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Maxwell theory on a compact manifold as a topologically ordered system

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We study novel types of contributions to the entropy of the Maxwell system defined on a compact manifold such as a torus. These new terms are not related to the physical propagating photons. Rather, these novel contributions emerge as a result of tunneling events when transitions occur between topologically different but physically identical vacuum winding states. We compute two new (topologically protected) types of contributions to the entropy in this system resulting from this dynamics. The first contribution has a negative sign, expressed in terms of the magnetic susceptibility, and it is similar in spirit to topological entanglement entropy discussed in condensed matter systems. The second contribution with a positive sign results from the emergent degeneracy which occurs when the system is placed into a background of external magnetic field. This degeneracy resembles a similar effect that occurs at $\theta = \pi$ in topological insulators. Based on these computations we claim that the Maxwell system defined on the four torus behaves in many respects as a topologically ordered system.

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I. INTRODUCTION: MOTIVATION

The main motivation for the present studies is as follows. It has been recently argued [1] that if the Maxwell theory is defined on a small compact manifold then some novel terms in the partition function will emerge. These terms are not related to the propagating photons with two transverse physical polarizations. Rather, these novel terms occur as a result of tunneling events between topologically different but physically identical states. These states play no role when the system is defined in Minkowski space-time $\mathbb{R}_{1,3}$. But these states become important when the system is defined on a finite compact manifold such as torus $\mathbb{T}^4$.

In particular, it has been explicitly shown in [1] that these novel terms lead to fundamentally new contributions to the Casimir vacuum pressure, which cannot be expressed in terms of conventional propagating physical degrees of freedom. Instead, the new contributions appear as a result of tunneling events between different topological sectors $|k\rangle$. Mathematically, these sectors emerge as a result of nontriviality of the fundamental group $\pi_1[U(1)] \cong \mathbb{Z}$ when the system is defined on a torus.

The crucial observation for the present studies is as follows. While the Maxwell electrodynamics is the theory of massless particles (photons), the topological portion of the system decouples from dynamics of these massless propagating photons. Indeed, as we discuss below, the total partition function $\mathcal{Z}$ can be represented as a product $\mathcal{Z} = \mathcal{Z}_0 \times \mathcal{Z}_{\rm top}$. The conventional partition function $\mathcal{Z}_0$ describing physical photons is not sensitive to the topological sectors $|k\rangle$ of the system, which itself is described by $\mathcal{Z}_{\rm top}$. The topological portion of the partition function $\mathcal{Z}_{\rm top}$ behaves very much as a topological quantum field theory (TQFT) as we argue below. Furthermore, it demonstrates many features of topologically ordered systems, which were initially introduced in the context of condensed matter systems (see original papers [2–5] and recent reviews [6–8]).

As a result of these similarities, the key question addressed in the present work is as follows. It has been known since [9,10] that the topologically ordered systems can be characterized by the so-called topological entanglement entropy (TEE). While the TEE is a subleading contribution to the entanglement entropy, it is nevertheless a topologically protected universal constant of the system which can serve as a probe of the topological order.

We formulate a similar question for the Maxwell system defined on a nontrivial compact manifold: is there a similar universal contribution to the entropy that is topologically protected and that can serve as a probe of the topological order? Our ultimate answer is "yes," as our explicit computations below show. Furthermore, this universal constant contribution to the entropy cannot be expressed in terms of physical propagating photons. Instead, it is formulated in terms of the magnetic susceptibility, which itself is a topologically protected object, and which is saturated by the "instantons," rather than by propagating degrees of freedom. In many respects this object is similar to well-known topological susceptibility in QCD. As we shall see below this object does not vanish exclusively as a result of the dynamics of the topological sectors described by $\mathcal{Z}_{\rm top}$.

The second question we address in this work can be formulated as follows. It is known that the main feature of a topologically ordered system is the presence of a degeneracy of the ground state which cannot be described in terms of any local observables. We formulate a similar question for the Maxwell system defined on a nontrivial compact manifold: is there a similar degeneracy that can be described by some global, rather than local, characteristics? Our ultimate answer is "yes" again, as our explicit computations below show. This degeneracy resembles a similar feature that occurs at $\theta = \pi$ in topological insulators.

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One may wonder at this point what went wrong with the standard and generically accepted arguments suggesting that all physical effects in Maxwell theory can be formulated in terms of physical propagating photons that are completely described by conventional $\mathcal{Z}_0$. Yet all the effects discussed in the present work are formulated in fundamentally different terms coded by $\mathcal{Z}_{\rm top}$. The point is that the standard description is not quite complete when four-dimensional Maxwell theory is formulated on a non-simply connected, compact manifold. The standard "naive" argument neglects the topological sectors, which are indeed absent when the theory is formulated in the topologically trivial Minkowski space-time. However, these topological sectors become important when the theory is formulated on a nontrivial manifold.

When one attempts to remove all unphysical degrees of freedom by a gauge fixing, the physics related to pure gauge configurations describing the topological sectors of the theory does not go away; instead, this physics reappears in a much more complicated form where the so-called Gribov's ambiguities [11] emerge in the Maxwell system formulated on a compact manifold [12–15] (see some comments on this matter in concluding Sec. V). In this work we opt to keep some gauge freedom and study these topological sectors explicitly, rather than deal with a (technically complicated) analysis of the Gribov's copies.

The structure of our presentation is as follows. In Sec. II, we review the relevant parts of the two-dimensional Maxwell "empty" theory which does not have any physical propagating degrees of freedom. Still, it demonstrates a number of very nontrivial topological features present in the system. In Sec. III we generalize our computations for four-dimensional Maxwell theory defined on the four torus. We find two types of novel contributions to the entropy in this system. The first contribution with a negative sign is very similar to topological entanglement entropy well studied in topologically ordered condensed matter systems. The second contribution with a positive sign results from emergent degeneracy, which occurs when the system is placed into a background of external magnetic field that resembles a behavior of topological insulators with $\theta = \pi$.

II. MAXWELL THEORY IN TWO DIMENSIONS AS TOPOLOGICAL QFT

The two-dimensional Maxwell model has been solved numerous times using very different techniques (see, e.g., [16–18]). It is known that this is an "empty" theory in the sense that it does not support any propagating degrees of freedom in the bulk of space-time. It is also known that this model can be treated as a conventional TQFT. In particular, this model can be formulated in terms of the so-called background field (BF) action involving no metric. Furthermore, this model exhibits many other features such as fractional edge observables that are typical for TQFT (see, e.g., [17]). We emphasize these properties of the two-dimensional Maxwell theory because the topological portion of the partition function $\mathcal{Z}_{\rm top}$ in our description of the four-dimensional Maxwell system, given in Sec. III, is identically the same as the partition function of the two-dimensional Maxwell system. As we already mentioned, such a relation between the two different systems is a result of decoupling of physical propagating photons from the topological sectors in the four-dimensional system.

Our goal here is to review this "empty" two-dimensional Maxwell theory with nontrivial dynamics of the topological sectors when conventional propagating degrees of freedom are not supported by this system.

A. Partition function

We consider the two-dimensional Maxwell theory defined on the Euclidean torus $S^1 \times S^1$ with lengths L and β, respectively. In the Hamiltonian framework we choose a $A_0 = 0$ gauge along with $\partial_1 A_1 = 0$. This implies that $A_1(t)$ is the only dynamical variable of the system with $E = \dot{A}_1$.

The spectrum for θ vacua is well known [16] and it is given by $E_n(\theta) = \frac{1}{2}(n + \frac{\theta}{2\pi})^2 e^2 L$, such that the corresponding partition function takes the form

$$\mathcal{Z}(V, \theta) = \sum_{n\in\mathbb{Z}} e^{-\frac{e^2 V}{2}(n+\frac{\theta}{2\pi})^2}, \tag{1}$$

where $V = \beta L$ is the two-volume of the system.

We want to reproduce Eq. (1) using a different approach based on Euclidean path integral computations because it can be easily generalized to similar computations defined by four-dimensional Maxwell theory on the four torus. Our goal here is to understand the physical meaning of Eq. (1) in terms of the path integral computations.

To proceed with path integral computations, one considers the "instanton" configurations on the two-dimensional Euclidean torus with total area $V = L\beta$ described as follows [18],

$$\int d^2x Q(x) = k, \qquad eE^{(k)} = \frac{2\pi k}{V}, \tag{2}$$

where $Q = \frac{e}{2\pi} E$ is the topological charge density and k is the integer-valued topological charge in the two-dimensional $U(1)$ gauge theory, and $E(x) = \partial_0 A_1 - \partial_1 A_0$ is the field strength. The action of this classical configuration is

$$\frac{1}{2}\int d^2x E^2 = \frac{2\pi^2 k^2}{e^2 V}. \tag{3}$$

This configuration corresponds to the topological charge k as defined by (2). The next step is to compute the partition function defined as follows,

$$\mathcal{Z}(\theta) = \sum_{k\in\mathbb{Z}} \int \mathcal{D}A^{(k)} e^{-\frac{1}{2}\int d^2x E^2 + \int d^2x L_\theta}, \tag{4}$$

where θ is the standard theta parameter that defines the $|\theta\rangle$ ground state and that enters the action with topological density operator

$$L_\theta = i\theta \int d^2x Q(x) = i\theta \frac{e}{2\pi} \int d^2x E(x). \quad (5)$$

All integrals in this partition function are Gaussian and can be easily evaluated using the technique developed in [18]. The result is

$$\mathcal{Z}(V,\theta) = \sqrt{\frac{2\pi}{e^2V}} \sum_{k\in\mathbb{Z}} e^{-\frac{2\pi^2k^2}{e^2V}+ik\theta}, \quad (6)$$

where the expression in the exponent represents the classical instanton configurations with action (3) and topological charge (2), while the factor in front is due to the fluctuations (see [1] with some technical details and relevant references). While expressions (1) and (6) look different, they are actually identically the same, as the Poisson summation formula states,

$$\mathcal{Z}(\theta) = \sum_{n\in\mathbb{Z}} e^{-\frac{e^2V}{2}(n+\frac{\theta}{2\pi})^2} = \sqrt{\frac{2\pi}{e^2V}} \sum_{k\in\mathbb{Z}} e^{-\frac{2\pi^2k^2}{e^2V}+ik\theta}. \quad (7)$$

Therefore, we reproduce the original expression (1) using the path integral approach.

The crucial observation for our present study is that this naively "empty" theory, which has no physical propagating degrees of freedom, nevertheless shows some very nontrivial features of the ground state related to the topological properties of the theory. These new properties are formulated in terms of different topological vacuum sectors of the system $|k\rangle$, which have identical physical properties as they are connected to each other by large gauge transformation operator $\mathcal{T}$ commuting with the Hamiltonian $[\mathcal{T}, H] = 0$. As explained in detail in [1], the corresponding dynamics of this "empty" theory represented by partition function (7) should be interpreted as a result of tunneling events between these "degenerate" winding $|k\rangle$ states, which correspond to one and the same physical state.

It is known that this model can be treated as TQFT; e.g., it supports edge observables that may assume the fractional values and shows many other features that are typical for a TQFT (see [17] and references therein). The presence of the topological features of the model can be easily understood from observation that the entire dynamics of the system is due to the transitions between the topological sectors, which themselves are determined by the behavior of surface integrals at infinity $\oint A_\mu dx^\mu$. These sectors are classified by integer numbers and they are not sensitive to specific details of the system such as the geometrical shape of the system. Therefore, it is not really a surprise that the system is not sensitive to specific geometrical details and can be treated as TQFT.

The important point we would like to make is that our analysis of the topological portion $\mathcal{Z}_{\rm top}$ of the partition function for the four-dimensional Maxwell system defined on $\mathbb{T}^4$ assumes exactly the same form (7) as a result of the decoupling of propagating photons from the topological part of the partition function, as will be discussed in Sec. III. As a result of this decoupling, the topological portion of the four-dimensional Maxwell system behaves in very much the same way as the two-dimensional "empty" theory. Therefore, one should not be very surprised that this four-dimensional system also demonstrates some topological features, similar to the two-dimensional system reviewed in this section.

Before we proceed with computations of the topological entropy, we make a short detour on properties of the topological susceptibility in this model, as it plays an important role in our discussions of the entropy.

B. Topological susceptibility

The topological susceptibility χ is defined as follows,

$$\chi \equiv \lim_{k\to 0} \int d^2x\, e^{ikx} \langle TQ(x)Q(0)\rangle, \quad (8)$$

where Q is the topological charge density operator normalized according to Eq. (2). The χ measures response of the free energy to the introduction of a source term defined by Eq. (5). The computations of χ in this simple "empty" model can be easily carried out as the partition function $\mathcal{Z}(\theta)$ defined by (4) is known exactly (7). To compute χ we should simply differentiate the partition function twice with respect to θ. It leads to the following well-known expression for χ, which is finite in the infinite volume limit [18,19],

$$\chi(V\to\infty) = -\frac{1}{V}\cdot \frac{\partial^2 \ln \mathcal{Z}(\theta)}{\partial\theta^2}\bigg|_{\theta=0} = \frac{e^2}{4\pi^2}. \quad (9)$$

A typical value of the topological charge k which saturates the topological susceptibility χ in the large volume limit is very large, $k \sim \sqrt{e^2V} \gg 1$.

A few comments are in order. First, any physical state contributes to χ with a negative sign,

$$\chi_{\rm dispersive} \sim \lim_{k\to 0} \sum_n \frac{\langle 0|\frac{e}{2\pi}E|n\rangle\langle n|\frac{e}{2\pi}E|0\rangle}{-k^2 - m_n^2} < 0, \quad (10)$$

while (9) has a positive sign. Therefore, this nondispersive (contact) term (9) cannot be identified according to (10) with any contribution from any asymptotic state even when physical degrees of freedom, such as fermions, are included into the system. This term has a fundamentally different, nondispersive nature. In fact it is ultimately related to different topological sectors as our computation (9) shows. Second, the integrand for the topological susceptibility (8) demonstrates a singular behavior,

$$\langle Q(x)Q(0)\rangle = \frac{e^2}{4\pi^2}\delta^2(x), \quad (11)$$

which is not a specific property of this "empty" theory, but in fact a very generic feature that is present in many gauge theories; it represents the contact term, which is not related to any propagating degrees of freedom. In particular, such singular behavior (11) is known to occur in QCD and its modifications and is well supported by the QCD lattice Monte Carlo simulations (see [19] for the details and related references).

The $\delta^2(x)$ function in (11) should be understood as the total divergence related to the infrared (IR) physics, rather than to the ultraviolet (UV) behavior. Indeed,

$$\chi = \frac{e^2}{4\pi^2}\int \delta^2(x)\mathrm{d}^2x = \frac{e^2}{4\pi^2}\int \mathrm{d}^2x\partial_\mu\left(\frac{x^\mu}{2\pi x^2}\right) = \frac{e^2}{4\pi^2}\oint_{S_1}\mathrm{d}l_\mu\left(\frac{x^\mu}{2\pi x^2}\right). \tag{12}$$

In other words, the nondispersive contact term with the "wrong" sign (11) is determined by IR physics at arbitrary large distances rather than UV physics, which can be erroneously assumed to be a source of $\delta^2(x)$ behavior in (11). Our computations in terms of the delocalized instantons (2) explicitly show that all observables in this system are originated from the IR physics.

Our final comment here is that the same contact term (9) and its local expression (11) can also be computed using the auxiliary ghost field, the so-called Kogut-Susskind (KS) ghost, as has been done originally in Ref. [20] (see also [19] for relevant discussions in the present context). This auxiliary KS ghost field provides the required "wrong" sign (9) as a consequence of the negative sign of the kinetic term in the corresponding effective Lagrangian [20]. This unphysical ghost field does not violate unitarity or any other important properties of the theory as a consequence of the Gupta-Bleuler–like condition on the physical Hilbert space [19,20]. This description in terms of the KS ghost implicitly takes into account the presence of topological sectors in the system. The same property is explicitly reflected by summation over topological sectors $k \in \mathbb{Z}$ in direct computations (4) and (6) without introducing any auxiliary fields.

It is interesting to note that a similar structure also emerges in other gauge theories, e.g., in the so-called "deformed QCD." In that case the topological sectors also produce the $\delta(x)$ function behavior for the topological susceptibility, similar to Eq. (11). Furthermore, this contact term in that model can also be described by a ghost, which turns out to be an auxiliary topological field described by the Chern-Simons topological action [21].

In the next section we shall see that a similar contact term with $\delta(x)$ function behavior also emerges in the four-dimensional Maxwell system defined on a compact manifold. Furthermore, this contact term in the four-dimensional system can be understood in terms of auxiliary topological fields in BF formulation as discussed in Sec. IV B. The corresponding auxiliary nonpropagating fields play the same role as KS ghost fields in two-dimensional QED [19,20] and topological Chern-Simons fields in "deformed QCD" as presented in [21].

C. Entropy in the two-dimensional Maxwell system

The partition function (1), (6), and (7) computed above allows us to compute the entropy of the system. However, before we proceed, we want to make a short historical detour on the entropy studies in this "empty" model.

It has been claimed [22] that using the so-called conical method, the black hole entropy is equal to the entropy of entanglement for spins zero and one-half fields (at least at one loop level). However, for gauge Maxwell field, the entropy has an extra term describing the contact interaction with the horizon. While the entropy is a positively defined entity, the Kabat contact term is negative [22]. Furthermore, this term being a total divergence can be represented as a surface integral determined by the behavior of the theory at arbitrarily large distances; i.e., it obviously has an infrared (IR) origin. More recently, it has been conjectured [19] that the Kabat contact term is originated from the same topological gauge sectors and tunneling transitions that cannot be associated with any physical degrees of freedom as there are none in the "empty" two-dimensional theory. The next step in this development was explicit demonstration [23] that the computation of the entropy in the two-dimensional Maxwell system is highly sensitive to the IR physics. Therefore, the IR regularization should be treated very carefully. Appropriate treatment has been suggested in [23] by defining the system in a large box size V. With this regularization the computation for the entropy can be easily performed, as the corresponding partition function for the two-dimensional Maxwell system defined in a box is known (1), (6), and (7). The conical entropy with this regularization coincides with conventional thermodynamical entropy since the volume of the Euclidean manifold is linear in the deficit angle,

$$S(V) = \frac{\partial}{\partial T}(T\ln \mathcal{Z}) = \left(1 - V\frac{\partial}{\partial V}\right)\ln \mathcal{Z}(V), \tag{13}$$

such that the final expression for $S(V)$ can be represented in the following form [23] (see also [24,25] with related discussions):

$$S(V) = \left(\ln \mathcal{Z}(V) + \frac{1}{2}\right) - \frac{1}{2}\left(\frac{4\pi^2}{e^2}\right)\chi(V). \tag{14}$$

In this formula, $\chi(V)$ is the topological susceptibility (8) and $\mathcal{Z}(V)$ is the partition function determined by (1), (6), and (7) at $\theta = 0$ (see Sec. II D for generalization on $\theta \neq 0$).

A few comments are in order. First, the entropy $S(V)\to 0$ approaches zero in the large volume limit $V \to \infty$ as it should since there are no propagating degrees of freedom present in the system. However, entropy vanishes in a quite nontrivial way: a conventional contribution from (14) $\sim \ln \mathcal{Z}$ approaches zero as $\mathcal{Z} \to 1$ according to (1).

At the same time, the negative contribution due to the topological term [which is determined by its asymptotic value $\chi \to e^2/4\pi^2$, see Eq. (9)] exactly cancels with the positive contribution $1/2$ originated from the quantum fluctuations. Second, one can explicitly see from (14) that the gauge-invariant negative contribution is indeed present in this expression for the entropy. This term with the "wrong" sign in Eq. (14) is exactly proportional to the topological susceptibility (8) in agreement with conjecture [19]. Third, this term can be represented as a surface integral because $Q = \frac{e}{2\pi}E$ entering (8) is the topological charge density operator which is a total divergence. Fourth, while the term $\sim\chi(V)$ in Eq. (14) can be represented as a surface integral, the entropy itself does not possess such a surface representation. Nevertheless, both these entities (S and χ) are separately gauge-invariant observables.

Furthermore, the entropy (14) is obviously a positively defined function at any finite V and can be interpreted as the entanglement entropy [23]. Indeed, the only local observable is E, which is constant over space. It implies that the measurements of E will be perfectly correlated on the opposite sides of the system. We interpret the same feature of entanglement in a different way. Our interpretation is based on Euclidean formulation of the system when the partition function (4) and (6) can be interpreted as a probability of tunneling events between different topological sectors $|k\rangle$ in volume V. Once a tunneling event happens, the corresponding boundary conditions of the gauge filed are entangled on opposite sides of the system as topological charge (2) is unambiguously determined by these boundary conditions.

Finally, we want to emphasize that the presence of the topologically protected term in Eq. (14) proportional to the topological susceptibility (8) is correlated with the fact that this "empty" system, in fact, is the topological quantum field theory. We shall see that a similar correlation also holds for the four-dimensional Maxwell system which also shows a number of other manifestations being typical for topologically ordered systems.

We conclude this subsection with the following short comment on terminology. The term "entanglement entropy" is normally used to describe the entangled properties of physical propagating degrees of freedom. Our entropy (14) in "empty" theory has fundamentally different nature as it does not correspond to any propagating states. Rather, entropy (14) results from topologically different but physically identical quantum winding states $|k\rangle$ and their dynamics (tunneling transitions between them). Therefore, it is more appropriate to coin this entropy as "homotopical entropy" as corresponding expression (14) is in fact a direct consequence of nontrivial homotopy of the gauge group $\pi_1[U(1)] = \mathbb{Z}$ in two-dimensional QED. It can only emerge in gauge systems with nontrivial topological features. In particular, it cannot occur in scalar field theories. To simplify terminology we shall refer to (14) as topological entropy[1] (TE). Our ultimate goal of this work is to compute the TE in four dimensional QED defined on the four torus, similar to Eq. (14) describing two-dimensional QED. As we shall see in Sec. III, the corresponding properties of the TE are fundamentally different from conventional thermodynamical entropy. In fact, the TE more resembles the topological entanglement entropy introduced in condensed matter literature to study the topologically ordered systems rather than conventional thermodynamical entropy describing propagating degrees of freedom.

D. Topological entropy at $\theta \neq 0$

Now we want to generalize the results of Sec. II C to include $\theta \neq 0$. We want to see how the system varies when $\theta \neq 0$ and how the topological entropy reflects the corresponding variations. This generalization for $\theta \neq 0$ will play an important role in our discussions of four-dimensional Maxwell system in Sec. III.

First of all, it has been known for many years [26] that a nonvanishing $\theta \neq 0$ is equivalent to the uniform electric field present in the system. The simplest way to see this in the Hamiltonian approach is to observe that $\theta \neq 0$ produces the shift of the electric field $n \to (n + \theta/2\pi)$ in formula (1), i.e., a nonvanishing θ corresponds to the background electric field

$$\langle E\rangle_{\rm Mink} = \frac{e\theta}{2\pi}. \tag{15}$$

The same result can be easily reproduced in Euclidean path integral approach by differentiating partition function $\mathcal{Z}(\theta, V)$ with respect to θ according to the definition (4) and (5). Assuming that $|\theta| < \pi$ ($\theta = \pi$ requires a special treatment, and will be presented at the end of the section) we get:

$$\langle iQ\rangle_{\rm Eucl} = \left\langle\frac{ieE}{2\pi}\right\rangle_{\rm Eucl} = \frac{\partial \ln \mathcal{Z}(\theta, V)}{V\partial\theta}\bigg|_{V\to\infty} = -\frac{\theta e^2}{4\pi^2}, \tag{16}$$

[1]We define the TE as conventional thermodynamical entropy (13) with the only difference is that the partition function $\mathcal{Z}$ in (13) describes the dynamics of degenerate winding states rather than the dynamics of real degrees of freedom. In two-dimensional QED this is the only dynamics which is present in the system such that TE identically coincides with the conventional thermodynamical entropy. In four-dimensional QED discussed in next section the conventional contribution $\sim\mathcal{Z}_0$ due to physical photons decouple from the topological contribution $\mathcal{Z}_{\rm top}$, i.e., $\mathcal{Z} = \mathcal{Z}_0 \times \mathcal{Z}_{\rm top}$. Therefore, the TE derived from $\mathcal{Z}_{\rm top}$ normally represents a small correction to conventional thermodynamical entropy (which is typically proportional to the volume of the system) derived from $\mathcal{Z}_0$. However, the main point of this work is that while the TE is a parametrically small portion in terms of a magnitude, it produces a topologically protected contribution to the entropy, similar to analogous feature of the topological entanglement entropy computed for topologically ordered condensed matter systems (see Sec. III C with more comments).

which coincides with (15) when written in Minkowski notations. As a consequence of nonzero background field (15) and (16) in the system the expression for the topological entropy for $\theta \neq 0$ is slightly modified in comparison with (14). Now it can be written in the following form

$$S(\theta, V) = \left(\ln \mathcal{Z}(\theta, V) + \frac{1}{2}\right) - \frac{1}{2}\left(\frac{4\pi^2}{e^2}\right)\chi(\theta, V) - \frac{1}{2}\left(\frac{4\pi^2}{e^2}\right) \cdot V \cdot \langle Q\rangle \cdot \langle Q\rangle, \tag{17}$$

where new term appears as a result of nonvanishing background field. In this formula $\langle Q\rangle$ and χ are defined as before [see Eqs. (9) and (16) in terms of first and second derivatives of $\ln \mathcal{Z}(\theta)$ with respect to θ correspondingly], i.e.,

$$\langle iQ\rangle = \frac{1}{V}\frac{\partial \ln \mathcal{Z}(\theta)}{\partial\theta}, \quad \chi(\theta, V) = -\frac{1}{V}\frac{\partial^2 \ln \mathcal{Z}(\theta)}{\partial\theta^2}. \tag{18}$$

The topological entropy (17) approaches zero in large volume limit as it should. But this vanishing result is realized quite differently when the background field (16) is present in the system. Indeed, the partition function (1) in large volume limit, up to exponentially small corrections, can be approximated as

$$\ln \mathcal{Z}(\theta, V \to \infty) \simeq -\frac{e^2 V\theta^2}{8\pi^2}, \tag{19}$$

which leads to the following asymptotic expressions for background field and topological susceptibility after differentiation according to definitions (18),

$$\langle Q\rangle_{\rm Eucl}(\theta, V \to \infty) = i\frac{\theta e^2}{4\pi^2}, \qquad \chi(\theta, V \to \infty) = \frac{e^2}{4\pi^2}. \tag{20}$$

By substituting (19) and (20) to Eq. (17) one can verify that the entropy indeed vanishes (with exception of a single degenerate point $\theta = \pi + 2\pi n$, see below) in infinite volume limit at arbitrary θ as it should. The entropy $S(\theta, V)$ approaches zero from above in this case as a result of cancellation between two terms proportional to the volume: conventional $\ln \mathcal{Z}$ contribution (19) cancels the background field contribution which is also proportional to the volume (17). It is instructive to represent the same formula (17) in somewhat different way

$$S(\theta, V) = \left(\ln \mathcal{Z}(\theta, V) + \frac{1}{2}\right) - \frac{1}{2}\left(\frac{4\pi^2}{e^2}\right) \times \int d^2x \langle Q^{\rm tot}(x), Q^{\rm tot}(0)\rangle, \tag{21}$$

where $Q^{\rm tot} = \langle Q\rangle + Q$ represents the total topological density operator including its background portion (16) proportional to the constant electric field. One should note that the correlation function which appears in (21) cannot be represented as second derivative of $\ln \mathcal{Z}(\theta)$ with respect to the θ as defined by Eq. (18). Instead, this term can be represented as second derivative of $\mathcal{Z}$ itself,

$$\int d^2x \langle Q^{\rm tot}(x), Q^{\rm tot}(0)\rangle = -\frac{1}{V\mathcal{Z}(\theta)}\frac{\partial^2 \mathcal{Z}(\theta)}{\partial\theta^2}. \tag{22}$$

To conclude this section we want to mention that the physics is perfectly 2π periodic with respect to θ as partition function (1) and (6) is obviously a 2π periodic function. In particular $\theta = 2\pi$ is identically the same as $\theta = 0$ state. At the same time at $\theta = \pi$ the double degenerate states appear in the system. This time this degeneracy corresponds to physical degeneracy when two distinct vacuum states have equal energies. One can see the emergence of this degeneracy by approaching $\theta = \pi$ point from opposite sides, i.e., $\theta = \pi \pm \epsilon$ with $\epsilon \to 0$. These two degenerate states are classified by different directions of the electric field characterizing the system as we computed above,

$$\langle Q\rangle_{\rm Eucl} = \pm\frac{i\pi e^2}{4\pi^2}, \qquad \langle E\rangle_{\rm Mink} = \pm\frac{e\pi}{2\pi} = \pm\frac{e}{2}. \tag{23}$$

One should mention that similar formulas for degenerate states with $\theta = \pi$ in this model were also recently discussed in [27] in context of topological insulators.

The expression for topological entropy (17) receives a crucial modification as partition function (1) has two identical terms for $\theta = \pi$ which correspond to the physical degeneracy mentioned above. Indeed, in large volume limit the partition function at $\theta = \pi \pm \epsilon$ can be approximated as

$$\mathcal{Z}(\theta = \pi \pm \epsilon) \simeq [e^{-\frac{e^2V}{2}(\frac{\pi\pm\epsilon}{2\pi})^2} + e^{-\frac{e^2V}{2}(1-\frac{\pi\pm\epsilon}{2\pi})^2}], \tag{24}$$

such that two terms contribute with equal weight at $\theta = \pi$. As a consequence, the entropy which can be easily computed by substituting the corresponding expressions to general formula (17) does not vanish at $\theta = \pi$, but rather assumes the following value:

$$S(\theta = \pi) = \ln 2. \tag{25}$$

The interpretation of this result is amazingly simple and straightforward. While our system is indeed "empty" and does not support any propagating degrees of freedom in the bulk, the ground state is in fact degenerate, and it is characterized by the vacuum expectation value of electric field (23). Furthermore, this degeneracy which leads to extra term in the topological entropy (25) is, in fact, a volume independent phenomenon. Indeed, one can easily see that every term in the partition function (1) has its identical partner at $\theta = \pi$ at arbitrary volume V, which eventually leads to extra factor $\ln 2$ in expression (25). Therefore, the emergence of nonvanishing entropy (25) in this "empty" system is obviously a pure topological effect which reflects the two-fold vacuum degeneracy (23) present in the system. Conventional thermodynamical entropy must vanish at $T = 0$ which is obviously not the case for TE in thermodynamical limit at zero temperature given by Eq. (25).

III. TOPOLOGICAL ENTROPY IN MAXWELL THEORY IN FOUR DIMENSIONS

The main goal of this section is to derive formulas for the topological entropy in Maxwell theory defined on the four-dimensional torus, similar to (14) and (17) derived for the two-dimensional system. The corresponding expressions will be entirely due to the tunneling events similar to analysis of "empty" two-dimensional system. The interpretation of the corresponding formulas will be also very similar to our discussions of the two-dimensional system. Namely, we will interpret the corresponding corrections to the entropy as topological entropy, similar to our discussions in Sec. II C, because the relevant topological configurations describe the tunneling events are uniquely determined by the properties of the topological configurations, similar to two-dimensional analysis.

The crucial difference with two-dimensional studies is, of course, that Maxwell theory in four dimensions describes real physical photons with two transverse polarizations in contrast with our studies of "empty" two-dimensional theory in Sec. II. However, as we discuss below the propagating degrees of freedom with nonzero momentum completely decouple from topological contributions such that the partition function can be represented in the form $\mathcal{Z} = \mathcal{Z}_0 \times \mathcal{Z}_{\rm top}$.

The conventional, topologically trivial portion of entropy related to $\mathcal{Z}_0$ is well known for the Maxwell system. It is an extensive entity and it produces the vanishing entropy for zero temperature. The entanglement entropy for Maxwell theory is also known (see recent paper [24] with many references on previous works therein). The corresponding contribution to the entanglement entropy is entirely determined by two transverse photon's polarizations and proportional to the area, similar to scalar field theories,

$$S \sim A(\Sigma)(d-2), \tag{26}$$

where $A(\Sigma)$ is the area of surface Σ and $(d-2)$ is the number of the on-shell physical propagating degrees of freedom in d dimensional space. This leading term is related to physical propagating degrees of freedom. The conventional thermodynamical entropy $S \sim VT^3$ and entanglement entropy (26) are *not* subject of the present work as they completely decouple from topological contributions related to $\mathcal{Z}_{\rm top}$, which is the main subject of our present studies. To avoid confusion with terminology we also emphasize that these entropies (conventional thermodynamical entropy as well as entanglement entropy) do not depend on topological $|k\rangle$ sectors of the system due to linearity of the Maxwell theory (see details below). Therefore, they cannot depend on $|k\rangle$, nor they can carry any information about topological features of the system.

In the rest of this section we will concentrate on behavior of $\mathcal{Z}_{\rm top}$ and the expression for TE which follows from $\mathcal{Z}_{\rm top}$. In other words, our goal here is to study the corrections to the thermodynamical entropy due to topological configurations describing the tunneling events, rather than contributions related to the physical propagating photons, similar to our analysis of the "empty" theory in Sec. II. For these computations we use the same definition (13) for the thermodynamical entropy we have been using before in our studies of the two dimensional system.

A. Topological partition function $\mathcal{Z}_{\rm top}$

Our goal here is to define the Maxwell system on a Euclidean four torus with sizes $L_1 \times L_2 \times L_3 \times \beta$ in the respective directions. It provides the IR regularization of the system. As we discussed in Sec. II C this IR regularization plays a key role in proper treatment of the contact topological term which is related to tunneling events rather than the propagation of the physical photons with transverse polarizations.

We follow [1] in our construction of the partition function $\mathcal{Z}_{\rm top}$ where it was employed for computation of the corrections to the Casimir effect due to these novel type of topological fluctuations. The crucial point is that we impose the periodic boundary conditions on gauge A^μ field up to a large gauge transformation. In what follows we simplify our analysis by considering a clear case with winding topological sectors $|k\rangle$ in the z direction only. The classical configuration in Euclidean space which describes the corresponding tunneling transitions can be represented as follows:

$$\vec{B}_{\rm top} = \vec{\nabla} \times \vec{A}_{\rm top} = \left(0, 0, \frac{2\pi k}{eL_1L_2}\right), \qquad \Phi = e\int dx_1 dx_2 B^z_{\rm top} = 2\pi k \tag{27}$$

in close analogy with the two-dimensional case (2).

The Euclidean action of the system is quadratic and has the following form

$$\frac{1}{2}\int d^4x\{\vec{E}^2 + (\vec{B} + \vec{B}_{\rm top})^2\}, \tag{28}$$

where $\vec{E}$ and $\vec{B}$ are the dynamical quantum fluctuations of the gauge field. The key point is that the classical topological portion of the action decouples from quantum fluctuations, such that the quantum fluctuations do not depend on topological sector k and can be computed in topologically trivial sector $k = 0$. Indeed, the cross term

$$\int d^4x \vec{B}\cdot\vec{B}_{\rm top} = \frac{2\pi k}{eL_1L_2}\int d^4x B_z = 0 \tag{29}$$

vanishes because the magnetic portion of quantum fluctuations in the z-direction, represented by $B_z = \partial_x A_y - \partial_y A_x$, is a periodic function as $\vec{A}$ is periodic over the domain of integration. This technical remark in fact greatly simplifies our analysis as the contribution of the physical propagating photons is not sensitive to the topological sectors k. This is, of course, a specific feature of quadratic action (28), in contrast with non-Abelian and nonlinear gauge field theories where quantum fluctuations of course depend on topological k sectors. The authors of Ref. [27] arrived to the same conclusion (on decoupling of the topological terms from conventional fluctuating photons with nonzero momentum), though in a different context of topological insulators in the presence of the $\theta = \pi$ term.

The classical action for configuration (27) takes the form

$$\frac{1}{2}\int d^4x\vec{B}^2_{\rm top} = \frac{2\pi^2k^2\beta L_3}{e^2L_1L_2}. \tag{30}$$

To simplify our analysis further in computing $\mathcal{Z}_{\rm top}$ we consider a geometry where L_1, $L_2 \gg L_3$, β similar to construction relevant for the Casimir effect [1]. In this case our system is closely related to two-dimensional Maxwell theory by dimensional reduction: taking a slice of the four-dimensional system in the xy plane will yield precisely the topological features of the two-dimensional torus considered in Sec. II. Furthermore, with this geometry our simplification (27) when we consider exclusively the magnetic fluxes in z direction is justified as the corresponding classical action (30) assumes a minimal possible values.[2] With this assumption we can consider very small temperature, but still we cannot take a formal limit $\beta \to \infty$ in our final expressions as a result of our technical constraints in the system.

With these additional simplifications the topological partition function becomes [1]:

$$\mathcal{Z}_{\rm top} = \sqrt{\frac{2\pi\beta L_3}{e^2L_1L_2}}\sum_{k\in\mathbb{Z}} e^{-\frac{2\pi^2k^2\beta L_3}{e^2L_1L_2}}, \tag{31}$$

which is essentially the dimensionally reduced expression for the topological partition function (6) for the two-dimensional Maxwell theory analyzed in Sec. II.

One should note that the dimensional reduction which is employed here is not the most generic one. In fact, one can impose a nontrivial boundary condition on every slice in the four-dimensional torus (see comments in footnote 2 and Ref. [15] for most generic construction). However, the main goal of this work is not a generic classification. Rather, we wish to discuss the contact term, topological entropy, degeneracy and other nontrivial features by considering the simplest possible setup (27) when physics can be easily understood and analyzed. In other words, we wish to consider a nontrivial BC imposed on a single slice, while keeping the trivial periodic BC for other slices.

We follow [1] and introduce the dimensionless parameter

$$\tau \equiv 2\beta L_3/e^2L_1L_2 \tag{32}$$

such that the partition function $\mathcal{Z}_{\rm top}$ can be written in the dual form:

$$\mathcal{Z}_{\rm top}(\tau) = \sqrt{\pi\tau}\sum_{k\in\mathbb{Z}} e^{-\pi^2\tau k^2} = \sum_{n\in\mathbb{Z}} e^{-\frac{n^2}{\tau}}, \tag{33}$$

where the Poisson summation formula (7) is used again. Our normalization of the partition function $\mathcal{Z}_{\rm top}$ is such that in the limit $L_1L_2 \to \infty(\tau \to 0)$ the topological portion of the partition function $\mathcal{Z}_{\rm top} \to 1$ as one can see from the dual representation (33). In this limit the dimensional reduction is justified and we recover the conventional physics which is encoded in $\mathcal{Z}_0$. This is a result of the same decoupling of the topological transitions from physics related to propagating photons, as we discussed above when $\mathcal{Z} = \mathcal{Z}_0 \times \mathcal{Z}_{\rm top}$ and $\mathcal{Z}_{\rm top} \to 1$ in this limit. One should note that the normalization factor $\sqrt{\pi\tau}$ which appears in Eq. (31) does not depend on topological sector k, and essentially it represents our convention of the normalization $\mathcal{Z}_{\rm top} \to 1$ in the limit $L_1L_2 \to \infty$.

B. Topological entropy and magnetic susceptibility in the four-dimensional Maxwell system

We are in position now to compute the TE associated with $\mathcal{Z}_{\rm top}$. As we mentioned previously, we shall use the same definition for the entropy we used previously in two-dimensional studies (13), i.e.,

$$\begin{aligned} S_{\rm top}(\tau) &= \left(1 - \beta\frac{\partial}{\partial\beta}\right)\ln \mathcal{Z}_{\rm top}(\tau) \\ &= \ln \mathcal{Z}_{\rm top}(\tau) - \frac{1}{\tau}\cdot\frac{1}{\mathcal{Z}_{\rm top}(\tau)}\cdot\sum_{n\in\mathbb{Z}} n^2 e^{-\frac{n^2}{\tau}}, \end{aligned} \tag{34}$$

[2] There are also electric fluxes Φ_E in the system in description of the Euclidean path integral. The corresponding electric fluxes are originated from the requirement that the electromagnetic potential $\vec{A}$ satisfies the periodic boundary conditions up to the large gauge transformations, i.e., $A_3(\beta) = A_3(0) + \frac{2\pi l}{eL_3}$ which corresponds to the electric flux with uniform electric field $E_3 = \frac{2\pi l}{e\beta L_3}$. The Euclidean action for corresponding configurations is parametrically $(L_1L_2/\beta L_3)$ larger than the magnetic classical action (30), and therefore it is consistently neglected in our analysis. One should also note that the electric field in the Euclidean classical action must not be confused with electric field in the Hamiltonian formulation where it is a constant of motion [see, e.g., [27,28] where the electric fluxes emerge in Hamiltonian description in quite different context (the $U(1)$ spin liquid and topological insulators correspondingly)]. The corresponding electric fields in these two descriptions (Euclidean path integral approach vs Hamiltonian approach) are in fact related by the duality transformation, similar to the two-dimensional analysis in Sec. II where electric field in Hamiltonian formulation enters formula (1), while electric flux in Euclidean path integral formulation enters Eqs. (2) and (6) [see [1] for more details and references].

where we use the dual representation (33) for the partition function $Z_{\rm top}(\tau)$.

Our next step is to represent the second term in (34) in a form, similar to expression for the entropy (14) in terms of the topological susceptibility in two dimensional theory. To achieve this goal we formally introduce $\theta_{\rm eff}$ into the dual representation (33) for the partition function as follows

$$Z_{\rm top}(\tau, \theta_{\rm eff}) = \sum_{n\in\mathbb{Z}} e^{-\frac{n^2}{\tau}+in\theta_{\rm eff}}. \quad (35)$$

One should emphasize that the $\theta_{\rm eff}$ parameter introduced in (35) is not a fundamental θ parameter normally introduced into the Lagrangian in front of $\vec{E}\cdot\vec{B}$ operator. Furthermore, integer number n which appears in front of $\theta_{\rm eff}$ in (35) is not the magnetic flux k defined by Eq. (27) which enters original partition function (31). Rather, integer n appears in the dual form (33) for this partition function. Nevertheless, as we discuss in Sec. III D the parameter $\theta_{\rm eff}$ has a perfect physical meaning related to the external magnetic flux through the xy-plane applied to the system,

$$\theta_{\rm eff} = B_z^{\rm ext} L_1 L_2 e. \quad (36)$$

Now, the second term in (34) can be formally represented as the second derivative of $Z_{\rm top}(\tau, \theta_{\rm eff})$ with respect to $\theta_{\rm eff}$. Indeed, using identity

$$\frac{1}{Z_{\rm top}(\tau)}\cdot\sum_{n\in\mathbb{Z}} n^2 e^{-\frac{n^2}{\tau}} = -\frac{\partial^2 \ln Z_{\rm top}(\tau,\theta_{\rm eff})}{\partial\theta_{\rm eff}^2}\bigg|_{\theta_{\rm eff}=0} \quad (37)$$

one can rewrite the expression for entropy (34) in the following form

$$S_{\rm top}(\tau) = \ln Z_{\rm top}(\tau) - \frac{1}{2}\chi_{\rm mag}(\tau),$$
$$\chi_{\rm mag}(\tau) \equiv -\frac{2}{\tau}\frac{\partial^2 \ln Z_{\rm top}(\tau,\theta_{\rm eff})}{\partial\theta_{\rm eff}^2}\bigg|_{\theta_{\rm eff}=0}. \quad (38)$$

Significance of this representation is that $\chi_{\rm mag}(\tau)$ entering the expression (38) has many features similar to the topological susceptibility entering (14) in two dimensional theory. In particular, $\chi_{\rm mag}(\tau)$ can be represented as a surface integral and it assumes a finite value in the thermodynamical limit. Furthermore, $\chi_{\rm mag}(\tau)$ is not sensitive to any specific details in the bulk of the system, nor its boundary's geometrical shape. Rather it is only sensitive to the boundary conditions which globally classify the topological sectors of the system.

Furthermore, as we shall see in a moment $\chi_{\rm mag}(\tau)$ is in fact the conventional magnetic susceptibility which measures response of the free energy to the introduction of arbitrary small external magnetic field. This is because the formal parameter $\theta_{\rm eff}$ entering (35) is related to the physical external field $B_z^{\rm ext}$ as Eq. (36) states. As a result of this relation, the differentiation of $\ln Z_{\rm top}(\tau, \theta_{\rm eff})$ with respect to $\theta_{\rm eff}$ is equivalent to differentiation with respect to external field $B_z^{\rm ext}$ which is, by definition, the conventional magnetic susceptibility,

$$\chi_{\rm mag}(\tau) = -\frac{2}{\tau}\frac{\partial^2 \ln Z_{\rm top}(\tau,\theta_{\rm eff})}{\partial\theta_{\rm eff}^2}\bigg|_{\theta_{\rm eff}=0} = -\frac{1}{\beta V}\frac{\partial^2 \ln Z_{\rm top}}{\partial B_{\rm ext}^2}\bigg|_{B_{\rm ext}=0} = \int d^4x \langle B_z(x), B_z(0)\rangle. \quad (39)$$

Representation (39) for $\chi_{\rm mag}(\tau)$ entering the expression for the entropy (38) obviously implies that this term is a total divergence, as $B^i = \epsilon^{ijk}\partial_j A_k$. In other words, this topological contribution to the entropy is determined by the behavior of the gauge fields at arbitrary large distances, similar to our studies of the entropy in Sec. II in two-dimensional "empty" theory with relation (14) being a precise analog of Eq. (38).

The crucial difference between these two cases is, of course, that the four-dimensional Maxwell system describes real physical massless photons, in contrast with "empty" two-dimensional theory. However, the topological contribution, which is main subject of the present work, behaves in Maxwell theory formulated on the four-dimensional torus very much in the same way as in the two-dimensional case. Furthermore, the interpretation of these topological terms in the four-dimensional theory is also very much the same as in the two-dimensional case. To be more specific, we interpret (38) as TE (which is a subleading contribution to the thermodynamical entropy) resulting from tunneling processes. This term is always much smaller than the leading term $S \sim VT^3$ originating from conventional propagating physical photons with two transverse polarizations.

To get some feeling on numerical (un)importance of the topological terms (38) in comparison with conventional leading term we consider τ parameter defined by Eq. (32) to be very large $\tau \gg 1$ assuming the thermodynamical limit at very low (but nonvanishing) temperature.[3] In this case

$$Z_{\rm top}(\tau \gg 1) \to \sqrt{\pi\tau}, \quad (40)$$

while the topological entropy assumes the following asymptotic value

$$S_{\rm top}(\tau \gg 1) \to \left[\frac{1}{2}\ln(\pi\tau) - \frac{1}{2}\right]. \quad (41)$$

The magnetic susceptibility asymptotically approaches unity in this limit,

[3]One should note that large $\tau \gg 1$ is consistent with our "technical" simplification related to the dimensional reduction employed in (31). In particular, $\tau \gg 1$ can always be arranged by considering very small $e \to 0$ in Eq. (32) before considering the dimensional reduction employed in (31). Still, we cannot put $\tau \sim \beta = \infty$ because our simplified computations based on dimensional reduction require $L_1L_2 \gg \beta L_3$.

$$\chi_{\mathrm{mag}}(\tau \gg 1) \to 1. \quad (42)$$

It is very instructive to compare the behavior (41) with similar formula (14) in the two-dimensional case. In both cases the topologically protected nondispersive contact contribution (a contribution which cannot be expressed in terms of physical propagating degrees of freedom) approaches one and the same constant $(-1/2)$. In the two-dimensional case this contact term is related to the topological susceptibility (9) while in the four-dimensional case, the contact terms is formulated in terms of the magnetic susceptibility (39). In the two-dimensional case this term cancels with another positive contribution as the total entropy is determined by one and the same partition function $\mathcal{Z}$ (see Sec. II C). In the four-dimensional case, the same factor $(-1/2)$ remains untouched and stays in Eq. (41) as it represents only the topological portion of the entropy, not the total entropy. One can argue that the total entropy in the limit $\beta \to \infty$ also vanishes [15], similar to the two-dimensional case. However, in the four-dimensional case it vanishes as a result of cancellation of the topological term (41) with the conventional contribution computed on the four torus and related to the physical propagating photons described by $\mathcal{Z}_0$.

One can argue that χ_{mag} from (42) is saturated by a nondispersive contact term which cannot be associated with any physical propagating degrees of freedom, similar to the two-dimensional expression (11). To be more precise, the integrand for χ_{mag} is expected to have the following structure

$$\langle B_z(x), B_z(0)\rangle = \frac{\delta^2(x)}{L_3\beta}, \quad \chi_{\mathrm{mag}} = \int_{\mathbb{T}^4} \frac{\delta^2(x) d^4x}{L_3\beta} = 1, \quad (43)$$

where $\delta^2(x)$ should be understood as the discretized version of the delta function defined on the torus

$$\delta^2(x) = \frac{1}{L_1 L_2}\sum_{n_1 n_2} e^{2\pi i(\frac{n_1 x_1}{L_1} + \frac{n_2 x_2}{L_2})}. \quad (44)$$

Our argument supporting $\delta^2(x)$ function behavior in Eq. (43) is based on observation that the magnetic susceptibility has nondispersive nature. Indeed, χ_{mag} is derived from topological partition function $\mathcal{Z}_{\mathrm{top}}(\tau)$ which completely decouples from $\mathcal{Z}_0$ describing propagating photons with physical transverse polarizations. Therefore, any nonvanishing correlation function, including (43) must be expressed in terms of a contact term with structure (43) similar to the contact term (11) in the two-dimensional system. Explicit computations in terms of the auxiliary fields using the so called "BF" formulation in Sec. IV B also supports the structure (43). Furthermore, one can argue that the nondispersive contact term (43) is related to the IR physics at large distances rather than UV physics, in close analogy to our discussions of the two-dimensional case (12).

Does it make any sense to keep this subleading term $(-1/2)$ in Eq. (41) in the presence of much greater conventional contribution $S \sim VT^3$ and $\ln(\pi\tau)$ also entering (41)? Our ultimate answer is "yes" as this constant factor proportional to the magnetic susceptibility $-1/2\chi_{\mathrm{mag}}$ has some universal topological properties as we shall argue below.

C. Similarities and differences between TE and topological entanglement entropy in CM systems

Before we proceed with our arguments we want to make a short detour on topological entanglement entropy in the three-dimensional as well as in the four-dimensional cases in condensed matter (CM) systems.[4] The main purpose for this detour is to present some analogies and similarities between these two very different entities: TE discussed in last section vs topological entanglement entropy introduced in Refs. [9,10].

The entanglement entropy in arbitrary number of dimensions is proportional to the surface area of the boundary. In the four-dimensional case it corresponds to the area law (26). In the three-dimensional system, the leading term is proportional to the length L. We shall not discuss this leading term in the present work. A portion of the entanglement entropy which is important for our discussions is in fact a subleading term which may emerge in some systems. A well-known example is a three-dimensional system in a topologically ordered phase, where the first subleading term is universal constant, the so-called TEE and independent of the size or shape of L as argued in Refs. [9,10], i.e.,

$$S = aL - \ln \mathcal{D}, \quad (45)$$

where a is a nonuniversal and cutoff-dependent coefficient, while $\mathcal{D}$ is the so-called total quantum dimension of the topological phase, and it is universal constant [9,10]. In other words, any small variations of the system do not change $\mathcal{D}$. It has been argued that the presence of such term is potentially very useful probe of a topological phase (see, e.g., original papers [2–6] and recent reviews [7,8,29] with large number of original references therein). Similar studies in four dimensions had received much less attention in the past. Still, it is known that a constant term similar to $\ln \mathcal{D}$ in (45) can appear in a four-dimensional system even for a nontopologically ordered phase. However, when the system is defined on flat space-time (e.g., has the topology of a torus, which is precisely our case) the constant term in

[4] not to be confused with conventional CM notations, where it is a customary to count the spatial number of dimensions, rather than total number of dimensions, such that our four-dimensional system corresponds to $(D+1)$ Maxwell theory with $D = 3$ in CM notations.

the entropy would signal that the system is in fact in topologically ordered phase [30].

After this short detour we return to our Maxwell system in four dimensions. The topological part of the system characterized by partition function (31) and (33) obviously describes some sort of entanglement, similar to the two-dimensional "empty" theory as discussed in Sec. II and Ref. [23]. This is because the fluxes that saturate the partition function (31) and (33) are constant over space, which means that the measurement of the field will be perfectly correlated on the opposite sides of the system, similar to arguments of Ref. [23] presented for two-dimensional "empty" theory. However, this is not a conventional entanglement describing physical propagating degrees of freedom in CM systems. Rather, our system is formulated in terms of "instantons" (instead of propagating quasiparticles in CM systems) in Euclidean space-time with action (30). These pseudoparticles saturate the topological portion of the partition function (31) and (33). Such topological fluctuations occur even when no propagating degrees of freedom exist in the system as the two-dimensional example from Sec. II shows.

As a result of this difference we cannot use many standard tools which normally would detect the topological order. For example, we cannot compute the braiding phases of charges and vortices which are normally used in CM systems simply because our system does not support such kind of excitations. Furthermore, the "degeneracy" in our system is related to degenerate of winding states $|k\rangle$ which are connected to each other by large gauge transformation, and therefore must be identified as they correspond to the same physical state. It is very different from conventional term "degeneracy" in topologically ordered CM systems when *distinct* degenerate states are present in the system as a result of formulation of a theory on a topologically nontrivial manifold such as torus. Finally, our system supports conventional massless photons with physical polarizations, in contrast with conventional topologically ordered phases characterized by a gap. However, these massless degrees of freedom completely decouple from our topological fluctuations according to Eq. (29). Formally, this decoupling is expressed as $\mathcal{Z} = \mathcal{Z}_0 \times \mathcal{Z}_{\rm top}$ as discussed after Eq. (29). Therefore, these massless physical photons can be completely ignored in our discussions of the topological properties of the partition function $\mathcal{Z}_{\rm top}$. In this respect it is very similar to topologically ordered superconductors [5] when massless phonons always exist in the system, but nevertheless, they completely decouple from relevant dynamics, and can be ignored in discussions of the topological features of the model.

In spite of the differences mentioned above, it is very instructive to compare the topological portions of the TE given by coefficient $-1/2$ in Eq. (41) for Maxwell system and $-\ln \mathcal{D}$ in Eq. (45) for CM system. In both cases these terms are topologically protected, i.e., they are not sensitive to any specific details in the bulk of the system, nor the boundary's geometrical shapes. Rather these terms are determined by the global properties of the systems. Indeed, in the case of the four-dimensional Maxwell system, the coefficient $-1/2$ in Eq. (41) is expressed in terms of the magnetic susceptibility (42) which itself is represented by a surface integral, while in CM systems the topological protection was advocated in Refs. [9,10]. Furthermore, in both cases these topological contributions, being the subleading terms, have a negative sign in comparison with the leading terms.

Therefore, the TE behaves very much in the same way as topological entanglement entropy does, though TE cannot be interpreted in terms of the quantum dimension $\mathcal{D}$ entering (45) as in CM systems. This is because, as we already mentioned, the quasiparticles which can propagate simply do not exist in this Euclidean Maxwell system. Nevertheless, if we equalize $-\ln \mathcal{D}$ from Eq. (45) and $-1/2$ from (41), we arrive to a formal relation

$$- \ln \mathcal{D} = - \frac{\chi_{\rm mag}(\tau \gg 1)}{2} = - \frac{1}{2} \to \mathcal{D} = \sqrt{e}, \qquad (46)$$

which should be compared with conventional $\mathcal{D} = \sqrt{m}$ for a Laughlin state in a fractional quantum Hall system with filling factor $\nu = 1/m$, or $\mathcal{D} = 2$ for $p + ip$ superconductor, or any other similar systems (see recent reviews [7,8] for the details and original references). The emergence of the exponential function $e^{1/2}$ in (46) instead of $m^{1/2}$ hints that TE in our system is originated from tunneling transitions rather than from dynamics of quasiparticles in the system described by quantum dimensions $\mathcal{D}$. This interpretation is obviously consistent with our construction of the partition function $\mathcal{Z}_{\rm top}(\tau)$ describing the tunneling events between topologically different but physically identical $|k\rangle$ states. Still, these different entities $-\ln \mathcal{D}$ in Eq. (45) and $-\frac{1}{2}\chi_{\rm mag}(\tau)$ in (38) in very different systems behave very similarly under small variations of the systems, which justifies our comparison in form of equalizing these two different things represented by Eq. (46).

The main message of this subsection is as follows. We observe that a subleading correction $-1/2\chi_{\rm mag}(\tau)$ in (38) to thermodynamical entropy $S \sim VT^3$ is topologically protected in the same way as TEE is protected in CM systems (45). Therefore, this subleading term might be signalling that our system behaves as a topologically ordered CM system.[5] Furthermore, our system demonstrates a property of physical degeneracy of the ground state when $\theta_{\rm eff} = \pi$, similar to the two-dimensional case (23), as we shall discuss below. The emergence of such degeneracy in a system

[5]This is in spite of the fact that our system of course supports massless photons in contrast with fully gapped CM systems. However, as explained above the conventional massless degrees of freedom completely decouple from topological contributions as Eq. (29) states.

is a typical manifestation of a topological order in CM systems. In Sec. IV B we will reformulate the same Maxwell system in terms of the so-called "BF" action. A similar "BF" structure in CM systems is known to describe a large distance behavior in a topologically ordered phases. Therefore, such a BF representation of the Maxwell system in Sec. IV B is an additional argument supporting our claim that the Maxwell system defined on a compact torus belongs to a topologically ordered phase.

D. Topological entropy in the background of a magnetic field

In this section we want to generalize our results on TE for Euclidean Maxwell system in the presence of the external magnetic field. Normally, in the conventional quantization of electromagnetic fields in Minkowski space, there is no *direct* coupling between fluctuating vacuum photons and an external magnetic field as a consequence of linearity of the Maxwell system. The coupling with fermions generates a negligible effect $\sim\alpha^2 B_{\rm ext}^2/m_e^4$ as the nonlinear Euler-Heisenberg Effective Lagrangian suggests (see [1] for the details). The interaction of the external magnetic field with topological fluctuations (27), in contrast with coupling with conventional photons, will lead to the effects of order of unity as a result of interference of the external magnetic field with fluxes-instantons.

The corresponding partition function can be easily constructed for external magnetic field $B_z^{\rm ext}$ pointing along z direction, as the crucial technical element on decoupling of the background fields from quantum fluctuations assumes the same form (29). In other words, the physical propagating photons with nonvanishing momenta are not sensitive to the topological k sectors, nor to the external uniform magnetic field, similar to our discussions after (29).

The classical action for configuration in the presence of the uniform external magnetic field $B_z^{\rm ext}$ therefore takes the form

$$\frac{1}{2}\int d^4x(\vec{B}_{\rm ext}+\vec{B}_{\rm top})^2=\pi^2\tau\left(k+\frac{\theta_{\rm eff}}{2\pi}\right)^2, \qquad (47)$$

where τ is defined by (32) and the effective theta parameter $\theta_{\rm eff}$ is expressed in terms of the original external magnetic field (36). Therefore, the partition function in the presence of the uniform magnetic field can be reconstructed from (31) and it is given by [1],

$$\mathcal{Z}_{\rm top}(\tau,\theta_{\rm eff})=\sqrt{\pi\tau}\sum_{k\in\mathbb{Z}}\exp\left[-\pi^2\tau\left(k+\frac{\theta_{\rm eff}}{2\pi}\right)^2\right]. \qquad (48)$$

The dual representation for this partition function is obtained by applying the Poisson summation formula (7),

$$\begin{aligned}\mathcal{Z}_{\rm top}(\tau,\theta_{\rm eff})&=\sqrt{\pi\tau}\sum_{k\in\mathbb{Z}}\exp\left[-\pi^2\tau\left(k+\frac{\theta_{\rm eff}}{2\pi}\right)^2\right]\\&=\sum_{n\in\mathbb{Z}}\exp\left[-\frac{n^2}{\tau}+in\cdot\theta_{\rm eff}\right],\end{aligned} \qquad (49)$$

which justifies our notation for the effective theta parameter $\theta_{\rm eff}$ as it enters the partition function in combination with integer number n. One should emphasize that integer number n in the dual representation (49) is not the integer magnetic flux k defined by Eq. (27) which enters original partition function (31). Furthermore, the $\theta_{\rm eff}$ parameter which enters (48) and (49) is not a fundamental θ parameter which is normally introduced into the Lagrangian in front of $\vec{E}\cdot\vec{B}$ operator. Rather, this parameter $\theta_{\rm eff}$ should be understood as an effective parameter representing the construction of the $\theta_{\rm eff}$ state for each slice in four dimensional system. In fact, there are three such $\theta_{\rm eff}$ parameters representing different slices and corresponding external magnetic fluxes. There are similar three θ_i parameters representing the external electric fluxes [15]. This problem of classification shall not be elaborated in the present work, as our goal here is to understand and analyze the simplest possible topological configurations. We leave the corresponding classification problem which would include a combination of different BC imposed on different slices for future studies.

Now we are in position to compute the TE for our system in case of nonvanishing external field. The corresponding generalization of formula (38) is given by

$$\begin{aligned}S_{\rm top}(\tau,\theta_{\rm eff})&=\ln\mathcal{Z}_{\rm top}(\tau,\theta_{\rm eff})-\frac{1}{2}\chi_{\rm mag}(\tau,\theta_{\rm eff})\\&\quad+\frac{V\beta}{2}\langle B_{\rm ind}(\tau,\theta_{\rm eff})\rangle^2,\end{aligned} \qquad (50)$$

where $V\equiv L_1L_2L_3$ is the three volume of the system and $\langle B_{\rm ind}(\tau,\theta_{\rm eff})\rangle$ is the induced magnetic field defined as follows [1]:

$$\begin{aligned}&\langle B_{\rm ind}(\tau,\theta_{\rm eff})\rangle\\&\quad=-\frac{1}{\beta V}\frac{\partial\ln\mathcal{Z}_{\rm top}(\tau,\theta_{\rm eff})}{\partial B_{\rm ext}}\\&\quad=\frac{\sqrt{\tau\pi}}{\mathcal{Z}_{\rm top}}\sum_{k\in\mathbb{Z}}\left(B_{\rm ext}+\frac{2\pi k}{L_1L_2e}\right)\exp\left[-\tau\pi^2\left(k+\frac{\theta_{\rm eff}}{2\pi}\right)^2\right].\end{aligned} \qquad (51)$$

Magnetic susceptibility $\chi_{\rm mag}(\tau,\theta_{\rm eff})$ in (50) is defined similarly to Eq. (38) with the only difference being that one should keep $\theta_{\rm eff}\neq 0$ after taking the derivatives, i.e.,

$$\chi_{\rm mag}(\tau,\theta_{\rm eff})\equiv-\frac{2}{\tau}\frac{\partial^2\ln\mathcal{Z}_{\rm top}(\tau,\theta_{\rm eff})}{\partial\theta_{\rm eff}^2}. \qquad (52)$$

One can see from (51) that our definition of the induced field accounts for the total field which includes both terms:

the external part as well as the induced topological portion of the field. In the absence of the external field when $B_{\rm ext} = 0$, the series is antisymmetric under $k \to -k$ and $\langle B_{\rm ind}(\theta_{\rm eff} = 0)\rangle$ vanishes. This feature is similar to the vanishing expectation value of the topological density (16) in two-dimensional gauge theory when $\theta = 0$. One could anticipate this result from the very beginning as the theory must respect $\mathcal{P}$ invariance at $\theta_{\rm eff} = 0$, and therefore $\langle B_{\rm ind}\rangle$ must vanish at $\theta_{\rm eff} = 0$.

Now it is easy compute all the ingredients which enter the expression for entropy (50) at nonvanishing external field in large $\tau \gg 1$ limit similar to our computations (41) and (42). The corresponding asymptotic expressions for $\theta_{\rm eff} \neq 0$ with exponential accuracy can be represented as follows,

$$\ln \mathcal{Z}_{\rm top}(\tau \gg 1, \theta_{\rm eff}) \to \frac{1}{2}\ln(\pi\tau) - \pi^2\tau\left(\frac{\theta_{\rm eff}}{2\pi}\right)^2$$
$$\langle B_{\rm ind}(\tau \gg 1, \theta_{\rm eff})\rangle \to \frac{\theta_{\rm eff}}{eL_1L_2} \qquad \chi_{\rm mag}(\tau \gg 1, \theta_{\rm eff}) \to 1, \tag{53}$$

where we assume that $|\theta_{\rm eff}| < \pi$. The degenerate case $\theta_{\rm eff} = \pi$ requires a special treatment, similar to the two-dimensional analysis presented in Sec. II D, and will be discussed at the very end of this section. We substitute (53) to general expression for the entropy (50) to arrive at

$$S_{\rm top}(\tau \gg 1, \theta_{\rm eff}) \to \left[\frac{1}{2}\ln(\pi\tau) - \frac{1}{2}\right], \tag{54}$$

where $-1/2$ in Eq. (54) is due to the topologically protected $\chi_{\rm mag}$ similar to previous formula (41) derived for vanishing background field, $\theta_{\rm eff} = 0$.

The expression (54) for TE at asymptotically large τ is independent on $\theta_{\rm eff}$, similar to our previous studies in two dimensional "empty" gauge theory in Sec. II D. It implies that the topologically protected contribution in Eq. (54) assumes exactly the same value (46) independently of a magnitude of the external field. Therefore, we interpret $-1/2$ in Eq. (54) as topological entropy similar to our discussions leading to Eq. (46). We claim that the relation (46) holds even in the presence of external field when $\theta_{\rm eff} \neq 0$.

E. Degeneracy at $\theta_{\rm eff} = \pi$

In this subsection we want to analyze a special but important case with $\theta_{\rm eff} = \pi$ when the system becomes degenerate. This case is very similar to our previously studied system of the two-dimensional "empty" theory discussed at the end of Sec. II D. The crucial element is that our system is 2π periodic as explicit expression for the partition function (49) shows. At the same time the point $\theta_{\rm eff} = \pi$ requires a special treatment as the system shows two-fold degeneracy at this point. Indeed, the partition function in vicinity $\theta_{\rm eff} \simeq \pi \pm \epsilon$ at large τ can be approximated as

$$\mathcal{Z}_{\rm top}(\tau \gg 1, \theta_{\rm eff} = \pi \pm \epsilon) = \sqrt{\pi\tau}\left[e^{-\pi^2\tau(\frac{\pi\pm\epsilon}{2\pi})^2} + e^{-\pi^2\tau(1-\frac{\pi\pm\epsilon}{2\pi})^2}\right], \tag{55}$$

where we keep only two leading terms at large τ. One can explicitly see that these two terms identically coincide when $\theta_{\rm eff} = \pi$ which implies the degeneracy of the system. These two degenerate states are classified by different directions of the induced magnetic field characterizing the system. Indeed, with exponential accuracy one gets

$$\langle B_{\rm ind}(\tau \gg 1, \theta_{\rm eff} = \pi - \epsilon)\rangle = \frac{\pi}{eL_1L_2}$$
$$\langle B_{\rm ind}(\tau \gg 1, \theta_{\rm eff} = \pi + \epsilon)\rangle = -\frac{\pi}{eL_1L_2}, \tag{56}$$

where two terms in (56) are originated from different terms in Eq. (55) by approaching $\theta_{\rm eff} = \pi$ from different sides. The effect of degeneracy is very similar in spirit to our studies of two dimensional "empty" theory at the end of Sec. II D. Using an explicit expression for the partition function (55) in vicinity $\theta_{\rm eff} \simeq \pi$ at large τ one can arrive to the following additional term to the entropy (54) at $\theta_{\rm eff} = \pi$,

$$\Delta S_{\rm top}(\tau \gg 1, \theta_{\rm eff} = \pi) = \ln 2. \tag{57}$$

A few comments are in order. First, we should emphasize that the fact of degeneracy itself does not actually depend on magnitude of τ, though the expression (57) is computed in the limit of large $\tau \gg 1$. Indeed, from initial formula (48) one can explicitly see that for $\theta_{\rm eff} = \pi$ an each term with given positive k has its partner $(-k-1)$ which produces an identical contribution to $\mathcal{Z}_{\rm top}(\tau, \theta_{\rm eff} = \pi)$. This obviously implies the emergence of degeneracy in the system at $\theta_{\rm eff} = \pi$ which is reflected by (56) and (57). Second, in the limit $L_1L_2 \to \infty$ in Eq. (56) the expectation values of the local operator in these degenerate states are the same (they vanish, $\langle B_{\rm ind}\rangle = 0$). A proper interpretation in this limit should be formulated in terms of total flux determined by the global behavior, rather than by local expectation values (56):

$$\frac{eL_1L_2}{2\pi}\langle B_{\rm ind}\rangle_+ = \left\langle \frac{e}{2\pi}\oint A_i dx_i\right\rangle_{\theta_{\rm eff}=\pi-\epsilon} = +\frac{1}{2}$$
$$\frac{eL_1L_2}{2\pi}\langle B_{\rm ind}\rangle_- = \left\langle \frac{e}{2\pi}\oint A_i dx_i\right\rangle_{\theta_{\rm eff}=\pi+\epsilon} = -\frac{1}{2}. \tag{58}$$

In other words, these degenerate states cannot be distinguished locally; they are classified by the global characteristics (58), similar to topologically ordered CM systems [2–8]. Third, the entropy (57) does not vanish in thermodynamical limit in the presence of the external magnetic field at $\theta_{\rm eff} = \pi$ as a reflection of the degeneracy of the ground state (58). This feature is a direct analog of the degeneracy discussed in Ref. [15] in the presence of the electric fluxes. As we already mentioned after Eq. (49) one should expect six different θ's parameters corresponding three different magnetic fluxes and three different electric fluxes. There will be extra degeneracy (and extra $\ln 2$ contribution to the entropy) when each flux assumes $1/2$ of its value.

Finally, a similar degeneracy does not occur for another $\mathcal{CP}$ even state with $\theta_{\mathrm{eff}} = 0$. Indeed, while each contribution with positive $k \neq 0$ has its partner with negative $-k$ in Eq. (48) there is a unique single term with $k = 0$ which does not have its partner. This single term prevents the degeneracy to occur in the system with $\theta_{\mathrm{eff}} = 0$.

IV. "BF" FORMULATION OF THE MAXWELL SYSTEM

In the previous section we presented a number of arguments suggesting that the Maxwell system defined on a compact manifold behaves very much in the same way as a topologically ordered system. The arguments include the analysis of such "signatures" of a topological phase as the degeneracy and the topologically protected finite correction to the entropy. Still, it would be highly desirable to describe the same system in more conventional way in terms of auxiliary fields governed by the topological Chern-Simons action. In this case our claim (that the Maxwell system defined on a compact manifold is a topologically ordered system) would be less puzzling and mysterious notion. We should remark here that our system supports conventional massless photons with physical polarizations, in apparent contrast with conventional description of topologically ordered systems which normally are characterized by a gap. However, as we already mentioned the massless photons with physical transverse polarizations completely decouple from our topological fluctuations described by the topological portion of the partition function $\mathcal{Z}_{\mathrm{top}}$ according to Eq. (29). Therefore, these massless photons can be completely ignored in our studies of the topological properties of the partition function, similar to decoupling of the massless phonons (which are in fact responsible for the mere existence of a gap) in treatment of the topologically ordered superconductors [5].

A. Partition function in "BF" formulation

We wish to derive the topological action for the Maxwell system by using the same conventional technique exploited e.g., in [21] for the so-called "deformed QCD" and in [5] for the Higgs model. Our starting point is to insert the delta function into the path integral with the field $b^z(\mathbf{x})$ acting as a Lagrange multiplier

$$\delta[B^z(\mathbf{x}) - \epsilon^{zjk}\partial_j a_k(\mathbf{x})] \sim \int \mathcal{D}[b_z] e^{iL_3\beta \int d^2\mathbf{x}\, b_z(\mathbf{x})\cdot[B^z(\mathbf{x}) - \epsilon^{zjk}\partial_j a_k(\mathbf{x})]}, \quad (59)$$

where $B^z(\mathbf{x})$ in this formula is treated as the original expression for the field operator entering the action (28), including all classical k-instanton configurations (27) and (30) and quantum fluctuations surrounding these classical configurations. In other words, we treat $B^z(\mathbf{x})$ as fast degrees of freedom. At the same time $a_k(\mathbf{x})$ is treated as a slow-varying external source effectively describing the large distance physics for a given instanton configuration. Our task now is to integrate out the original fast "instantons" and describe the large distance physics in terms of slow varying fields $b_z(\mathbf{x})$, $a_k(\mathbf{x})$ in form of the effective action. We use the same procedure by summation over k instantons as before which is expressed in terms of the partition function (31). The only new element in comparison with the previous computations is that the fast degrees of freedom must be integrated out in the presence of the new slow varying background fields $b_z(\mathbf{x})$, $a_k(\mathbf{x})$ which appear in Eq. (59). Fortunately, the computations can be easily performed if one notices that the background field $b_z(\mathbf{x})$ enters Eq. (59) exactly in the same manner as external magnetic field enters (48). Therefore, assuming that $b_z(\mathbf{x})$, $a_k(\mathbf{x})$ are slow varying background fields we arrive to the following expression for the partition function:

$$\mathcal{Z}_{\mathrm{top}} = \sqrt{\pi\tau} \int \mathcal{D}[b_z]\mathcal{D}[a] e^{-\pi^2\tau\cdot \int \frac{d^2\mathbf{x}}{L_1 L_2}(\frac{\phi(\mathbf{x})}{2\pi})^2 - S_{\mathrm{top}}}, \quad (60)$$

where $\phi(\mathbf{x}) \equiv eL_1L_2 b_z(\mathbf{x})$ represents the slow varying background auxiliary b_z field which is assumed to lie in the lowest $k = 0$ branch, $|\phi(\mathbf{x})| < \pi$. Correspondingly, in formula (60) we kept only asymptotically leading term with $k = 0$ in the series (48) at large $\tau \gg 1$. The topological term $S_{\mathrm{top}}[b_z, a_k]$ in Eq. (60) reads

$$S_{\mathrm{top}}[b_z, a_k] = iL_3\beta \int d^2\mathbf{x}[b_z(\mathbf{x})\epsilon^{zjk}\partial_j a_k(\mathbf{x})]. \quad (61)$$

Our observation here is as follows. The topological term (61) which emerges as an effective description of our system is in fact a Chern-Simons–like topological action. In our simplified setting we limited ourself by considering the fluxes-instantons along z direction only. It is naturally to assume that a more general construction would include fluxes-instantons in all three directions which leads to a generalization of action (61). It is quite natural to expect that the action in this case would assume a Chern-Simons–like form $i\beta \int d^3\mathbf{x}[\epsilon^{ijk} b_i(\mathbf{x})\partial_j a_k(\mathbf{x})]$ which replaces (61). A similar structure in CM systems is known to describe a topologically ordered phase. Therefore, it is not really a surprise that we found in Sec. III some signatures of the topological phases (such as degeneracy and topological entropy) in the Maxwell system defined on a compact manifold. The emergence of the topological Chern-Simons action (61) further supports our basic claim that the Maxwell system on a compact manifold belongs to a topologically ordered phase.

B. Magnetic susceptibility in "BF" formulation

Our goal here is to consider a simplest application of the effective low energy topological action constructed above

(60). To be more specific, we want to reproduce our expression for the magnetic susceptibility (39) and (42) by integrating out the b_z and a_k fields using low energy effective description (60):

$$\langle B_z(x), B_z(0)\rangle = \frac{1}{\mathcal{Z}}\int \mathcal{D}[b_z]\mathcal{D}[a]e^{-S_{\rm tot}[b_z,a_k]} \cdot [\epsilon^{zjk}\partial_j a_k(\mathbf{x}), \epsilon^{zj'k'}\partial_{j'}a_{k'}(\mathbf{0})], \quad (62)$$

where $S_{\rm tot}[b_z, a_k]$ determines the dynamics of auxiliary b_z and a_k fields, and it is given by

$$S_{\rm tot} = L_3\beta\int d^2\mathbf{x}\left[\frac{1}{2}b_z^2(\mathbf{x}) + ib_z(\mathbf{x})\epsilon^{zjk}\partial_j a_k(\mathbf{x})\right]. \quad (63)$$

The obtained Gaussian integral (62) over $\int \mathcal{D}[b_z]$ can be explicitly executed, and we are left with the following integral over $\int \mathcal{D}[a]$

$$\langle B_z(x), B_z(0)\rangle = \frac{1}{\mathcal{Z}}\int \mathcal{D}[a]e^{-\frac{L_3\beta}{2}\int d^2\mathbf{x}[\epsilon^{zjk}\partial_j a_k(\mathbf{x})]^2} \cdot [\epsilon^{zjk}\partial_j a_k(\mathbf{x}), \epsilon^{zj'k'}\partial_{j'}a_{k'}(\mathbf{0})]. \quad (64)$$

The integral (64) is also gaussian and can be explicitly evaluated with the following final result

$$\langle B_z(x), B_z(0)\rangle = \frac{1}{\beta L_3}\delta^2(\mathbf{x}). \quad (65)$$

Formula (65) precisely reproduces our previous expression (43) derived by explicit summation over fluxes-instantons, and without even mentioning any auxiliary topological fields $b_z(\mathbf{x})$, $a_k(\mathbf{x})$. It obviously demonstrates a self-consistency of our formal manipulations with auxiliary topological fields.

Few comments are in order. First of all, the expression (43) and (65) for the magnetic susceptibility represents the contact nondispersive term which cannot be associated with any physical propagating degrees of freedom as it has a "wrong sign," similar to our discussions for two-dimensional QED (10). The nature of this contact term is very much the same as in two-dimensional QED (11), and it results from tunneling transitions between topologically different but physically identical states. As we mentioned in Sec. II B, this contact term in two-dimensional QED can also be understood in terms of KS ghost [19,20]. Secondly, this term is responsible for the topologically protected contribution to the entropy as Eq. (38) states, and serves as a signal of a topologically ordered phase. Finally, as we already mentioned in Sec. II B an analogous construction also emerges in "deformed QCD" [21] where the auxiliary topological fields, similar in spirit to $b_z(\mathbf{x})$, $a_k(\mathbf{x})$ fields from (63), and which saturate the "wrong sign" in topological susceptibility can be identified with the so-called Veneziano ghost. Our observation here is that in all considered cases the presence of a nonvanishing contact term in expression for the entropy and some manifestations of a topologically ordered phase are somehow related. A deep understanding for such a correlation is still lacking.

V. CONCLUSION

Before we formulate the main results of this work, we want to make few general comments on connection with other related studies.

A. Connection with other related studies

In this work we discussed a number of very unusual effects in Maxwell theory formulated on a compact manifold such as the four torus. All these effects are originated from the topological portion of the partition function $\mathcal{Z}_{\rm top}(\tau, \theta_{\rm eff})$ and cannot be formulated in terms of conventional $E\&M$ propagating photons with two physical polarizations. In fact, a strong hint that something is missing in attempt to describe everything in terms of the propagating degrees of freedom (d.o.f.) comes from study of the "empty" two-dimensional gauge theory discussed in Sec. II when the system cannot support any physical propagating d.o.f. Still, all physical effects relevant for this work are already present in the two-dimensional system. The same comment also holds for the four-dimensional Maxwell theory when photons with two physical polarizations are present in the system. However, their contribution completely decouple from topological effects studied in the present work.

The source of these unusual effects is as follows. When the Maxwell system is quantized on a compact manifold one cannot completely remove all unphysical degrees of freedom from the system as it would result in emergence of the so-called Gribov's ambiguities [11] (see recent paper [15] and also some previous relevant discussions [12–14]). These ambiguities were originally discussed for non-Abelian gauge theories in Minkowski space when one tries to completely remove all unphysical degrees of freedom in the Coulomb gauge [11], but similar ambiguities also emerge in Abelian Maxwell theory defined on a nontrivial manifold [12–14].

In the present work we opted to keep some gauge freedom in our analysis. The corresponding construction is implemented by allowing the boundary conditions to be periodic up to large gauge transformations, which are precisely reflected by the presence of the "instantons" (27) interpolating between topologically different, but physically identical, pure gauge configurations. The same topological construction has been used previously in four dimensions in study of the topological Casimir effect [1] where it has been claimed that there is an additional contribution to the Casimir force in Maxwell theory which cannot be accounted for by conventional propagating photons with two physical polarizations. Similar in spirit computations were also carried out in [21] in weakly coupled "deformed QCD" where fractionally charged monopoles-instantons describe the tunneling events between topologically different, but physically identical winding states. In this case one can also argue that the gapped "deformed QCD" belongs to a topologically ordered phase, related to the "degeneracy" of these winding states.

These computations imply that an extra energy (and entropy), not associated with any physical propagating degrees of freedom, may appear in some gauge systems. The extra energy in all these cases emerges as a result of dynamics of pure gauge configurations at very large distances. This unique feature of the system when extra energy is not related to any physical propagating degrees of freedom was the main motivation for a proposal [19,31] that the observed dark energy in the Universe may have precisely such nondispersive nature. Essentially, the proposal [19,31] identifies the observed dark energy with the Casimir type energy, which however is originated not from dynamics of the physical propagating degrees of freedom, but rather, from the dynamics of the topological sectors which are always present in gauge systems. A de Sitter behavior of the Universe in this case can be formulated in terms of the auxiliary topological fields which are similar in spirit to $b_z(\mathbf{x})$, $a_k(\mathbf{x})$ fields from (63) and which effectively describe the dynamics of the topological sectors in the expanding background [32]. It would be very exciting if this new type of energy not associated with propagating particles could be experimentally measured in a laboratory as suggested in [1]. Furthermore, one could argue that these finite contributions (to the entropy and to the energy), not related to any propagating degrees of freedom cannot be removed by any means such as subtraction or redefinition of observables, (see Appendix of Ref. [1] with corresponding arguments).

B. Main results

In this work we have discussed two novel (topologically protected) contributions to the thermodynamical entropy. The analysis of these additional contributions to the entropy, not related to the physical propagating photons, represents the main result of the present work. There are two types of extra terms to the entropy which are related to each other as they both originated from topological portion of the partition function $\mathcal{Z}_{\rm top}(\tau, \theta_{\rm eff})$, but nevertheless the physical meaning of these contributions are very different.

The contribution of the first type always enters the entropy with the negative sign. This term is expressed in terms of the topologically protected magnetic susceptibility $\chi_{\rm mag}$, can be represented as the infrared-sensitive surface integral, and emerges even in the absence of external fields as discussed in Sec. III C. Such contributions are similar in spirit to the topological entanglement entropy in condensed matter systems, which are normally expressed in terms of the quantum dimension $\mathcal{D}$.

The arguments that such nondispersive contact contributions to the entropy may emerge in the gauge systems have been discussed in the literature long ago (see original Ref. [22] and a few comments and references on the recent development at the beginning of Sec. II C). These contributions to the entropy are obviously very unique. It is important to emphasize that these terms do not contradict to any fundamental principles as we discussed in this work. In particular, the total entropy is always positive function, and the ground-state entropy vanishes in the thermodynamical limit (outside of the degeneracy points, see next paragraph). Nonetheless, the presence of these contact nondispersive terms expressed in terms of the infrared-sensitive surface integrals signalling that some kind of long range order, not related to propagating degrees of freedom (irrespectively whether they are massless or massive) may emerge in the system.

The contribution to the entropy of the second type is also related to the topological portion of the partition function $\mathcal{Z}_{\rm top}(\tau, \theta_{\rm eff})$. However, these terms emerge only in the presence of external field represented by $\theta_{\rm eff} = \pi$. In this case this system becomes degenerate as discussed in Sec. III E and, as a result of this degeneracy the entropy receives an additional topologically protected contribution with the positive sign as Eq. (57) states (see also footnote 2 on generalization of this construction when the external electric fluxes along with magnetic fluxes are included). This degeneracy can be interpreted as a result of a spontaneous symmetry breaking of $\mathcal{P}$ parity when the induced magnetic field may choose one of two possible directions. The effect can be observed only globally (58) rather than locally.

Such behavior of the system should be contrasted with conventional picture when physical photons do not directly couple to an external field, especially when it is rather small, within a mG range which corresponds to $L_1 \sim L_2 \sim 10^{-2}$ cm in Eq. (36). The coupling with physical photons occurs only through the fermion loop, which leads to enormous suppression for all effects normally expressed in terms of a nonlinear effective Lagrangian. At the same time the coupling of such small external fields with "instanton-fluxes" is always of order of unity even in absence of any charged fermions. Eventually, it results to the large effects such as emergence of the degeneracy in the system (57) and (58).

Another result of this work is an explicit demonstration that the topological portion of the partition function $\mathcal{Z}_{\rm top}(\tau, \theta_{\rm eff})$ can be reformulated in conventional terms using the Chern-Simons–like effective description in Sec. IV B. We reproduced the contact term in magnetic susceptibility using this effective description. Such a formulation once again supports our main claim that the Maxwell system formulated on the four torus belongs to a topologically ordered phase.

It remains to be seen if the system discussed in the present work can be used as a platform for quantum computations similar to the previous well-known suggestion [33] as there are many formal similarities between our system and topologically ordered condensed matter systems (see Sec. III C with some comments on these analogies). The crucial question is, of course, if one can manipulate with topological "fluxes-instantons" from Sec. III in the same way as with real quasiparticles in CM systems using the external magnetic field $\sim\theta_{\mathrm{eff}}$ from Eq. (36), or varying the boundary conditions. The point is that such manipulations may become efficient only for sufficiently small systems as one should deal with a magnitude of a single flux, as numerical estimates for the topological Casimir effect show [1]. Another way to manipulate the topological "fluxes-instantons" is to couple the topological fields with real propagating quasiparticles. This coupling is likely to create the so-called "edge states," which is a typical manifestation of topological phases in CM physics. We leave all these interesting subjects for the future studies.

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