Branching fractions and direct *CP* violation in charmless three-body decays of *B* mesons

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Charmless three-body decays of B mesons are studied using a simple model based on the framework of the factorization approach. Hadronic three-body decays receive both resonant and nonresonant contributions. Dominant nonresonant contributions to tree-dominated three-body decays arise from the $b \to u$ tree transition which can be evaluated using heavy-meson chiral perturbation theory valid in the soft-meson limit. For penguin-dominated decays, nonresonant signals come mainly from the penguin amplitude governed by the matrix elements of scalar densities $\langle M_1M_2|\bar{q}_1q_2|0\rangle$. We use the measurements of $\bar{B}^0 \to K_SK_SK_S$ to constrain the nonresonant component of $\langle K\bar{K}|\bar{s}s|0\rangle$. The intermediate vector-meson contributions to three-body decays are identified through the vector current, while the scalar-meson resonances are mainly associated with the scalar density. While the calculated direct CP violation in $B^- \to K^+K^-K^-$ and $B^- \to \pi^+\pi^-\pi^-$ decays agrees well with experiment in both magnitude and sign, the predicted CP asymmetries in $B^- \to \pi^-K^+K^-$ and $B^- \to K^-\pi^+\pi^-$ have incorrect signs when confronted with experiment. It has been conjectured recently that a possible resolution to this CP puzzle may rely on final-state rescattering of $\pi^+\pi^-$ and K^+K^-. Assuming a large strong phase associated with the matrix element $\langle K\pi|\bar{s}q|0\rangle$ arising from some sort of power corrections, we fit it to the data of $K^-\pi^+\pi^-$ and find a correct sign for $\pi^-K^+K^-$. We predict some testable CP violation in $\bar{B}^0 \to K^+K^-\pi^0$ and $K^+K^-K_S$. In the low-mass regions of the Dalitz plot, we find that the regional CP violation is indeed largely enhanced with respect to the inclusive one, though it is still significantly below the data. In this work, strong phases arise from effective Wilson coefficients, propagators of resonances, and the matrix element of the scalar density $\langle M_1M_2|\bar{q}_1q_2|0\rangle$.

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I. INTRODUCTION

Recently, LHCb has measured direct CP violation in charmless three-body decays of B mesons [1–3] and found evidence of CP asymmetries in $B^+ \to \pi^+\pi^+\pi^-$ (4.9σ), $B^+ \to K^+K^+K^-$ (3.7σ), and $B^+ \to K^+K^-\pi^+$ (3.2σ), and a 2.8σ signal of CP violation in $B^+ \to K^+\pi^+\pi^-$ (see Table I). Direct CP violation in two-body resonances in the Dalitz plot has been seen at B factories. For example, both *BABAR* [5] and Belle [6] have claimed evidence of partial rate asymmetries in the channel $B^\pm \to \rho^0(770)K^\pm$ in the Dalitz-plot analysis of $B^\pm \to K^\pm\pi^\mp\pi^\pm$. The inclusive CP asymmetry in three-body decays results from the interference of the two-body resonances and three-body nonresonant (NR) decays and through the tree-penguin interference. CP asymmetries in certain local regions of the phase space are likely to be greater than the integrated inclusive ones. Indeed, LHCb has also observed large asymmetries in localized regions of phase space [1–3]. For example,

$$\mathcal{A}_{CP}^{\rm region}(K^+K^-K^-) = -0.226 \pm 0.020 \pm 0.004 \pm 0.007 \tag{1.1}$$

for $m^2_{K^+K^-\,{\rm high}} < 15\,{\rm GeV}^2$ and $1.2 < m^2_{K^+K^-\,{\rm low}} < 2.0\,{\rm GeV}^2$,

$$\mathcal{A}_{CP}^{\rm region}(K^-\pi^+\pi^-) = 0.678 \pm 0.078 \pm 0.032 \pm 0.007 \tag{1.2}$$

for $m^2_{K^-\pi^+\,{\rm high}} < 15\,{\rm GeV}^2$ and $0.08 < m^2_{\pi^+\pi^-\,{\rm low}} < 0.66\,{\rm GeV}^2$,

$$\mathcal{A}_{CP}^{\rm region}(K^+K^-\pi^-) = -0.648 \pm 0.070 \pm 0.013 \pm 0.007 \tag{1.3}$$

for $m^2_{K^+K^-} < 1.5\ {\rm GeV}^2$, and

$$\mathcal{A}_{CP}^{\rm region}(\pi^+\pi^-\pi^-) = 0.584 \pm 0.082 \pm 0.027 \pm 0.007 \tag{1.4}$$

for $m^2_{\pi^-\pi^-\,{\rm low}} < 0.4\ {\rm GeV}^2$ and $m^2_{\pi^+\pi^-\,{\rm high}} > 15\ {\rm GeV}^2$. Hence, significant signatures of CP violation were found in the above-mentioned low-mass regions devoid of most of the known resonances.

Three-body decays of heavy mesons are more complicated than the two-body case as they receive both resonant and nonresonant contributions. The analysis of these decays using the Dalitz-plot technique enables one to study the properties of various vector and scalar resonances. Indeed, most of the quasi-two-B decays are extracted from the Dalitz-plot analysis of three-body ones. Three-body hadronic B decays involving a vector meson or a

TABLE I. Experimental results of direct CP asymmetries (in %) for various charmless three-body B decays [1,2,4].

Final state	*BABAR*	Belle	LHCb	Average
$K^+K^-K^-$	$-1.7^{+1.9}_{-1.4} \pm 1.4$		$-4.3 \pm 0.9 \pm 0.3 \pm 0.7$	-3.7 ± 1.0
$(K^+K^-K^-)_{\rm NR}$	$6.0 \pm 4.4 \pm 1.3$			6.0 ± 4.8
$K^-K_SK_S$	$4^{+4}_{-5} \pm 2$			4^{+4}_{-5}
$K^+K^-\pi^-$	$0 \pm 10 \pm 3$		$-14.1 \pm 4.0 \pm 1.8 \pm 0.7$	-11.9 ± 4.1
$K^-\pi^+\pi^-$	$2.8 \pm 2.0 \pm 2.0 \pm 1.2$	$4.9 \pm 2.6 \pm 2.0$	$3.2 \pm 0.8 \pm 0.4 \pm 0.7$	3.3 ± 1.0
$K^-\pi^+\pi^0$	$-3.0^{+4.5}_{-5.1} \pm 5.5$	$7 \pm 11 \pm 1$		$0.0^{+5.9}_{-6.1}$
$(K^-\pi^+\pi^0)_{\rm NR}$	$10 \pm 16 \pm 8$			10 ± 18
$K^-\pi^0\pi^0$	$-6 \pm 6 \pm 4$			-6 ± 7
$\bar{K}^0\pi^+\pi^-$	$-1 \pm 5 \pm 1$			-1 ± 5
$\pi^+\pi^-\pi^-$	$3.2 \pm 4.4 \pm 3.1^{+2.5}_{-2.0}$		$11.7 \pm 2.1 \pm 0.9 \pm 0.7$	10.5 ± 2.2
$(\pi^+\pi^-\pi^-)_{\rm NR}$	$-14 \pm 14^{+18}_{-8}$			-14^{+23}_{-16}

charmed meson in the final state also have been observed at B factories. In this work we shall focus on charmless B decays into three pseudoscalar mesons.

It is known that the nonresonant signal in charm decays is small, less than 10% [7]. In the past years, many of the charmless B-to-three-body decay modes have been measured at B factories and studied using the Dalitz-plot analysis. The measured fractions and the corresponding branching fractions of nonresonant components for some of the three-body B decay modes are summarized in Table II. We see that the nonresonant fraction is about $\sim$(70–100)% in $B \to KKK$ decays, $\sim$(17–40)% in $B \to K\pi\pi$ decays, and $\sim$35% in the $B \to \pi\pi\pi$ decay. Hence, the nonresonant three-body decays play an essential role in penguin-dominated B decays. While this is striking in view of the rather small nonresonant background in three-body charm decays, it is not entirely unexpected because the energy release scale in weak B decays is of order 5 GeV, whereas the major resonances lie in the energy region of 0.77 to 1.6 GeV. Consequently, it is likely that three-body B decays may receive sizable nonresonant contributions. It is important to understand and identify the underlying mechanism for nonresonant decays.

Consider the nonresonant contributions to the three-body B decay $B \to P_1P_2P_3$. Under the factorization hypothesis, one of the nonresonant components arises from the transitions $B \to P_1P_2$ with an emission of P_3. The nonresonant background in charmless three-body B decays due to the transition $B \to P_1P_2$ has been studied extensively [22–27] based on heavy-meson chiral perturbation theory (HMChPT) [28–30]. However, the predicted rates of nonresonant decays due to the $B \to P_1P_2$ transition alone already exceed the measured total branching fractions for the tree-dominated modes, e.g., $\pi^-\pi^+\pi^-$ and $\pi^-K^+K^-$. For example, the branching fraction of the nonresonant rate of $B^- \to \pi^+\pi^-\pi^-$ estimated using HMChPT is found to be of order 75×10^{-6}, which is even larger than the total branching fraction of order 15×10^{-6} (see Table II). The issue has to do with the applicability of HMChPT. When it is applied to three-body decays, two of the final-state pseudoscalars have to be soft. If the soft-meson result is assumed to be the same in the whole Dalitz plot, the decay rate will be greatly overestimated. To overcome this issue, we have proposed in Ref. [31] to parametrize the momentum dependence of the nonresonant amplitudes induced by the $b \to u$ transition in an exponential form so that the HMChPT results are recovered in the soft pseudoscalar meson limit.

However, the nonresonant background in the $B \to P_1P_2$ transition does not suffice to account for the experimental observation that nonresonant contributions dominate in the penguin-dominated decays $B \to KKK$ and $B \to K\pi\pi$. As we have emphasized in Ref. [31], this implies that the nonresonant amplitude is also penguin dominated and governed by the matrix elements, e.g., $\langle K\bar{K}|\bar{s}s|0\rangle$ and $\langle K\pi|\bar{s}q|0\rangle$. That is, the matrix element of the scalar density should have a large nonresonant component. In Ref. [31] we have used the $\bar{B}^0 \to K_SK_SK_S$ mode in conjunction with the mass spectrum in $\bar{B}^0 \to K^+K^-\bar{K}^0$ to fix the nonresonant contribution to $\langle K\bar{K}|\bar{s}s|0\rangle$.

Besides the nonresonant background, it is necessary to study resonant contributions to three-body decays. The intermediate vector-meson contributions to three-body decays are identified through the vector current, while the scalar-meson resonances are mainly associated with the scalar density. They can also contribute to the three-body matrix element $\langle P_1P_2|J_\mu|B\rangle$. Resonant effects are conventionally described in terms of the usual Breit-Wigner formalism. In this manner we are able to identify the relevant resonances which contribute to the three-body decays of interest and compute the rates of $B \to VP$ and $B \to SP$. In conjunction with the nonresonant contribution, we are ready to calculate the total rates for three-body decays.

There are several competing approaches for describing charmless hadronic two-body decays of B mesons, such as QCD factorization (QCDF) [32], perturbative QCD (pQCD) [33], and soft-collinear effective theory [34]. Measurements of CP asymmetries will allow us to

TABLE II. Branching fractions of various charmless three-body decays of B mesons. The fractions and the corresponding branching fractions of nonresonant components are included whenever available. The first, second, and third entries are *BABAR*, Belle, and LHCb results, respectively.

Decay	$\mathcal{B}(10^{-6})$	$\mathcal{B}_{\rm NR}(10^{-6})$	NR fraction(%)	Resonances	Reference
$B^- \to \pi^+\pi^-\pi^-$	$15.2 \pm 0.6 \pm 1.3$	$5.3 \pm 0.7^{+1.3}_{-0.8}$	$34.9 \pm 4.2^{+8.0}_{-4.5}$	ρ^0, $\rho^0(1450)$	[8]
	–	–	–	$f_0(1370)$, $f_2(1270)$	
$B^- \to K^-\pi^+\pi^-$	$54.4 \pm 1.1 \pm 4.6$	$9.3 \pm 1.0^{+6.9}_{-1.7}$[a]	$17.1 \pm 1.7^{+12.4}_{-1.8}$	K^{*0}, K_0^{*0}, ρ^0, ω	[5]
	$48.8 \pm 1.1 \pm 3.6$	$16.9 \pm 1.3^{+1.7}_{-1.6}$	$34.0 \pm 2.2^{+2.1}_{-1.8}$	$f_0(980)$, K_2^{*0}, $f_2(1270)$	[6]
$B^- \to K^-\pi^0\pi^0$	$16.2 \pm 1.2 \pm 1.5$			K^{*-}, $f_0(980)$	[9]
	–	–	–		
$B^- \to K^+K^-\pi^-$	$5.0 \pm 0.5 \pm 0.5$				[10]
	<13				[11]
$B^- \to K^+K^-K^-$	$33.4 \pm 0.5 \pm 0.9$[b]	$22.8 \pm 2.7 \pm 7.6$	$68.3 \pm 8.1 \pm 22.8$	ϕ, $f_0(980)$, $f_0(1500)$	[12]
	$30.6 \pm 1.2 \pm 2.3$[b]	$24.0 \pm 1.5 \pm 1.5$	$78.4 \pm 5.8 \pm 7.7$	$f_0(1710)$, $f_2'(1525)$	[13]
$B^- \to K^-K_SK_S$	$10.1 \pm 0.5 \pm 0.3$	$19.8 \pm 3.7 \pm 2.5$	~ 196	$f_0(980)$, $f_0(1500)$	[12]
	$13.4 \pm 1.9 \pm 1.5$			$f_0(1710)$, $f_2'(1525)$	[11]
$\bar{B}^0 \to \bar{K}^0\pi^+\pi^-$	$50.2 \pm 1.5 \pm 1.8$	$11.1^{+2.5}_{-1.0} \pm 0.9$	$22.1^{+2.8}_{-2.0} \pm 2.2$	$f_0(980)$, ρ^0, K^{*+}	[14]
	$47.5 \pm 2.4 \pm 3.7$	$19.9 \pm 2.5^{+1.7}_{-2.0}$	$41.9 \pm 5.1^{+1.5}_{-2.6}$	K_0^{*+}, $f_2(1270)$	[15]
$\bar{B}^0 \to K^-\pi^+\pi^0$	$38.5 \pm 1.0 \pm 3.9$	$7.6 \pm 0.5 \pm 1.0$[c]	$19.7 \pm 1.4 \pm 3.3$	ρ^+, $\rho^+(1450)$	[16]
	$36.6^{+4.2}_{-4.3} \pm 3.0$	$5.7^{+2.7+0.5}_{-2.5-0.4} < 9.4$	<25.7	$K^{*(0,-)}$, $K_0^{*(0,-)}$	[17]
$\bar{B}^0 \to \overset{(-)}{K}{}^0 K^\mp\pi^\pm$[d]	$6.4 \pm 1.0 \pm 0.6$				[18]
	<18				[11]
	$6.4 \pm 0.9 \pm 0.4 \pm 0.3$				[19]
$\bar{B}^0 \to K^+K^-\pi^0$	–				
	$2.17 \pm 0.60 \pm 0.24$				[20]
$\bar{B}^0 \to K^+K^-\bar{K}^0$	$25.4 \pm 0.9 \pm 0.8$[b]	$33 \pm 5 \pm 9$	~ 130	ϕ, $f_0(980)$, $f_0(1500)$	[12]
	$28.3 \pm 3.3 \pm 4.0$			$f_0(1710)$, $f_2'(1525)$	[11]
	$19.1 \pm 1.5 \pm 1.1 \pm 0.8$				[19]
$\bar{B}^0 \to K_SK_SK_S$	$6.19 \pm 0.48 \pm 0.19$	$13.3^{+2.2}_{-2.3} \pm 2.2$	~ 215	$f_0(980)$, $f_0(1710)$	[21]
	$4.2^{+1.6}_{-1.3} \pm 0.8$			$f_2(2010)$	[11]

[a]The branching fraction for the phase-space nonresonant is $(2.4 \pm 0.5^{+1.3}_{-1.5}) \times 10^{-6}$.
[b]Contributions from χ_{c0} are excluded.
[c]The branching fraction for the phase-space nonresonant is $(2.8 \pm 0.5 \pm 0.4) \times 10^{-6}$.
[d]It is the sum of $\bar{K}^0K^+\pi^-$ and $K^0K^-\pi^+$.

discriminate between different models and improve the approach. For example, in the heavy-quark limit, the predicted *CP* asymmetries for the penguin-dominated modes $\bar{B}^0 \to K^-\pi^+$, $K^{*-}\pi^+$, $K^-\rho^+$, and $\bar{B}_s^0 \to K^+\pi^-$ have incorrect signs when confronted with experiment [35,36]. In the approach of QCDF, their signs can be flipped into the right direction by considering $1/m_b$ power corrections from penguin annihilation. Therefore, even an information on the sign of *CP* asymmetries will be very valuable.

The recent LHCb measurements of inclusive and local direct *CP* asymmetries in charmless $B \to P_1P_2P_3$ decays [1–3] provide a new test ground of the factorization approach. Let us first check the signs of *CP* violation. The observed negative relative sign of *CP* asymmetries between $B^- \to \pi^-\pi^+\pi^-$ and $B^- \to K^-K^+K^-$ and between $B^- \to K^-\pi^+\pi^-$ and $B^- \to \pi^-K^+K^-$ is in accordance with that expected from U-spin symmetry, which enables us to relate the $\Delta S = 0$ amplitude to the $\Delta S = 1$ one. However, symmetry arguments alone do not tell us the relative sign of *CP* asymmetries between $\pi^-\pi^+\pi^-$ and $\pi^-K^+K^-$ and between $K^-\pi^+\pi^-$ and $K^-K^+K^-$. Based on a realistic model calculation we find positive relative signs, which are in contradiction to the LHCb experiment. How to resolve this *CP* enigma becomes a very important issue in the study of hadronic three-body decays. The LHCb observation of the correlation of the *CP* violation between the decays, $\mathcal{A}_{CP}(\pi^-\pi^+\pi^-) \approx -\mathcal{A}_{CP}(\pi^-K^+K^-)$ and $\mathcal{A}_{CP}(K^-\pi^+\pi^-) \approx -\mathcal{A}_{CP}(K^-K^+K^-)$ has led to the conjecture that $\pi^+\pi^- \leftrightarrow K^+K^-$ rescattering may play an important role in the generation of the strong phase difference needed for such a violation to occur.

In this work we shall follow the framework of Ref. [31] to update the analysis of three-body decays and explore inclusive and regional *CP* violation in detail. We take the factorization approximation as a working hypothesis rather than a first-principles starting point as factorization has not been proven for three-body B decays. Therefore, we shall work in the phenomenological factorization model rather than in the established theories, such as QCDF, pQCD, or soft-collinear effective theory.[1] For CP violation, we will focus on direct CP asymmetry and will not discuss mixing-induced CP violation in, for example, $B^0 \to K^+K^-K_S$ and $K_SK_SK_S$. This topic has been discussed in Refs. [31,39].

The layout of the present paper is as follows. We shall first discuss the decay $B \to \pi\pi\pi$ in Sec. II in order to fix the parameter for describing the nonresonant background at the tree level. We discuss this mode in detail to set up the framework for studying resonant and nonresonant contributions. Then in Sec. III we proceed to $B \to KKK$ decays to emphasize the importance of nonresonant penguin contributions to penguin-dominated modes. The three-body channels $B \to K\pi\pi$ and $B \to KK\pi$ are discussed in Secs. IV and V, respectively. In Sec. VI, we determine the rates for $B \to VP$ and $B \to SP$ and compare our results with the approach of QCD factorization. Inclusive and localized CP asymmetries are addressed in Sec. VII. Section VIII contains our conclusions. Some of the input parameters used in this work are collected in Appendix A. Factorizable amplitudes for some of the $B \to PPP$ decays not discussed previously in Ref. [31] are shown in Appendix B.

II. $B \to \pi\pi\pi$ DECAYS

For three-body B decays, the $b \to sq\bar{q}$ penguin transitions contribute to the final states with an odd number of kaons, namely, KKK and $K\pi\pi$, while $b \to uq\bar{q}$ tree and $b \to dq\bar{q}$ penguin transitions contribute to final states with an even number of kaons, e.g., $KK\pi$ and $\pi\pi\pi$. We shall discuss the decay $B \to \pi\pi\pi$ first in order to fix the parameter needed for describing the nonresonant background at the tree level and then $B \to KKK$ to fix the unknown parameter for the nonresonant penguin contribution. Finally, we proceed to discuss the $B \to K\pi\pi$ and $B \to KK\pi$ channels.

Under the factorization hypothesis, the decay amplitudes are given by

$$\langle P_1P_2P_3|\mathcal{H}_{\rm eff}|B\rangle = \frac{G_F}{\sqrt{2}}\sum_{p=u,c}\lambda_p^{(r)}\langle P_1P_2P_3|T_p^{(r)}|B\rangle, \qquad (2.1)$$

where $\lambda_p^{(r)} \equiv V_{pb}V_{pr}^*$, with $r = d, s$. For the KKK and $K\pi\pi$ modes $r = s$, and for the $KK\pi$ and $\pi\pi\pi$ channels $r = d$. The Hamiltonian $T_p^{(r)}$ has the expression [32]

$$\begin{aligned} T_p^{(r)} = {}& a_1\delta_{pu}(\bar{u}b)_{V-A}\otimes(\bar{r}u)_{V-A} + a_2\delta_{pu}(\bar{r}b)_{V-A}\otimes(\bar{u}u)_{V-A} + a_3(\bar{r}b)_{V-A}\otimes\sum_q(\bar{q}q)_{V-A} \\ & + a_4^p\sum_q(\bar{q}b)_{V-A}\otimes(\bar{r}q)_{V-A} + a_5(\bar{r}b)_{V-A}\otimes\sum_q(\bar{q}q)_{V+A} - 2a_6^p\sum_q(\bar{q}b)_{S-P}\otimes(\bar{r}q)_{S+P} + a_7(\bar{r}b)_{V-A}\otimes\sum_q\frac{3}{2}e_q(\bar{q}q)_{V+A} \\ & - 2a_8^p\sum_q(\bar{q}b)_{S-P}\otimes\frac{3}{2}e_q(\bar{r}q)_{S+P} + a_9(\bar{r}b)_{V-A}\otimes\sum_q\frac{3}{2}e_q(\bar{q}q)_{V-A} + a_{10}^p\sum_q(\bar{q}b)_{V-A}\otimes\frac{3}{2}e_q(\bar{r}q)_{V-A}, \end{aligned} \qquad (2.2)$$

with $(\bar{q}q')_{V\pm A} \equiv \bar{q}\gamma_\mu(1\pm\gamma_5)q'$, $(\bar{q}q')_{S\pm P} \equiv \bar{q}(1\pm\gamma_5)q'$, and a summation over $q = u, d, s$ is implied. For the effective Wilson coefficients, we shall follow Ref. [31] and use

$$\begin{aligned} & a_1 \approx 0.99 \pm 0.037i, \quad a_2 \approx 0.19 - 0.11i, \quad a_3 \approx -0.002 + 0.004i, \quad a_5 \approx 0.0054 - 0.005i, \quad a_4^u \approx -0.03 - 0.02i, \\ & a_4^c \approx -0.04 - 0.008i, \quad a_6^u \approx -0.06 - 0.02i, \quad a_6^c \approx -0.06 - 0.006i, \quad a_7 \approx 0.54\times10^{-4}i, \quad a_8^u \approx (4.5 - 0.5i)\times10^{-4}, \\ & a_8^c \approx (4.4 - 0.3i)\times10^{-4}, \quad a_9 \approx -0.010 - 0.0002i, \quad a_{10}^u \approx (-58.3 + 86.1i)\times10^{-5}, \quad a_{10}^c \approx (-60.3 + 88.8i)\times10^{-5}, \end{aligned} \qquad (2.3)$$

for typical a_i at the renormalization scale $\mu = m_b/2 = 2.1$ GeV. The strong phases of the effective Wilson coefficients arise from vertex corrections and penguin contractions calculated in the QCD factorization approach [32].

A. $B^- \to \pi^+\pi^-\pi^-$ decay

The factorizable tree-dominated $B^- \to \pi^+\pi^-\pi^-$ decay reads

[1]For the study of $B \to PPP$ decays in different approaches, the reader is referred to Refs. [37,38].

$$\begin{aligned}\langle\pi^+\pi^-\pi^-|T_p|B^-\rangle &= \langle\pi^+\pi^-|(\bar{u}b)_{V-A}|B^-\rangle\langle\pi^-|(\bar{d}u)_{V-A}|0\rangle[a_1\delta_{pu}+a_4^p+a_{10}^p-(a_6^p+a_8^p)r_\chi^\pi]\\ &+\langle\pi^-|(\bar{d}b)_{V-A}|B^-\rangle\langle\pi^+\pi^-|(\bar{u}u)_{V-A}|0\rangle[a_2\delta_{pu}+a_3+a_5+a_7+a_9]\\ &+\langle\pi^-|(\bar{d}b)_{V-A}|B^-\rangle\langle\pi^+\pi^-|(\bar{d}d)_{V-A}|0\rangle\left[a_3+a_4^p+a_5-\frac{1}{2}(a_7+a_9+a_{10}^p)\right]\\ &+\langle\pi^-|(\bar{d}b)_{V-A}|B^-\rangle\langle\pi^+\pi^-|(\bar{s}s)_{V-A}|0\rangle\left[a_3+a_5-\frac{1}{2}(a_7+a_9)\right]\\ &+\langle\pi^-|\bar{d}b|B^-\rangle\langle\pi^+\pi^-|\bar{d}d|0\rangle(-2a_6^p+a_8^p)\\ &+\langle\pi^-\pi^+\pi^-|(\bar{d}u)_{V-A}|0\rangle\langle0|(\bar{u}b)_{V-A}|B^-\rangle(a_1\delta_{pu}+a_4^p+a_{10}^p)\\ &+\langle\pi^-\pi^+\pi^-|\bar{d}(1+\gamma_5)u|0\rangle\langle0|\bar{u}\gamma_5 b|B^-\rangle(2a_6^p+2a_8^p),\end{aligned}\tag{2.4}$$

where $r_\chi^\pi(\mu)=2\frac{m_\pi^2}{m_b(\mu)(m_d(\mu)-m_u(\mu))}$. Since there are two identical π^- mesons in this decay, one should take into account the identical-particle effects. For example,

$$\begin{aligned}&\langle\pi^+\pi^-|(\bar{u}b)_{V-A}|B^-\rangle\langle\pi^-|(\bar{d}u)_{V-A}|0\rangle\\ &=\langle\pi^+(p_1)\pi^-(p_2)|(\bar{u}b)_{V-A}|B^-\rangle\langle\pi^-(p_3)|(\bar{d}u)_{V-A}|0\rangle\\ &\quad+\langle\pi^+(p_1)\pi^-(p_3)|(\bar{u}b)_{V-A}|B^-\rangle\langle\pi^-(p_2)|(\bar{d}u)_{V-A}|0\rangle,\end{aligned}\tag{2.5}$$

and a factor of $\frac{1}{2}$ should be put in the decay rate. Note that $\langle\pi^+\pi^-|(\bar{d}d)_{V-A}|0\rangle=-\langle\pi^+\pi^-|(\bar{u}u)_{V-A}|0\rangle$ due to isospin symmetry. The matrix element $\langle\pi^+\pi^-|(\bar{s}s)_{V-A}|0\rangle$ is suppressed by the Okubo-Zweig-Iizuka (OZI) rule.

Under the factorization approach, the $B^-\to\pi^+\pi^-\pi^-$ decay amplitude consists of three distinct factorizable terms: (i) the current-induced process with a meson emission, $\langle B^-\to\pi^+\pi^-\rangle\times\langle0\to\pi^-\rangle$, (ii) the transition process, $\langle B^-\to\pi^-\rangle\times\langle0\to\pi^+\pi^-\rangle$, and (iii) the annihilation process $\langle B^-\to0\rangle\times\langle0\to\pi^+\pi^-\pi^-\rangle$, where $\langle A\to B\rangle$ denotes a $A\to B$ transition matrix element. We shall consider the nonresonant background and resonant contributions separately.

1. Nonresonant background

For the current-induced process, the three-body matrix element $\langle\pi^+\pi^-|(\bar{u}b)_{V-A}|B^-\rangle$ has the general expression [40]

$$\begin{aligned}&\langle\pi^+(p_1)\pi^-(p_2)|(\bar{u}b)_{V-A}|B^-\rangle\\ &=ir(p_B-p_1-p_2)_\mu+i\omega_+(p_2+p_1)_\mu\\ &\quad+i\omega_-(p_2-p_1)_\mu+h\epsilon_{\mu\nu\alpha\beta}p_B^\nu(p_2+p_1)^\alpha(p_2-p_1)^\beta.\end{aligned}\tag{2.6}$$

The form factors r, $\omega_\pm$, and h can be evaluated in the framework of HMChPT [40]. However, this will lead to decay rates that are too large, in disagreement with experiment [41]. The heavy-meson chiral Lagrangian given in Refs. [28–30] is needed to compute the strong B^*BP, B^*B^*P, and $BBPP$ vertices. The results for the form factors read [23,40]

$$\begin{aligned}\omega_+&=-\frac{g}{f_\pi^2}\frac{f_{B^*}m_{B^*}\sqrt{m_Bm_{B^*}}}{s_{23}-m_{B^*}^2}\left[1-\frac{(p_B-p_1)\cdot p_1}{m_{B^*}^2}\right]+\frac{f_B}{2f_\pi^2},\\ \omega_-&=\frac{g}{f_\pi^2}\frac{f_{B^*}m_{B^*}\sqrt{m_Bm_{B^*}}}{s_{23}-m_{B^*}^2}\left[1+\frac{(p_B-p_1)\cdot p_1}{m_{B^*}^2}\right],\\ r&=\frac{f_B}{2f_\pi^2}-\frac{f_B}{f_\pi^2}\frac{p_B\cdot(p_2-p_1)}{(p_B-p_1-p_2)^2-m_B^2}\\ &\quad+\frac{2gf_{B^*}}{f_\pi^2}\sqrt{\frac{m_B}{m_{B^*}}}\frac{(p_B-p_1)\cdot p_1}{s_{23}-m_{B^*}^2}\\ &\quad-\frac{4g^2f_B}{f_\pi^2}\frac{m_Bm_{B^*}}{(p_B-p_1-p_2)^2-m_B^2}\\ &\quad\times\frac{p_1\cdot p_2-p_1\cdot(p_B-p_1)p_2\cdot(p_B-p_1)/m_{B^*}^2}{s_{23}-m_{B^*}^2},\end{aligned}\tag{2.7}$$

where $s_{ij}\equiv(p_i+p_j)^2$, $f_\pi=132$ MeV, and g is a heavy-flavor-independent strong coupling which can be extracted from the CLEO measurement of the D^{*+} decay width, $|g|=0.59\pm0.01\pm0.07$ [42]. We shall follow Ref. [28] to fix its sign to be negative. It follows that

$$\begin{aligned}A_{\text{current-ind}}^{\text{HMChPT}}&\equiv\langle\pi^-(p_3)|(\bar{s}u)_{V-A}|0\rangle\\ &\quad\times\langle\pi^+(p_1)\pi^-(p_2)|(\bar{u}b)_{V-A}|B^-\rangle\\ &=-\frac{f_\pi}{2}[2m_3^2r+(m_B^2-s_{12}-m_3^2)\omega_+\\ &\quad+(s_{23}-s_{13}-m_2^2+m_1^2)\omega_-].\end{aligned}\tag{2.8}$$

However, as pointed out before, the predicted nonresonant rates based on HMChPT are unexpectedly too large for tree-dominated decays. For example, the branching fraction of nonresonant $B^-\to\pi^+\pi^-\pi^-$ is found to be of order 75×10^{-6}, which is one order of magnitude larger than the *BABAR* result of $\sim5.3\times10^{-6}$ (see Table II). The issue has to do with the applicability of HMChPT. In order to apply this approach, two of the final-state pseudoscalars in the $B\to P_1P_2$ transition have to be soft. The momentum of the soft pseudoscalar should be smaller than the chiral-symmetry-breaking scale of order 1 GeV. For three-body charmless B decays, the available phase space where chiral perturbation theory is applicable is only a small fraction of the whole Dalitz plot. Therefore, it is not justified to apply chiral and heavy-quark symmetries to a certain kinematic region and then generalize it to the region beyond its validity. If the soft-meson result is assumed to be the same in the whole Dalitz plot, the decay rate will be greatly overestimated. Following Ref. [31], we shall assume the momentum dependence of nonresonant amplitudes in an exponential form, namely,

$$A_{\text{current-ind}} = A_{\text{current-ind}}^{\text{HMChPT}} e^{-\alpha_{\text{NR}} p_B \cdot (p_1+p_2)} e^{i\phi_{12}}, \tag{2.9}$$

so that the HMChPT results are recovered in the soft-meson limit $p_1, p_2 \to 0$. That is, the nonresonant amplitude in the soft-meson region is described by HMChPT, but its energy dependence beyond the chiral limit is governed by the exponential term $e^{-\alpha_{\text{NR}} p_B \cdot (p_1+p_2)}$. In what follows, we shall use the tree-dominated $B^- \to \pi^+\pi^-\pi^-$ decay data to fix the unknown parameter α_{NR}. Besides the nonresonant contribution from the current-induced process, the matrix elements $\langle \pi^+\pi^-|\bar{q}\gamma_\mu q|0\rangle$ and $\langle \pi^+\pi^-|\bar{d}d|0\rangle$ also receive nonresonant contributions. In principle, the weak vector form factor of the former matrix element can be related to the charged pion electromagnetic (e.m.) form factors. However, unlike the kaon case (which will be discussed below), the time-like e.m. form factors of the pions are not measured well enough to allow us to determine the nonresonant parts. Therefore, we shall only consider the resonant contribution to $\langle \pi^+\pi^-|\bar{q}\gamma_\mu q|0\rangle$. As for the matrix element $\langle \pi^+\pi^-|\bar{d}d|0\rangle$, it can be related to $\langle K^+K^-|\bar{s}s|0\rangle$ (to be discussed below) via SU(3) flavor symmetry. Nevertheless, it is suppressed by the smallness of the penguin Wilson coefficients a_6 and a_8. Therefore, the nonresonant component of $B^- \to \pi^-\pi^+\pi^-$ is predominated by the current-induced process, and its measurement provides an ideal place to constrain the parameter α_{NR}, which turns out to be

$$\alpha_{\text{NR}} = 0.081^{+0.015}_{-0.009}\ \text{GeV}^{-2}. \tag{2.10}$$

This is very close to the naive expectation of $\alpha_{\text{NR}} \sim \mathcal{O}(1/(2m_B\Lambda_\chi))$ based on the dimensional argument. The phase ϕ_{12} of the nonresonant amplitude in the $(\pi^+\pi^-)$ system will be set to zero for simplicity.

2. Resonant contributions

In general, vector-meson and scalar resonances contribute to the two-body matrix elements $\langle P_1P_2|V_\mu|0\rangle$ and $\langle P_1P_2|S|0\rangle$, respectively.[2] They can also contribute to the three-body matrix element $\langle P_1P_2|J_\mu|B\rangle$. Resonant effects are described in terms of the usual Breit-Wigner formalism. More precisely,

$$\begin{aligned}
&\langle \pi^+(p_1)\pi^-(p_2)|(\bar{u}b)_{V-A}|B^-\rangle^R \\
&\quad = \sum_i \langle \pi^+\pi^-|V_i\rangle \frac{1}{s - m_{V_i}^2 + im_{V_i}\Gamma_{V_i}} \langle V_i|(\bar{u}b)_{V-A}|B^-\rangle \\
&\qquad + \sum_i \langle \pi^+\pi^-|S_i\rangle \frac{-1}{s - m_{S_i}^2 + im_{S_i}\Gamma_{S_i}} \langle S_i|(\bar{u}b)_{V-A}|B^-\rangle, \\
&\langle \pi^+\pi^-|\bar{q}\gamma_\mu q|0\rangle^R \\
&\quad = \sum_i \langle \pi^+\pi^-|V_i\rangle \frac{1}{s - m_{V_i}^2 + im_{V_i}\Gamma_{V_i}} \langle V_i|\bar{q}\gamma_\mu q|0\rangle, \\
&\langle \pi^+\pi^-|\bar{d}d|0\rangle^R \\
&\quad = \sum_i \langle \pi^+\pi^-|S_i\rangle \frac{-1}{s - m_{S_i}^2 + im_{S_i}\Gamma_{S_i}} \langle S_i|\bar{d}d|0\rangle,
\end{aligned} \tag{2.11}$$

where $V_i = \phi, \rho, \omega, \ldots$ and $S_i = f_0(980), f_0(1370), f_0(1500), \ldots$. It follows that

$$\begin{aligned}
&\langle \pi^+(p_1)\pi^-(p_2)|(\bar{u}b)_{V-A}|B^-\rangle^R \\
&\quad = \sum_i \frac{g^{V_i\to\pi^+\pi^-}}{s_{12} - m_{V_i}^2 + im_{V_i}\Gamma_{V_i}} \\
&\qquad \times \sum_{\text{pol}} \varepsilon^* \cdot (p_1 - p_2)\langle V_i|(\bar{u}b)_{V-A}|B^-\rangle \\
&\qquad - \sum_i \frac{g^{S_i\to\pi^+\pi^-}}{s_{12} - m_{S_i}^2 + im_{S_i}\Gamma_{S_i}} \langle S_i|(\bar{u}b)_{V-A}|B^-\rangle, \\
&\langle \pi^+(p_1)\pi^-(p_2)|\bar{q}\gamma_\mu q|0\rangle^R \\
&\quad = \sum_i \frac{g^{V_i\to\pi^+\pi^-}}{s - m_{V_i}^2 + im_{V_i}\Gamma_{V_i}} \sum_{\text{pol}} \varepsilon^* \cdot (p_1 - p_2)\langle V_i|\bar{q}\gamma_\mu q|0\rangle, \\
&\langle \pi^+\pi^-|\bar{d}d|0\rangle^R \\
&\quad = -\sum_i \frac{g^{S_i\to\pi^+\pi^-}}{s - m_{S_i}^2 + im_{S_i}\Gamma_{S_i}} \langle S_i|\bar{d}d|0\rangle.
\end{aligned} \tag{2.12}$$

Using the decay constants defined by

$$\langle S|\bar{q}_2q_1|0\rangle = m_S\bar{f}_S, \qquad \langle P(p)|\bar{q}_2\gamma_\mu\gamma_5 q_1|0\rangle = -if_P p_\mu, \qquad \langle V(p)|\bar{q}_2\gamma_\mu q_1|0\rangle = f_V m_V \varepsilon_\mu^*, \tag{2.13}$$

and the form factors defined by[3]

[2]The two-body matrix element $\langle P_1P_2|V_\mu|0\rangle$ sometimes can also receive contributions from scalar resonances. For example, both K^* and $K_0^*(1430)$ contribute to the matrix element $\langle K^-\pi^+|(\bar{s}d)_{V-A}|0\rangle$; see Eq. (2.12).

[3]We follow Ref. [43] for the $B \to P$ and $B \to V$ transition form factors. The form factors for the $B \to S$ transitions are defined in Ref. [44].

$$\begin{aligned}
\langle P(p')|V_\mu|B(p)\rangle &= \left((p+p')_\mu - \frac{m_B^2-m_P^2}{q^2}q_\mu\right)F_1^{BP}(q^2) + \frac{m_B^2-m_P^2}{q^2}q_\mu F_0^{BP}(q^2),\\
\langle S(p')|A_\mu|B(p)\rangle &= -i\left[\left((p+p')_\mu - \frac{m_B^2-m_S^2}{q^2}q_\mu\right)F_1^{BS}(q^2) + \frac{m_B^2-m_S^2}{q^2}q_\mu F_0^{BS}(q^2)\right],\\
\langle V(p',\varepsilon)|V_\mu|B(p)\rangle &= \frac{2}{m_B+m_V}\epsilon_{\mu\nu\alpha\beta}\varepsilon^{*\nu}p^\alpha p'^\beta V(q^2),\\
\langle V(p',\varepsilon)|A_\mu|B(p)\rangle &= i\left[(m_B+m_V)\varepsilon_\mu^* A_1^{BV}(q^2) - \frac{\varepsilon^*\cdot p}{m_B+m_V}(p+p')_\mu A_2^{BV}(q^2) - 2m_V\frac{\varepsilon^*\cdot p}{q^2}q_\mu[A_3^{BV}(q^2)-A_0^{BV}(q^2)]\right],
\end{aligned}\tag{2.14}$$

where $P_\mu=(p+p')_\mu$, $q_\mu=(p-p')_\mu$, $A_3^{BV}(0)=A_0^{BV}(0)$, and

$$A_3^{BV}(q^2) = \frac{m_B+m_V}{2m_V}A_1^{BV}(q^2) - \frac{m_B-m_V}{2m_V}A_2^{BV}(q^2),\tag{2.15}$$

we are led to

$$\begin{aligned}
&\langle\pi^+(p_1)\pi^-(p_2)|(\bar ub)_{V-A}|B^-\rangle^R\langle\pi^-(p_3)|(\bar du)_{V-A}|0\rangle\\
&\quad= -\sum_i\frac{f_\pi}{2\sqrt2}\frac{g^{\rho_i\to\pi^+\pi^-}}{s_{12}-m_{\rho_i}^2+im_{\rho_i}\Gamma_{\rho_i}}(s_{13}-s_{23})\left[(m_B+m_{\rho_i})A_1^{B\rho_i}(q^2) - \frac{A_2^{B\rho_i}(q^2)}{m_B+m_{\rho_i}}(m_B^2-s_{12}) - 2m_{\rho_i}[A_3^{B\rho_i}(q^2)-A_0^{B\rho_i}(q^2)]\right]\\
&\qquad-\sum_i f_\pi\frac{g^{f_{0i}\to\pi^+\pi^-}}{s_{12}-m_{f_{0i}}^2+im_{f_{0i}}\Gamma_{f_{0i}}}(m_B^2-s_{12})F_0^{Bf_0^u}(q^2),
\end{aligned}\tag{2.16}$$

with $q^2=(p_B-p_1-p_2)^2=p_3^2$, and

$$\begin{aligned}
&\langle\pi^+(p_1)\pi^-(p_2)|\bar u\gamma_\mu u|0\rangle^R\\
&\quad= -\frac{1}{\sqrt2}\sum_i\frac{m_{\rho_i}f_{\rho_i}g^{\rho_i\to\pi^+\pi^-}}{s_{12}-m_{\rho_i}^2+im_{\rho_i}\Gamma_{\rho_i}}(p_1-p_2)_\mu,\\
&\langle\pi^+(p_1)\pi^-(p_2)|\bar dd|0\rangle^R = -\sum_i\frac{m_{f_{0i}}\bar f_{f_{0i}}^d g^{f_{0i}\to\pi^+\pi^-}}{s_{12}-m_{f_{0i}}^2+im_{f_{0i}}\Gamma_{f_{0i}}},
\end{aligned}\tag{2.17}$$

where the scalar decay constant $\bar f_{f_{0i}}^q$ is defined by $\langle f_{0i}|\bar qq|0\rangle = m_{f_{0i}}\bar f_{f_{0i}}^q$, $g^{f_{0i}\to\pi^+\pi^-}$ is the $f_{0i}\to\pi^+\pi^-$ strong coupling. Hence, the relevant transition amplitudes are

$$\begin{aligned}
&\langle\pi^+(p_1)\pi^-(p_2)|(\bar uu)_{V-A}|0\rangle^R\langle\pi^-(p_3)|(\bar db)_{V-A}|B^-\rangle\\
&\quad= -F_1^{B\pi}(s_{12})F_R^{\pi^+\pi^-}(s_{12})(s_{13}-s_{23}),\\
&\langle\pi^+(p_1)\pi^-(p_2)|\bar dd|0\rangle^R\langle\pi^-|\bar db|B^-\rangle\\
&\quad= -\frac{m_B^2-m_\pi^2}{m_b-m_d}F_0^{B\pi}(s_{12})\sum_i\frac{m_{f_{0i}}\bar f_{f_{0i}}^d g^{f_{0i}\to\pi^+\pi^-}}{s_{12}-m_{f_{0i}}^2+im_{f_{0i}}\Gamma_{f_{0i}}},
\end{aligned}\tag{2.18}$$

with

$$F_R^{\pi^+\pi^-}(s) = \frac{1}{\sqrt2}\sum_i\frac{m_{\rho_i}f_{\rho_i}g^{\rho_i\to\pi^+\pi^-}}{s-m_{\rho_i}^2+im_{\rho_i}\Gamma_{\rho_i}}.\tag{2.19}$$

3. Numerical results

The strong coupling constants, such as $g^{\rho(770)\to\pi^+\pi^-}$ and $g^{f_0(980)\to\pi^+\pi^-}$, are determined from the measured partial widths through the relations

$$\Gamma_{S\to P_1P_2} = \frac{p_c}{8\pi m_S^2}g_{S\to P_1P_2}^2,\qquad \Gamma_{V\to P_1P_2} = \frac{2}{3}\frac{p_c^3}{4\pi m_V^2}g_{V\to P_1P_2}^2\tag{2.20}$$

for scalar and vector mesons, respectively, where p_c is the c.m. momentum. The numerical results are

$$\begin{aligned}
&g^{\rho(770)\to\pi^+\pi^-} = 6.0,\qquad g^{K^*(892)\to K^+\pi^-} = 4.59,\\
&g^{f_0(980)\to\pi^+\pi^-} = 1.33^{+0.29}_{-0.26}\ \text{GeV},\\
&g^{K_0^*(1430)\to K^+\pi^-} = 3.84\ \text{GeV}.
\end{aligned}\tag{2.21}$$

Note that the neutral ρ meson cannot decay into $\pi^0\pi^0$ owing to isospin invariance. In determining the coupling of $f_0\to\pi^+\pi^-$, we have used the partial width

$$\Gamma(f_0(980)\to\pi^+\pi^-) = (34.2^{+13.9+8.8}_{-11.8-2.5})\ \text{MeV}\tag{2.22}$$

measured by Belle [45]. In this work, we shall specifically use $g^{f_0(980)\to\pi^+\pi^-} = 1.18$ GeV to have a better description of $B\to f_0(980)K$ channels in $B\to K\pi\pi$ decays.

The calculated branching fractions of resonant and nonresonant contributions to $B^-\to\pi^+\pi^-\pi^-$ are summarized in Table III. The theoretical errors shown there are from the uncertainties in (i) the parameter $\alpha_{\rm NR}$ [see Eq. (2.10)] which governs the momentum dependence of the nonresonant amplitude, (ii) the strange-quark mass m_s for decay

TABLE III. Branching fractions (in units of 10^{-6}) of resonant and nonresonant contributions to $B^- \to \pi^+\pi^-\pi^-$. The nonresonant background is used as an input to fix the parameter α_{NR} defined in Eq. (2.9). Theoretical errors correspond to the uncertainties in (i) α_{NR}, (ii) $F_0^{B\pi}$, σ_{NR}, and $m_s(\mu) = (90 \pm 20)$ MeV at $\mu = 2.1$ GeV, and (iii) $\gamma = (69.7^{+1.3}_{-2.8})^\circ$. Experimental results are taken from Table II.

Decay mode	*BABAR* [8]	Theory
$\rho^0\pi^-$	$8.1 \pm 0.7 \pm 1.2^{+0.4}_{-1.1}$	$6.7^{+0.0+0.4+0.1}_{-0.0-0.4-0.1}$
$\rho^0(1450)\pi^-$	$1.4 \pm 0.4 \pm 0.4^{+0.3}_{-0.7}$	
$f_0(1370)\pi^-$	$2.9 \pm 0.5 \pm 0.5^{+0.7}_{-0.5}$	$1.6^{+0.0+0.0+0.0}_{-0.0-0.0-0.0}$
$f_0(980)\pi^-$	<1.5	$0.2^{+0.0+0.0+0.0}_{-0.0-0.0-0.0}$
NR	$5.3 \pm 0.7 \pm 0.6^{+1.1}_{-0.5}$	input
Total	$15.2 \pm 0.6 \pm 1.2^{+0.4}_{-0.3}$	$16.1^{+1.9+1.0+0.2}_{-2.3-0.8-0.2}$

modes involving kaon(s), the form factor $F_0^{B\pi}$, and the nonresonant parameter σ_{NR} to be introduced below in Eq. (3.11), and (iii) the unitarity angle γ.

TABLE IV. Predicted branching fractions (in units of 10^{-6}) of resonant and nonresonant contributions to $\bar{B}^0 \to \pi^+\pi^-\pi^0$.

Decay mode	Theory	Decay mode	Theory
$\rho^+\pi^-$	$3.8^{+0.0+0.4+0.0}_{-0.0-0.3-0.0}$	$\rho^0\pi^0$	$1.0^{+0.0+0.2+0.0}_{-0.0-0.1-0.0}$
$\rho^-\pi^+$	$13.8^{+0.0+3.5+0.1}_{-0.0-3.1-0.1}$	$f_0(980)\pi^0$	$0.004^{+0.000+0.001+0.000}_{-0.000-0.001-0.000}$
$\rho^\pm\pi^\mp$	$17.8^{+0.0+3.6+0.1}_{-0.0-3.1-0.1}$	NR	$1.6^{+0.5+0.0+0.0}_{-0.6-0.0-0.0}$
Total	$20.1^{+0.3+3.7+0.1}_{-0.3-3.3-0.1}$		

We see from Table III that the decay $B^- \to \pi^+\pi^-\pi^-$ is dominated by the ρ^0 pole and the nonresonant contribution. The calculated total branching fraction $(16.1^{+1.9}_{-2.3}) \times 10^{-6}$ agrees well with experiment.

B. $\bar{B}^0 \to \pi^+\pi^-\pi^0$ decay

The factorizable amplitude of $\bar{B}^0 \to \pi^+\pi^-\pi^0$ is given by

$$\begin{aligned}\langle \pi^0\pi^+\pi^-|T_p|\bar{B}^0\rangle &= \langle \pi^+\pi^0|(\bar{u}b)_{V-A}|\bar{B}^0\rangle\langle \pi^-|(\bar{d}u)_{V-A}|0\rangle[a_1\delta_{pu} + a_4^p + a_{10}^p - (a_6^p + a_8^p)r_\chi^\pi] \\ &+ \langle \pi^+\pi^-|(\bar{d}b)_{V-A}|\bar{B}^0\rangle\langle \pi^0|(\bar{u}u)_{V-A}|0\rangle\left[a_2\delta_{pu} - a_4^p + \left(a_6^p - \frac{1}{2}a_8^p\right)r_\chi^\pi + \frac{3}{2}(a_7 + a_9) + \frac{1}{2}a_{10}^p\right] \\ &+ \langle \pi^+|(\bar{u}b)_{V-A}|\bar{B}^0\rangle\langle \pi^-\pi^0|(\bar{d}u)_{V-A}|0\rangle[a_1\delta_{pu} + a_4^p + a_{10}^p] \\ &+ \langle \pi^0|(\bar{d}b)_{V-A}|\bar{B}^0\rangle\langle \pi^+\pi^-|(\bar{u}u)_{V-A}|0\rangle\left[a_2\delta_{pu} - a_4^p + \frac{3}{2}(a_7 + a_9) + \frac{1}{2}a_{10}^p\right] \\ &+ \langle \pi^0|\bar{d}b|\bar{B}^0\rangle\langle \pi^+\pi^-|\bar{d}d|0\rangle(-2a_6^p + a_8^p). \end{aligned} \tag{2.23}$$

It is obvious that while $B^- \to \pi^+\pi^-\pi^-$ is dominated by the ρ^0 resonance, the decay $\bar{B}^0 \to \pi^+\pi^-\pi^0$ receives intermediate $\rho^\pm$ and ρ^0 pole contributions. As a consequence, the $\pi^+\pi^-\pi^0$ mode has a rate larger than $\pi^+\pi^-\pi^-$ even though the former does not have two identical particles in the final state and moreover it involves a π^0 meson. Note that the calculated branching fractions of $\bar{B}^0 \to \rho^\pm\pi^\mp$, $\rho^0\pi^0$ shown in Table IV are consistent with the data (in units of 10^{-6}), 23.0 ± 2.3 and 2.0 ± 0.5, respectively, measured from other processes [4]. The nonresonant rate in $\bar{B}^0 \to \pi^+\pi^-\pi^0$ is fairly small because it is expected to be about four times smaller than that in $B^- \to \pi^+\pi^-\pi^-$. This is confirmed by a realistic calculation.

In Sec. V C we shall explore the possibility of the large rate of $\bar{B}^0 \to K^+K^-\pi^0$ observed recently by Belle [20] can arise from the decay $\bar{B}^0 \to \pi^+\pi^-\pi^0$ followed by final-state rescattering of $\pi^+\pi^- \to K^+K^-$.

III. $B \to KKK$ DECAYS

A. $B^- \to K^+K^-K^-$ decay

The factorizable penguin-dominated $B^- \to K^+K^-K^-$ decay amplitude is given by

$$\begin{aligned}\langle K^+K^-K^-|T_p|B^-\rangle &= \langle K^+K^-|(\bar{u}b)_{V-A}|B^-\rangle\langle K^-|(\bar{s}u)_{V-A}|0\rangle[a_1\delta_{pu} + a_4^p + a_{10}^p - (a_6^p + a_8^p)r_\chi^K] \\ &+ \langle K^-|(\bar{s}b)_{V-A}|B^-\rangle\langle K^+K^-|(\bar{u}u)_{V-A}|0\rangle(a_2\delta_{pu} + a_3 + a_5 + a_7 + a_9) \\ &+ \langle K^-|(\bar{s}b)_{V-A}|B^-\rangle\langle K^+K^-|(\bar{d}d)_{V-A}|0\rangle\left[a_3 + a_5 - \frac{1}{2}(a_7 + a_9)\right] \\ &+ \langle K^-|(\bar{s}b)_{V-A}|B^-\rangle\langle K^+K^-|(\bar{s}s)_{V-A}|0\rangle\left[a_3 + a_4^p + a_5 - \frac{1}{2}(a_7 + a_9 + a_{10}^p)\right] \\ &+ \langle K^-|\bar{s}b|B^-\rangle\langle K^+K^-|\bar{s}s|0\rangle(-2a_6^p + a_8^p) \\ &+ \langle K^+K^-K^-|(\bar{s}u)_{V-A}|0\rangle\langle 0|(\bar{u}b)_{V-A}|B^-\rangle(a_1\delta_{pu} + a_4^p + a_{10}^p) \\ &+ \langle K^+K^-K^-|\bar{s}(1+\gamma_5)u|0\rangle\langle 0|\bar{u}\gamma_5 b|B^-\rangle(2a_6^p + 2a_8^p). \end{aligned} \tag{3.1}$$

For the current-induced process with a kaon emission, the form factors r, $\omega_\pm$, and h for the three-body matrix element $\langle K^+K^-|(\bar{u}b)_{V-A}|B^-\rangle$ [see Eq. (2.6)] evaluated in the framework of HMChPT are the same as that of Eq. (2.7) except that B^* is replaced by B_s^*. As explained in the last section, the available phase space where chiral perturbation theory is applicable is only a small fraction of the whole Dalitz plot. Therefore, we have proposed to parametrize the $b \to u$ transition-induced nonresonant amplitude in an exponent form given in Eq. (2.9). The unknown parameter $\alpha_{\rm NR}$ is determined from the data of the tree-dominated decay $B^- \to \pi^+\pi^-\pi^-$ and is given by Eq. (2.10).

In addition to the $b \to u$ tree transition, we need to consider the nonresonant contributions to the $b \to s$ penguin amplitude,

$$A_1 = \langle K^-(p_1)|(\bar{s}b)_{V-A}|B^-\rangle\langle K^+(p_2)K^-(p_3)|(\bar{q}q)_{V-A}|0\rangle,$$
$$A_2 = \langle K^-(p_1)|\bar{s}b|B^-\rangle\langle K^+(p_2)K^-(p_3)|\bar{s}s|0\rangle. \tag{3.2}$$

The two-kaon creation matrix element can be expressed in terms of time-like kaon current form factors as

$$\langle K^+(p_{K^+})K^-(p_{K^-})|\bar{q}\gamma_\mu q|0\rangle = (p_{K^+} - p_{K^-})_\mu F_q^{K^+K^-},$$
$$\langle K^0(p_{K^0})\bar{K}^0(p_{\bar{K}^0})|\bar{q}\gamma_\mu q|0\rangle = (p_{K^0} - p_{\bar{K}^0})_\mu F_q^{K^0\bar{K}^0}. \tag{3.3}$$

The weak vector form factors $F_q^{K^+K^-}$ and $F_q^{K^0\bar{K}^0}$ can be related to the kaon e.m. form factors $F_{\rm em}^{K^+K^-}$ and $F_{\rm em}^{K^0\bar{K}^0}$ for the charged and neutral kaons, respectively. Phenomenologically, the e.m. form factors receive resonant and nonresonant contributions and can be expressed by

$$F_{\rm em}^{K^+K^-} = F_\rho^{KK} + F_\omega^{KK} + F_\phi^{KK} + F_{\rm NR},$$
$$F_{\rm em}^{K^0\bar{K}^0} = -F_\rho^{KK} + F_\omega^{KK} + F_\phi^{KK} + F'_{\rm NR}. \tag{3.4}$$

It follows from Eqs. (3.3) and (3.4) that

$$F_u^{K^+K^-} = F_d^{K^0\bar{K}^0} = F_\rho^{KK} + 3F_\omega^{KK} + \frac{1}{3}(3F_{\rm NR} - F'_{\rm NR}),$$
$$F_d^{K^+K^-} = F_u^{K^0\bar{K}^0} = -F_\rho^{KK} + 3F_\omega^{KK}, \tag{3.5}$$
$$F_s^{K^+K^-} = F_s^{K^0\bar{K}^0} = -3F_\phi^{KK} - \frac{1}{3}(3F_{\rm NR} + 2F'_{\rm NR}),$$

where isospin symmetry has been used.

The resonant and nonresonant terms in Eq. (3.4) can be parametrized as

$$F_h(s_{23}) = \frac{c_h}{m_h^2 - s_{23} - im_h\Gamma_h},$$
$$F_{\rm NR}^{(\prime)}(s_{23}) = \left(\frac{x_1^{(\prime)}}{s_{23}} + \frac{x_2^{(\prime)}}{s_{23}^2}\right)\left[\ln\left(\frac{s_{23}}{\tilde{\Lambda}^2}\right)\right]^{-1}, \tag{3.6}$$

with $\tilde{\Lambda} \approx 0.3$ GeV. The expression for the nonresonant form factor is motivated by the asymptotic constraint from pQCD, namely, $F(t) \to (1/t)[\ln(t/\tilde{\Lambda}^2)]^{-1}$ in the large-t limit [46]. The unknown parameters c_h, x_i, and x_i' are fitted from the kaon e.m. data, giving the best-fit values (in units of GeV^2 for c_h) [47]

$$c_\rho = 3c_\omega = c_\phi = 0.363, \quad c_{\rho(1450)} = 7.98 \times 10^{-3},$$
$$c_{\rho(1700)} = 1.71 \times 10^{-3}, \quad c_{\omega(1420)} = -7.64 \times 10^{-2},$$
$$c_{\omega(1650)} = -0.116, \quad c_{\phi(1680)} = -2.0 \times 10^{-2}, \tag{3.7}$$

and

$$x_1 = -3.26\ {\rm GeV}^2, \quad x_2 = 5.02\ {\rm GeV}^4,$$
$$x_1' = 0.47\ {\rm GeV}^2, \quad x_2' = 0. \tag{3.8}$$

Note that the form factors $F_{\rho,\omega,\phi}$ in Eqs. (3.4) and (3.5) include the contributions from the vector mesons $\rho(770)$, $\rho(1450)$, $\rho(1700)$, $\omega(782)$, $\omega(1420)$, $\omega(1650)$, $\phi(1020)$, and $\phi(1680)$. As a cross-check, following the derivation of the resonant component of $\langle\pi^+\pi^-|\bar{u}\gamma_\mu u|0\rangle$ in Eq. (2.17) we obtain the resonant contributions to the K^+K^- transition form factors,

$$F_{u,R}^{K^+K^-}(s) = -\frac{1}{\sqrt{2}}\left(\sum_i \frac{m_{\rho_i}f_{\rho_i}g^{\rho_i\to K^+K^-}}{s - m_{\rho_i}^2 + im_{\rho_i}\Gamma_{\rho_i}} + \sum_i \frac{m_{\omega_i}f_{\omega_i}g^{\omega_i\to K^+K^-}}{s - m_{\omega_i}^2 + im_{\omega_i}\Gamma_{\omega_i}}\right),$$
$$F_{d,R}^{K^+K^-}(s) = \frac{1}{\sqrt{2}}\left(\sum_i \frac{m_{\rho_i}f_{\rho_i}g^{\rho_i\to K^+K^-}}{s - m_{\rho_i}^2 + im_{\rho_i}\Gamma_{\rho_i}} - \sum_i \frac{m_{\omega_i}f_{\omega_i}g^{\omega_i\to K^+K^-}}{s - m_{\omega_i}^2 + im_{\omega_i}\Gamma_{\omega_i}}\right), \tag{3.9}$$
$$F_{s,R}^{K^+K^-}(s) = -\sum_i \frac{m_{\phi_i}f_{\phi_i}g^{\phi_i\to K^+K^-}}{s - m_{\phi_i}^2 + im_{\phi_i}\Gamma_{\phi_i}}.$$

Using the quark-model result $g^{\rho\to K^+K^-}:g^{\omega\to K^+K^-}:g^{\phi\to K^+K^-} = 1:1:-1/\sqrt{2}$ to fix the relative sign of the strong couplings and noting that $g^{\phi\to K^+K^-} = -4.54$ (determined from the measured $\phi \to K^+K^-$ rate), we find $c_\phi = -\frac{1}{3}m_\phi f_\phi g^{\phi\to K^+K^-} = 0.340$, in agreement with $c_\phi = 0.363$ obtained from a fit to the kaon e.m. data.

The use of the equation of motion thus leads to

$$A_1 = (s_{12} - s_{13})F_1^{BK}(s_{23})F_q^{K^+K^-}(s_{23}),$$
$$A_2 = \frac{m_B^2 - m_K^2}{m_b - m_s}F_0^{BK}(s_{23})f_s^{K^+K^-}(s_{23}), \tag{3.10}$$

where the matrix element $f_s^{K^+K^-}$ receives both resonant and nonresonant contributions,

$$\langle K^+(p_2)K^-(p_3)|\bar{s}s|0\rangle \equiv f_s^{K^+K^-}(s_{23}) = \sum_i \frac{m_{f_{0i}}\bar{f}_{f_{0i}}^s g^{f_{0i}\to K^+K^-}}{m_{f_{0i}}^2 - s_{23} - im_{f_{0i}}\Gamma_{f_{0i}}} + f_s^{\rm NR},$$
$$f_s^{\rm NR} = \frac{v}{3}(3F_{\rm NR} + 2F'_{\rm NR}) + \sigma_{\rm NR}e^{-\alpha s_{23}}, \tag{3.11}$$

with

$$v = \frac{m_{K^+}^2}{m_u + m_s} = \frac{m_K^2 - m_\pi^2}{m_s - m_d} \qquad (3.12)$$

characterizing the quark-order parameter $\langle \bar{q}q \rangle$ which spontaneously breaks the chiral symmetry. The nonresonant $\sigma_{\rm NR}$ term is introduced for the following reason. Although the nonresonant contributions to f_s^{KK} and F_s^{KK} are related through the equation of motion, the resonant ones are different and not related *a priori*. As stressed in Ref. [48], to apply the equation of motion, the form factors should be away from the resonant region. In the presence of resonances, we thus need to introduce a nonresonant $\sigma_{\rm NR}$ term which can be constrained by the measured $\bar{B}^0 \to K_S K_S K_S$ rate and the K^+K^- mass spectrum measured in $\bar{B}^0 \to K^+K^-K_S$ [31]. The parameter α appearing in the same equation should be close to the value of $\alpha_{\rm NR}$ given in Eq. (2.10). We will use the experimental measurement $\alpha = (0.14 \pm 0.02)\ {\rm GeV}^{-2}$ [49].

It is known that in the narrow-width approximation, the three-body decay rate obeys the factorization relation

$$\Gamma(B \to RP \to P_1P_2P) = \Gamma(B \to RP)\mathcal{B}(R \to P_1P_2), \qquad (3.13)$$

with R being a resonance. This means that the amplitudes $A(B \to RP \to P_1P_2P)$ and $A(B \to RP)$ should have the same expressions apart from some factors. Hence, using the known results for the quasi-two-body decay amplitude $A(B \to RP)$, one can have a cross-check on the three-body decay amplitude of $B \to RP \to P_1P_2P$. For example, the factorizable amplitude of the scalar $f_0(980)$ contribution to $B^- \to K^+K^-K^-$ derived from Eq. (3.1) is given by

$$\begin{aligned}
&\langle K^+K^-K^- | T_p | B^- \rangle_{f_0} \\
&= \frac{g^{f_0(980)\to K^+K^-}}{m_{f_0}^2 - s_{23} - i m_{f_0}\Gamma_{f_0}} \\
&\quad \times \Big\{ -\bar{r}_\chi^{f_0} \bar{f}_{f_0}^s F_0^{BK}(m_{f_0}^2)(m_B^2 - m_K^2)\Big(a_6^p - \frac{1}{2}a_8^p\Big) \\
&\quad + f_K F_0^{Bf_0^u}(m_K^2)(m_B^2 - m_{f_0}^2) \\
&\quad \times [a_1\delta_u^p + a_4^p + a_{10}^p - (a_6^p + a_8^p) r_\chi^K] \Big\}. \qquad (3.14)
\end{aligned}$$

Comparing this equation with Eq. (A6) of Ref. [50], we see that the expression inside $\{\cdots\}$ is identical to that of $B^- \to f_0(980)K^-$, as it should be.[4] In the above equation, $\bar{r}_\chi^{f_0} = 2m_{f_0}/m_b(\mu)$. The superscript u of the form factor $F_0^{Bf_0^u}$ reminds us that it is the $u\bar{u}$ quark content that gets involved in the B-to-f_0 form-factor transition.

We digress for a moment to discuss the wave function of the $f_0(980)$. The quark structure of the light scalar mesons below or near 1 GeV has been quite controversial. In this work we shall consider the conventional $q\bar{q}$ assignment for the $f_0(980)$. In the naive quark model, the flavor wave functions of the $f_0(980)$ and $f_0(500)$ (or σ meson) read

$$f_0(500) = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}), \qquad f_0(980) = s\bar{s}, \qquad (3.15)$$

where ideal mixing for $f_0(980)$ and $f_0(500)$ has been assumed. In this picture, $f_0(980)$ is purely an $s\bar{s}$ state. However, there also exist some experimental evidences indicating that $f_0(980)$ is not purely an $s\bar{s}$ state. First, the observation of $\Gamma(J/\psi \to f_0\omega) \approx \frac{1}{2}\Gamma(J/\psi \to f_0\phi)$ [7] clearly indicates the existence of the nonstrange- and strange-quark content in $f_0(980)$. Second, the fact that $f_0(980)$ and $a_0(980)$ have similar widths and that the $f_0(980)$ width is dominated by $\pi\pi$ also suggests the composition of $u\bar{u}$ and $d\bar{d}$ pairs in $f_0(980)$; that is, $f_0(980) \to \pi\pi$ should not be OZI suppressed relative to $a_0(980) \to \pi\eta$. Therefore, the isoscalars $f_0(500)$ and $f_0(980)$ must have a mixing

$$\begin{aligned}
|f_0(500)\rangle &= -|s\bar{s}\rangle \sin\theta + |n\bar{n}\rangle \cos\theta, \\
|f_0(980)\rangle &= |s\bar{s}\rangle \cos\theta + |n\bar{n}\rangle \sin\theta,
\end{aligned} \qquad (3.16)$$

with $n\bar{n} \equiv (\bar{u}u + \bar{d}d)/\sqrt{2}$. Experimental implications for the $f_0(980) - f_0(500)$ mixing angle have been discussed in detail in Ref. [52]. Assuming two-quark bound states for $f_0(980)$ and $f_0(500)$, the observed large rates of the $B^- \to f_0(980)K$ and $f_0(980)K^*$ modes can be explained in QCDF with the mixing angle θ in the vicinity of 20° [51]. In this work, we shall use $\theta = 20°$.

Finally, the matrix elements involving three-kaon creation are given by [41]

$$\begin{aligned}
&\langle \bar{K}^0(p_1)K^+(p_2)K^-(p_3)|(\bar{s}d)_{V-A}|0\rangle\langle 0|(\bar{d}b)_{V-A}|\bar{B}^0\rangle \approx 0, \\
&\langle \bar{K}^0(p_1)K^+(p_2)K^-(p_3)|\bar{s}\gamma_5 d|0\rangle\langle 0|\bar{d}\gamma_5 b|\bar{B}^0\rangle \\
&\quad = v\frac{f_B m_B^2}{f_\pi m_b}\Big(1 - \frac{s_{13} - m_1^2 - m_3^2}{m_B^2 - m_K^2}\Big)F^{KKK}(m_B^2). \qquad (3.17)
\end{aligned}$$

Both relations in Eq. (3.17) are originally derived in the chiral limit [41] and hence the quark masses appearing in Eq. (3.12) are referred to the scale ~1 GeV. The first relation reflects helicity suppression which is expected to be even more effective for energetic kaons. For the second relation, we introduce the form factor F^{KKK} to extrapolate the chiral result to the physical region. Following Ref. [41] we shall take $F^{KKK}(q^2) = 1/[1 - (q^2/\Lambda_\chi^2)]$, with $\Lambda_\chi = 0.83$ GeV being a chiral-symmetry-breaking scale.

To proceed with the numerical calculations, we shall assume that the main scalar-meson contributions are those that have dominant $s\bar{s}$ content and large coupling to $K\bar{K}$.

[4] There are some sign typos in Eq. (A6) of Ref. [50] including the one in the amplitude of $B^- \to f_0K^-$. When comparing Eq. (3.14) with Eq. (A1) of Ref. [51], we see that some terms are missing in Eq. (3.14). This is because one has to consider the convolution with the light-cone distribution amplitude of the $f_0(980)$ in the approach of QCDF. As a consequence, the amplitude for f_0 emission does not vanish in QCDF. We will not consider these subtitles in the simple factorization approach adapted here.

We consider the scalar mesons $f_0(980)$, $f_0(1500)$, and $f_0(1710)$, which are supposed to have the largest couplings with the $K\bar{K}$ pair. More specifically, we shall use $g^{f_0(980)\to K^+K^-} = 3.7$ GeV, $g^{f_0(1500)\to K^+K^-} = 0.69$ GeV, $g^{f_0(1710)\to K^+K^-} = 1.6$ GeV, $\Gamma_{f_0(980)} = 80$ MeV, $\Gamma_{f_0(1500)} = 0.109$ GeV, $\Gamma_{f_0(1710)} = 0.135$ GeV, $\bar{f}_{f_0(980)}(\mu = m_b/2) \simeq 0.46$ GeV [53], $\bar{f}_{f_0(1500)} \simeq 0.30$ GeV, and $\bar{f}_{f_0(1710)} \simeq 0.17$ GeV. As for the parameter σ_{NR} in Eq. (3.11), its magnitude can be determined from the measured $K_SK_SK_S$ rate, namely, $\mathcal{B}(\bar{B}^0 \to K_SK_SK_S) = (6.1 \pm 0.5) \times 10^{-6}$ [4]. As for the strong phase ϕ_r we follow Ref. [31] to take $\phi_\sigma \approx \pi/4$, which yields a K^+K^- mass spectrum in $\bar{B}^0 \to K^+K^-K_S$ consistent with the data,

$$\sigma_{NR} = e^{i\pi/4}(3.39^{+0.18}_{-0.21}) \text{ GeV}. \tag{3.18}$$

The calculated branching fractions of resonant and nonresonant contributions to $B^- \to K^+K^-K^-$, $\bar{B}^0 \to K^+K^-K^0$, $B^- \to K^-K_SK_S$, and $\bar{B}^0 \to K_SK_SK_S$ are depicted in Table V. The factorizable amplitudes of the last three modes can be found in Appendix A of Ref. [31]. Note that both *BABAR* and Belle used to see a broad scalar resonance $f_X(1500)$ in $B \to K^+K^+K^-$, $K^+K^-K_S$, and $K^+K^-\pi^+$ decays at energies around 1.5 GeV. However, the nature of $f_X(1500)$ is not clear as it cannot be identified with the well-known scaler meson $f_0(1500)$. Nevertheless, the recent angular-momentum analysis of

TABLE V. Branching fractions (in units of 10^{-6}) of resonant and nonresonant contributions to $B^- \to K^+K^-K^-$, $\bar{B}^0 \to K^+K^-K^0$, $B^- \to K^-K_SK_S$, and $\bar{B}^0 \to K_SK_SK_S$.

Decay mode	*BABAR* [12]	Belle [13]	Theory
$B^- \to K^+K^-K^-$			
ϕK^-	$4.48 \pm 0.22^{+0.33}_{-0.24}$	$4.72 \pm 0.45 \pm 0.35^{+0.39}_{-0.22}$	$2.9^{+0.0+0.5+0.0}_{-0.0-0.5-0.0}$
$f_0(980)K^-$	$9.4 \pm 1.6 \pm 2.8$	<2.9	$11.0^{+0.0+2.6+0.0}_{-0.0-2.1-0.0}$
$f_0(1500)K^-$	$0.74 \pm 0.18 \pm 0.52$		$0.62^{+0.0+0.11+0.0}_{-0.0-0.10-0.0}$
$f_0(1710)K^-$	$1.12 \pm 0.25 \pm 0.50$		$1.1^{+0+0.2+0}_{-0-0.2-0}$
$f_2'(1525)K^-$	$0.69 \pm 0.16 \pm 0.13$		
NR	$22.8 \pm 2.7 \pm 7.6$	$24.0 \pm 1.5 \pm 1.8^{+1.9}_{-5.7}$	$21.8^{+0.8+7.6+0.1}_{-1.1-5.9-0.1}$
Total	$33.4 \pm 0.5 \pm 0.9$	$30.6 \pm 1.2 \pm 2.3$	$26.9^{+0.4+7.5+0.1}_{-0.5-6.1-0.1}$
$\bar{B}^0 \to K^+K^-\bar{K}^0$			
Decay mode	*BABAR* [12]	Belle [11]	Theory
$\phi\bar{K}^0$	$3.48 \pm 0.28^{+0.21}_{-0.14}$		$2.6^{+0.0+0.4+0.0}_{-0.0-0.4-0.0}$
$f_0(980)\bar{K}^0$	$7.0^{+2.6}_{-1.8} \pm 2.4$		$9.1^{+0.0+1.7+0.0}_{-0.0-1.4-0.0}$
$f_0(1500)\bar{K}^0$	$0.57^{+0.25}_{-0.19} \pm 0.12$		$0.55^{+0.0+0.10+0.0}_{-0.0-0.09-0.0}$
$f_0(1710)\bar{K}^0$	$4.4 \pm 0.7 \pm 0.5$		$1.0^{+0.0+0.2+0.0}_{-0.0-0.2-0.0}$
$f_2'(1525)\bar{K}^0$	$0.13^{+0.12}_{-0.08} \pm 0.16$		
NR	$33 \pm 5 \pm 9$		$12.0^{+0.4+2.8+0.1}_{-0.5-2.4-0.1}$
Total[a]	$25.4 \pm 0.9 \pm 0.8$	$28.3 \pm 3.3 \pm 4.0$	$18.7^{+0.2+3.5+0.0}_{-0.3-3.1-0.0}$
$B^- \to K^-K_SK_S$			
Decay mode	*BABAR* [12]	Belle [11]	Theory
$f_0(980)K^-$	$14.7 \pm 2.8 \pm 1.8$		$8.7^{+0.0+2.1+0.0}_{-0.0-1.6-0.0}$
$f_0(1500)K^-$	$0.42 \pm 0.22 \pm 0.58$		$0.59^{+0.00+0.10+0.00}_{-0.00-0.09-0.00}$
$f_0(1710)K^-$	$0.48^{+0.40}_{-0.24} \pm 0.11$		$1.08^{+0.00+0.18+0.00}_{-0.00-0.17-0.00}$
$f_2'(1525)K^-$	$0.61 \pm 0.21^{+0.12}_{-0.09}$		
NR	$19.8 \pm 3.7 \pm 2.5$		$11.3^{+0.2+3.7+0.0}_{-0.3-3.0-0.0}$
Total	$10.1 \pm 0.5 \pm 0.3$	$13.4 \pm 1.9 \pm 1.5$	$15.1^{+0.0+3.7+0.0}_{-0.0-3.2-0.0}$
$\bar{B}^0 \to K_SK_SK_S$			
Decay mode	*BABAR* [21]	Belle [11]	Theory
$f_0(980)K_S$	$2.7^{+1.3}_{-1.2} \pm 0.4 \pm 1.2$		$2.4^{+0.0+0.6+0.0}_{-0.0-0.5-0.0}$
$f_0(1500)K_S$			$0.15^{+0.00+0.03+0.00}_{-0.00-0.02-0.00}$
$f_0(1710)K_S$	$0.50^{+0.46}_{-0.24} \pm 0.04 \pm 0.10$		$0.28^{+0.00+0.05+0.00}_{-0.00-0.04-0.00}$
$f_2(2010)K_S$	$0.54^{+0.21}_{-0.20} \pm 0.03 \pm 0.52$		
NR	$13.3^{+2.2}_{-2.3} \pm 0.6 \pm 2.1$		$6.58^{+0.09+2.04+0.01}_{-0.12-1.70-0.01}$
Total	$6.19 \pm 0.48 \pm 0.15 \pm 0.12$	$4.2^{+1.6}_{-1.3} \pm 0.8$	$6.19^{+0.01+1.62+0.01}_{-0.02-1.42-0.01}$

[a]The LHCb measurement is $\mathcal{B}(\bar{B}^0 \to K^+K^-\bar{K}^0) = (19.1 \pm 1.5 \pm 1.1 \pm 0.8) \times 10^{-6}$ [19].

the above-mentioned three channels by *BABAR* [12] shows that the $f_X(1500)$ state is not a single scalar resonance, but instead can be described by the sum of the well-established resonances $f_0(1500)$, $f_0(1710)$, and $f_2'(1525)$.

From Table V it is obvious that the predicted rates for resonant and nonresonant components are consistent with experiment within errors. It is known that the calculated $\mathcal{B}(B\to\phi K)$ is smaller than experiment and this rate-deficit problem calls for the $1/m_b$ power corrections from penguin annihilation. A unique feature of hadronic $B\to KKK$ decays is that they are predominated by the nonresonant contributions with a nonresonant fraction of order 80%. The nonresonant background due to the current-induced process through the $B\to KK$ transition accounts for only 5% of the observed nonresonant contributions as it is suppressed by the parameter $\alpha_{\rm NR}$. This implies that the two-body matrix element of scalar densities, e.g., $\langle K\bar K|\bar ss|0\rangle$ induced from the penguin diagram should have a large nonresonant component. This is plausible because the decay $B\to KKK$ is dominated by the $b\to s$ penguin transition. Consequently, it is natural to expect that the nonresonant contribution to this decay is also penguin dominated.

IV. $B\to K\pi\pi$ DECAYS

The factorizable penguin-dominated $B^-\to K^-\pi^+\pi^-$ decay amplitude has the expression

$$\begin{aligned}\langle K^-\pi^+\pi^-|T_p|B^-\rangle=&\langle\pi^+\pi^-|(\bar ub)_{V-A}|B^-\rangle\langle K^-|(\bar su)_{V-A}|0\rangle[a_1\delta_{pu}+a_4^p+a_{10}^p-(a_6^p+a_8^p)r_\chi^K]\\&+\langle K^-|(\bar sb)_{V-A}|B^-\rangle\langle\pi^+\pi^-|(\bar uu)_{V-A}|0\rangle[a_2\delta_{pu}+a_3+a_5+a_7+a_9]\\&+\langle K^-|(\bar sb)_{V-A}|B^-\rangle\langle\pi^+\pi^-|(\bar dd)_{V-A}|0\rangle\left[a_3+a_5-\frac{1}{2}(a_7+a_9)\right]\\&+\langle K^-|(\bar sb)_{V-A}|B^-\rangle\langle\pi^+\pi^-|(\bar ss)_{V-A}|0\rangle\left[a_3+a_4^p+a_5-\frac{1}{2}(a_7+a_9+a_{10}^9)\right]\\&+\langle K^-|\bar sb|B^-\rangle\langle\pi^+\pi^-|\bar ss|0\rangle(-2a_6^p+a_8^p)+\langle\pi^-|(\bar db)_{V-A}|B^-\rangle\langle K^-\pi^+|(\bar sd)_{V-A}|0\rangle\left(a_4^p-\frac{1}{2}a_{10}^p\right)\\&+\langle\pi^-|\bar db|B^-\rangle\langle K^-\pi^+|\bar sd|0\rangle(-2a_6^p+a_8^p)+\langle K^-\pi^+\pi^-|(\bar su)_{V-A}|0\rangle\langle 0|(\bar ub)_{V-A}|B^-\rangle(a_1\delta_{pu}+a_4^p+a_{10}^p)\\&+\langle K^-\pi^+\pi^-|\bar s(1+\gamma_5)u|0\rangle\langle 0|\bar u\gamma_5 b|B^-\rangle(2a_6^p+2a_8^p).\end{aligned}\tag{4.1}$$

The factorizable amplitudes for other $\bar B\to\bar K\pi\pi$ modes, such as $B^-\to\bar K^0\pi^-\pi^0$, $\bar B^0\to K^-\pi^+\pi^0$, $\bar K^0\pi^+\pi^-$, and $\bar K^0\pi^0\pi^0$, can be found in Appendix A of Ref. [31]. The expression for $A(B^-\to K^-\pi^0\pi^0)$ is given in Eq. (B1). All six channels have the three-body matrix element $\langle\pi\pi|(\bar qb)_{V-A}|B\rangle$ which has the similar expression as Eqs. (2.7) and (2.8). The three-body matrix elements also receive resonant contributions; for example,

$$\begin{aligned}&\langle K^-(p_1)\pi^+(p_2)|(\bar sb)_{V-A}|\bar B^0\rangle^R\\&\quad=\sum_i\frac{g^{K_i^*\to K^-\pi^+}}{s_{12}-m_{K_i^*}^2+im_{K_i^*}\Gamma_{K_i^*}}\\&\qquad\times\sum_{\rm pol}\varepsilon^*\cdot(p_1-p_2)\langle\bar K_i^{*0}|(\bar sb)_{V-A}|\bar B^0\rangle,\\&\qquad-\frac{g^{K_0^*\to K^-\pi^+}}{s_{12}-m_{K_0^*}^2+im_{K_0^*}\Gamma_{K_0^*}}\langle\bar K_0^{*0}|(\bar sb)_{V-A}|\bar B^0\rangle,\end{aligned}\tag{4.2}$$

with $K_i^*=K^*(892), K^*(1410), K^*(1680),\ldots,$ and $K_0^*=K_0^*(1430)$.

For the two-body matrix elements $\langle\pi^+K^-|(\bar sd)_{V-A}|0\rangle$, $\langle\pi^+\pi^-|(\bar uu)_{V-A}|0\rangle$, and $\langle\pi^+\pi^-|\bar ss|0\rangle$, we note that

$$\begin{aligned}&\langle K^-(p_1)\pi^+(p_2)|(\bar sd)_{V-A}|0\rangle\\&\quad=\langle\pi^+(p_2)|(\bar sd)_{V-A}|K^+(-p_1)\rangle\\&\quad=(p_1-p_2)_\mu F_1^{K\pi}(s_{12})\\&\qquad+\frac{m_K^2-m_\pi^2}{s_{12}}(p_1+p_2)_\mu[-F_1^{K\pi}(s_{12})+F_0^{K\pi}(s_{12})],\end{aligned}\tag{4.3}$$

where we have taken into account the sign flip arising from interchanging the operators $s\leftrightarrow d$. The resonant contributions are

$$\begin{aligned}&\langle K^-(p_1)\pi^+(p_2)|(\bar sd)_{V-A}|0\rangle^R\\&\quad=\sum_i\frac{g^{K_i^*\to K^-\pi^+}}{s_{12}-m_{K_i^*}^2+im_{K_i^*}\Gamma_{K_i^*}}\\&\qquad\times\sum_{\rm pol}\varepsilon^*\cdot(p_1-p_2)\langle K_i^*|(\bar sd)_{V-A}|0\rangle\\&\qquad-\sum_i\frac{g^{K_{0i}^*\to K^-\pi^+}}{s_{12}-m_{K_{0i}^*}^2+im_{K_{0i}^*}\Gamma_{K_{0i}^*}}\langle K_{0i}^*|(\bar sd)_{V-A}|0\rangle.\end{aligned}\tag{4.4}$$

Hence, the form factors $F_1^{K\pi}$ and $(-F_1^{K\pi}+F_0^{K\pi})$ receive the following resonant contributions:

$$(F_1^{K\pi}(s))^R = \sum_i \frac{m_{K_i^*} f_{K_i^*} g^{K_i^* \to K\pi}}{m_{K_i^*}^2 - s - i m_{K_i^*}\Gamma_{K_i^*}},$$

$$(-F_1^{K\pi}(s) + F_0^{K\pi}(s))^R = \sum_i \frac{m_{K_{0i}^*} f_{K_{0i}^*} g^{K_{0i}^* \to K\pi}}{m_{K_{0i}^*}^2 - s - i m_{K_{0i}^*}\Gamma_{K_{0i}^*}} \frac{s}{m_K^2 - m_\pi^2} - \sum_i \frac{m_{K_i^*} f_{K_i^*} g^{K_i^* \to K\pi}}{m_{K_i^*}^2 - s - i m_{K_i^*}\Gamma_{K_i^*}} \frac{s}{m_{K_i^*}^2}. \quad (4.5)$$

Note that for the scalar meson the decay constant $\bar{f}_S$ is defined in Eq. (2.13), while f_S is defined by $\langle S(p)|\bar{q}_2\gamma_\mu q_1|0\rangle = f_S p_\mu$. The two decay constants are related by the equations of motion [50]

$$\mu_S f_S = \bar{f}_S, \quad \text{with} \quad \mu_S = \frac{m_S}{m_2(\mu) - m_1(\mu)}, \quad (4.6)$$

where m_2 and m_1 are the running current-quark masses. The nonresonant contribution $\langle \pi^+(p_2)\pi^-(p_3)|\bar{s}s|0\rangle^{\rm NR}$ vanishes under the OZI rule.

Now, the amplitude $\langle K^-\pi^+|(\bar{s}d)_{V-A}|0\rangle \times \langle \pi^-|(\bar{d}b)_{V-A}|B^-\rangle$ in Eq. (4.1) has the expression

$$\begin{aligned}&\langle K^-(p_1)\pi^+(p_2)|(\bar{s}d)_{V-A}|0\rangle\langle \pi^-(p_3)|(\bar{d}b)_{V-A}|B^-\rangle \\ &= F_1^{B\pi}(s_{12})F_1^{K\pi}(s_{12}) \\ &\quad\times\left[s_{13} - s_{23} - \frac{(m_B^2 - m_\pi^2)(m_K^2 - m_\pi^2)}{s_{12}}\right] \\ &\quad + F_0^{B\pi}(s_{12})F_0^{K\pi}(s_{12})\frac{(m_B^2 - m_\pi^2)(m_K^2 - m_\pi^2)}{s_{12}}, \quad (4.7)\end{aligned}$$

with

$$\begin{aligned}&\langle K^-(p_1)\pi^+(p_2)|\bar{s}d|0\rangle \\ &= \sum_i \frac{m_{K_{0i}^*}\bar{f}_{K_{0i}^*} g^{K_{0i}^* \to K^-\pi^+}}{m_{K_{0i}^*}^2 - s_{12} - i m_{K_{0i}^*}\Gamma_{K_{0i}^*}} + \langle K^-(p_1)\pi^+(p_2)|\bar{s}d|0\rangle^{\rm NR}. \end{aligned} \quad (4.8)$$

We consider the factorizable amplitude of the weak decay $B^- \to K_0^{*0}(1430)\pi^-$ followed by the strong decay $K_0^{*0}(1430) \to K^-\pi^+$ as a cross-check on the three-body decay amplitude of $B \to RP \to P_1P_2P$. From Eq. (4.1) we obtain

$$\begin{aligned}&\langle K^-(p_1)\pi^+(p_2)\pi^-(p_3)|T_p|B^-\rangle_{K_0^{*0}(1430)} \\ &= \frac{g^{K_0^{*0}(1430)\to K^-\pi^+}}{m_{K_0^*}^2 - s_{12} - i m_{K_0^*}\Gamma_{K_0^*}} \\ &\quad\times\left\{\left(a_4^p - r_\chi^{K_0^*} a_6^p - \frac{1}{2}\left(a_{10}^p - r_\chi^{K_0^*} a_8^p\right)\right)\right. \\ &\quad\left.\times f_{K_0^*} F_0^{B\pi}(m_{K_0^*}^2)(m_B^2 - m_\pi^2)\right\}, \quad (4.9)\end{aligned}$$

where

$$r_\chi^{K_0^*}(\mu) = \frac{2m_{K_0^*}^2}{m_b(\mu)(m_s(\mu) - m_q(\mu))}. \quad (4.10)$$

The expression inside $\{\cdots\}$ agrees with the amplitude of $\bar{B}^0 \to \bar{K}_0^{*0}(1430)\pi^0$ given in Eq. (A6) of Ref. [50].

The momentum dependence of the weak form factor $F^{K\pi}(q^2)$ is parametrized as

$$F^{K\pi}(q^2) = \frac{F^{K\pi}(0)}{1 - q^2/\Lambda_\chi^2 + i\Gamma_R/\Lambda_\chi}, \quad (4.11)$$

with Γ_R being the width of the relevant resonance, which is taken to be 200 MeV [41].

It should be stressed that the nonresonant branching fraction $(2.4 \pm 0.5^{+1.3}_{-1.5}) \times 10^{-6}$ in $B^- \to K^-\pi^+\pi^-$ reported by *BABAR* [5] is much smaller than the one $(16.9 \pm 1.3^{+1.7}_{-1.6}) \times 10^{-6}$ measured by Belle (see Table VI). Since the *BABAR* and Belle definitions of the $K_0^*(1430)$ and nonresonant contribution differ, it does not make sense to compare the branching fractions and phases directly. While Belle (see, e.g., Ref. [6]) employed an exponential parametrization to describe the nonresonant contribution, *BABAR* [5] used the LASS parametrization to describe the $K\pi$ *S*-wave and the nonresonant component by a single amplitude suggested by the LASS collaboration. While this approach is experimentally motivated, the use of the LASS parametrization is limited to the elastic region of $M(K\pi) \lesssim 2.0$ GeV, and an additional amplitude is still required for a satisfactory description of the data. In short, the *BABAR* definition for the $K_0^*(1430)$ includes an effective range term to account for the low-energy $K\pi$ *S*-wave, while for the Belle parametrization, this component is absorbed into the nonresonant piece. For the example at hand, the aforementioned *BABAR* result $\mathcal{B}(B^- \to K^-\pi^+\pi^-)_{\rm NR}$ is solely due to the phase-space nonresonant piece. It is clear that part of the LASS shape is really nonresonant, which has a substantial mixing with $K_0^*(1430)$. In principle, this should be added to the phase-space nonresonant piece to get the total nonresonant contribution. Indeed, by combining coherently the nonresonant part of the LASS parametrization and the phase-space nonresonant, *BABAR* found the total nonresonant branching fraction to be $(9.3 \pm 1.0 \pm 1.2^{+6.8}_{-1.3}) \times 10^{-6}$. We see from Table VI that the *BABAR* result is now consistent with Belle within errors, though the agreement is not perfect. Likewise, the branching fraction $(2.8 \pm 0.5 \pm 0.4) \times 10^{-6}$ of the phase-space nonresonant contribution to $\bar{B}^0 \to K^-\pi^+\pi^0$ measured by *BABAR* [16] is now modified to $(7.6 \pm 0.5 \pm 1.0) \times 10^{-6}$ when the nonresonant part of the LASS parametrization is added coherently to the phase-space nonresonant piece (see Table VI).

For the resonant contributions from $K_0^*(1430)$, the branching fractions of the quasi-two-body decays $B \to K_0^*(1430)\pi$ can be inferred from Table VI and the results are shown in Table IX below. From the table we see that the measured branching fractions of the $K_0^{*-}(1430)\pi^+$ and

TABLE VI. Branching fractions (in units of 10^{-6}) of resonant and nonresonant contributions to $B^- \to K^-\pi^+\pi^-$, $B^- \to K^-\pi^0\pi^0$, $\bar{B}^0 \to \bar{K}^0\pi^+\pi^-$, and $\bar{B}^0 \to K^-\pi^+\pi^0$. Note that the *BABAR* result for $\bar{K}_0^{*0}(1430)\pi^-$ in Ref. [5], $K_0^{*-}(1430)\pi^+$ in Ref. [14], all the *BABAR* results in Ref. [16], and the Belle results in Ref. [17] are their absolute ones. We have converted them into the product branching fractions, namely, $\mathcal{B}(B \to Rh) \times \mathcal{B}(R \to hh)$.

Decay mode	*BABAR* [5]	Belle [6]	Theory
$B^- \to K^-\pi^+\pi^-$			
$\bar{K}^{*0}\pi^-$	$7.2 \pm 0.4 \pm 0.7^{+0.3}_{-0.5}$	$6.45 \pm 0.43 \pm 0.48^{+0.25}_{-0.35}$	$2.4^{+0.0+0.6+0.0}_{-0.0-0.5-0.0}$
$\bar{K}_0^{*0}(1430)\pi^-$	$19.8 \pm 0.7 \pm 1.7^{+5.6}_{-0.9} \pm 3.2$	$32.0 \pm 1.0 \pm 2.4^{+1.1}_{-1.9}$	$11.3^{+0.0+3.3+0.1}_{-0.0-2.8-0.1}$
$\rho^0 K^-$	$3.56 \pm 0.45 \pm 0.43^{+0.38}_{-0.15}$	$3.89 \pm 0.47 \pm 0.29^{+0.32}_{-0.29}$	$0.65^{+0.00+0.69+0.01}_{-0.00-0.19-0.01}$
$f_0(980)K^-$	$10.3 \pm 0.5 \pm 1.3^{+1.5}_{-0.4}$	$8.78 \pm 0.82 \pm 0.65^{+0.55}_{-1.64}$	$6.6^{+0.0+1.6+0.0}_{-0.0-1.3-0.0}$
NR	$9.3 \pm 1.0 \pm 1.2^{+6.7}_{-0.4} \pm 1.2$[a]	$16.9 \pm 1.3 \pm 1.3^{+1.1}_{-0.9}$	$15.5^{+0.0+8.0+0.0}_{-0.0-5.1-0.0}$
Total	$54.4 \pm 1.1 \pm 4.6$	$48.8 \pm 1.1 \pm 3.6$	$33.1^{+0.2+14.3+0.0}_{-0.2-9.2-0.0}$
$B^- \to K^-\pi^0\pi^0$			
Decay mode	*BABAR* [9]	Belle	Theory
$K^{*-}\pi^0$	$2.7 \pm 0.5 \pm 0.4$		$0.91^{+0.00+0.18+0.03}_{-0.00-0.17-0.03}$
$K_0^{*-}(1430)\pi^0$			$2.4^{+0.0+0.8+0.0}_{-0.0-0.7-0.0}$
$f_0(980)K^-$	$2.8 \pm 0.6 \pm 0.5$		$3.3^{+0.0+0.8+0.0}_{-0.0-0.6-0.0}$
NR			$5.9^{+0.0+2.5+0.0}_{-0.0-1.8-0.0}$
Total	$16.2 \pm 1.2 \pm 1.5$		$11.7^{+0.1+4.2+0.0}_{-0.0-3.1-0.0}$
$\bar{B}^0 \to \bar{K}^0\pi^+\pi^-$			
Decay mode	*BABAR* [14]	Belle [15]	Theory
$K^{*-}\pi^+$	$5.52^{+0.61}_{-0.54} \pm 0.35 \pm 0.41$	$5.6 \pm 0.7 \pm 0.5^{+0.4}_{-0.3}$	$2.0^{+0.0+0.5+0.1}_{-0.0-0.5-0.1}$
$K_0^{*-}(1430)\pi^+$	$18.5^{+1.4}_{-1.1} \pm 1.0 \pm 0.4 \pm 2.0$	$30.8 \pm 2.4 \pm 2.4^{+0.8}_{-3.0}$	$10.3^{+0.0+2.9+0.0}_{-0.0-2.5-0.0}$
$\rho^0\bar{K}^0$	$4.37^{+0.70}_{-0.61} \pm 0.29 \pm 0.12$	$6.1 \pm 1.0 \pm 0.5^{+1.0}_{-1.1}$	$0.12^{+0.00+0.49+0.00}_{-0.00-0.07-0.00}$
$f_0(980)\bar{K}^0$	$6.92 \pm 0.77 \pm 0.46 \pm 0.32$	$7.6 \pm 1.7 \pm 0.7^{+0.5}_{-0.7}$	$5.9^{+0.0+1.5+0.0}_{-0.0-1.5-0.0}$
$f_2(1270)\bar{K}^0$	$1.15^{+0.42}_{-0.35} \pm 0.11 \pm 0.35$		
NR	$11.1^{+2.5}_{-1.0} \pm 0.9$	$19.9 \pm 2.5 \pm 1.6^{+0.7}_{-1.2}$	$15.0^{+0.2+7.8+0.0}_{-0.2-5.1-0.0}$
Total	$50.2 \pm 1.5 \pm 1.8$	$47.5 \pm 2.4 \pm 3.7$	$30.6^{+0.1+13.7+0.0}_{-0.1-8.9-0.0}$
$\bar{B}^0 \to K^-\pi^+\pi^0$			
Decay mode	*BABAR* [16]	Belle [17]	Theory
$K^{*-}\pi^+$	$2.7 \pm 0.4 \pm 0.3$	$4.9^{+1.5+0.5+0.8}_{-1.5-0.3-0.3}$	$1.0^{+0.0+0.3+0.0}_{-0.0-0.2-0.0}$
$\bar{K}^{*0}\pi^0$	$2.2 \pm 0.3 \pm 0.3$	<2.3	$0.7^{+0.0+0.2+0.0}_{-0.0-0.2-0.0}$
$K_0^{*-}(1430)\pi^+$	$8.6 \pm 0.8 \pm 1.0$	[b]	$5.0^{+0.0+1.5+0.1}_{-0.0-1.2-0.1}$
$\bar{K}_0^{*0}(1430)\pi^0$	$4.3 \pm 0.3 \pm 0.7$	[b]	$4.1^{+0.0+1.4+0.0}_{-0.0-1.2-0.0}$
ρ^+K^-	$6.6 \pm 0.5 \pm 0.8$	$15.1^{+3.4+1.4+2.0}_{-3.3-1.5-2.1}$	$2.4^{+0.0+2.6+0.1}_{-0.0-1.1-0.1}$
NR	$7.6 \pm 0.5 \pm 1.0$[c]	$5.7^{+2.7+0.5}_{-2.5-0.4} < 9.4$	$9.0^{+0.3+5.8+0.0}_{-0.3-3.3-0.0}$
Total	$38.5 \pm 1.0 \pm 3.9$	$36.6^{+4.2}_{-4.1} \pm 3.0$	$18.6^{+0.4+11.9+0.1}_{-0.4-6.7-0.1}$

[a]The branching fraction $(2.4 \pm 0.5 + 1.3 - 1.5) \times 10^{-6}$ given in Table II of Ref. [5] is for the phase-space nonresonant contribution to $B^- \to K^-\pi^+\pi^-$.
[b]What Belle has measured is for $K_x^*\pi$ where K_x^* is not specified, though it could be $K_0^*(1430)$ [17].
[c]The branching fraction $(2.8 \pm 0.5 \pm 0.4) \times 10^{-6}$ given in Table VI of Ref. [16] is for the phase-space nonresonant contribution to $\bar{B}^0 \to K^-\pi^+\pi^0$.

$\bar{K}_0^{*0}(1430)\pi^-$ channels are of order 30×10^{-6} by *BABAR* and 50×10^{-6} by Belle. Note that the *BABAR* results are obtained from $(K\pi)_0^{*0}\pi^-$ and $(K\pi)_0^{*-}\pi^+$ by subtracting the elastic range term from the $K\pi$ S-wave [5,14]. For example, the *BABAR* result shown in Table VI for the branching fraction of $\bar{K}_0^{*0}(1430)\pi^-$ comes only from the Breit-Wigner component of the LASS parametrization, while the nonresonant contribution includes both the nonresonant part of the LASS shape and the phase-space nonresonant piece. Nevertheless, the discrepancy between *BABAR* and Belle for the $K_0^*\pi$ modes still remains and it is crucial to resolve this important issue.

Experimentally, the nonresonant rates in $B^- \to K^-\pi^+\pi^-$ and $\bar{B}^0 \to \bar{K}^0\pi^+\pi^-$ are of the same order of magnitude as that in $B \to KKK$ decays (see Tables V and VI). Indeed, this is what we would expect. The nonresonant components of $B \to KKK$ are governed by the $K\bar{K}$ matrix element $\langle K\bar{K}|\bar{s}s|0\rangle$. By the same token, the nonresonant contribution to the penguin-dominated $B \to K\pi\pi$ decays should also be dominated by the $K\pi$ matrix element,

TABLE VII. Branching fractions (in units of 10^{-6}) of resonant and nonresonant contributions to $B^-\to\bar K^0\pi^-\pi^0$ and $\bar B^0\to\bar K^0\pi^0\pi^0$.

Decay mode	Theory	Decay mode	Theory
$B^-\to\bar K^0\pi^-\pi^0$			
$K^{*-}\pi^0$	$1.7^{+0.0+0.3+0.2}_{-0.0-0.3-0.2}$	$\bar K^{*0}\pi^-$	$1.2^{+0.0+0.3+0.0}_{-0.0-0.3-0.0}$
$K_0^{*-}(1430)\pi^0$	$5.4^{+0.0+1.6+0.1}_{-0.0-1.4-0.1}$	$\bar K_0^{*0}(1430)\pi^-$	$5.3^{+0.0+1.6+0.0}_{-0.0-1.4-0.0}$
$\rho^-\bar K^0$	$1.5^{+0.0+2.5+0.0}_{-0.0-0.9-0.0}$	NR	$9.4^{+0.3+6.2+0.0}_{-0.3-3.6-0.0}$
Total	$16.6^{+0.2+10.3+0.0}_{-0.2-5.8-0.0}$		
$\bar B^0\to\bar K^0\pi^0\pi^0$			
$f_0(980)\bar K^0$	$3.0^{+0.0+0.7+0.0}_{-0.0-0.6-0.0}$	$\bar K^{*0}\pi^0$	$0.88^{+0.00+0.18+0.00}_{-0.00-0.16-0.00}$
$\bar K_0^{*0}(1430)\pi^0$	$2.3^{+0.0+0.8+0.0}_{-0.0-0.6-0.0}$	NR	$5.5^{+0.0+2.3+0.0}_{-0.0-1.7-0.0}$
Total	$10.8^{+0.1+3.9+0.0}_{-0.0-2.9-0.0}$		

namely, $\langle K\pi|\bar s q|0\rangle$. Its precise expression will be given in Eq. (7.11) below. The reason why the nonresonant fraction is as large as 90% in KKK decays but becomes only $(17\sim 40)\%$ in $K\pi\pi$ channels (see Table II) can be explained as follows. The nonresonant rates in the $K^-\pi^+\pi^-$ and $\bar K^0\pi^+\pi^-$ modes should be similar to that in $K^+K^-\bar K^0$ or $K^+K^-K^-$. Since the KKK channel receives resonant contributions only from ϕ and f_0 mesons, while K^*, K_0^*, ρ, f_0 resonances contribute to $K\pi\pi$ modes, this explains why the nonresonant fraction is of order 90% in the former and becomes of order 40% or smaller in the latter.

The results of our calculation are shown in Tables VI and VII. It is obvious that except for $f_0(980)K$ the predicted rates for $K^*\pi$, $K_0^*(1430)\pi$, and ρK are smaller than the data. Indeed, the predictions based on QCD factorization for these decays are also generally smaller than experiment by a factor of $2\sim 5$. This will be discussed in more detail in Sec. VI. As a result, this also explains why our predictions of the total branching fractions of $B\to K\pi\pi$ are smaller than experiment.

V. $B\to KK\pi$ DECAYS

In this section we turn to the three-body decay modes $KK\pi$ dominated by $b\to u$ tree and $b\to d$ penguin transitions.

A. $B^-\to K^+K^-\pi^-$ decay

The factorizable tree-dominated $B^-\to K^+K^-\pi^-$ decay amplitude reads

$$\begin{aligned}
\langle\pi^-K^+K^-|T_p|B^-\rangle &= \langle K^+K^-|(\bar u b)_{V-A}|B^-\rangle\langle\pi^-|(\bar d u)_{V-A}|0\rangle[a_1\delta_{pu}+a_4^p+a_{10}^p-(a_6^p+a_8^p)r_\chi^\pi]\\
&+\langle\pi^-|(\bar d b)_{V-A}|B^-\rangle\langle K^+K^-|(\bar u u)_{V-A}|0\rangle(a_2\delta_{pu}+a_3+a_5+a_7+a_9)\\
&+\langle\pi^-|(\bar d b)_{V-A}|B^-\rangle\langle K^+K^-|(\bar d d)_{V-A}|0\rangle\left[a_3+a_4^p+a_5-\frac{1}{2}(a_7+a_9+a_{10}^p)\right]\\
&+\langle\pi^-|(\bar d b)_{V-A}|B^-\rangle\langle K^+K^-|(\bar s s)_{V-A}|0\rangle\left[a_3+a_5-\frac{1}{2}(a_7+a_9)\right]\\
&+\langle\pi^-|\bar d b|B^-\rangle\langle K^+K^-|\bar d d|0\rangle(-2a_6^p+a_8^p)+\langle K^-|(\bar s b)_{V-A}|B^-\rangle\langle K^+\pi^-|(\bar d s)_{V-A}|0\rangle\left(a_4^p-\frac{1}{2}a_{10}^p\right)\\
&+\langle K^-|\bar s b|B^-\rangle\langle K^+\pi^-|\bar d s|0\rangle(-2a_6^p+a_8^p)+\langle K^+K^-\pi^-|(\bar d u)_{V-A}|0\rangle\langle 0|(\bar u b)_{V-A}|B^-\rangle\\
&\times(a_1\delta_{pu}+a_4^p+a_{10}^p)+\langle K^+K^-\pi^-|\bar d(1+\gamma_5)u|0\rangle\langle 0|\bar u\gamma_5 b|B^-\rangle(2a_6^p+2a_8^p). \qquad (5.1)
\end{aligned}$$

Just as with the $B^-\to\pi^-\pi^+\pi^-$ decay, the branching fraction of the nonresonant contribution due to the $b\to u$ tree transition will be too large—of order 42×10^{-6}—if it is evaluated solely based on HMChPT. Hence, the momentum dependence of nonresonant amplitudes in an exponential form given by Eq. (2.9) has to be introduced.

Note that we have included the matrix element $\langle K^+K^-|\bar d d|0\rangle$. Although its nonresonant contribution vanishes as K^+ and K^- do not contain the valence d or $\bar d$ quark, this matrix element does receive a nonresonant contribution from the scalar f_0 pole,

$$\langle K^+(p_2)K^-(p_3)|\bar d d|0\rangle^R = \sum_i \frac{m_{f_{0i}}\bar f^d_{f_{0i}} g^{f_{0i}\to\pi^+\pi^-}}{m^2_{f_{0i}}-s_{23}-im_{f_{0i}}\Gamma_{f_{0i}}}, \qquad (5.2)$$

where $\langle f_0|\bar d d|0\rangle = m_{f_0}\bar f^d_{f_0}$. In the two-quark model for $f_0(980)$, $\bar f^d_{f_0(980)}=\bar f_{f_0(980)}\sin\theta/\sqrt{2}$. Also note that the matrix element $\langle K^-(p_3)|(\bar s b)_{V-A}|B^-\rangle\times\langle\pi^-(p_1)K^+(p_2)|(\bar d s)_{V-A}|0\rangle$ has a similar expression as Eq. (4.7),

$$\begin{aligned}
&\langle K^-(p_3)|(\bar s b)_{V-A}|B^-\rangle\langle\pi^-(p_1)K^+(p_2)|(\bar d s)_{V-A}|0\rangle\\
&\quad=-F_1^{BK}(s_{12})F_1^{K\pi}(s_{12})\\
&\qquad\times\left[s_{13}-s_{23}-\frac{(m_B^2-m_K^2)(m_K^2-m_\pi^2)}{s_{12}}\right]\\
&\qquad-F_0^{BK}(s_{12})F_0^{K\pi}(s_{12})\frac{(m_B^2-m_K^2)(m_K^2-m_\pi^2)}{s_{12}}. \qquad (5.3)
\end{aligned}$$

As in Eq. (4.5), the form factor $F_1^{K\pi}$ receives a resonant contribution for the K^* pole. The nonresonant and various

TABLE VIII. Predicted branching fractions (in units of 10^{-6}) of resonant and nonresonant contributions to $B^- \to K^+K^-\pi^-$ and $\bar{B}^0 \to K_S K^\pm \pi^\mp$. Experimental results are taken from Table II.

Decay mode		Decay mode	
$B^- \to K^+K^-\pi^-$			
$K^{*0}K^-$	$0.22^{+0.00+0.04+0.01}_{-0.00-0.04-0.01}$	$K_0^{*0}(1430)K^-$	$1.0^{+0.0+0.2+0.0}_{-0.0-0.2-0.0}$
$f_0(980)\pi^-$	$0.23^{+0.00+0.01+0.01}_{-0.00-0.01-0.01}$	NR	$2.9^{+0.7+0.7+0.0}_{-0.7-0.4-0.0}$
Total(theory)	$5.1^{+0.7+1.1+0.0}_{-0.8-0.7-0.0}$	Total(expt.)	5.0 ± 0.7
$\bar{B}^0 \to \overset{(-)0}{K} K^\mp \pi^\pm$			
$K^{*0}\bar{K}^0$	$0.20^{+0.00+0.04+0.00}_{-0.00-0.03-0.00}$	$K_0^{*0}(1430)\bar{K}^0$	$1.3^{+0.0+0.4+0.0}_{-0.0-0.3-0.0}$
NR	$4.2^{+0.7+1.9+0.1}_{-0.8-0.9-0.1}$		
Total(theory)	$6.2^{+0.7+2.6+0.1}_{-0.8-1.6-0.1}$	Total(expt.)	6.4 ± 0.8

resonant contributions to $B^- \to K^+K^-\pi^-$ are shown in Table VIII. The predicted total rate agrees well with experiment.

Note that no clear $\phi(1020)$ signature is observed in the mass region $m^2_{K^+K^-}$ around 1 GeV^2 [2]. Indeed, the branching fraction of the two-body decay $B^- \to \phi\pi^-$ is expected to be very small, of order 4.3×10^{-8}. It is induced mainly from $B^- \to \omega\pi^-$ followed by a small $\omega - \phi$ mixing [36].

B. $\bar{B}^0 \to K_S K^\pm \pi^\mp$ decay

The factorizable $\bar{B}^0 \to \overset{(-)0}{K} K^\mp \pi^\pm$ decay amplitude is given in Eq. (B2). The calculated branching fraction $(6.3^{+2.8}_{-1.8}) \times 10^{-6}$ is in good agreement with the current average of *BABAR* [18] and LHCb [19], namely, $(6.4 \pm 0.8) \times 10^{-6}$. The resonant states K^{*-} and $K_0^{*-}(1430)$ are absent in this decay because the quasi-two-body decays $\bar{B}^0 \to K^\pm K^{*\mp}$ and $K^\pm K_0^{*\mp}(1430)$ can proceed only through the W-exchange diagram and hence they are very suppressed.

C. $\bar{B}^0 \to K^+K^-\pi^0$ decay

The factorizable amplitude of $\bar{B}^0 \to K^+K^-\pi^0$ can be found in Eq. (B3). Since $\mathcal{B}(B^- \to K^+K^-\pi^-) = (5.0 \pm 0.7) \times 10^{-6}$ [10], it has been conjectured that the branching fraction of $\bar{B}^0 \to K^+K^-\pi^0$ should be of order 2.5×10^{-6}, which is indeed very close to the Belle measurement $(2.17 \pm 0.65) \times 10^{-6}$ [20]. However, a detailed study indicates that $\mathcal{B}(\bar{B}^0 \to K^+K^-\pi^0)$ is very small, of order 5×10^{-8}. This is mainly because the short-distance contribution to this mode is much smaller than the $K^+K^-\pi^-$ one because the latter is governed by the external pion-emission tree amplitude, while the former is dominated by the internal pion emission. As a result, $A(\bar{B}^0 \to K^+K^-\pi^0)/A(B^- \to K^+K^-\pi^-) \approx a_2/(\sqrt{2}a_1)$. The experimental observation of a sizable rate for $K^+K^-\pi^0$ implies that this mode should receive dominant long-distance contributions. Since the branching fraction of $\bar{B}^0 \to \pi^+\pi^-\pi^0$ is of order 20×10^{-6} (see Table IV), it is tempting to consider a final-state rescattering of $\pi^+\pi^-$ into K^+K^- that may substantially enhance the rate of $\bar{B}^0 \to K^+K^-\pi^0$. To estimate the effect of $\pi^+\pi^- \to K^+K^-$ rescattering, we work in the framework of Ref. [54] and note that in the quasi-elastic rescattering in $B \to PP$ modes, the corresponding rescattering amplitude is governed by the so-called annihilation rescatterings. The K^+K^- amplitude receives contributions from the $\pi^+\pi^-$ amplitude with a rescattering factor of $i(r_a^{(1/2)} + r_t^{(1/2)})$, where r_a and r_e, respectively, correspond to annihilation and total-annihilation rescatterting parameters [see Figs. 1(c), 1(d), and Eqs. (8) and (10) of Ref. [54]]. This factor is highly constrained by the $\bar{B}^0 \to K^+K^-$ rate and is found to be 0.15 in magnitude and $-144°$ in phase [54]. Consequently, the contribution to the $K^+K^-\pi^0$ rate from $\pi^+\pi^-\pi^0$ rescattering is estimated to be 0.5×10^{-6}, which is too small to account for the observed rate. Of course, rescattering in three-body decays is not necessarily the same as in two-body decays, but in general we do not expect a sizable change from the above estimation. Therefore, the unexpectedly large rate of $\bar{B}^0 \to K^+K^-\pi^0$ still remains unexplained.

VI. TWO-BODY $B \to VP$ AND $B \to SP$ DECAYS

So far we have considered the branching fraction products $\mathcal{B}(B \to Rh_1)\mathcal{B}(R \to h_2h_3)$ with the resonance R being a vector meson or a scalar meson. Using the experimental information on $\mathcal{B}(R \to h_2h_3)$ [7],

$$\begin{aligned}
\mathcal{B}(K^{*0} \to K^+\pi^-) &= \mathcal{B}(K^{*+} \to K^0\pi^+) \\
&= 2\mathcal{B}(K^{*+} \to K^+\pi^0) = \frac{2}{3}, \\
\mathcal{B}(K_0^{*0}(1430) \to K^+\pi^-) &= 2\mathcal{B}(K_0^{*+}(1430) \to K^+\pi^0) \\
&= \frac{2}{3}(0.93 \pm 0.10), \\
\mathcal{B}(\phi \to K^+K^-) &= 0.489 \pm 0.005, \qquad (6.1)
\end{aligned}$$

and applying the narrow-width approximation (3.13), one can extract the branching fractions of $B \to VP$ and $B \to SP$. The results are summarized in Table IX. Except for the channels $\rho^-\bar{K}^0$ from *BABAR*, ϕK^0, $\rho^0\pi^-$ from Belle, and $\rho^0\pi^0$ and $\rho^\pm\pi^\mp$ from both *BABAR* and Belle, all the experimental results are obtained from the three-body Dalitz-plot analyses shown in previous tables.

We see that except for the $\rho\pi$ and $f_0(980)K$ modes, the naive factorization predictions for penguin-dominated decays such as $B \to \phi K$, $K^*\pi$, $K_0^*(1430)\pi$ are usually too small by a factor of 2–3 and further suppressed for $B \to \rho K$ when confronted with experiment. This calls for $1/m_b$ power corrections to solve the rate-deficit problem. Within the framework of QCD factorization, we have considered two different types

TABLE IX. Branching fractions (in units of 10^{-6}) of quasi-two-body decays $B \to VP$ and $B \to SP$ obtained from the studies of three-body decays based on the factorization approach. Unless specified, the experimental results are obtained from the three-body Dalitz-plot analyses given in previous tables. Theoretical uncertainties have been added in quadrature. QCDF predictions taken from Ref. [36] for VP modes and from Ref. [51] for SP channels are shown here for comparison.

Decay mode	*BABAR*	Belle	QCDF	This work
ϕK^-	$9.2 \pm 0.4^{+0.7}_{-0.5}$	$9.6 \pm 0.9^{+1.1}_{-0.8}$	$8.8^{+2.8+4.7}_{-2.7-3.6}$	$5.8^{+1.1}_{-1.0}$
ϕK^0	$7.1 \pm 0.6^{+0.4}_{-0.3}$	$9.0^{+2.2}_{-1.8} \pm 0.7^{a}$	$8.1^{+2.6+4.4}_{-2.5-3.3}$	$5.3^{+0.9}_{-0.8}$
$\bar{K}^{*0}\pi^-$	$10.8 \pm 0.6^{+1.2}_{-1.4}$	$9.7 \pm 0.6^{+0.8}_{-0.9}$	$10.4^{+1.3+4.3}_{-1.5-3.9}$	$3.6^{+0.9}_{-0.8}$
$\bar{K}^{*0}\pi^0$	$3.3 \pm 0.5 \pm 0.4$	$0.4^{+1.9}_{-1.7} \pm 0.1$	$3.5^{+0.4+1.6}_{-0.4-1.4}$	$1.0^{+0.3}_{-0.3}$
$K^{*-}\pi^+$	8.4 ± 0.8	$8.4 \pm 1.1^{+0.9}_{-0.8}$	$9.2^{+1.0+3.7}_{-1.0-3.3}$	$3.1^{+0.8}_{-0.7}$
$K^{*-}\pi^0$	$8.2 \pm 1.5 \pm 1.1$		$6.7^{+0.7+2.4}_{-0.7-2.2}$	$2.7^{+0.6}_{-0.5}$
$K^{*0}K^-$	<1.1		$0.80^{+0.20+0.31}_{-0.17-0.38}$	$0.33^{+0.06}_{-0.05}$
$\rho^0 K^-$	$3.56 \pm 0.45^{+0.57}_{-0.46}$	$3.89 \pm 0.47^{+0.43}_{-0.41}$	$3.5^{+2.9+2.9}_{-1.2-1.8}$	$0.65^{+0.69}_{-0.19}$
$\rho^0 \bar{K}^0$	$4.4 \pm 0.7 \pm 0.3$	$6.1 \pm 1.0^{+1.1}_{-1.2}$	$5.4^{+3.4+4.3}_{-1.7-2.8}$	$0.1^{+0.5}_{-0.1}$
$\rho^+ K^-$	$6.6 \pm 0.5 \pm 0.8$	$15.1^{+3.4+2.4}_{-3.3-2.6}$	$8.6^{+5.7+7.4}_{-2.8-4.5}$	$2.4^{+2.6}_{-1.1}$
$\rho^- \bar{K}^0$	$8.0^{+1.4}_{-1.3} \pm 0.6^{a}$		$7.8^{+6.3+7.3}_{-2.9-4.4}$	$1.5^{+2.5}_{-0.9}$
$\rho^0\pi^-$	$8.1 \pm 0.7^{+1.3}_{-1.6}$	$8.0^{+2.3}_{-2.0} \pm 0.7^{a}$	$8.7^{+2.7+1.7}_{-1.3-1.4}$	$6.7^{+0.4}_{-0.4}$
$\rho^\pm\pi^\mp$	$22.6 \pm 1.8 \pm 2.2^{a}$	$22.6 \pm 1.1 \pm 4.4^{a}$	$25.1^{+1.5+1.4}_{-2.2-1.8}$	$17.8^{+3.6}_{-3.2}$
$\rho^0\pi^0$	$1.4 \pm 0.6 \pm 0.3^{a}$	$3.0 \pm 0.5 \pm 0.7^{a}$	$1.3^{+1.7+1.2}_{-0.6-0.6}$	$1.0^{+0.2}_{-0.1}$
$f_0(980)K^-; f_0 \to \pi^+\pi^-$	$10.3 \pm 0.5^{+2.0b}_{-1.4}$	$8.8 \pm 0.8^{+0.9}_{-1.8}$	$8.1^{+1.0+15.4c}_{-0.9-5.5}$	$6.6^{+1.6}_{-1.3}$
$f_0(980)K^0; f_0 \to \pi^+\pi^-$	$6.9 \pm 0.8 \pm 0.6$	$7.6 \pm 1.7^{+0.8}_{-0.9}$	$7.4^{+0.9+14.3c}_{-0.8-5.1}$	$5.9^{+1.5}_{-1.5}$
$f_0(980)K^-; f_0 \to K^+K^-$	$9.4 \pm 1.6 \pm 2.8$	<2.9		$11.0^{+2.6}_{-2.1}$
$f_0(980)K^0; f_0 \to K^+K^-$	$7.0^{+2.6}_{-1.8} \pm 2.4$			$9.1^{+1.7}_{-1.4}$
$f_0(980)\pi^-; f_0 \to \pi^+\pi^-$	<1.5		$0.13^{+0.02+0.09c}_{-0.02-0.06}$	$0.20^{+0.01}_{-0.01}$
$\bar{K}_0^{*0}(1430)\pi^-$	$32.0 \pm 1.2^{+10.8}_{-6.0}$	$51.6 \pm 1.7^{+7.0}_{-7.5}$	$12.9^{+4.6}_{-3.7}$	$18.3^{+8.1}_{-6.5}$
$\bar{K}_0^{*0}(1430)\pi^0$	$7.0 \pm 0.5 \pm 1.1$		$5.6^{+2.6}_{-1.3}$	$6.7^{+3.3}_{-2.7}$
$K_0^{*-}(1430)\pi^+$	$29.9^{+2.3}_{-1.7} \pm 3.6^{d}$	$49.7 \pm 3.8^{+6.8}_{-8.2}$	$13.8^{+4.5}_{-3.6}$	$16.7^{+7.3}_{-5.9}$

[a]Not determined directly from the Dalitz-plot analysis of three-body decays.
[b]The *BABAR* measurement $\mathcal{B}(B^- \to f_0(980)K^-; f_0(980) \to \pi^0\pi^0) = (2.8 \pm 0.6 \pm 0.3) \times 10^{-6}$ is not consistent with another *BABAR* result, $\mathcal{B}(B^- \to f_0(980)K^-; f_0(980) \to \pi^+\pi^-) = (10.3 \pm 0.5^{+2.0}_{-1.4}) \times 10^{-6}$, in view of the fact that $\mathcal{B}(f_0 \to \pi^+\pi^-) = 2\mathcal{B}(f_0 \to \pi^0\pi^0)$.
[c]We have assumed $\mathcal{B}(f_0(980) \to \pi^+\pi^-) = 0.50$ for the QCDF calculation.
[d]Another *BABAR* measurement of $\bar{B}^0 \to K^-\pi^+\pi^0$ (see Table VI) leads to $\mathcal{B}(\bar{B}^0 \to K_0^{*-}(1430)\pi^+) = 27.8 \pm 2.5 \pm 3.3$.

of power-correction effects in order to resolve the *CP* puzzles and rate-deficit problems with penguin-dominated two-body decays of B mesons and color-suppressed tree-dominated $\pi^0\pi^0$ and $\rho^0\pi^0$ modes: penguin annihilation and soft corrections to the color-suppressed tree amplitude [36]. However, the consideration of these power corrections for three-body B decays is beyond the scope of this work.

VII. DIRECT *CP* ASYMMETRIES

A. Inclusive *CP* asymmetries

Experimental measurements of direct *CP* violation for various charmless three-body B decays are collected in Table I. We notice that *CP* asymmetries of the pair $\pi^-\pi^+\pi^-$ and $K^-K^+K^-$ are of opposite signs, and likewise for the pair $K^-\pi^+\pi^-$ and $\pi^-K^+K^-$. This can be understood in terms of U-spin symmetry. In the limit of U-spin symmetry, $\Delta S = 0$ B^- decays can be related to the $\Delta S = 1$ one. For example,

$$\begin{aligned} A(B^- \to \pi^-\pi^+\pi^-) &= V_{ub}^* V_{ud}\langle \pi^-\pi^+\pi^-|O_d^u|B^-\rangle \\ &\quad + V_{cb}^* V_{cd}\langle \pi^-\pi^+\pi^-|O_d^c|B^-\rangle, \\ A(B^- \to K^-K^+K^-) &= V_{ub}^* V_{us}\langle K^-K^+K^-|O_s^u|B^-\rangle \\ &\quad + V_{cb}^* V_{cs}\langle K^-K^+K^-|O_s^c|B^-\rangle, \end{aligned} \tag{7.1}$$

where the four-quark operator O_s is for the $b \to sq_1\bar{q}_2$ transition and O_d is for the $b \to dq_1\bar{q}_2$ transition. The assumption of U-spin symmetry implies that under $d \leftrightarrow s$ transitions

$$\begin{aligned} \langle K^-K^+K^-|O_s^u|B^-\rangle &= \langle \pi^-\pi^+\pi^-|O_d^u|B^-\rangle, \\ \langle K^-K^+K^-|O_s^c|B^-\rangle &= \langle \pi^-\pi^+\pi^-|O_d^c|B^-\rangle, \end{aligned} \tag{7.2}$$

which can be checked from Eqs. (2.4) and (3.1). Using the relation for the Cabibbo-Kobayashi-Maskawa (CKM) matrix [55]

$$\mathrm{Im}(V_{ub}^* V_{ud} V_{cb} V_{cd}^*) = -\mathrm{Im}(V_{ub}^* V_{us} V_{cb} V_{cs}^*), \tag{7.3}$$

it is straightforward to show that

$$|A(B^- \to K^-K^+K^-)|^2 - |A(B^+ \to K^+K^-K^+)|^2 = |A(B^- \to \pi^-\pi^+\pi^-)|^2 - |A(B^+ \to \pi^+\pi^-\pi^+)|^2. \tag{7.4}$$

Hence, U-spin symmetry leads to the relation [56]

$$R_1 \equiv \frac{\mathcal{A}_{CP}(B^- \to \pi^-\pi^+\pi^-)}{\mathcal{A}_{CP}(B^- \to K^-K^+K^-)} = -\frac{\Gamma(B^- \to K^-K^+K^-)}{\Gamma(B^- \to \pi^-\pi^+\pi^-)}. \tag{7.5}$$

Likewise,

$$R_2 \equiv \frac{\mathcal{A}_{CP}(B^- \to \pi^-K^+K^-)}{\mathcal{A}_{CP}(B^- \to K^-\pi^+\pi^-)} = -\frac{\Gamma(B^- \to K^-\pi^+\pi^-)}{\Gamma(B^- \to \pi^-K^+K^-)}. \tag{7.6}$$

The predicted signs of the ratios R_1 and R_2 are confirmed by experiment.

What is the relative sign between $\mathcal{A}_{CP}(B^- \to \pi^-K^+K^-)$ and $\mathcal{A}_{CP}(B^- \to \pi^-\pi^+\pi^-)$? Applying U-spin symmetry to two of the mesons in the final states—one with positive charge and the other with negative charge—we obtain from Eqs. (2.4) and (5.1) that

$$A(B^- \to \pi^-\pi^-\pi^+)_{p_1p_2p_3} = A(B^- \to \pi^-K^-K^+)_{p_1p_2p_3} + A(B^- \to \pi^-K^-K^+)_{p_2p_1p_3}, \tag{7.7}$$

where the subscript $p_1p_2p_3$ denotes the momentum of the corresponding meson in order. Similarly,

$$A(B^- \to K^-K^-K^+)_{p_1p_2p_3} = A(B^- \to K^-\pi^-\pi^+)_{p_1p_2p_3} + A(B^- \to K^-\pi^-\pi^+)_{p_2p_1p_3}. \tag{7.8}$$

The above two relations agree with Ref. [57]. Because of the momentum dependence of decay amplitudes, the CP rate difference in $\pi^-\pi^-\pi^+$ ($K^-K^+K^-$) cannot be related to $\pi^-K^+K^-$ ($K^-\pi^-\pi^+$). Therefore, U-spin or flavor SU(3) symmetry does not lead to any testable relations between $\mathcal{A}_{CP}(\pi^-K^+K^-)$ and $\mathcal{A}_{CP}(\pi^-\pi^+\pi^-)$ and between $\mathcal{A}_{CP}(K^-\pi^+\pi^-)$ and $\mathcal{A}_{CP}(K^+K^-K^-)$.

Although symmetry arguments alone do not give hints at the relative sign of CP asymmetries in the pair of $\Delta S = 0$ and $\Delta S = 1$ decays, a realistic model calculation in the framework of this work shows a positive relative sign. When the unknown two-body matrix elements of scalar densities $\langle K\pi|\bar{s}q|0\rangle$—such as $\langle K^-\pi^+|\bar{s}d|0\rangle$ and $\langle \bar{K}^0\pi^-|\bar{s}u|0\rangle$, or $\langle K^-\pi^0|\bar{s}u|0\rangle$ and $\langle \bar{K}^0\pi^0|\bar{s}d|0\rangle$—are related to $\langle K^+K^-|\bar{s}s|0\rangle$ via SU(3) symmetry, e.g.,

$$\langle K^-(p_1)\pi^+(p_2)|\bar{s}d|0\rangle^{\rm NR} = \langle K^+(p_1)K^-(p_2)|\bar{s}s|0\rangle^{\rm NR} = f_s^{\rm NR}(s_{12}), \tag{7.9}$$

with the expression for $f_s^{\rm NR}$ given in Eq. (3.11), we find $\mathcal{A}_{CP}(K^-\pi^+\pi^-) \approx -3.7\%$ and $\mathcal{A}_{CP}(K^+K^-\pi^-) \approx 13.1\%$. Hence, they are of the same sign as $\mathcal{A}_{CP}(K^-K^+K^-)$ and $\mathcal{A}_{CP}(\pi^+\pi^-\pi^-)$, respectively. However, the naive predictions have incorrect signs when confronted with the corresponding data, $(3.3 \pm 1.0)\%$ and $(-11.9 \pm 4.1)\%$. That is, the data in Table I indicate that CP asymmetries of the pair $K^-K^+K^-$ and $K^-\pi^+\pi^-$ are of similar magnitude but opposite in sign and likewise for the pair $\pi^-K^+K^-$ and $\pi^-\pi^+\pi^-$. They have the common feature that when K^+K^- is replaced by $\pi^+\pi^-$, the sign of the CP asymmetry flips.

Recently, it has been conjectured that maybe the final rescattering between $\pi^+\pi^-$ and K^+K^- in conjunction with CPT invariance is responsible for the sign change [56,58,59]. As was stressed in Ref. [60], the presence of final-state interactions (FSIs) can have an interesting impact on the direct CP violation phenomenology. Long-distance final-state rescattering effects, in general, will lead to a different pattern of CP violation, namely, "compound" CP violation. Predictions of simple CP violation are quite distinct from that of compound CP violation. Moreover, the sign of CP asymmetry can be easily flipped by long-distance rescattering effects [60]. A well-known example is the direct CP violation in $\bar{B}^0 \to K^-\pi^+$. In the heavy-quark limit, the decay amplitudes of charmless two-body decays of B mesons can be described in terms of decay constants and form factors. However, the predicted direct CP-violating asymmetries for $\bar{B}^0 \to K^-\pi^+$ and $\bar{B}_s^0 \to K^+\pi^-$ have different signs than those measured by experiment [61]. This calls for the necessity of going beyond the leading $1/m_b$ power expansion. Possible $1/m_b$ power corrections to QCD penguin amplitudes include long-distance charming penguins, final-state interactions, and penguin annihilation. Because of possible "double-counting" problems, one should not take into account all power-correction effects simultaneously. It has been shown explicitly in Ref. [60] that FSIs can account for the sign flip of CP asymmetry and the rate deficit of $\bar{B}^0 \to K^-\pi^+$. More precisely, the decays $\bar{B}^0 \to D^{(*)}\bar{D}_s^{(*)}$ followed by the final-state rescattering $D^{(*)}\bar{D}_s^{(*)} \to K^-\pi^+$ will give a sizable and negative long-distance contribution $\mathcal{A}_{CP}^{\rm LD}$, so that the net CP asymmetry $\mathcal{A}_{CP} = \mathcal{A}_{CP}^{\rm SD} + \mathcal{A}_{CP}^{\rm LD}$ is negative for $\bar{B}^0 \to K^-\pi^+$ (for details, see Ref. [60]). In the QCD factorization approach [32], a sign flip can be caused by penguin annihilation parametrized in terms of two unknown parameters ρ_A and ϕ_A.

It is known how to explicitly take into account the constraints from the CPT theorem when computing partial-rate asymmetries for inclusive decays at the quark level [62,63] (for a review, see Ref. [64]). However, the implication of the CPT theorem for CP asymmetries at the hadron level in exclusive or semi-inclusive reactions is more complicated and remains mostly unclear [65].

Taking the cue from the LHCb observation of $\mathcal{A}_{CP}(\pi^-\pi^+\pi^-) \approx -\mathcal{A}_{CP}(\pi^- K^+ K^-)$ and $\mathcal{A}_{CP}(K^-\pi^+\pi^-) \approx -\mathcal{A}_{CP}(K^- K^+ K^-)$, it is conceivable that final-state rescattering may play an important role for direct *CP* violation. In the absence of a detailed model of final-state interactions for the pair $B^- \to K^-\pi^+\pi^-$ and $\pi^- K^+ K^-$, we shall assume that FSIs amount to giving a large strong phase δ to the nonresonant component of the matrix element of the scalar density $\langle K^-\pi^+|\bar{s}d|0\rangle$,

$$\langle K^-(p_1)\pi^+(p_2)|\bar{s}d|0\rangle^{\rm NR} = \frac{v}{3}(3F_{\rm NR} + 2F'_{\rm NR}) + \sigma_{\rm NR} e^{-\alpha s_{12}} e^{i\delta}. \tag{7.10}$$

Since *CP* violation arises from the interference between tree and penguin amplitudes and since nonresonant penguin contributions to the penguin-dominated decay $K^-\pi^+\pi^-$ are governed by the matrix element $\langle K^-\pi^+|\bar{s}d|0\rangle$, it is plausible that a strong phase in $\langle K^-\pi^+|\bar{s}d|0\rangle$ induced from FSIs might flip the sign of *CP* asymmetry. A fit to the data of $K^-\pi^+\pi^-$ yields

$$\langle K^-(p_1)\pi^+(p_2)|\bar{s}d|0\rangle^{\rm NR} \approx \frac{v}{3}(3F_{\rm NR} + 2F'_{\rm NR}) + \sigma_{\rm NR} e^{-\alpha s_{12}} e^{i\pi}\left(1 + 4\frac{m_K^2 - m_\pi^2}{s_{12}}\right), \tag{7.11}$$

with the parameter $\sigma_{\rm NR}$ given in Eq. (3.18). It follows from *U*-spin symmetry that

$$\langle K^+(p_1)\pi^-(p_2)|\bar{d}s|0\rangle^{\rm NR} \approx \frac{v}{3}(3F_{\rm NR} + 2F'_{\rm NR}) + \sigma_{\rm NR} e^{-\alpha s_{12}} e^{i\pi}\left(1 - 4\frac{m_K^2 - m_\pi^2}{s_{12}}\right), \tag{7.12}$$

which will be used to describe $B \to K\bar{K}\pi$ decays. Note that we have implicitly assumed that power corrections will not affect *CP* violation in $\pi^+\pi^-\pi^-$ and $K^+K^-K^-$.

The major uncertainty with direct *CP* violation comes from the strong phases which are needed to induce partial-rate *CP* asymmetries. In this work, the strong phases arise from the effective Wilson coefficients a_i^p listed in Eq. (2.3), the Breit-Wigner formalism for resonances, and the penguin matrix elements of scalar densities. Since direct *CP* violation in charmless two-body *B* decays can be significantly affected by final-state rescattering [60], it is natural to extend the study of final-state rescattering effects to the case of three-body *B* decays. We will leave this to a future investigation.

The calculated inclusive *CP* asymmetries $(8.7^{+1.7}_{-1.9})\%$ for $\pi^+\pi^-\pi^-$ and $(-7.1^{+2.4}_{-1.7})\%$ for $K^+K^-K^-$ (see Table X) are consistent with LHC measurements in both sign and magnitude (see Table I). As noted in passing, if we set $\delta = 0$ in Eq. (7.10) so that $\langle K^+\pi^-|\bar{d}s|0\rangle = \langle K^+K^-|\bar{s}s|0\rangle$, the predicted *CP* violation $\mathcal{A}_{CP}(K^-\pi^+\pi^-) = (-3.8^{+1.2}_{-0.7})\%$ will have the wrong sign. If a strong phase δ is allowed due to some power corrections such as FSIs, we obtain $\mathcal{A}_{CP}(K^-\pi^+\pi^-) = (2.6^{+0.6}_{-0.9})\%$ provided that the modified matrix element Eq. (7.11) is applied. Using Eq. (7.12), which follows from Eq. (7.11) via *U*-spin symmetry, we then predict $\mathcal{A}_{CP}(K^+K^-\pi^-) = (-13.4^{+1.9}_{-2.5})\%$, in agreement with experiment.

TABLE X. Direct *CP* asymmetries (in %) for various charmless three-body *B* decays. Experimental results are taken from Ref. [4] and Refs. [1,2]. The mass regions for local *CP* asymmetries are specified in Eqs. (1.1), (1.2), (1.3), and (1.4).

Final state	Theory	Experiment
$K^+K^-K^-$	$-7.1^{+2.0+1.0+0.1}_{-1.4-1.1-0.1}$	-3.7 ± 1.0
$(K^+K^-K^-)_{\rm region}$	$-17.7^{+3.8+2.9+0.3}_{-2.5-3.2-0.3}$	-22.6 ± 2.2
$K^+K^-\pi^-$	$-10.0^{+1.5+1.4+0.1}_{-2.4-1.3-0.1}$	-12.4 ± 4.5
$(K^+K^-\pi^-)_{\rm region}$	$-18.2^{+0.7+1.7+0.1}_{-1.0-1.5-0.1}$	-64.8 ± 7.2
$K^-\pi^+\pi^-$	$2.7^{+0.1+0.7+0.0}_{-0.2-0.8-0.0}$	3.3 ± 1.0
$(K^-\pi^+\pi^-)_{\rm region}$	$14.1^{+0.2+13.9+0.4}_{-0.2-11.7-0.4}$	67.8 ± 8.5
$\pi^+\pi^-\pi^-$	$8.7^{+0.5+1.6+0.0}_{-1.1-1.5-0.0}$	10.3 ± 2.5
$(\pi^+\pi^-\pi^-)_{\rm region}$	$22.5^{+0.5+2.9+0.1}_{-0.4-3.3-0.1}$	58.4 ± 8.7
$K^+K^-K_S$	$-5.5^{+1.4+0.5+0.1}_{-1.0-0.5-0.1}$	
$K_SK_SK_S$	$0.74^{+0.01+0.00+0.01}_{-0.01-0.00-0.01}$	17 ± 18
$K^-K_SK_S$	$3.5^{+0.0+0.3+0.1}_{-0.0-0.2-0.1}$	4^{+4}_{-5}
$K^+K^-\pi^0$	$-9.2^{+0.0+0.0+0.0}_{-0.0-0.0-0.0}$	
$K_SK^\pm\pi^\mp$	$1.8^{+1.7+1.5+0.0}_{-2.9-2.5-0.0}$	
$\bar{K}^0\pi^+\pi^-$	$-0.83^{+0.03+0.12+0.01}_{-0.02-0.14-0.01}$	-1 ± 5
$\bar{K}^0\pi^-\pi^0$	$0.64^{+0.06+0.04+0.01}_{-0.04-0.06-0.01}$	
$\pi^+\pi^-\pi^0$	$-1.4^{+0.3+0.5+0.0}_{-0.2-0.7-0.0}$	

Besides direct *CP* violation in the $K^+K^-K^-$, $K^+K^-\pi^-$, $K^-\pi^+\pi^-$, and $\pi^-\pi^+\pi^-$ modes, we have calculated *CP*-violating asymmetries in other three-body *B* decays, as summarized in Table X. It is expected that $\bar{B}^0 \to K^+K^+\pi^0$ and $K^+K^-K_S$ can have sizable asymmetries.

B. Regional *CP* asymmetries

Large local *CP* asymmetries in three-body charged *B* decays have been observed by LHCb in the low-mass regions specified in Eqs. (1.1), (1.2), (1.3), and (1.4) [1–3]. If intermediate resonant states are not associated in these low-mass regions, it is natural to expect that the Dalitz plot is governed by nonresonant contributions. In this case direct *CP* violation arises solely from the interference of tree and penguin nonresonant amplitudes. For example, in the absence of resonances, *CP* asymmetry in $B^- \to K^-\pi^+\pi^-$ stems mainly from the interference of the nonresonant tree amplitude $\langle\pi^+\pi^-|(\bar{u}b)_{V-A}|B^-\rangle\langle K^-|(\bar{s}u)_{V-A}|0\rangle$ and the nonresonant penguin amplitude $\langle\pi^-|\bar{d}b|B^-\rangle\langle K^-\pi^+|\bar{s}d|0\rangle$. The results of the calculated local *CP* asymmetries are shown in Table XI. It is evident that except for the mode $K^+K^-\pi^-$ regional *CP* violation is indeed dominated by the nonresonant background.

TABLE XI. Predicted direct CP asymmetries (in %) due to nonresonant contributions to various charmless three-body charged B decays. The mass regions for local CP asymmetries are specified in Eqs. (1.1), (1.2), (1.3), and (1.4). LHCb measurements [1–3] are shown for comparison.

	$\pi^-\pi^+\pi^-$	$K^-\pi^+\pi^-$	$K^+K^-\pi^-$	$K^+K^-K^-$
$(\mathcal{A}_{CP}^{\rm region})_{\rm NR}$	$57.4^{+3.2+2.6+1.1}_{-3.4-4.0-1.1}$	$49.0^{+7.0+7.7+0.3}_{-10.5-8.4-0.4}$	$-25.8^{+2.9+2.8+0.4}_{-5.6-2.5-0.4}$	$-13.2^{+2.0+2.9+0.3}_{-1.2-3.3-0.3}$
$(\mathcal{A}_{CP}^{\rm region})_{\rm expt}$	58.4 ± 8.7	67.8 ± 8.5	-64.8 ± 7.2	-22.6 ± 2.2

A realistic and straightforward calculation of regional CP asymmetries in our model yields the results shown in Table X. We see in this table that while regional CP violation of $K^+K^-K^-$ agrees with experiment within errors, the predicted local asymmetries of order -19%, 20%, and 23% for $K^+K^-\pi^-$, $K^-\pi^+\pi^-$, and $\pi^+\pi^-\pi^-$, respectively, are indeed greatly enhanced with respect to the inclusive ones, though they are still significantly below the corresponding data of order -65%, 68%, and 58%. The reader may wonder why the realistic calculation yields results different from the naive expectation. We will come to this point later.

It has been claimed recently that the observed large localized CP violation in $B^- \to \pi^+\pi^-\pi^-$ may result from the interference of a light scalar meson $f_0(500)$ and the vector $\rho^0(770)$ resonance [56,66], even though the latter resonance is not covered in the low-mass region $m^2_{\pi^-\pi^-{\rm low}} < 0.4\ {\rm GeV}^2$. Let us first consider the vector-meson resonance ρ^0 in $B^- \to \pi^+\pi^-\pi^-$ decay. As pointed out in Sec. II A, the calculated $\mathcal{B}(B^- \to \rho^0\pi^-) = (6.8 \pm 0.4) \times 10^{-6}$ is consistent with the world average $(8.3^{+1.2}_{-1.3}) \times 10^{-6}$ [4] within errors. Its CP asymmetry is found to be $\mathcal{A}_{CP}(\rho^0\pi^-) = 0.059^{+0.012}_{-0.010}$. At first sight, this seems to be in agreement in sign with the *BABAR* measurement $0.18 \pm 0.07^{+0.05}_{-0.15}$ [8]. However, theoretical predictions based on QCDF, pQCD, and soft-collinear effective theory all lead to a negative CP asymmetry for $B^\mp \to \rho^0\pi^\mp$ (see Table XIII of Ref. [36]). As was shown explicitly in Table IV of Ref. [36], within the framework of QCDF the inclusion of $1/m_b$ power corrections to penguin annihilation is responsible for the sign flip of $\mathcal{A}_{CP}(\rho^0\pi^-)$ to the right one. The consideration of power corrections is however beyond the scope of this work based on a simple factorization approach.

As for the scalar resonance $f_0(500)$, if we assume the form factor $F_0^{B\sigma}(0) = 0.25$ and take the mixing angle $\theta = 20^0$ in Eq. (3.16), we find the branching fraction of $B^- \to f_0(500)\pi^-$ to be order of 2.6×10^{-6}, but its CP violation is very small, of order -1%. In our model calculation, we find that the local asymmetry due to $\rho^0(770)$ and $f_0(500)$ resonances is $(\mathcal{A}_{CP}^{\rm region})_{\rho+\sigma} \approx -0.02$. Of course, the magnitude and even the sign might get modified if the model is improved to yield a negative CP violation for $B^\mp \to \rho^0\pi^\mp$, as discussed above.

Even the low-mass region $m^2_{\pi^-\pi^-{\rm low}} < 0.4\ {\rm GeV}^2$ is below the resonance $\rho^0(770)$, we find in our calculation that $\rho^0(770)$ makes sizable contributions to the rate and CP violation of $\pi^-\pi^+\pi^-$. Indeed, the fraction of nonresonant contribution to the total rate is found to be only 10%. Therefore, a reliable estimate of CP violation in the local regions of the Dalitz plot needs to take into account the effects of nearby resonances. As remarked before, our simple factorization model perhaps does not produce the "right" CP asymmetry of $B^- \to \rho^0\pi^-$; this may explain why our prediction of $\mathcal{A}_{CP}^{\rm region}$ for $\pi^+\pi^-\pi^-$ is below the LHCb measurement.

For the decay $B^- \to K^+K^-\pi^-$, the resonance $f_0(980)$ is in the low-mass region $m^2_{K^+K^-} < 1.5\ {\rm GeV}^2$, but it is not clear if the intermediate states $K^*(892)$ and $K_0^*(1430)$ are excluded. As a result, it is not surprising that the measured (and also the calculated) local asymmetry in this mode is very different from the one arising solely from the nonresonant contribution.

C. Comments on other works

CP violation in three-body decays of the charged B meson has been investigated in Refs. [56,59,66–68]. The authors of Refs. [56,66] considered the possibility of having a large local CP violation in $B^- \to \pi^+\pi^-\pi^-$ resulting from the interference of the resonances $f_0(500)$ and $\rho^0(770)$. A similar mechanism has been applied to the decay $B^- \to K^-\pi^+\pi^-$ [68]. Studies of flavor SU(3) symmetry imposed on the nonresonant decay amplitudes and its implication on CP violation were elaborated on in Ref. [67]. In our work, we have taken into account both resonant and nonresonant amplitudes simultaneously and worked out their contributions to branching fractions and CP violation in detail. We found that even in the absence of the $f_0(500)$ resonance, local CP asymmetry in $\pi^+\pi^-\pi^-$ can already reach the level of 23% due to nonresonant and other resonant contributions. Moreover, the regional asymmetry induced solely by the nonresonant component can be as large as 57% in our calculation.

The strong coupling between the K^+K^- and $\pi^+\pi^-$ channels was studied in Ref. [59] to explain the observed asymmetries in $B^- \to K^-K^+K^-$ and $B^- \to K^-\pi^+\pi^-$. Just as with the example of $\bar{B}^0 \to K^-\pi^+$—whose CP violation is originally predicted to have the wrong sign in naive factorization and gets a correct sign after power corrections, such as final-state interactions or penguin annihilation, are taken into account—it will be very interesting to see an explicit demonstration of the sign flip of $\mathcal{A}_{CP}(K^-\pi^+\pi^-)$ and $\mathcal{A}_{CP}(\pi^-K^+K^-)$ when the final-state rescattering of $\pi\pi \leftrightarrow K\bar{K}$ is turned on.

VIII. CONCLUSIONS

We have presented in this work a study of charmless three-body decays of B mesons within the framework of a simple model based on the factorization approach. Our main results are as follows.

(i) Dominant nonresonant contributions to tree-dominated three-body decays arise from the $b \to u$ tree transition which can be evaluated using heavy-meson chiral perturbation theory valid in the soft-meson limit. The momentum dependence of nonresonant $b \to u$ transition amplitudes is parametrized in an exponential form $e^{-\alpha_{\rm NR} p_B \cdot (p_i + p_j)}$ so that the HMChPT results are recovered in the soft-meson limit $p_i, p_j \to 0$. The parameter $\alpha_{\rm NR}$ is fixed by the measured nonresonant rate in $B^- \to \pi^+\pi^-\pi^-$.

(ii) A unique feature of hadronic $B \to KKK$ decays is that they are predominated by the nonresonant contributions with a nonresonant fraction of order (70–90)%. It follows that nonresonant contributions to the penguin-dominated modes should also be dominated by the penguin mechanism. Hence, nonresonant signals must come mainly from the penguin amplitude governed by the matrix element of scalar densities $\langle M_1M_2|\bar{q}_1q_2|0\rangle$. We used the measurements of $\bar{B}^0 \to K_SK_SK_S$ to constrain the nonresonant component of $\langle K\bar{K}|\bar{s}s|0\rangle$.

(iii) The branching fraction of nonresonant contributions is of order $(15\text{–}20) \times 10^{-6}$ in penguin-dominated decays $B^- \to K^+K^-K^-$, $K^-\pi^+\pi^-$, and of order $(3\text{–}5) \times 10^{-6}$ in tree-dominated decays $B^- \to \pi^+\pi^-\pi^-$, $K^+K^-\pi^-$. The nonresonant fraction is predicted to be around 60% in $B \to K\bar{K}\pi$ decays.

(iv) The intermediate vector-meson contributions to three-body decays are identified through the vector current, while the scalar-meson resonances are mainly associated with the scalar density. Both scalar and vector resonances can contribute to the three-body matrix element $\langle P_1P_2|J_\mu|B\rangle$.

(v) The $\pi^+\pi^-\pi^0$ mode is predicted to have a rate larger than $\pi^+\pi^-\pi^-$ even though the former involves a π^0 and has no identical particles in the final state. This is because while the latter is dominated by the ρ^0 pole, the former receives $\rho^\pm$ and ρ^0 resonant contributions.

(vi) We have made predictions for the resonant and nonresonant contributions to $\bar{B}^0 \to \pi^+\pi^-\pi^0$, $\bar{K}^0\pi^0\pi^0$, $K_SK^\pm\pi^\mp$, and $B^- \to \bar{K}^0\pi^-\pi^0$.

(vii) We emphasize that the seemingly huge difference between *BABAR* and Belle for the nonresonant contributions to $B^- \to K^-\pi^+\pi^-$ and $\bar{B}^0 \to K^-\pi^+\pi^0$ is now relieved when the nonresonant part of the LASS parametrization adapted by *BABAR* for the description of the $K\pi$ S-wave is added coherently to the phase-space nonresonant piece.

(viii) The surprisingly large rate of $\bar{B}^0 \to K^+K^-\pi^0$ observed by Belle is bigger than the naive expectation by two orders of magnitude. It implies that this mode should be dominated by long-distance contributions. It may arise from the decay $\bar{B}^0 \to \pi^+\pi^-\pi^0$ followed by the final-state rescattering of $\pi^+\pi^-$ into K^+K^-. However, an estimation based on the two-body FSI model shows that $\mathcal{B}(\bar{B}^0 \to K^+K^-\pi^0)$ can be enhanced via final-state rescattering only up to the level of 0.5×10^{-6}. Therefore, the unexpectedly large rate of $\bar{B}^0 \to K^+K^-\pi^0$ still remains unexplained.

(ix) Based on the factorization approach, we have computed the resonant contributions to three-body decays and determined the rates for the quasi-two-body decays $B \to VP$ and $B \to SP$. The predicted $\rho\pi$, $f_0(980)K$, and $f_0(980)\pi$ rates are consistent with experiment, while the calculated ϕK, $K^*\pi$, ρK, and $K_0^*(1430)\pi$ are too small compared to the data.

(x) While the calculated direct CP asymmetries for the $K^+K^-K^-$ and $\pi^+\pi^-\pi^-$ modes are in good agreement with experiment in both magnitude and sign, the predicted CP asymmetries in $B^- \to \pi^-K^+K^-$ and $B^- \to K^-\pi^+\pi^-$ have the wrong signs when confronted with experiment. It has been conjectured recently that a possible resolution to this CP puzzle relies on final-state rescattering of $\pi^+\pi^-$ and K^+K^-. Assuming a large strong phase associated with $\langle K\pi|\bar{s}q|0\rangle$ arising from some sort of power corrections, we fit it to the data of $K^-\pi^+\pi^-$ and get correct signs for both the $\pi^-K^+K^-$ and $K^-\pi^+\pi^-$ modes. We predict some testable CP violation in $\bar{B}^0 \to K^+K^-\pi^0$ and $K^+K^-K_S$.

(xi) In this work there are three sources of strong phases: effective Wilson coefficients, propagators of resonances, and the matrix element of the scalar density $\langle M_1M_2|\bar{q}_1q_2|0\rangle$.

(xii) In the low-mass regions devoid of the known resonances, direct CP violation is naively expected to be dominated by nonresonant contributions. We found that—except for the $K^+K^-\pi^-$ mode where resonances are not excluded in the local region—partial-rate asymmetries due to the nonresonant background are fairly close to the LHCb measurements. However, realistic model calculations show that resonances near the localized region can make sizable contribution to the total rates and asymmetries. At any rate, we have shown that the regional CP violation is indeed largely enhanced with respect to the inclusive one, though it is still significantly below the data.

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APPENDIX A: INPUT PARAMETERS

Many of the input parameters for the decay constants of pseudoscalar and vector mesons and form factors for $B \to P$, V transitions can be found in Ref. [36] where uncertainties in form factors are shown. The reader is referred to Ref. [51] for decay constants and form factors related to scalar mesons.

For the CKM matrix elements, we use the updated Wolfenstein parameters $A = 0.823$, $\lambda = 0.22457$, $\bar\rho = 0.1289$, and $\bar\eta = 0.348$ [69]. The corresponding CKM angles are $\sin 2\beta = 0.689 \pm 0.019$ and $\gamma = (69.7^{+1.3}_{-2.8})^\circ$ [69]. For the running quark masses we shall use [7,70]

$$\begin{aligned} &m_b(m_b) = 4.2\ \text{GeV}, && m_b(2.1\ \text{GeV}) = 4.94\ \text{GeV},\\ &m_b(1\ \text{GeV}) = 6.34\ \text{GeV}, && m_c(m_b) = 0.91\ \text{GeV},\\ &m_c(2.1\ \text{GeV}) = 1.06\ \text{GeV}, && m_c(1\ \text{GeV}) = 1.32\ \text{GeV},\\ &m_s(2.1\ \text{GeV}) = 95\ \text{MeV}, && m_s(1\ \text{GeV}) = 118\ \text{MeV},\\ &m_d(2.1\ \text{GeV}) = 5.0\ \text{MeV}, && m_u(2.1\ \text{GeV}) = 2.2\ \text{MeV}. \end{aligned} \tag{A1}$$

Among the quarks, the strange quark gives the major theoretical uncertainty to the decay amplitude. Hence, we will only consider the uncertainty in the strange-quark mass given by $m_s(2.1\ \text{GeV}) = 95 \pm 5$ MeV.

APPENDIX B: DECAY AMPLITUDES OF $B \to PPP$ DECAYS

Most of the factorizable decay amplitudes of $\Delta S = 0$ and $\Delta S = 1$ three-body decays of B mesons are already collected in Appendix A of Ref. [31]. In this work, we have shown the factorizable decay amplitudes of $B^- \to K^+K^-K^-$, $K^-K^+\pi^-$, $K^-\pi^+\pi^-$, and $\pi^+\pi^-\pi^-$ for the purpose of discussion and for corrections. In the following we write down the factorizable amplitudes of $B^- \to K^-\pi^0\pi^0$ and $\bar B^0 \to K_S K^\pm \pi^\mp$, $K^+K^-\pi^0$:

$$\begin{aligned} \langle K^-\pi^0\pi^0|T_p|B^-\rangle = {}& \langle \pi^0\pi^0|(\bar u b)_{V-A}|B^-\rangle\langle K^-|(\bar s u)_{V-A}|0\rangle[a_1\delta_{pu} + a_4^p + a_{10}^p - (a_6^p + a_8^p)r_\chi^K]\\ &+ \langle K^-\pi^0|(\bar s b)_{V-A}|B^-\rangle\langle \pi^0|(\bar u u)_{V-A}|0\rangle\left[a_2\delta_{pu} + \frac{3}{2}(-a_7 + a_9)\right]\\ &+ \langle K^-|(\bar s b)_{V-A}|B^-\rangle\langle \pi^0\pi^0|(\bar u u)_{V-A}|0\rangle[a_2\delta_{pu} + a_3 + a_5 + a_7 + a_9]\\ &+ \langle K^-|(\bar s b)_{V-A}|B^-\rangle\langle \pi^0\pi^0|(\bar d d)_{V-A}|0\rangle\left[a_3 + a_5 - \frac{1}{2}(a_7 + a_9)\right]\\ &+ \langle K^-|(\bar s b)_{V-A}|B^-\rangle\langle \pi^0\pi^0|(\bar s s)_{V-A}|0\rangle\left[a_3 + a_4^p + a_5 - \frac{1}{2}(a_7 + a_9 + a_{10}^9)\right]\\ &+ \langle K^-|\bar s b|B^-\rangle\langle \pi^0\pi^0|\bar s s|0\rangle(-2a_6^p + a_8^p) + \langle \pi^0|(\bar u b)_{V-A}|B^-\rangle\langle K^-\pi^0|(\bar s u)_{V-A}|0\rangle(a_4^p + a_{10}^p)\\ &+ \langle \pi^0|\bar u b|B^-\rangle\langle K^-\pi^0|\bar s u|0\rangle(-2a_6^p - 2a_8^p) + \langle K^-\pi^0\pi^0|(\bar s u)_{V-A}|0\rangle\langle 0|(\bar u b)_{V-A}|B^-\rangle(a_1\delta_{pu} + a_4^p + a_{10}^p)\\ &+ \langle K^-\pi^0\pi^0|\bar s(1+\gamma_5)u|0\rangle\langle 0|\bar u\gamma_5 b|B^-\rangle(2a_6^p + 2a_8^p), \end{aligned} \tag{B1}$$

$$\begin{aligned} \langle \overset{(-)0}{K} K^\mp\pi^\pm|T_p|\bar B^0\rangle = {}& \langle K^+\bar K^0|(\bar u b)_{V-A}|\bar B^0\rangle\langle \pi^-|(\bar d u)_{V-A}|0\rangle[a_1\delta_{pu} + a_4^p + a_{10}^p - (a_6^p + a_8^p)r_\chi^\pi]\\ &+ \langle \pi^+|(\bar u b)_{V-A}|\bar B^0\rangle\langle K^-K^0|(\bar d u)_{V-A}|0\rangle(a_1\delta_{pu} + a_4^p + a_{10}^p)\\ &+ \langle K^-\pi^+|(\bar s b)_{V-A}|\bar B^0\rangle\langle K^0|(\bar d s)_{V-A}|0\rangle\left[a_4^p - \frac{1}{2}a_{10}^p - \left(a_6^p - \frac{1}{2}a_8^p\right)r_\chi^K\right]\\ &+ \langle \bar K^0|(\bar s b)_{V-A}|\bar B^0\rangle\langle K^+\pi^-|(\bar d s)_{V-A}|0\rangle\left(a_4^p - \frac{1}{2}a_{10}^p\right) + \langle \pi^+|\bar u b|\bar B^0\rangle\langle K^-K^0|\bar d u|0\rangle(-2a_6^p - 2a_8^p)\\ &+ \langle \bar K^0|\bar s b|\bar B^0\rangle\langle K^+\pi^-|\bar d s|0\rangle(-2a_6^p + a_8^p) + \langle \overset{(-)0}{K} K^\mp\pi^\pm|(\bar u u)_{V-A}|0\rangle\langle 0|(\bar d b)_{V-A}|\bar B^0\rangle(a_2\delta_{pu} + a_3 + a_5\\ &+ a_7 + a_9) + \langle \overset{(-)0}{K} K^\mp\pi^\pm|\bar d(1+\gamma_5)d|0\rangle\langle 0|\bar d\gamma_5 b|\bar B^0\rangle(2a_6^p - a_8^p), \end{aligned} \tag{B2}$$

$$\begin{aligned}\langle\pi^0K^+K^-|T_p|\bar{B}^0\rangle &= \langle\pi^0|(\bar{d}b)_{V-A}|\bar{B}^0\rangle\langle K^+K^-|(\bar{u}u)_{V-A}|0\rangle(a_2\delta_{pu}+a_3+a_5+a_7+a_9)\\ &+\langle\pi^0|\bar{d}b|\bar{B}^0\rangle\langle K^+K^-|\bar{d}d|0\rangle(-2a_6^p+a_8^p)\\ &+\langle\pi^0|(\bar{d}b)_{V-A}|\bar{B}^0\rangle\langle K^+K^-|(\bar{s}s)_{V-A}|0\rangle\Big[a_3+a_5-\frac{1}{2}(a_7+a_9)\Big]\\ &+\langle K^+K^-\pi^0|(\bar{u}u)_{V-A}|0\rangle\langle 0|(\bar{d}b)_{V-A}|\bar{B}^0\rangle(a_2\delta_{pu}+a_4^p+a_{10}^p)\\ &+\langle K^+K^-\pi^0|\bar{d}\gamma_5 d|0\rangle\langle 0|\bar{d}\gamma_5 b|\bar{B}^0\rangle(2a_6^p-a_8^p).\end{aligned} \tag{B3}$$

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