall the fitting procedure shows no preference for the model over the standard ΛCDM model.

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I. INTRODUCTION

Cosmological models with interacting dark components have gained interest because there is the possibility that the most abundant components in the present Universe, dark energy (DE) and dark matter (DM), interact one with each other, and some authors claim that some of these interaction terms are promising mechanisms to solve the problems of the standard cosmological model, the so-called ΛCDM model (see for instance [1–3] and references therein).

On the other hand, it has been known since before the discovery of the present accelerated expansion of the Universe that a bulk viscous fluid may induce an accelerating cosmology [4]. Hence, it has been proposed that the bulk viscous pressure can be one of the possible mechanisms to accelerate the Universe today (see for instance [5–8]). However, this idea still needs some physically motivated model to explain the origin of the bulk viscosity. In this sense some proposals have been already put forward in [9].

In the present work, we have the interest to explore and test an interacting dark sector model which also takes into account a bulk viscosity in the DM component. Similar combinations of two interacting *relativistic* fluids with bulk viscosity have been also proposed in studies for the origin of the large-scale temperature fluctuations of the cosmic microwave background [10]. Our purpose is twofold: first, we explore the general conditions for the model to have a complete cosmological dynamics, and second, we use cosmological observations to fit the free parameters of the model.

The full dynamics of the model is found through a dynamical system analysis, a common tool in the analysis of cosmological models [11,12], and then the DM-DE interaction term is chosen such as to allow the writing of the equations of motion as an autonomous set of differential equations. We are then able to write general conditions for the existence of radiation and matter eras at early times that are useful for a wide variety of interacting models.

The bulk viscous coefficient in our model is directly proportional to the Hubble parameter, and we impose upon it a constraint that comes from the local second law of thermodynamics (LSLT). In general, as it also happens for our model, this latter condition selects only positive definite values of the bulk viscous coefficient [13,14].

The model is also compared with different cosmological observations: type Ia supernovae, the acoustic peak of baryon acoustic oscillation (BAO), the Hubble parameter $H(z)$, and the location of the first peak l_1 of the CMB, in order to constrain its free parameters. As we shall show, the fitted values acquire different values depending on whether we use low-redshift or intermediate-redshift observations. In a similar way as in the condition for a complete cosmological dynamics, wrong conclusions may be obtained if the analysis is only made with observations in the lowest range of redshifts (late times).

One major difference with respect to other previous studies is that we have included as a requirement a so-called *complete cosmological dynamics*, which means that all physically viable models must allow the existence of radiation and matter domination eras at early enough times, so that the known processes of the early Universe are not significantly changed with respect to those of the standard big bang model. This seems to be an usually

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overlooked condition in most studies of alternative cosmological models, for which the primary concern is the present accelerated expansion of the Universe, and then it is commonly thought that a low-redshift analysis is quite enough for the task.

The paper is organized as follows. In Sec. II we present the full characteristics of the model, the main equations of motion, and the dynamical system analysis. The bulk viscosity of the model is represented by a single free parameter, whereas the DM-DE interaction term is considered a free function of the DM and DE density parameters, as long as the dynamical system of equations remains autonomous. The cosmological eras of the model are given in terms of the critical points of the dynamical system, whose existence conditions depend upon the values of the free parameters of the model. A detailed discussion about the existence or not of appropriate cosmological eras is provided in terms of the aforementioned constraint of a complete cosmological dynamics.

In Sec. III, we focus our attention in a particular form for the DM-DE interacting term that is directly proportional to the DE energy density. Full details are given about the existence and stability of the critical points, which are in turn transformed into conditions upon the free parameters of the model. Also, we show some particular examples of the dynamics of the model for selected values of the free parameters.

We explain in Sec. IV the cosmological probes that are used to constrain the model, and give separate examples of the fitting procedure for different subcases of the model. For completeness, we include here low- and intermediate-redshift constraints, so that we can track the changes in the values of the parameters for those cases. Finally, the main results are summarized and discussed in Sec. V.

II. INTERACTING BULK VISCOUS DARK FLUIDS

We study a cosmological model in a spatially flat Friedmann-Robertson-Walker (FRW) metric, in which the matter components are radiation, baryons, DM, and DE. Radiation and baryons are assumed to have the usual properties of perfect fluids, whereas DM is treated as an imperfect fluid having bulk viscosity, with a null hydrodynamical pressure, and in interaction with DE. This phenomenological model is a natural extension of that proposed by Kremer and Sobreiro [2].

The Friedmann constraint and the conservation equations for the matter fluids can be written as

$$H^2 = \frac{8\pi G}{3}(\rho_{\rm r} + \rho_{\rm b} + \rho_{\rm dm} + \rho_{\rm de}), \tag{1a}$$

$$\dot{\rho}_{\rm r} = -4H\rho_{\rm r}, \tag{1b}$$

$$\dot{\rho}_{\rm b} = -3H\rho_{\rm b}, \tag{1c}$$

$$\dot{\rho}_{\rm dm} = -3H\rho_{\rm dm} + Q + 9H^2 H\zeta, \tag{1d}$$

$$\dot{\rho}_{\rm de} = -3H\gamma_{\rm de}\rho_{\rm de} - Q, \tag{1e}$$

where G is the Newton gravitational constant, H the Hubble parameter, $(\rho_{\rm r}, \rho_{\rm b}, \rho_{\rm dm}, \rho_{\rm de})$ are the energy densities of the radiation, baryon, DM, and DE fluid components, respectively, and $\gamma_{\rm de}$ is the barotropic index of the equation of state (EOS) of DE, which is defined from the relationship $p_{\rm de} = (\gamma_{\rm de} - 1)\rho_{\rm de}$, where $p_{\rm de}$ is the pressure of DE. The term $9H^2\zeta$ in Eq. (1d) corresponds to the bulk viscous pressure of the dark matter fluid, with ζ the bulk viscous coefficient, whereas Q is the DM-DE interaction term.

Bulk viscosity arises typically in fluids that are composed of several types of particles [15–18], and is intrinsically related or produced by the interaction of expanding mixed fluids that are slightly out of local thermodynamic equilibrium during short periods of time. It is then considered that the bulk viscosity cannot depend on just one component (perhaps the bulk viscous component), but it has to depend on other cosmological components too, given that it typically arises in the interaction (or decay) between two or more different fluids that compose the total cosmological fluid.

Typical examples of this bulk viscous dissipative mechanism are found in a fluid built up of radiation and matter components [17] (actually many dissipative processes are well described by this mixture [15]), in a Boltzmann gas composed of low and high energy particles, and in mixtures of massive gauge bosons and ultrarelativistic particles [18]. Also, the decay of particles is another mechanism to produce bulk viscous dissipation. The nonconservation of the particle number increases the entropy, which then gives rise to a bulk viscosity.

Following the discussion above, we take the bulk viscous coefficient ζ to be proportional to the square root of the total matter density, $\rho_{\rm t} = \rho_{\rm r} + \rho_{\rm b} + \rho_{\rm dm} + \rho_{\rm de}$, in the form

$$\zeta = \frac{\zeta_0}{\sqrt{24\pi G}}\rho_{\rm t}^{1/2} = \left(\frac{1}{8\pi G}\right)H\zeta_0, \tag{2}$$

where ζ_0 is a dimensionless constant to be estimated from the comparison with cosmological observations. From Eq. (1a), we can see that this parametrization also corresponds to a bulk viscosity proportional to the expansion rate of the Universe, i.e., to the Hubble parameter. As explained before, a bulk viscosity proportional to the square root of the total energy density allows us precisely to consider the dependence with respect to all the other cosmological fluids. Finally, the Raychadury equation of the model is

$$\dot{H} = -4\pi G\left(\frac{4}{3}\rho_{\rm r} + \rho_{\rm b} + \rho_{\rm dm} + \gamma_{\rm de}\rho_{\rm de} - 3H\zeta\right). \tag{3}$$

In our analysis, we will take into account an important restriction over the bulk viscous coefficient that comes from the local second law of thermodynamics (LSLT). The *local* entropy production for a fluid on a FRW space-time is expressed as [14]

$$T\nabla_\nu s^\nu = \zeta(\nabla_\nu u^\nu)^2 = 9H^2\zeta, \tag{4}$$

where T is the temperature of the fluid, and $\nabla_\nu s^\nu$ is the rate of entropy production in a unit volume. Then, the second law of the thermodynamics can be stated as $T\nabla_\nu s^\nu \geq 0$; since the Hubble parameter H is positive for an expanding universe, Eq. (4) implies that $\zeta \geq 0$. For the present model, this inequality in turn becomes [see Eq. (2)]

$$\zeta_0 \geq 0. \tag{5}$$

A. The dynamical system perspective

In order to study all possible cosmological scenarios of the model, we proceed to a dynamical system analysis of Eqs. (1) and (3). Let us first define the set of dimensionless variables:

$$x = \frac{8\pi G}{3H^2}\rho_{\rm de}, \qquad y = \frac{8\pi G}{3H^2}\rho_{\rm dm}, \tag{6a}$$

$$u = \frac{8\pi G}{3H^2}\rho_{\rm b}, \qquad z = \frac{8\pi G}{3H^3}Q. \tag{6b}$$

Then, the equations of motion can be written in the following, equivalent, form:

$$\frac{dx}{dN} = -z + x(4 - u - y - 3\gamma_{\rm de} - 3\zeta_0) - x^2(4 - 3\gamma_{\rm de}), \tag{7a}$$

$$\frac{dy}{dN} = y(1 - u - y - x(4 - 3\gamma_{\rm de}) - 3\zeta_0) + z + 3\zeta_0, \tag{7b}$$

$$\frac{du}{dN} = u(1 - u - y - x(4 - 3\gamma_{\rm de}) - 3\zeta_0), \tag{7c}$$

where the derivatives are with respect to the e-folding number $N \equiv \ln a$. In term of the new variables, the Friedmann constraint (1a) can be written as

$$\Omega_{\rm r} = \frac{8\pi G}{3H^2}\rho_{\rm r} = 1 - x - y - u, \tag{8}$$

and then we can choose (x, y, u) as the only independent dynamical variables.

Taking into account that $0 \leq \Omega_{\rm r} \leq 1$, and imposing the conditions that both the DM and DE components are both positive definite and bounded at all times, we can define the phase space of Eq. (7) as

$$\Psi = \{(x, y, u): 0 \leq 1 - x - y - u \leq 1, 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq u < 1\}. \tag{9}$$

Other cosmological parameters of interest are the total effective EOS, $w_{\rm eff}$, and the deceleration parameter, $q = -(1 + \dot{H}/H^2)$, which can be written, respectively, as

$$w_{\rm eff} = \frac{1}{3}(1 - u - y - x(4 - 3\gamma_{\rm de}) - 3\zeta_0), \tag{10a}$$

$$q = \frac{1}{2}\{2 - u - y - x(4 - 3\gamma_{\rm de}) - 3\zeta_0\}. \tag{10b}$$

TABLE I. Some proposed forms of $Q(\rho_{\rm de}, \rho_{\rm dm})$ for which the dynamical system (7) becomes an autonomous system of differential equations.

Model	$Q(\rho_{\rm de}, \rho_{\rm dm})$	$z(x, y)$	References
i	$3H(\alpha_1\rho_{\rm de} + \alpha_2\rho_{\rm dm})$	$3(\alpha_1 x + \alpha_2 y)$	[12,19]
ii	$3H\lambda\frac{\rho_{\rm de}\rho_{\rm dm}}{\rho_{\rm de}+\rho_{\rm dm}}$	$3\lambda\frac{xy}{x+y}$	[20]
iii	$3H\lambda\rho_{\rm dm}$	$3\lambda y$	[21,22]

In order to obtain an autonomous system of ordinary differential equations from Eq. (7), we will focus our attention hereafter only in general interaction functions of the form $Q = 3Hf(\rho_{\rm dm}, \rho_{\rm de})$ that can lead to closed functions $z = z(x, y)$. As we shall see in the next section, this election will allow us to impose general conditions over the variable z (and on the Q term as well) in order to achieve a well behaved dynamics (see [12] for a similar exercise). Some examples of the interaction Q that lead to the desired form of z are listed in Table I.

B. General conditions for a complete cosmological dynamics

If the system of equations (7) is autonomous, one then expects that important stages in the evolution of the model be represented by critical points in phase space. We will work on this hypothesis here to make a description of the existence, or not, of the different domination eras that have to be present in any model of physical interest.

We then demand that our model must follow a complete cosmological dynamics: namely, it should start in a radiation dominated era (RDE), later enter into a matter dominated era (MDE), and finally enter into the present stage of accelerated expansion; every one of these statements can be translated in definite mathematical equations, which we are going to discuss in detail in the sections below.

Before that, we need to calculate the critical points (x_*, y_*, u_*) of the dynamical system (7), which are to be found from the conditions:

$$0 = -z_* + x_*(4 - u_* - y_* - 3\gamma_{\rm de} - 3\zeta_0) - x_*^2(4 - 3\gamma_{\rm de}), \tag{11a}$$

$$0 = y_*(1 - u_* - y_* - x_*(4 - 3\gamma_{\rm de}) - 3\zeta_0) + z_* + 3\zeta_0, \tag{11b}$$

$$0 = u_*(1 - u_* - y_* - x_*(4 - 3\gamma_{\rm de}) - 3\zeta_0), \tag{11c}$$

where $z_* \equiv z(x_*, y_*)$ is the interaction variable evaluated at the critical points, see Eq. (6).

1. Radiation domination

Let us start with the conditions for a purely RDE. According to the Friedmann constraint (8), $\Omega_r = 1$ corresponds to $(x_*, y_*, u_*) = (0, 0, 0)$, and then Eq. (11) further dictates that

$$z_* = 0, \qquad z_* = -3\zeta_0. \tag{12}$$

The first condition on the DM-DE interaction term holds for many of the interacting functions $z = z(x, y)$ in the specialized literature, like for those examples listed in Table I; but the second condition strongly implies that it is not possible to reconcile a purely RDE with a nonzero bulk viscosity, $\zeta_0 \neq 0$.

However, there are other less extreme possibilities for radiation domination in which a bulk viscosity exists, as long as we allow the coexistence of radiation and other matter components early in the evolution of the Universe.

As the bulk viscosity term only appears actively for the equation of motion of DM, see Eqs. (7b) and (11b), we see that the early presence of DM could allow the existence of bulk viscosity in a RDE. The critical point we are looking for is of the form $(x_*, y_*, u_*) = (0, y_*, 0)$, under the assumption $y_* \ll 1$, and then we obtain the following conditions:

$$y_* = -3\zeta_0, \qquad z_* = 0. \tag{13}$$

Thus, a RDE is possible as long as the DM-DE interacting term is null, and the bulk viscosity is negative, $\zeta_0 < 0$ (in order to preserve the condition $y \geq 0$). However, this is at variance with the condition from the LSLT in Eq. (5).

The null condition for the interaction term can be obtained if z is a function with mixed $x - y$ terms like that of model (ii) in Table I, or with a dependence only on x, an instance of which is model (i) with $\alpha_2 = 0$.

Another possible critical point for a RDE would be $(x_*, y_*, u_*) = (x_*, 0, 0)$, which by means of Eq. (11), leads to the conditions

$$x_* = \frac{-3\zeta_0}{4 - 3\gamma_{\rm de}}, \qquad z_* = -3\zeta_0. \tag{14}$$

As the DE EOS satisfies $\gamma_{\rm de} < 1$, then a RDE is achieved if $x_* \ll 1$ and $\zeta_0 < 0$, but the latter condition is again at variance with the LSLT in Eq. (5).

2. Matter domination

The existence of a MDE requires a scaling relation between the baryonic and CDM densities in the form $(x_*, y_*, u_*) = (0, \beta, 1 - \beta)$, where $\beta \in [0, 1]$,[1] so that $y_* + u_* = 1$, as dictated by the Friedmann constraint (8). This time, Eq. (11) dictates that

$$0 = -z_*, \qquad 0 = -3(1 - \beta)\zeta_0 \tag{15}$$

are the simultaneous independent conditions to fulfill a MDE.

The first condition requires again the interaction term z to be a function with mixed $x - y$ terms like that of model (ii) in Table I, or with a dependence only on x, like model (i) with $\alpha_2 = 0$. For this latter case, and also model (iii), a nonzero value of α_2 needs a baryon dominated critical point, $(x_*, y_*, u_*) = (0, 0, 1)$, which we consider as nonrealistic.

The second condition allows two possibilities:

(i) $\zeta_0 = 0$. As in the condition for a successful RDE, the model needs a null bulk viscosity to reach a correct MDE.

(ii) $\beta = 1$ [$\forall\ \zeta_0 \in [0, \infty)$] represents a critical point of pure CDM domination, which is at variance with the well established fact that baryons have a non-negligible contribution to the matter contents.

Another scenario to describe the MDE is a scaling relation among baryonic matter, CDM and DE. This requirement implies a fine-tuning over the very small amount of DE allowed for this period, without preventing or slowing structure formation. This translates into $(x_*, y_*, u_*) = (1 - y_* - u_*, y_*, u_*)$, so that $x_* + y_* + u_* = 1$, as indicated by the Friedmann constraint (8). With the above values, Eq. (11) leads to two independent possibilities. The first one is

$$z_* = 3(1 - y_*)((1 - \gamma_{\rm de})y_* - \zeta_0), \qquad u_* = 0, \tag{16}$$

where the null contribution of baryons, and the scaling relation between CDM and DE, suggest that it is impossible to recover a successful MDE, even though this critical point could correspond to a possible late time scenario. The second one is

$$z_* = -3\zeta_0, \qquad x_* = \zeta_0/(\gamma_{\rm de} - 1), \tag{17}$$

where we have either: $\zeta_0 > 0$ and $\gamma_{\rm de} > 1$, which agrees with the LSLT in Eq. (5), but corresponds to a nonrealistic DE EOS, $w_{\rm de} > 0$; or $\zeta_0 < 0$ and $\gamma_{\rm de} < 1$, which violates Eq. (5), but somehow allows a valid MDE if $x_* \ll 1$.

3. Accelerated expansion

In order to describe the present stage of accelerated expansion, and at the same time alleviate the coincidence problem, we need a scaling regime between the DM and DE components. This requirement leads to the critical point $(x_*, y_*, u_*) = (x_*, 1 - x_*, 0)$, and then Eq. (11) leads to the single condition:

$$z_* = 3x_*(1 - x_* - \gamma_{\rm de} + x_*\gamma_{\rm de} - \zeta_0). \tag{18}$$

This last equation can be solved once the interaction term is given for a particular model, and we can foresee that there must be valid solutions of it for any values, positive or negative, of the bulk viscosity constant ζ_0. Moreover, if we impose the condition for strict DE domination, $x_* = 1$, then $z_* = -3\zeta_0$; this can be possible, for instance, for model (i) in Table I.

4. Final comments

The requirement of a complete cosmological dynamics discussed above, from the dynamical system point of view,

[1] Only the values of β in the range [0, 1] belong to the phase space (9), and therefore make physical sense.

TABLE II. Location, existence conditions according to the physical phase space (9), and stability of the critical points of the autonomous system (7a)–(7c) under $Q = 3H\alpha\rho_{\rm de}$. The eigenvalues of the linear perturbation matrix associated to each of the following critical points are displayed in Table IV.

P_i	x	y	u	Existence	Stability
$1a$	0	$-3\zeta_0$	0	$-\frac{1}{3} \le \zeta_0 \le 0$	Unstable if $\zeta_0 > -\frac{1}{3}$, $\alpha < \frac{4}{3} - \gamma_{\rm de}$ Saddle if $\zeta_0 > -\frac{1}{3}$, $\alpha > \frac{4}{3} - \gamma_{\rm de}$ or
$1b$	x	$-x - 3\zeta_0$	0	$\gamma_{\rm de} = 1$, $\alpha = \frac{1}{3}$ and ($\zeta_0 = 0$, $x = 0$ or $-\frac{1}{3} \le \zeta_0 < 0$, $0 \le x \le -3\zeta_0$)	Removed from phase space. See discussion in Sec. III A.
$1c$	x	$x(-4 + 3\gamma_{\rm de}) - 3\zeta_0$	0	$\alpha = \frac{4}{3} - \gamma_{\rm de}$, together with those in Table III below.	Saddle if $\zeta_0 < -\frac{1}{3}$
$2a$	0	1	0	*Always*	Unstable if $\zeta_0 < -\frac{1}{3}$, $\alpha < 1 - \gamma_{\rm de} - \zeta_0$ Stable if $\zeta_0 > 0$, $\alpha > 1 - \gamma_{\rm de} - \zeta_0$ Saddle if $\zeta_0 < -\frac{1}{3}$, $\alpha > 1 - \gamma_{\rm de} - \zeta_0$ or $-\frac{1}{3} < \zeta_0 < 0$, $\alpha \ne 1 - \gamma_{\rm de} - \zeta_0$ or $\zeta_0 \ge 0$, $\alpha < 1 - \gamma_{\rm de} - \zeta_0$
$2b$	0	y	$1 - y$	$\zeta_0 = 0$, $0 < y \le 1$	Saddle if $\alpha < 1 - \gamma_{\rm de}$
$2c$	$\frac{\zeta_0}{\gamma_{\rm de} - 1}$	y	$1 - y - \frac{\zeta_0}{\gamma_{\rm de} - 1}$	$\alpha = 1 - \gamma_{\rm de}$ and ($\zeta_0 > 0$, $0 \le y < 1$, $\gamma_{\rm de} \ge 1 - \frac{\zeta_0}{y - 1}$ or, $\zeta_0 < 0$, $0 \le y < 1$, $\gamma_{\rm de} \le 1 - \frac{\zeta_0}{y - 1}$ or $\zeta_0 = 0$, $0 < y \le 1$, $\gamma_{\rm de} \ne 1$)	Saddle if $\zeta_0 > 0$
$2d$	x	y	$1 - x - y$	$\zeta_0 = \alpha = 0$, $\gamma_{\rm de} = 1$ and ($y = 1$, $x = 0$ or $y = 0$, $0 < x \le 1$ or $0 < y < 1$, $0 \le x \le 1 - y$)	Removed from phase space. See discussion in Sec. III A.
$3a$	$1 - \frac{\alpha + \zeta_0}{1 - \gamma_{\rm de}}$	$\frac{\alpha + \zeta_0}{1 - \gamma_{\rm de}}$	0	$\gamma_{\rm de} < 1$, $-\zeta_0 \le \alpha \le 1 - \gamma_{\rm de} - \zeta_0$ or $\gamma_{\rm de} > 1$, $1 - \gamma_{\rm de} - \zeta_0 \le \alpha \le -\zeta_0$	Unstable if $\zeta_0 > -\frac{1}{3}$, $\alpha > \frac{4}{3} - \gamma_{\rm de}$ or $\zeta_0 \le -\frac{1}{3}$, $\alpha > 1 - \gamma_{\rm de} - \zeta_0$ Stable if $\zeta_0 > 0$, $\alpha < 1 - \gamma_{\rm de} - \zeta_0$ or $\zeta_0 \le 0$, $\alpha < 1 - \gamma_{\rm de}$ Saddle if $\zeta_0 > 0$, $1 - \gamma_{\rm de} < \alpha < \frac{4}{3} - \gamma_{\rm de}$ or $\zeta_0 > 0$, $1 - \gamma_{\rm de} - \zeta_0 < \alpha < 1 - \gamma_{\rm de}$ or $-\frac{1}{3} < \zeta_0 \le 0$, $1 - \gamma_{\rm de} - \zeta_0 < \alpha < \frac{4}{3} - \gamma_{\rm de}$ or $-\frac{1}{3} < \zeta_0 < 0$, $1 - \gamma_{\rm de} < \alpha < 1 - \gamma_{\rm de} - \zeta_0$ or $\zeta_0 \le -\frac{1}{3}$, $1 - \gamma_{\rm de} < \alpha < \frac{4}{3} - \gamma_{\rm de}$ or $\zeta_0 < -\frac{1}{3}$, $\frac{4}{3} - \gamma_{\rm de} < \alpha < 1 - \gamma_{\rm de} - \zeta_0$
$3b$	$1 - y$	y	0	$\alpha = -\zeta_0$, $\gamma_{\rm de} = 1$	Removed from phase space. See discussion in Sec. III A.

rules out any model that obeys the equations of motion (1) with the ansatz (2), because the presence of the bulk viscosity blockades the existence of standard RDE and MDE, if we are to believe in the LSLT as stated in Eq. (5); this is true even for the noninteracting case, $z(x, y) \equiv 0$, as shown in conditions (12)–(17). It must be noticed, though, that an accelerated expansion of the Universe at low redshifts is indeed compatible with the presence of a bulk viscosity.

In Secs. III and IV below, we will perform a full dynamical system analysis of the field equations for the particular case $Q = 3H\alpha\rho_{\rm de}$, and then we will show the importance of taking into account the full evolution of the Universe to constraint cosmological models.

TABLE III. Existence conditions for the critical point P_{1c} according to the physical phase space (9).

P_i	Existence
$1c$	($\zeta_0 < -\frac{1}{3}$ and ($\gamma_{\rm de} < \frac{4}{3} - \zeta_0$, $\frac{1+3\zeta_0}{-3+3\gamma_{\rm de}} \le x \le \frac{3\zeta_0}{-4+3\gamma_{\rm de}}$ or $\gamma_{\rm de} = \frac{4}{3} - \zeta_0$, $x = \frac{3\zeta_0}{-4+3\gamma_{\rm de}}$)) or ($\zeta_0 = -\frac{1}{3}$ and ($\gamma_{\rm de} < 1$, $0 \le x \le \frac{1}{4-3\gamma_{\rm de}}$ or $\gamma_{\rm de} > 1$, $x = 0$)) or ($-\frac{1}{3} < \zeta_0 < 0$ and ($0 \le x \le 1$, $\gamma_{\rm de} = \frac{4}{3} + \zeta_0$ or $0 \le x \le \frac{3\zeta_0}{-4+3\gamma_{\rm de}}$, $1 < \gamma_{\rm de} < \frac{4}{3} + \zeta_0$ or $0 \le x \le \frac{3\zeta_0}{-4+3\gamma_{\rm de}}$, $\gamma_{\rm de} < 1$ or $0 \le x \le \frac{1+3\zeta_0}{-3+3\gamma_{\rm de}}$, $\gamma_{\rm de} > \frac{4}{3} + \zeta_0$)) or ($\zeta_0 = 0$ and ($x = 0$, $\gamma_{\rm de} < 1$ or $x = 0$, $1 < \gamma_{\rm de} < \frac{4}{3}$ or $0 \le x \le 1$, $\gamma_{\rm de} = \frac{4}{3}$ or $0 \le x \le \frac{1}{-3+3\gamma_{\rm de}}$)) or ($\zeta_0 > 0$ and ($x = 1$, $\gamma_{\rm de} = \frac{4}{3}$ or $\frac{3\zeta_0}{-4+3\gamma_{\rm de}} \le x \le \frac{1+3\zeta_0}{-3+3\gamma_{\rm de}}$, $\gamma_{\rm de} > \frac{4}{3} + \zeta_0$))

TABLE IV. Eigenvalues and some basic physical parameters for the critical points listed in Table II, see also Eqs. (6) and (10).

P_i	λ_1	λ_2	λ_3	$w_{\rm eff}$	Ω_r	q
$1a$	1	$4-3\gamma_{\rm de}-3\alpha$	$1+3\zeta_0$	$\frac{1}{3}$	$1+3\zeta_0$	1
$1b$	1	0	$1+3\zeta_0$	$\frac{1}{3}$	$1+3\zeta_0$	1
$1c$	1	0	$1+3\zeta_0$	$\frac{1}{3}$	$1-3x(\gamma_{\rm de}-1)+3\zeta_0$	1
$2a$	$-1-3\zeta_0$	$-3\zeta_0$	$-3(-1+\gamma_{\rm de}+\alpha+\zeta_0)$	$-\zeta_0$	0	$\frac{1}{2}(1-3\zeta_0)$
$2b$	-1	0	$-3(-1+\gamma_{\rm de}+\alpha)$	0	0	$\frac{1}{2}$
$2c$	-1	0	$3\zeta_0$	0	0	$\frac{1}{2}$
$2d$	-1	0	0	0	0	$\frac{1}{2}$
$3a$	$3(-1+\gamma_{\rm de}+\alpha)$	$-4+3\gamma_{\rm de}+3\alpha$	$3(-1+\gamma_{\rm de}+\alpha+\zeta_0)$	$-1+\gamma_{\rm de}+\alpha$	0	$\frac{1}{2}(-2+3\gamma_{\rm de}+3\alpha)$
$3b$	0	$-1-3\zeta_0$	$-3\zeta_0$	$-\zeta_0$	0	$\frac{1}{2}(1-3\zeta_0)$

III. THE CASE FOR $Q = 3H\alpha\rho_{\rm de}$

This model of interaction was studied by [2] in the context of interacting DM-DE with the presence of bulk viscosity. The model can be recovered from model (i) in Table I with $\alpha_1 = -\sqrt{3}\zeta_0$ and $\alpha_2 = 0$. Nonetheless, our study below generalizes the model in [2] by taking a general interaction constant α, and two new components in the cosmic inventory: radiation and baryonic matter. We will comment on the model of [2] at the end of this section.

The selected Q term leads to the following dimensionless interaction variable z:

$$z = 3\alpha x. \tag{19}$$

The nine critical points of the autonomous system (7), together with the interaction term in Eq. (19), are summarized in Table II, whereas details about their stability and relevance for cosmology are given in Table IV.

A. Critical points and stability

The first point P_{1a} corresponds to the coexistence of radiation and DM, and exists if the bulk viscosity takes values in the range $-\frac{1}{3} \le \zeta_0 \le 0$. It also represents a decelerating expansion solution with $q = 1$ and $w_{\rm eff} = 1/3$. Critical point P_1 exhibits two different stability behaviors:

(i) Unstable if $\zeta_0 > -\frac{1}{3}$ and $\alpha < \frac{4}{3} - \gamma_{\rm de}$;
Saddle if $\zeta_0 > -\frac{1}{3}$ and $\alpha > \frac{4}{3} - \gamma_{\rm de}$.

In this point, the dimensionless energy parameter for radiation and DM takes the following values $\Omega_r = 1 + 3\zeta_0$ and $\Omega_{\rm dm} = -3\zeta_0$ respectively, as shown in Table IV. Therefore this point could represent a true RDE if $\Omega_{\rm dm} \ll 1$, as long as ζ_0 takes a negative value very close to zero, or, in the most extreme case, if $\Omega_{\rm dm} = 0$ then $\zeta_0 = 0$, meaning no bulk viscosity. In both cases, the existence interval and the needed values for the bulk viscosity, to archive a successful RDE, are outside the region of validity of the LSLT ($\zeta_0 > 0$).[2]

The nonhyperbolic critical point P_{1b} exists if $\gamma_{\rm de} = 1$ and $\alpha = \frac{1}{3}$. The first one condition is at odds with our expectation of a genuine DE fluid ($\gamma_{\rm de} < 1$), and then we will not take into account this critical point in our analysis.

P_{1c} correspond to a decelerated solution ($q = 1$) in which there is radiation, DM, and DE. In effective terms, this point is able to mimic the behavior of a radiation fluid ($w_{\rm eff} = \frac{1}{3}$) but, a truly RDE is only reached if $x \ll 1$ and $-1 \ll \zeta_0 < 0$, being the latter condition in contradiction with the LSLT. If $x = \frac{-3\zeta_0}{4-3\gamma_{\rm de}}$, then this critical point reproduces the analysis developed in the previous section, see Eq. (14). Despite its nonhyperbolic nature, P_{1c} always has a saddle behavior if $\zeta_0 < -\frac{1}{3}$, since it possesses nonempty stable and unstable manifolds, see Table IV.

Critical point P_{2a} represents a pure DM domination solution ($\Omega_{\rm DM} = 1$) and always exists, this fact is motivated by a null contribution of baryonic matter. The stability of this fixed point is the following:

(i) Saddle if $\zeta_0 < -\frac{1}{3}$, $\alpha > 1 - \gamma_{\rm de} - \zeta_0$ or $-\frac{1}{3} < \zeta_0 < 0$, $\alpha \neq 1 - \gamma_{\rm de} - \zeta_0$ or $\zeta_0 \ge 0$, $\alpha < 1 - \gamma_{\rm de} - \zeta_0$
(ii) Stable if $\zeta_0 > 0$, $\alpha > 1 - \gamma_{\rm de} - \zeta_0$
(iii) Unstable if $\zeta_0 < -\frac{1}{3}$, $\alpha < 1 - \gamma_{\rm de} - \zeta_0$.

An interesting fact of P_{2a} is the value of the effective EOS parameter ($w_{\rm eff} = -\zeta_0$): because of the nonnegative value of the bulk viscosity constant required by the LSLT, $w_{\rm eff} \le 0$, which means that we cannot recover a standard DM dominated picture, unless $\zeta_0 = 0$. P_{2a} is represented by a red point in Fig. 1.

The nonhyperbolic fixed point P_{2b} represents a scaling relation between the baryonic and DM components. As we claimed before in Sec. II, this critical point behaves as a realistic MDE and exists only under a null bulk viscosity contribution ($\zeta_0 = 0$). If $\alpha < 1 - \gamma_{\rm de}$ this critical point has a saddle behavior.

P_{2c} is a scaling solution between three components: baryons, DM, and DE, and, unlike point P_{2b}, it exists for all values of ζ_0 (see Table II for the rest of existence conditions). This critical point could represent a feasible MDE if $x = \frac{\zeta_0}{\gamma_{\rm de}-1} \ll 1$, and then $0 < \zeta_0 \ll 1$. This implies a fine-tuning over the bulk viscosity parameter due to the negligible amount of DE that should exist during a MDE,

[2] $\zeta_0 > 0$, as required by the LSLT, implies that for this critical point $y = \Omega_{\rm dm} < 0$, and then we get a wrong RDE, see Fig. 1.

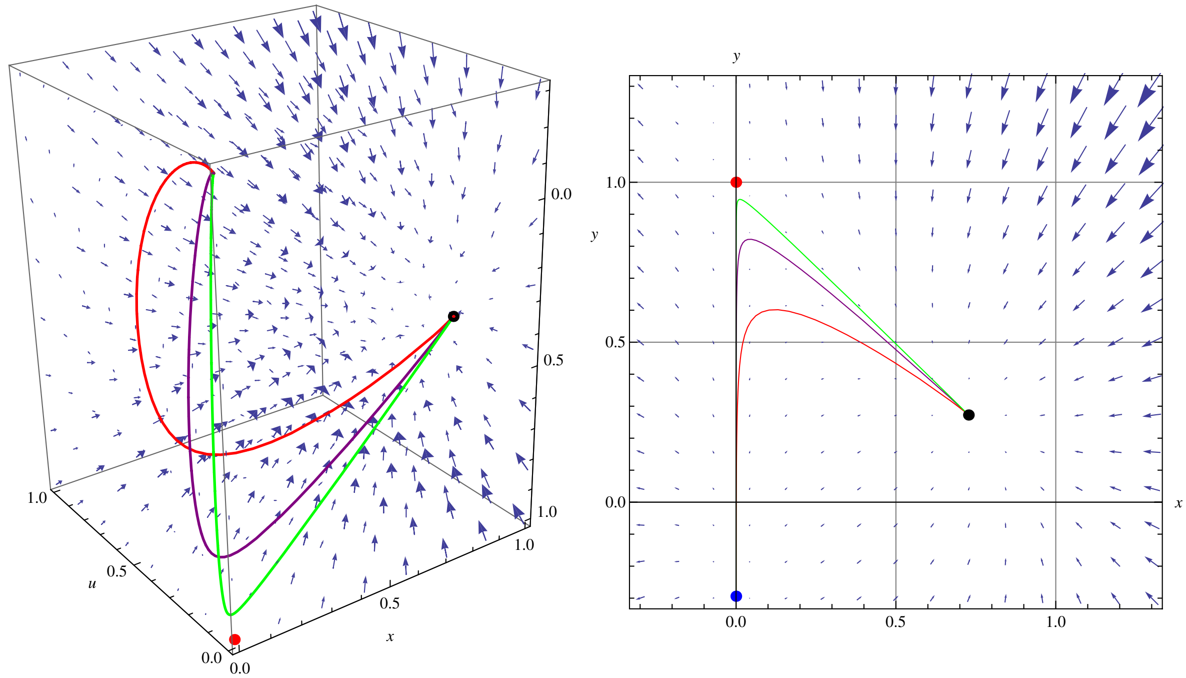

FIG. 1 (color online). Some orbits in the phase space for the choice $(\zeta_0, \gamma_{\rm de}, \alpha) = (0.098, 0.2, 0.12)$. This parameter election guarantees the saddle behavior of the pure DM dominated solution P_{2a} (red point) and the late time attractor nature of P_{3a}, black point. Because of the nonzero value of ζ_0 the early time unstable solution corresponds to a *wrong RDE* represented by the blue point.

which would render it almost indistinguishable from P_{2b} in the phase space. This critical point exists given that

(i) $\alpha = 1 - \gamma_{\rm de}$, $\zeta_0 > 0$, $0 \le y < 1$, $\gamma_{\rm de} \ge 1 - \frac{\zeta_0}{y-1}$. This region satisfies the LSLT (5), $\zeta_{\rm de} > 0$, but corresponds to a nontruly DE component, $w_{\rm de} > 0$.

(ii) $\alpha = 1 - \gamma_{\rm de}$, $\zeta_0 < 0$, $0 \le y < 1$, $\gamma_{\rm de} \le 1 - \frac{\zeta_0}{y-1}$. This region violates LSLT (5) but allows a valid MDE if the above condition, $0 < \zeta_0 \ll 1$, is satisfied.

(iii) $\alpha = 1 - \gamma_{\rm de}$, $0 < y \le 1$, $\gamma_{\rm de} \ne 1$ and $\zeta_0 = 0$.

Despite its nonhyperbolic nature, the critical point always exhibits a saddle behavior if $\zeta_0 > 0$ since it has nonempty stable and unstable manifolds.

Critical point P_{2d} corresponds to a very particular selection of the model parameters: $\alpha = \zeta_0 = 0$ and $\gamma_{\rm de} = 1$. These values represent a model with a null interaction between DM and DE, together with a null bulk viscosity contribution. The point P_{2d} will not appear in the phase space as long as we take $\alpha \ne 0$ and $\zeta_0 \ne 0$.

Point P_{3a} corresponds to a scaling solution between the DM and DE components. From Table II we can note that this point exists for any valid value of ζ_0, and it represents an accelerated solution if

$$\alpha < \frac{2}{3} - \gamma_{\rm de}. \tag{20}$$

P_{3a} exhibits an stable behavior if $\zeta_0 > 0$, $\alpha < 1 - \gamma_{\rm de} - \zeta_0$, or $\zeta_0 \le 0$, $\alpha < 1 - \gamma_{\rm de}$. The first one condition is supported by the LSLT, but the second is not. In the particular case $\alpha = -\zeta_0$ the strict DE domination is recovered ($\Omega_{\rm de} = x = 1$). The full set of stability conditions for this critical point is shown in Table IV.

If $\gamma_{\rm de} = 1$, $\alpha = -\zeta_0$ and $\zeta_0 > 0$, the critical point P_{3b} also appears in the phase space. However, the very first condition is at variance with our expectations of a truly DE fluid with $\gamma_{\rm de} < 1$. Hence, this critical point will be hereafter left out from our considerations.

B. Cosmology evolution from critical points

According to our complete cosmological dynamics criterion, one of the critical points of any physically viable model should correspond to a RDE at early enough times, and this point should be an unstable point; the unstable nature of this critical point guarantees that it can be the source of any orbits in the phase space. The only two possible candidates so far in our model are points P_{1a} and P_{1c}. Both cases require $-1 \ll \zeta_0 \le 0$ in order to be a true RDE point, but such a condition means a null contribution of bulk viscosity ($\zeta_0 = 0$), or else contradiction with the LSLT. Thus, we must conclude that no critical point exists in the model that can represent a RDE.

On the other hand, the evolution of the Universe requires the existence of a long enough matter dominated epoch, in which DM and baryons can be the dominant components. In our system, we need to look carefully at critical points P_2 to search for an appropriate candidate to be an unstable critical point dominated by the matter components.

In order to be in line with observations it is better to avoid those initial conditions that lead orbits to approach point P_{2a}, as it does not permit the presence of baryons and its effective EOS is negative in the region allowed by the LSTL, but it represents a point dominated solely by DM. Points P_{2b} and P_{2d} must also be discarded, as their existence always requires a null value of the viscosity coefficient, and P_{2d} even requires a null interaction between DM and DE.

The only possibility seems to be point P_{2c}, as long as observations could allow the presence of an early DE contribution to the energy density of the Universe. In such a case, the value of the viscosity coefficient ζ_0 would have to be finely adjusted. Unfortunately, as we showed in the previous discussion, this critical point requires a nonrealistic DE component with EOS $w_{\rm de} > 0$ ($\gamma_{\rm de} > 1$) in one case, and violation of the LSLT through a negative value of the bulk viscosity ($\zeta_0 < 0$) in the other.

Finally, we must get, as a possibility to alleviate the coincidence problem of DE, a scaling solution with a nearly constant ratio between the energy densities of DM and DE at late times, which should in turn correspond to a stable critical point; the only one at hand in our system that could fulfill those expectations is P_{3a}. For the allowed values of ζ_0, this point represents a scaling solution between the DE and DM components in the existence regions, and also admits a pure DE domination solution if only $\alpha = -\zeta_0$ ($\gamma \neq 1$). The required presence of the bulk viscosity limits the possibility of choosing initial conditions that lead orbits to connect MDE to DM-DE scaling solution to the following possibilities:

(i) Orbits that connect P_{2a} with P_{3a}. Despite the stable and accelerated nature of the scaling solution P_{3a}, it is not possible to recover the RDE and MDE as previously discussed. In Fig. 1 are shown some numerical integrations of the autonomous system (7a)–(7c), for the interaction function (19) with $(\zeta_0, \gamma_{\rm de}, \alpha) = (0.098, 0.2, 0.12)$. The orbits reveal that the P_{3a} solution is the future attractor whereas the *wrong* RDE (P_1) is the past attractor.

(ii) Orbits that connect P_{2c} with P_{3a}. The existence conditions of both critical points (see Table II) also imply that the DM-DE scaling solution mimics the behavior of pressureless matter ($w_{\rm eff} = 0$). In the same region, this solution is not accelerated ($q = \frac{1}{2}$) being impossible to explain the late-time behavior of the Universe. This result rules out those initial conditions leading to orbits connecting both critical points. Figure 2 shows some example orbits in the $x - y$ plane.

Unlike the above cases, the presence of non-null bulk viscosity entails no problem for a successful late-time accelerated evolution of the Universe but it is impossible to recover a well behaved picture of the whole history of the Universe without being at variance with the LSLT. The simultaneous presence of interaction between DM and DE and bulk viscosity results in a very restrictive condition for the model.

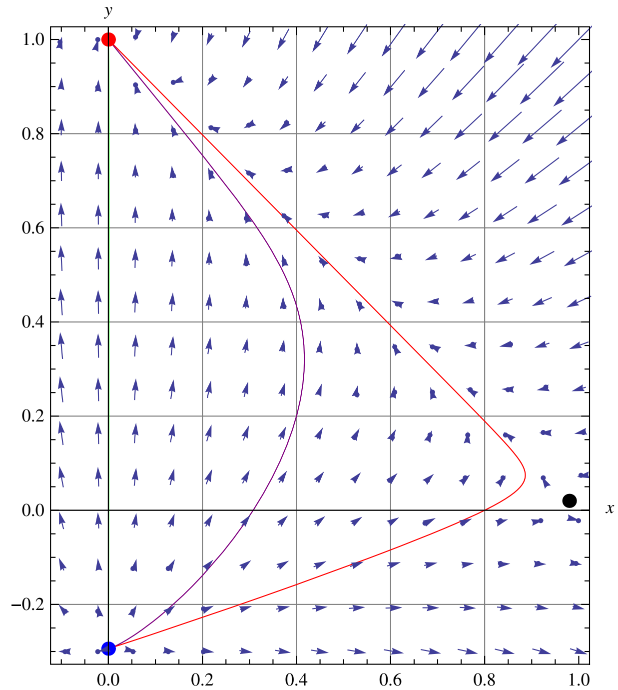

FIG. 2 (color online). Some orbits in the phase space for the choice $(\zeta_0, \gamma_{\rm de}, \alpha) = (0.098, 1.1, -0, 1)$. This parameter election guarantees the existence of the saddle critical point P_{2c} and at the same time changes the dynamics of the phase portrait: now the late time attractor is the DM dominated solution P_{2a} (red point) and the scaling solution between DM and DE, P_{3a} (black point), display a saddle-type behavior. As Table II shows, under this parameter choice P_{3a} is contained, as a particular case, in P_{2c}. Because of the nonzero value of ζ_0 the early time unstable solution corresponds to a *wrong RDE* represented by the blue point.

IV. COSMOLOGICAL CONSTRAINTS

We now proceed to constrain the values of $(\zeta_0, \gamma_{\rm de}, \alpha)$, compute their confidence intervals, and calculate their best estimated values, as we compare with different cosmological observations that measure the expansion history of the Universe. For future reference, we write here an explicit expression for the normalized Hubble parameter, which is an exact result for the model (1):

$$E^2(z) = \Omega_{r0}(1+z)^4 + \Omega_{b0}(1+z)^3 + \Omega_{de0}(1+z)^{3(\gamma_{\rm de}+\alpha)} + \hat{\Omega}_{\rm dm}(z), \tag{21}$$

where $E(z) \equiv H(z)/H_0$, and the cumbersome formula for $\hat{\Omega}_{\rm dm}(z)$ is given in Eq. (A10) of the Appendix, where all detailed calculations can be found.

A. Cosmological data and χ^2 functions

To perform all the numerical analyses, we assume for the baryon and radiation components (photons and relativistic neutrinos), the present values of $\Omega_{b0} = 0.04$ [23], and $\Omega_{r0} = 0.0000766$, respectively, where the latter value is computed from the expression [24]

$$\Omega_{r0} = \Omega_{\gamma 0}(1 + 0.2271 N_{\rm eff}). \quad (22)$$

Here, $N_{\rm eff} = 3.04$ is the standard number of effective neutrino species [23,25], and $\Omega_{\gamma 0} = 2.469 \times 10^{-5} h^{-2}$ corresponds to the present-day photon density parameter for a temperature of $T_{\rm cmb} = 2.725$ K [23], with h the dimensionless Hubble constant: $h \equiv H_0/(100\ {\rm km/s/Mpc})$.

1. Type Ia supernovae

The luminosity distance d_L in a spatially flat FRW universe is defined as

$$d_L(z, X, H_0) = \frac{c(1+z)}{H_0} \int_0^z \frac{dz'}{E(z', X)}, \quad (23)$$

where c corresponds to the speed of light in units of km/sec and $X = \{\gamma_{\rm de}, \alpha, \zeta_0\}$ is the parameter vector.

In order to marginalize analytically the nuisance parameter H_0, we define a new dimensionless "luminosity distance" as $D_L(z, X) \equiv H_0 \cdot d_L(z, X, H_0)/c$, so that $D_L(z, X)$ does not depend on H_0 anymore. The theoretical distance moduli can be written as

$$\mu^{\rm t}(z, X, H_0) = \tilde{\mu}^{\rm t}(z, X) - 5\log_{10}[\tilde{H}_0], \quad (24)$$

where we have defined $\tilde{\mu}^{\rm t}(z, X) \equiv 5\log_{10}[D_L(z, X)] + 25$ and a dimensionless Hubble parameter $\tilde{H}_0 \equiv H_0 \cdot {\rm Mpc}/c$. The superscript t stands for "theoretical." We construct the χ^2 function with these definitions as

$$\chi^2(X, H_0) = \sum_{i=1}^{n} \left(\frac{\tilde{\mu}^{\rm t}(z_i, X) - \mu_i^{\rm obs} - 5\log_{10}\tilde{H}_0}{\sigma_i} \right)^2, \quad (25)$$

where $\mu_i^{\rm obs}$ is the observed distance moduli and σ_i its variance. For our case, $n = 580$, as we are using the type Ia supernovae (SNe Ia) in the Union2.1 data set of the Supernova Cosmology Project, which is composed of 580 SNe Ia [26].

We rewrite the expression (25) as

$$\chi^2(X, H_0) = A - 2Bx + Cx^2, \quad (26)$$

with $x \equiv 5\log_{10}(\tilde{H}_0)$, and

$$A \equiv \sum_{i=1}^{n} \left(\frac{\tilde{\mu}_i^{\rm t} - \mu_i^{\rm obs}}{\sigma_i} \right)^2, \qquad B \equiv \sum_{i=1}^{n} \frac{\tilde{\mu}_i^{\rm t} - \mu_i^{\rm obs}}{\sigma_i^2}, \qquad C \equiv \sum_{i=1}^{n} \frac{1}{\sigma_i^2}. \quad (27)$$

The *posterior* probability distribution function (*pdf*) that is constructed from the χ^2 function (26) is

$$\mathbf{pdf}(X, \tilde{H}_0) = {\rm const} \cdot {\rm e}^{-\chi^2/2}, \quad (28)$$

where "const" is a normalization constant. We consider a *flat* prior distribution function for H_0. So, the marginalization over H_0 this $\mathbf{pdf}(X, \tilde{H}_0)$ corresponds to

$$\mathbf{pdf}(X) = {\rm const} \int_{-\infty}^{\infty} \mathbf{pdf}(X, \tilde{H}_0) d\tilde{H}_0. \quad (29)$$

Performing this integral analytically (see the Appendix A of [7] for a detailed explanation), we obtain

$$\mathbf{pdf}(X) = {\rm const} \left(\frac{\ln 10}{5} \right) \sqrt{\frac{2\pi}{C}} \exp\left[-\frac{1}{2}\left(A - \frac{\tilde{B}^2}{C} \right) \right], \quad (30)$$

so that $\mathbf{pdf}(X)$ does not depend on H_0 anymore. We express this $\mathbf{pdf}(X)$ in terms of a new $\chi^2_{\rm SNe}$ function as

$$\mathbf{pdf}(X) = b \cdot {\rm e}^{-\chi^2_{\rm SNe}/2}, \quad (31)$$

where $b \equiv {\rm const} \cdot \sqrt{2\pi} \ln 10/(5\sqrt{C})$, and then

$$\chi^2_{\rm SNe}(X) = A(X) - \frac{[B(X) + \ln(10)/5]^2}{C}. \quad (32)$$

2. Hubble expansion rate $H(z)$

For the Hubble parameter $H(z)$ at different redshifts, we use the 21 available data listed in Table I of Farooq *et al.* [27]. The χ^2 function is defined as

$$\chi^2_{\rm H}(X, H_0) = \sum_{i=1}^{21} \left(\frac{H^t(X, H_0, z_i) - H_i^{\rm obs}}{\sigma_H} \right)^2, \quad (33)$$

where $H^t(X, H_0, z_i)$ is the theoretical value predicted by the model given by Eq. (21), and $H_i^{\rm obs}$ is the observed value with a standard deviation σ_H.

In order to marginalize over the nuisance parameter H_0, we rewrite Eq. (33) as

$$\chi^2_{\rm H}(X, H_0) = H_0^2 \sum_i \frac{E^2(X, z_i)}{\sigma_i^2} - 2H_0 \sum_i \frac{E(X, z_i) \cdot H_i^{\rm obs}}{\sigma_i^2} + \sum_i \left(\frac{H_i^{\rm obs}}{\sigma_i} \right)^2, \quad (34)$$

where $E(X, z) \equiv H(X, z)/H_0$ is given by Eq. (21). We assume a Gaussian prior distribution for the possible value of H_0, centered at $\tilde{H}_0 = 73.8\ {\rm km\,Mpc^{-1}\,s^{-1}}$, with a standard deviation of $\sigma_H = 2.4\ {\rm km\,Mpc^{-1}\,s^{-1}}$, as measured by Riess *et al.* [28].

The *posterior* probability distribution function that is constructed from this χ^2 function and Gaussian prior is

$$\mathbf{pdf}_H(X, H_0) = cte \cdot e^{-\chi^2_{\rm H}(X, H_0)/2} \cdot e^{-(H_0 - \tilde{H}_0)^2/(2\sigma_H^2)}, \quad (35)$$

where cte is a normalization constant. Marginalizing over H_0 with

$$\mathbf{pdf}_H(X) = cte \int_{-\infty}^{\infty} \mathbf{pdf}(X, H_0) dH_0, \tag{36}$$

we obtain

$$\mathbf{pdf}_H(X) = cte \cdot \frac{1}{2\sigma_H \sqrt{\eta}} \left[1 + \mathrm{erf}\left(\frac{\beta}{\sqrt{2\eta}}\right)\right] \times \exp\left[-\frac{1}{2}\left(\gamma - \frac{\beta^2}{\eta}\right)\right], \tag{37}$$

where "erf" is the error function,[3] and

$$\eta \equiv \frac{1}{\sigma_H^2} + \sum_i \frac{E^2(X, z_i)}{\sigma_i^2}, \tag{38a}$$

$$\beta \equiv \frac{\tilde{H}_0}{\sigma_H^2} + \sum_i \frac{E(X, z_i) \cdot H_i^{\mathrm{obs}}}{\sigma_i^2}, \tag{38b}$$

$$\gamma \equiv \frac{\tilde{H}_0^2}{\sigma_H^2} + \sum_i \left(\frac{H_i^{\mathrm{obs}}}{\sigma_i}\right)^2. \tag{38c}$$

Writing the posterior **pdf** in terms of the likelihood distribution with flat priors for the parameters X, we have

$$\mathbf{pdf}_H(X) = \mathrm{const} \cdot \mathrm{e}^{-\tilde{\chi}_{\mathrm{H}}^2/2}. \tag{39}$$

Then, solving for the new $\tilde{\chi}^2$ function for H we obtain

$$\tilde{\chi}_{\mathrm{H}}^2(X) = 2\ln 2 + \ln(\eta\sigma_H^2) - -2\ln\left[1 + \mathrm{erf}\left(\frac{\beta}{\sqrt{2\eta}}\right)\right] + \left(\gamma - \frac{\beta^2}{\eta}\right). \tag{40}$$

Given that we are interested in the *location* of the point X_{best} in the parameter space that corresponds to the minimum of the χ^2 function, and that to construct the credible regions for pairs of set of $(\gamma_{\mathrm{de}}, \alpha, \zeta_0)$ we are interested only on *differences* $\Delta\chi^2$, then the constant "$2\ln 2$" may be discarded from the expression (40).

3. Baryon acoustic oscillations

To constrain our model using the baryon acoustic oscillation (BAO), we use a combination of six data listed in Table V [29–31], which are implemented in two different ways, as described below.

The WiggleZ Dark Energy Survey [31] reports three data using the acoustic parameter $A(z)$ defined as [32]

$$A(z, X) \equiv \frac{H_0 D_V(z, X)\sqrt{\Omega_{\mathrm{m}}}}{cz}, \tag{41}$$

where Ω_{m} it the total nonrelativistic matter (baryon and dark matter), and $D_V(z)$ is defined as (spatially flat Universe)

$$D_V(z, X) = c\left[\left(\int_0^z \frac{dz'}{H(z', X)}\right)^2 \frac{z}{H(z, X)}\right]^{1/3}. \tag{42}$$

[3] $\mathrm{erf}(x) \equiv (2/\sqrt{\pi}) \int_0^x e^{-t^2} dt$.

Note that $A(z, X)$ does not depend on H_0. With the acoustic parameter we define a χ^2 function as

$$\chi_A^2(X) = \sum_i^3 \left(\frac{A(z_i, X) - A^{\mathrm{obs}}(z_i)}{\sigma_A}\right)^2, \tag{43}$$

where $A^{\mathrm{obs}}(z_i)$ and σ_A correspond to the observed values for $A(z_i)$ and their standard deviations respectively, reported in Table III of Blake *et al.* [31].

On the other hand, SDSS (two data) [30] and 6dFGS (one datum) [29] collaborations report the indirect observed values of distance ratio d_z at different redshifts, defined as

$$d_z(X) \equiv \frac{r_s(z_d, X)}{D_V(z, X)}, \tag{44}$$

where $r_s(z)$ corresponds to the comoving sound horizon given by

$$r_s(z, X) = \frac{c}{\sqrt{3}} \int_0^{1/(1+z)} \frac{da}{a^2 H(a, X)\sqrt{1 + (3\Omega_{\mathrm{b}0}/4\Omega_{\gamma 0})a}}. \tag{45}$$

As mentioned above, we take the following values of the density parameters: $\Omega_{\gamma 0} = 2.469 \times 10^{-5} h^{-2}$ for photons, and $\Omega_{b0} = 0.02255 h^{-2}$ for baryons [23]. z_d is the redshift at the baryon drag epoch computed from the fitting formula [33]

$$z_d = 1291 \frac{(\Omega_{\mathrm{m}0} h^2)^{0.251}}{1 + 0.659(\Omega_{\mathrm{m}0} h^2)^{0.828}} [1 + b_1(\Omega_{\mathrm{m}0} h^2)^{b_2}], \tag{46a}$$

$$b_1 = 0.313(\Omega_{\mathrm{m}0} h^2)^{-0.419}[1 + 0.607(\Omega_{\mathrm{m}0} h^2)^{0.674}], \tag{46b}$$

$$b_2 = 0.238(\Omega_{\mathrm{m}0} h^2)^{0.223}. \tag{46c}$$

Note again that $d_z(X)$ does not depend on H_0

With the distance ratio we define a χ^2 function as

$$\chi_d^2(X) = \sum_i^3 \left(\frac{d_{z_i}(X) - d_{z_i}^{\mathrm{obs}}}{\sigma_{d_z}}\right)^2, \tag{47}$$

where $d_{z_i}^{\mathrm{obs}}$ and σ_{d_z} correspond to the observed values for d_{z_i} and their standard deviations respectively, shown in Table III of Blake *et al.* [29–31].

With the χ^2 functions in expressions (43) and (47), we define the total BAO χ^2 function as

$$\chi_{\mathrm{BAO}}^2(X) \equiv \chi_A^2(X) + \chi_d^2(X). \tag{48}$$

4. Cosmic microwave background radiation

We use the parameter θ_A proposed by Vonlanthen *et al.* in [34] that is defined as

$$\theta_A = \frac{r_{\mathrm{sp}}(z_*)}{D_A(z_*)}, \tag{49}$$

where $r_{sp}(z_*)$ and $D_A(z_*)$ are the *physical* sound horizon and the proper angular diameter distance between today and the redshift of decoupling z_*, respectively. θ_A is directly related to the position of the first peak l_1 in the power spectrum of the CMB. It has been shown in [34] that this parameter is suitable to test cosmological models of dark energy different to the ΛCDM model, given that it is an almost model-independent parameter for late time models. They argue that any dark energy model must predict a value of (given in radians)

$$\theta_A = 0.01035 \pm 0.00001745 \tag{50}$$

at the redshift of decoupling $z_* = 1094$. For an additional discussion on the use of the position of the first peak l_1 of the CMB for testing dark energy models, see also [35].

In a spatially flat universe D_A is given by

$$D_A(z) = \frac{c}{(1+z)H_0}\int_0^z \frac{dz'}{E(z')}. \tag{51}$$

The physical sound horizon corresponds to $r_{sp}(z) = r_s(z)/(1+z)$, where $r_s(z)$ is given by Eq. (45).

Note that the ratio $r_s(z)/D_A(z)$ cancels out H_0, so that θ_A does not depend on this nuisance parameter.

We define a χ^2 function as

$$\chi^2_{\rm CMB}(X) \equiv \left(\frac{\theta_A(X) - \theta_A^{\rm obs}}{\sigma_\theta}\right)^2, \tag{52}$$

where $\theta_A(X)$ is given by Eq. (49), and $\theta_A^{\rm obs}$, σ_θ corresponds to the values of the expression (50).

B. Observational constraints

Finally, with each of the χ^2 functions defined above we construct the total χ^2 function given by

$$\chi^2 = \chi^2_{\rm SNe} + \chi^2_{\rm BAO} + \chi^2_{\rm H} + \chi^2_{\rm CMB}. \tag{53}$$

We minimize this function with respect to the parameters $(\gamma_{\rm de}, \alpha, \zeta_0)$ to compute their best estimated values and confidence intervals.

There are four special cases we will discuss here, whose parameters are described and estimated in Table VI, and in the confidence intervals (CI) in Figs. 3, 5, 7, and 9.

Some general comments are in turn before the detailed explanation of the different models. First, we have noticed, for the quantities reported in Table VI, that there is a qualitative change in the models if only the low-redshift data sets are taken into account; in our case, these data sets are those of the supernovae (SNe) and the Hubble parameter $[H(z)]$. Such change is particularly acute in the case of the bulk viscosity ζ_0: it is consistently positive definite whenever the interaction parameter α is set free, like in model I. If α is fixed to be equal to ζ_0, or to have a null value, then the bulk viscosity is negative definite, like in models III and IV.

However, when high-redshift measurements are included in the analysis, we have the opposite behavior; all models consistently point to a negative value of the bulk viscosity

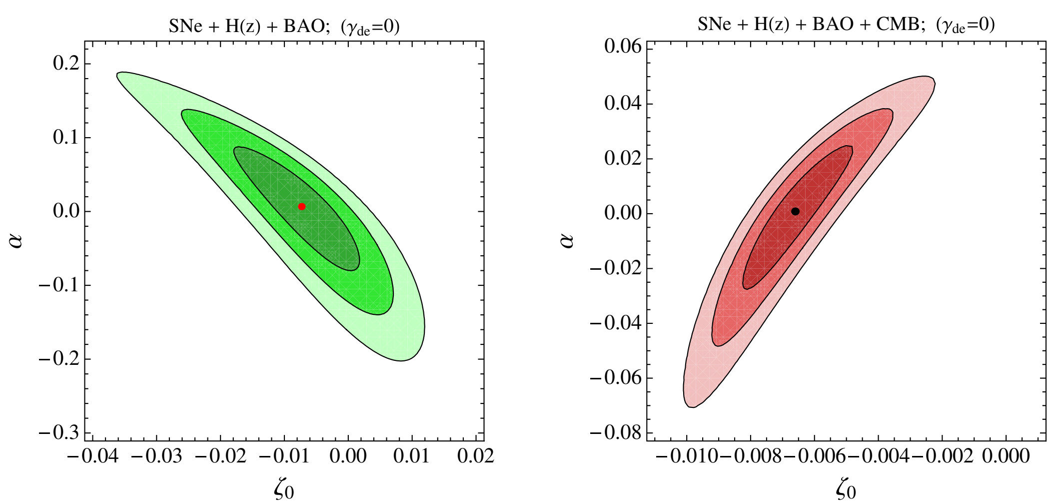

FIG. 3 (color online). Credible regions (CR) for model I: (ζ_0, α) as free parameters, and $\gamma_{\rm DE} = 0$. The CR shown correspond to the 68.3%, 95.4%, and 99.7% confidence level. The left and right panels correspond to the constraints on (ζ_0, α) that come from the use of the combined "SNe + BAO + $H(z)$" (SBH) and "SNe + BAO + $H(z)$ + CMB" (SBHC) data sets, respectively. For the left panel we find at 99.7% (3σ) confidence level that $-0.036 < \zeta_0 < 0.012$ and $-0.2026 < \alpha < 0.188$. For the right panel $-0.01 < \zeta_0 < -0.0022$ and $-0.07 < \alpha < 0.05$, when (ζ_0, α) are constrained simultaneously. The marginal best estimated values for each parameter individually are $\zeta_0 = -0.0073^{+0.067}_{-0.006}$ and $\alpha = 0.0067^{+0.056}_{-0.053}$ for SBH, and $\zeta_0 = -0.0066^{+0.016}_{-0.018}$ and $\alpha = 0.00082 \pm 0.001$ for SBHC, where the errors are given to 68.3% of confidence, see also Table VI. The bulk viscosity is constrained to small but *negative* values; however, $\zeta_0 < 0$ is forbidden by the LSLT, and then model I is ruled out with at least 99.7% of probability (3σ) when using SBHC, or with about 60% when using SBH.

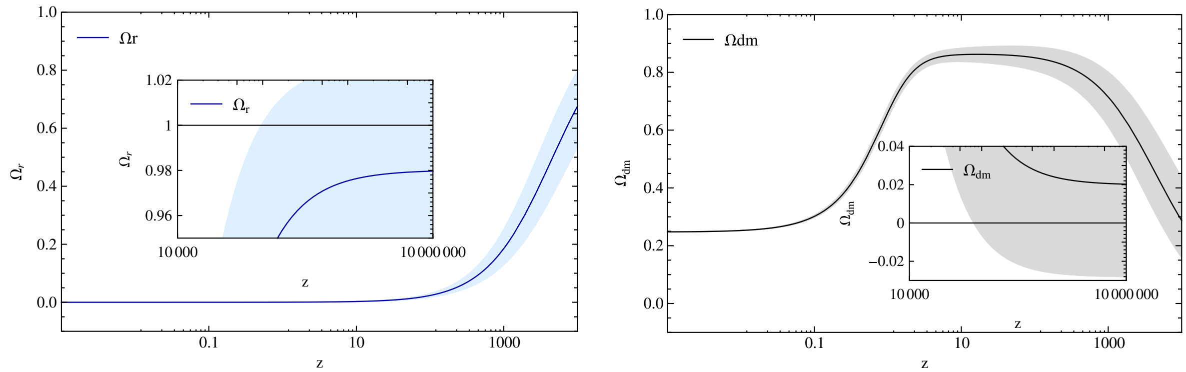

FIG. 4 (color online). Evolution of the dimensionless energy densities Ω_r and $\Omega_{\rm dm}$ as a function of the redshift z for Model I. The central thick lines comes from the evaluation of both observables at the best estimated values for $(\gamma_{\rm de}, \alpha, \zeta_0)$ (see Table VI). At high-redshift the contribution of the DM is about 2% for the best estimated values and, at 1σ the model can even describe a *wrong* RDE due to the unphysical values for $\Omega_r > 1$ and $\Omega_{\rm dm} < 1$. The transition from the RDE to MDE (dark matter + baryons) occurs at $z_{\rm eq} = 4607.80$ for the best estimated values. The error bands are given at 68.3% (1σ) of confidence level.

whenever it is freely fitted, like in models I, III, and IV. This means, actually, that all models with bulk viscosity as a free parameter are at variance with the LSLT when they are fitted to the sample set of cosmological observations.

Our second general comment is that none of the models are consistent with our so-called statement of complete cosmological dynamics presented and discussed in Sec. II B. The main reason being that we cannot recover an appropriate RDE at early times. One must notice, though, that our data sets cannot cover high enough redshifts in order to properly sample the early RDE of the Universe, but it is nonetheless significant that the estimated values of the free parameters already indicate a nonrecovery of an RDE. Such a difficulty was already observed in models with bulk viscosity [36,37], but it has not been sufficiently remarked in models with a DM-DE interaction [12,19,38].

One last comment regards that of the nature of DE in all models: we have consistently found that phantom DE [39] is slightly favored by all data sets whenever the DE EOS is freely varied, and that in most of the models an energy transfer from DM to DE is preferred.

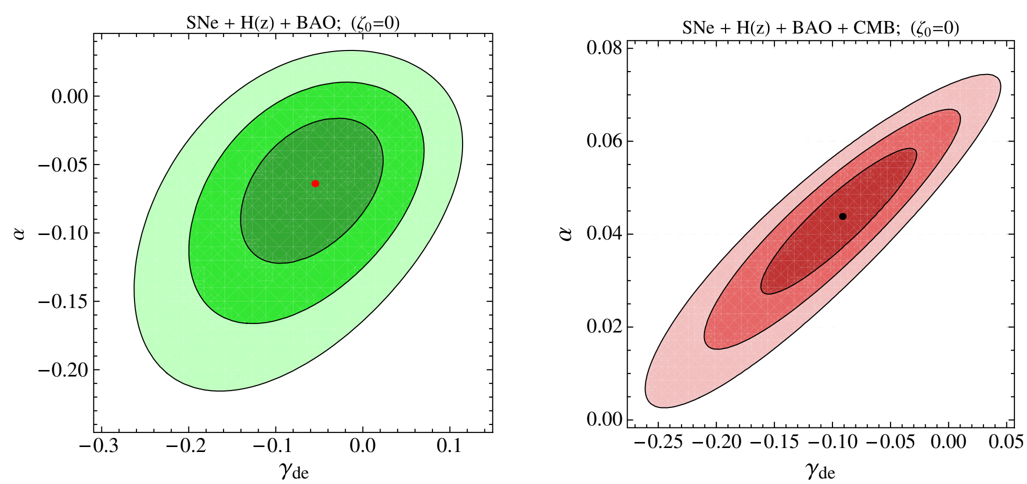

FIG. 5 (color online). Credible regions (CR) for model II (DM-DE interacting model *without* bulk viscosity): $(\gamma_{\rm de}, \alpha)$ as free parameters, and $\zeta_0 = 0$. The CI correspond to 68.3%, 95.4%, and 99.7% confidence level. The left and right panels correspond to the constraints on $(\gamma_{\rm de}, \alpha)$ that come from the use of the combined "SNe + BAO + $H(z)$" (SBH) and "SNe + BAO + $H(z)$ + CMB" (SBHC) data sets, respectively. For the left panel we find at 99.7% (3σ) confidence level that $-0.262 < \gamma_{\rm de} < 0.1154$ and $-0.216 < \alpha < 0.033$. For the right panel $-0.261 < \gamma_{\rm de} < 0.045$ and $0.0025 < \alpha < 0.074$, when $(\gamma_{\rm de}, \alpha)$ are constrained simultaneously. The marginal best estimated values for each parameter individually are $\gamma_{\rm de} = -0.054^{+0.55}_{-0.52}$ and $\alpha = -0.064^{+0.037}_{-0.033}$ for SBH, and $\gamma_{\rm de} = -0.091^{+0.043}_{-0.046}$ and $\alpha = 0.044 \pm 0.01$ for SBHC, where the errors are given to 68.3% of confidence, see also Table VI. We find that using the SBHC the interacting parameter α is positive with at least 99.7% of probability, favoring an interaction between the dark components, where the energy transfer goes from the dark matter to the dark energy. For the barotropic index $\gamma_{\rm de}$ the data favor negative, values with about 93% of probability, which corresponds to a *phantom* DE.

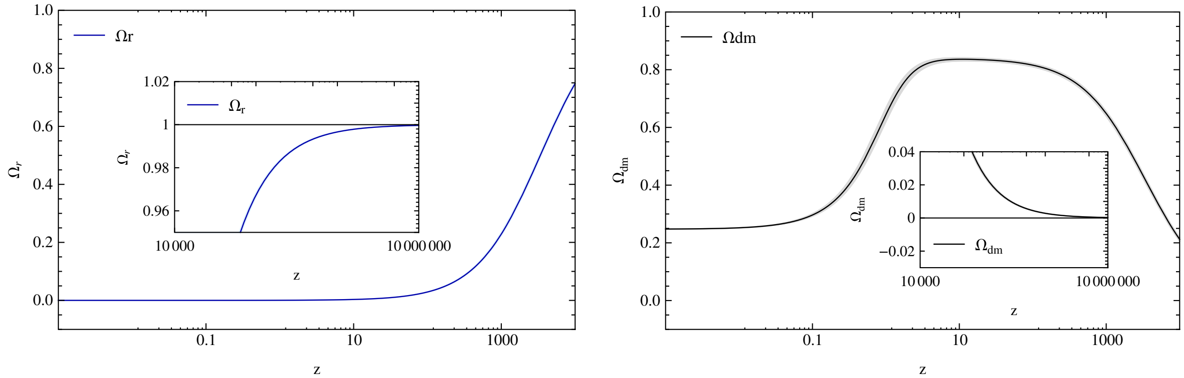

FIG. 6 (color online). Evolution of the dimensionless energy densities Ω_r and $\Omega_{\rm dm}$ as a function of the redshift z for model II. The central thick lines come from the evaluation of both observables at the best estimated values for $(\gamma_{\rm de}, \alpha, \zeta_0)$ (see Table VI). At high redshift a true RDE is recovery with $\Omega_r \rightarrow 1$ and $\Omega_{\rm dm} \rightarrow 0$. As was shown in the previous section, this model describes a *complete cosmological dynamics* since is possible to choose initial conditions that lead orbits to connect a true RDE → MDE → accelerated late time solution. The transition from the radiation to the matter dominated (dark matter + baryons) epoch occurs at $z_{\rm eq} = 3378.69$ for the best estimated values. The error bands are given at 68.3% (1σ) confidence level.

1. Model I

Model I corresponds to $\gamma_{\rm DE} = 0$, whereas α and ζ_0 are free parameters; that is, this case corresponds to a DE-DM interacting model, in which DM is a dust fluid with bulk viscosity, and DE is a cosmological constant, see for instance [2,40,41] and references therein for similar models.

According to the values presented in Table VI and in Fig. 3, the bulk viscosity is positive for low-redshift data sets, but it takes small negative values when the full data set is considered. However, we must recall that $\zeta_0 < 0$ is forbidden by the LSLT, see Eq. (5), and because of this model I would then be ruled out with at least 68% of probability (1σ). We notice that there is a slight preference for small but positive values of α, i.e, from Fig. 3 we see that the 68% contour region lies in the positive region for α. Finally, the CI for (ζ_0, α) parameters lie almost completely in a region that is not consistent with a well behaved cosmology defined by the dynamical system analysis in Sec. II B as Fig. 4 shows, because none of the critical points P_{1a} or P_{1c}, see Table II, is a suitable

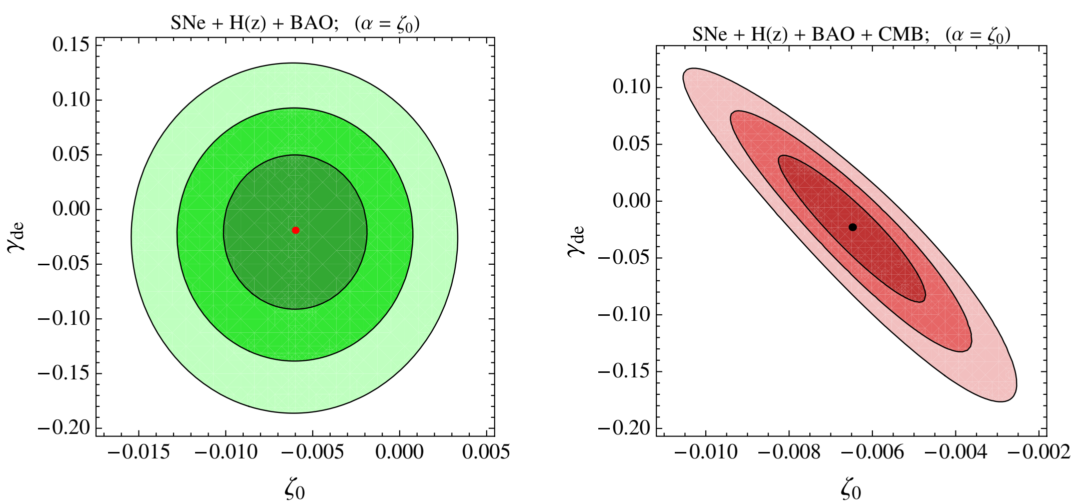

FIG. 7 (color online). Credible regions (CR) for model III: $(\zeta_0, \gamma_{\rm de})$ as free parameters, and $\alpha = \zeta_0$. The left and right panels correspond to the constraints on $(\zeta_0, \gamma_{\rm de})$ that come from the use of the combined "SNe + BAO + $H(z)$" (SBH) and "SNe + BAO + $H(z)$ + CMB" (SBHC) data sets, respectively. For the left panel we find at 99.7% (3σ) confidence level that $-0.0154 < \zeta_0 < 0.0033$ and $-0.186 < \gamma_{\rm de} < 0.133$. For the right panel $-0.0105 < \zeta_0 < -0.0025$ and $-0.177 < \gamma_{\rm de} < 0.117$, when $(\zeta_0, \gamma_{\rm de})$ are constrained simultaneously. The marginal best estimated values for each parameter individually are $\zeta_0 = -0.006 \pm 0.0027$ and $\gamma_{\rm de} = -0.019^{+0.047}_{-0.045}$ for SBH, and $\zeta_0 = -0.0065 \pm 0.0011$ and $\gamma_{\rm de} = -0.023^{+0.042}_{-0.043}$ for SBHC, where the errors are given to 68.3% of confidence, see also Table VI. We find that the bulk viscosity CR lie in the negative region, $\zeta_0 < 0$, with at least 99.7% of probability for SBHC, and therefore ruled out because this is at variance with the LSLT ($\zeta_0 < 0$). Even just using SBH it turns out that $\zeta_0 < 0$ with about 90% of probability. For $\gamma_{\rm de}$, there is a slight preference of the CR to lie in the negative region, i.e., $\gamma_{\rm de} < 0$ that corresponds to phantom DE.

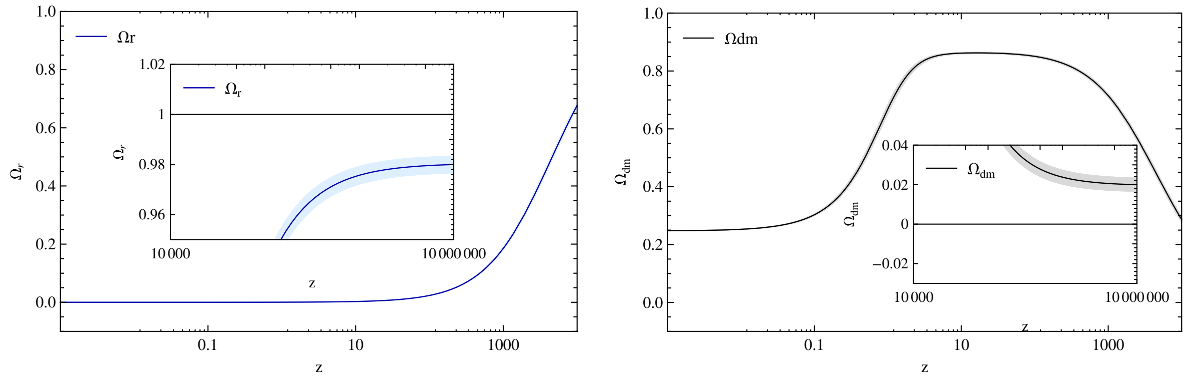

FIG. 8 (color online). Evolution of the dimensionless energy densities Ω_r and $\Omega_{\rm dm}$ as a function of the redshift z for model III. The central thick lines comes from the evaluation of both observables at the best estimated values for $(\gamma_{\rm de}, \alpha, \zeta_0)$ (see Table VI). Once again the negatives values of the bulk viscosity leads to a non-negligible contribution of the DM of about 2% at high redshift. The transition from the radiation to the matter dominated (dark matter + baryons) era occurs at $z_{\rm eq} = 4671.87$ for the best estimated values. The error bands are given at 68.3% (1σ) of confidence level.

point for a RDE; this fact then adds for the ruling out of this model.

2. Model II

Model II corresponds to $\zeta_0 = 0$, whereas $\gamma_{\rm DE}$ and α are free parameters; that is, it corresponds to a purely DM-DE interacting model, see for instance [12,19,36] and references therein. By definition, this model is in agreement with the LSLT.

Results for the free parameters are presented in Fig. 5. Both parameters $(\gamma_{\rm de}, \alpha)$ are close to zero, but *phantom* DE is slightly favored, $(\gamma_{\rm DE} < 0)$ at about 68.3% (1σ), as also is $\alpha < 0$, which corresponds to energy transfer from DM to DE. In both regions, as shown in Table II, the model describes a *complete cosmological dynamics* since it is possible to choose initial conditions that lead orbits to connect $P_{1a} \rightarrow P_{2b} \rightarrow P_{3a}$, see also Fig. 6.

3. Model III

Model III corresponds to $\alpha = \zeta_0$, whereas $\gamma_{\rm DE}$ and ζ_0 are free parameters; that is, this case corresponds to an *interacting* bulk viscous DM-DE model, where the

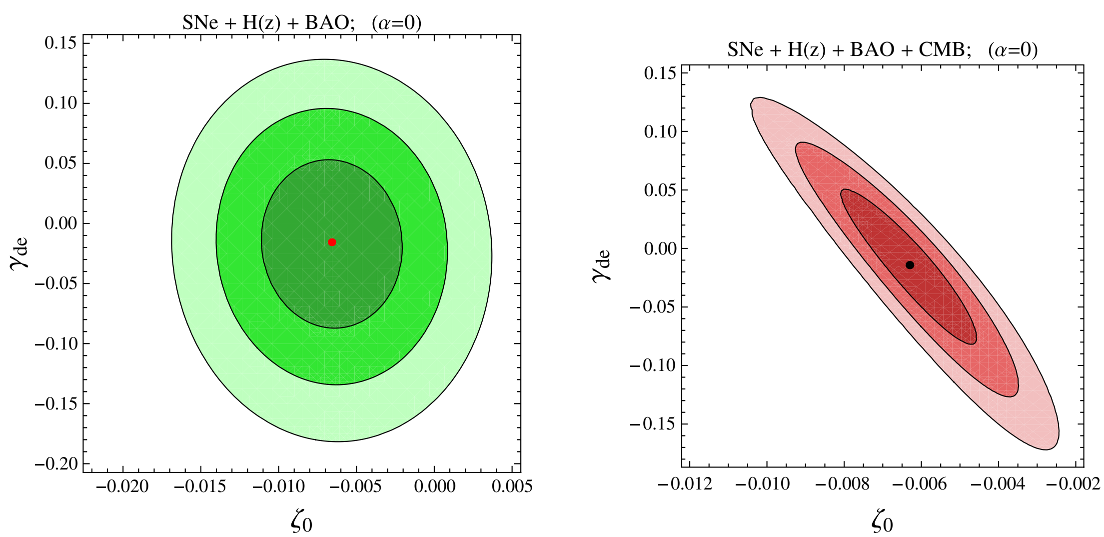

FIG. 9 (color online). Credible regions (CR) for model IV: $(\zeta_0, \gamma_{\rm de})$ as free parameters, and $\alpha = 0$. See Fig. 7 too. For the left panel we find at 99.7% (3σ) confidence level that $-0.0168 < \zeta_0 < 0.00375$ and $-0.1821 < \gamma_{\rm de} < 0.1365$. For the right panel $-0.0104 < \zeta_0 < -0.00242$ and $-0.1723 < \gamma_{\rm de} < 0.129$, when $(\zeta_0, \gamma_{\rm de})$ are constrained simultaneously. The marginal best estimated values for each parameter individually are $\zeta_0 = -0.0065 \pm 0.003$ and $\gamma_{\rm de} = -0.0156^{+0.047}_{-0.045}$ for SBH, and $\zeta_0 = -0.0063 \pm 0.0011$ and $\gamma_{\rm de} = -0.014^{+0.043}_{-0.044}$ for SBHC, where the errors are given to 68.3% of confidence, see also Table VI. We find interesting that the CRs are almost identical to the case $\alpha = 0$ (see Fig. 7), suggesting that the value of the bulk viscosity and the nature of the DE in this model is insensitive to the assumption of the interaction. The conclusions in this case are the same as for Fig. 7; this case is ruled out given that $\zeta_0 < 0$, with at least 99.7% of probability for SBHC, and there is a slight preference of the CR to lie in the negative region, i.e., $\gamma_{\rm de} < 0$ that corresponds to phantom DE.

TABLE V. BAO data.

Sample	z	d_z	$A(z)$
6dFGS [29]	0.106	0.336 ± 0.015	$\cdots$
SDSS [30]	0.2	0.1905 ± 0.0061	$\cdots$
SDSS [30]	0.35	0.1097 ± 0.0036	$\cdots$
WiggleZ [31]	0.44	$\cdots$	0.474 ± 0.034
WiggleZ [31]	0.6	$\cdots$	0.442 ± 0.020
WiggleZ [31]	0.73	$\cdots$	0.424 ± 0.021

interacting parameter is directly proportional to the bulk viscosity. A related model was studied by Kremer and Sobreiro [2] where they assume $\alpha = -\zeta_0$.

We find it interesting that the CIs presented in Fig. 7 are almost identical to those of model II (for $\alpha = 0$), see Sec. IV B 2 and Fig. 5, suggesting that the value of the bulk viscosity and the nature of the DE component in model III is insensitive to the assumption of the DM-DE interaction. Moreover, model III is not compatible with a RDE, see Table II and Fig. 8. Regardless of the initial conditions, P_{3a} is the only possible late time attractor of model III, but, as Table VI shows, observations favor negative values for α (and hence for ζ_0) and, for those negative values P_{3a} does not belong to the phase space (9) of the model. Thus, the model is ruled out because it is neither compatible with a *complete cosmological dynamics* nor with the LSLT.

4. Model IV

Model IV corresponds to $\alpha = 0$, whereas γ_{DE} and ζ_0 are free parameters; that is, it corresponds to a *noninteracting* DM-DE model, in which DM has bulk viscosity. See for instance [5,42] and references therein.

From the CR of the joint SNe + CMB + BAO + $H(z)$ data sets shown in Fig. 9, we find that the bulk viscosity is constrained again to small values and mainly in the negative region, with almost 68% of probability (1σ). The negative value of the bulk viscosity allows a RDE

TABLE VI. Marginal best estimated values of the parameters (γ_{de}, α, ζ_0) for the different models discussed in the text; notice that the DM barotropic index is that of a dust fluid for all cases, $\gamma_{DM} = 1$. The asterisk superscript indicates the cases when the zero value of one of the parameters was assumed *a priori*. The top (middle) part of the table only considers SNe [$H(z)$] observations, whereas the bottom part of the table corresponds to the use of the combined SNe + CMB + BAO + $H(z)$ data sets together, see Sec. IV. The fourth and fifth columns correspond to the minimum value of the χ^2 function, $\chi^2_{\rm min}$, and the χ^2 by degrees of freedom, $\chi^2_{\rm d.o.f.}$, respectively. The latter is defined as $\chi^2_{\rm d.o.f.} = \chi^2_{\rm min}/(n-p)$, where n is the number of data and p the number of free parameters. The next-to-last row (Model V) corresponds to best estimates of the three parameters (γ_{de}, α, ζ_0) computed simultaneously. H_0 was marginalized assuming a flat prior distribution. The last row, with ($\gamma_{de} = 0$, $\alpha = 0$, $\zeta_0 = 0$) corresponds to the value that we obtain for the ΛCDM model, using the same procedure and data sets, in order to compare our results. According to the value $\chi^2_{\rm d.o.f.}$, we find that all our cases fit the data sets as well as ΛCDM does. The last columns indicate the type of DE, the energy transfer direction, the consistency with the local second law of thermodynamics [LSLT, Eq. (5)], and with a complete cosmological dynamics (CCD) as discussed in Sec. II B. See Figs. 3 to 9 for their corresponding confidence intervals, and the text for more details.

Model	γ_{de}	α	ζ_0	$\chi^2_{\rm min}$	$\chi^2_{\rm d.o.f.}$	DE	Energy Transfer	LSLT	CCD
			SNe						
I	0^*	$-0.0132^{+0.22}_{-0.37}$	$0.0017^{+0.097}_{-0.075}$	562.223	0.972	Λ	DE ← DM	√	×
II	$-0.0011^{+0.1}_{-0.11}$	$-0.0086^{+0.1}_{-0.11}$	0^*	562.224	0.972	Phantom	DE ← DM	√	√
III	-0.0040 ± 0.14	$\alpha = \zeta_0$	$-0.0026^{+0.035}_{-0.032}$	562.224	0.972	Phantom	DE ← DM	×	×
IV	-0.0052 ± 0.18	0^*	$-0.0037^{+0.055}_{-0.051}$	562.225	0.972	Phantom	None	×	×
			SNe + BAO + $H(z)$						
I	0^*	$0.0067^{+0.056}_{-0.053}$	$-0.0073^{+0.0067}_{-0.0060}$	583.886	0.965	Λ	DE → DM	×	×
II	$-0.0548^{+0.55}_{-0.52}$	$-0.0639^{+0.037}_{-0.033}$	0^*	584.235	0.965	Phantom	DE ← DM	√	√
III	$-0.019^{+0.047}_{-0.045}$	$\alpha = \zeta_0$	-0.0060 ± 0.0027	583.765	0.964	Phantom	DE ← DM	×	×
IV	$-0.0156^{+0.047}_{-0.045}$	0^*	-0.00658 ± 0.0029	583.785	0.964	Phantom	None	×	×
V	-0.0235 ± 0.134	-0.0130 ± 0.0996	-0.0053 ± 0.010	583.757	0.965	Phantom	DE ← DM	×	×
ΛCDM	0^*	0^*	0^*	588.857	0.970	Λ	None	√	√
			SNe + BAO + $H(z)$ + CMB						
I	0^*	0.00082 ± 0.0011	$-0.0066^{+0.016}_{-0.018}$	583.899	0.963	Λ	DE → DM	×	×
II	$-0.0913^{+0.043}_{-0.046}$	0.0438 ± 0.010	0^*	603.166	0.995	Phantom	DE → DM	√	√
III	$-0.023^{+0.042}_{-0.043}$	$\alpha = \zeta_0$	-0.0065 ± 0.0011	583.803	0.963	Phantom	DE → DM	×	×
IV	$-0.014^{+0.043}_{-0.044}$	0^*	-0.0063 ± 0.0011	583.795	0.963	Phantom	None	×	×
V	-0.0180 ± 0.122	-0.003 ± 0.022	-0.0064 ± 0.0026	583.775	0.964	Phantom	DE → DM	×	×
ΛCDM	0^*	0^*	0^*	588.87	0.973	Λ	None	√	√

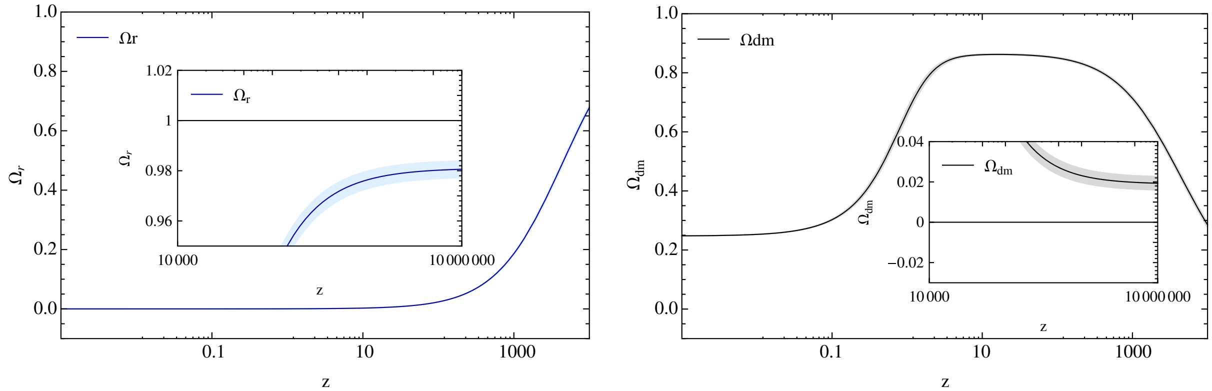

FIG. 10 (color online). Evolution of the dimensionless energy densities Ω_r and $\Omega_{\rm dm}$ as a function of the redshift z for model IV. The central thick lines come from the evaluation of both observables at the best estimated values for $(\gamma_{\rm de}, \alpha, \zeta_0)$ (see Table VI). At high redshift the contribution of the DM is about 2%. The transition from RDE to MDE (dark matter + baryons) occurs at $z_{\rm eq} = 4572.67$ for the best estimated values. The error bands are given at 68.3% (1σ) confidence level.

but with a CDM contribution of about 2% for high redshifts, see Fig. 10. At the same time, the future attractor will be the *wrong* DM-DE scaling solution (P_{3a} in Table II) with $x = \Omega_{\rm de} > 1$ and $y = \Omega_{\rm dm} < 0$. The above reasons ruled out this model because it is in tension with the LSLT and our statement of a complete cosmological dynamics.

On the other hand, we find that values of $\gamma_{\rm de} < 0$ are preferred by the observations with at least 68% of probability, corresponding to *phantom* DE.

5. Model V and ΛCDM

Model V corresponds to the case in which all parameters are freely varied simultaneously, and as such is our most general case. As in previous cases, we find again that phantom DE is slightly preferred, as is also the energy transfer from DE to DM. However, the final output is not compatible with the LSLT nor with a *complete cosmological dynamics*. For the latter, a proper RDE cannot be recovered at early times, and, at late times, the attractor is a *wrong* DM-DE scaling solution with unphysical values for $\Omega_{\rm dm} < 1$ and $\Omega_{\rm de} > 1$.

Just for comparison, we have also fitted the ΛCDM model to the same data and using the procedure; notice that this model is also our null-hypothesis case, as it is recovered if all parameters are given null values. Interestingly enough, the good of fitness of our models is as good as that of ΛCDM, a fact that points out that the used data sets are not powerful enough to differentiate the models; this is why we had to consider other constraints from the theoretical point of view, like that of the LSLT in Eq. (5), and the complete cosmological dynamics reviewed in Sec. III B.

V. DISCUSSION AND CONCLUSIONS

In the present work we studied, in general terms, a cosmological model that includes DM with bulk viscosity, an interaction term between DM and DE, and a free barotropic equation of state $p = (\gamma_{\rm de} - 1)\rho_{\rm de}$ for the dark energy. The dissipation in the DM component was characterized by a bulk viscosity ζ directly proportional to the expansion rate of the Universe, i.e., $\zeta = H\zeta_0/(8\pi G)$, where ζ_0 is a dimensionless constant. Another important assumption was that, except the DM, all matter components are represented by perfect fluids, which in itself constrains the type of DE that are affected by our analysis.

First of all, we performed a detailed dynamical system analysis of the model in order to investigate its asymptotic evolution and behavior. In addition, we demanded that our model must follow what we called a *complete cosmological dynamics*: namely, the existence of a viable RDE and MDE prior to a late-time acceleration stage; these three different eras have to be present in any model of physical interest. The imposition of this requirement is so strong that practically rules out all cases studied in the present work that present a bulk viscosity in the DM sector. This results from the fact that the bulk viscosity needs to be negative definite in order to have standard RDE and MDE, but that is not possible if we are to believe in the LSLT. However, a negative definite bulk viscosity is compatible with the speed up of the Universe at low redshifts, which actually was one of the appealing aspects of these types of models.

For purposes of illustration, we have applied our general results to the specific interaction function: $Q = 3\alpha\rho_{\rm de}H$, where the parameter α quantifies the strength and direction of the DM-DE interaction. As said before, we found that the bulk viscosity parameter was the troublesome one, and that we could accommodate a complete cosmological dynamics as long as $\zeta_0 = 0$.

Also, we tested the model using cosmological observations to estimate the free parameters and set constraints on them. The three parameters $(\gamma_{\rm de}, \alpha, \zeta_0)$ allowed us to have a very rich diversity of possible models to study, from purely interacting models ($\zeta_0 = 0$), to purely viscous models ($\alpha = 0$), and even the case of ΛCDM, which then acted as our null hypothesis ($\zeta_0 = 0 = \alpha$).

Whenever we tested the model with a non-null bulk viscosity, we found that a *negative* value of it was preferred, with at least 1σ of confidence level. This result is a drawback of the model presented here, given that it is in tension with the LSLT that reads $\zeta_0 > 0$. It should be said, though, that such a result was obtained when high-redshift data were included in the analysis. Actually, low-redshift data seems to favor a positive definite value of the bulk viscosity, but that would have lead us to wrong conclusions about the viability of the model. It would be interesting to verify if these conclusions could be extended to other functional forms for the bulk viscosity. As for the interaction parameter α, we found that in general the data favor a negative value, indicating an energy transfer from the DM to DE.

On the other hand, it is interesting to notice that using the cosmological observations we consistently found *negative* values of the barotropic index $\gamma_{\rm de}$, suggesting a phantom nature for the DE fluid that is in agreement with recent results [43], even though such a setup is troublesome from the theoretical point of view (like in the violation of the null energy condition $\rho + p \geq 0$).

We computed also the $\chi^2_{\rm d.o.f.}$ of all the models, and found that the goodness-of-fit to the data were equally good for all of them. This fact seems to indicate that the inclusion of new free parameters did not significantly improve the viability of the models, nor did it help to distinguish them from the null hypothesis represented by the concordance ΛCDM model.

ACKNOWLEDGMENTS

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APPENDIX: THE HUBBLE PARAMETER

Here, we give details about the calculation of the parameter density of the dark matter, $\hat{\Omega}_{\rm dm}(z)$, that appears in the Hubble parameter, see Eq. (21), that is required in Sec. IV to compute the observational constraints.

The exact solutions of the conservation equations (1b) and (1c), are, respectively,

$$\rho_{\rm r}(a) = \rho_{\rm r0}/a^4, \qquad \rho_{\rm b}(a) = \rho_{\rm b0}/a^3, \tag{A1}$$

where a is the scale factor, and the subscript zero labels the present values of the energy densities. If we take the interaction term $Q = 3H\alpha\rho_{\rm de}$, the conservation equations (1d) and (1e) can be rewritten as

$$\dot{\rho}_{\rm dm} + 3H\gamma^e_{\rm dm}\rho_{\rm dm} = 0, \qquad \dot{\rho}_{\rm de} + 3H\gamma^e_{\rm de}\rho_{\rm de} = 0, \tag{A2}$$

where we have defined the effective barotropic indexes,

$$\gamma^e_{\rm dm} = \gamma_{\rm dm} + (\gamma_{\rm de} - \gamma^e_{\rm de})\frac{\rho_{\rm de}}{\rho_{\rm dm}} - \frac{3H\zeta}{\rho_{\rm dm}}, \tag{A3}$$

$$\gamma^e_{\rm de} = \gamma_{\rm de} + \alpha. \tag{A4}$$

As all parameters are constant, we can integrate the equation of motion for the DE energy density, see Eq. (A2), and obtain

$$\rho_{\rm de}(a) = \rho_{\rm de0}a^{-3(\gamma_{\rm de}+\alpha)}. \tag{A5}$$

Hence, the barotropic index of DM, see Eq. (A3), can be written as

$$\gamma^e_{\rm dm} = \gamma_{\rm dm} - \frac{1}{\rho_{\rm dm}}\left(\alpha\rho_{\rm de} + \zeta_0\frac{3H^2}{8\pi G}\right). \tag{A6}$$

With the help of the Friedmann constraint (1a), and the exact solutions of the energy densities, Eq. (A6) can be finally rewritten as

$$\gamma^e_{\rm dm} = \gamma_{\rm dm} - \frac{1}{\rho_{\rm dm}}\left[\zeta_0\left(\frac{\rho_{\rm r0}}{a^4} + \frac{\rho_{\rm b0}}{a^3} + \rho_{\rm dm}\right) + \frac{\rho_{\rm de0}}{a^{3(\gamma_{\rm de}+\alpha)}}(\alpha + \zeta_0)\right], \tag{A7}$$

and then the equation of motion (A2) for the DM energy density becomes

$$\dot{\rho}_{\rm dm} = -3H\left[\gamma_{\rm dm}\rho_{\rm dm} - \zeta_0\left(\frac{\rho_{\rm r0}}{a^4} + \frac{\rho_{\rm b0}}{a^3} + \rho_{\rm dm}\right) - \frac{\rho_{\rm de0}}{a^{3(\gamma_{\rm de}+\alpha)}}(\alpha + \zeta_0)\right]. \tag{A8}$$

Next, we take the dimensionless density parameters for all matter components, $\Omega_{i0} \equiv \rho_{i0}/\rho^0_{\rm crit}$ and $\hat{\Omega}_{\rm dm} \equiv \rho_{\rm dm}/\rho^0_{\rm crit}$, where $\rho^0_{\rm crit} \equiv 3H_0^2/(8\pi G)$ is the present critical density; thus, Eq. (A8) becomes

$$\frac{(1+z)}{3}\frac{d\hat{\Omega}_{\rm dm}}{dz} - \hat{\Omega}_{\rm dm}(\gamma_{\rm dm} - \zeta_0) + \Omega_{\rm de0}(\alpha + \zeta_0)(1+z)^{3(\gamma_{\rm de}+\alpha)} + \zeta_0(1+z)^3[\Omega_{\rm r0}(1+z) + \Omega_{\rm b0}] = 0, \tag{A9}$$

where z is the redshift, which is related to the scale factor through $a = 1/(1+z)$. The analytical solution of Eq. (A9) is

$$\hat{\Omega}_{\rm dm}(z) = \frac{1}{(1+z)^{3\zeta_0}(1+3\zeta_0)(\alpha+\gamma_{\rm de}+\zeta_0-1)}\{3\zeta_0^2[(\Omega_{\rm b0}+\Omega_{\rm dm0}+\Omega_{\rm r0}-1)(1+z)^{3(\alpha+\gamma_{\rm de}+\zeta_0)} - (1+z)^{3(1+\zeta_0)}((1+z)\Omega_{\rm r0}+\Omega_{\rm b0}) + (1+z)^3] + \alpha(3\zeta_0+1)(\Omega_{\rm b0}+\Omega_{\rm dm0}+\Omega_{\rm r0}-1)(1+z)^{3(\alpha+\gamma_{\rm de}+\zeta_0)} + \zeta_0[(\Omega_{\rm b0}+\Omega_{\rm dm0}+\Omega_{\rm r0}-1)(1+z)^{3(\alpha+\gamma_{\rm de}+\zeta_0)} + (1+z)^{3(1+\zeta_0)}((2-3\gamma_{\rm de})\Omega_{\rm b0} - 3(1+z)(\gamma_{\rm de}-1)\Omega_{\rm r0}) + (1+z)^3(3(\gamma_{\rm de}-1)(\Omega_{\rm b0}+\Omega_{\rm dm0}) + (3\gamma_{\rm de}-4)\Omega_{\rm r0}+1)] + \alpha(1+z)^3[(1+z)^{3\zeta_0}(-3(1+z)\zeta_0\Omega_{\rm r0} - (1+3\zeta_0)\Omega_{\rm b0}) + 3\zeta_0 - \Omega_{\rm r0} + 1] - (1+z)^3(\gamma_{\rm de}-1)[\Omega_{\rm b0}((1+z)^{3\zeta_0}-1) - \Omega_{\rm dm0}]\}, \tag{A10}$$

where we have set $\gamma_{\rm dm} = 1$, and made use of the present Friedmann constraint $\Omega_{\rm de0} = 1 - (\Omega_{\rm r0} - \Omega_{\rm b0} - \Omega_{\rm dm0})$. We assume $\Omega_{\rm dm0} = 0.23$ in the calculations.

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