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Spontaneously generated inhomogeneous phases via holography

James Alsup,[1,*] Eleftherios Papantonopoulos,[2,†] George Siopsis,[3,‡] and Kubra Yeter[3,§]

[1]*Engineering and Physics Department, Computer Science, The University of Michigan-Flint, Flint, Michigan 48502-1907, USA*
[2]*Department of Physics, National Technical University of Athens, Zografou Campus GR 157 73, Athens, Greece*
[3]*Department of Physics and Astronomy, The University of Tennessee, Knoxville, Tennessee 37996-1200, USA*

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We discuss a holographic model consisting of a $U(1)$ gauge field and a scalar field coupled to a charged AdS black hole under a spatially homogeneous chemical potential. By turning on a higher-derivative interaction term between the $U(1)$ gauge field and the scalar field, a spatially dependent profile of the scalar field is generated spontaneously. We calculate the critical temperature at which the transition to the inhomogeneous phase occurs for various values of the parameters of the system. We solve the equations of motion below the critical temperature, and show that the dual gauge theory on the boundary spontaneously develops a spatially inhomogeneous charge density.

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I. INTRODUCTION

There has been considerable recent activity studying phenomena at strong coupling using a weakly coupled dual gravity description. The tool to carry out such a study is the gauge/gravity duality. This holographic principle [1] has many applications in string theory, where it is well founded, but it has also been applied to other physical systems encountered in condensed matter physics. One of the most extensively studied condensed matter systems using the gauge/gravity duality is the holographic superconductor (for a review see [2]).

The gravity dual of a homogeneous superconductor consists of a system with a black hole and a charged scalar field. The black hole admits scalar hair at temperatures lower than a critical temperature [3], while there is no scalar hair at higher temperatures. According to the holographic principle, this breaking of the Abelian $U(1)$ symmetry corresponds in the boundary theory to a scalar operator which condenses at a critical temperature dependent on the charge density of the scalar potential. The fluctuations of the vector potential give the frequency dependent conductivity in the boundary theory [4]. Backreaction effects on the metric were studied in [5]. In [6] an exact gravity dual of a gapless superconductor was discussed in which the charged scalar field responsible for the condensation was an exact solution of the equations of motion, and below a critical temperature dressed a vacuum black hole with scalar hair.

Apart from holographic applications to conventional homogeneous superconductors, extensions to unconventional superconductors characterized by higher critical temperatures, such as cuprates and iron pnictides, have also been studied. Interesting new features of these systems include competing orders related to the breaking of lattice symmetries introducing inhomogeneities. A study of the effect on the pairing interaction in a weakly coupled BCS system was performed in [7]. Additionally, numerical studies of Hubbard models [8,9] suggest that inhomogeneity might play a role in high-T_c superconductivity.

The recent discovery of transport anomalies in $La_{2-x}Ba_xCuO_4$ might be explained under the assumption that the cuprate is a superconductor with a unidirectional charge density wave, i.e., a "striped" superconductor [10]. Other studies using mean-field theory have also shown that, unlike the homogeneous superconductor, the striped superconductor exhibits the existence of a Fermi surface in the ordered phase [11,12] and possesses complex sensitivity to quenched disorder [10]. Holographic striped superconductors were discussed in [13] by introducing a modulated chemical potential producing superconducting stripes below a critical temperature. Properties of the striped superconductors and backreaction effects were studied in [14,15]. Striped phases breaking parity and time-reversal invariance were found in electrically charged AdS-Reissner-Nordström black branes with neutral pseudoscalars [16]. In [17], it was shown that similar phases could be generated in Einstein-Maxwell-dilaton theories that leave parity and time-reversal invariance intact.

Inhomogeneities also appear in condensed matter systems other than superconductors. These systems are characterized by additional ordered states which compete or coexist with superconductivity [18,19]. The most important of these are charge and spin density waves (CDW and SDW, respectively) [20]. The development of these states corresponds to spontaneous modulation of the electronic charge and spin density, below a critical temperature T_c. Density waves are widely spread among different classes of materials. One may distinguish between types either orbitally [21], Zeeman driven [22], field-induced CDWs, confined [23], and even unconventional density waves [24].

The usual approach to study the effect of inhomogeneity at strong coupling is to introduce a modulated chemical

*jalsup@umflint.edu
†lpapa@central.ntua.gr
‡siopsis@tennessee.edu
§kyeter@tennessee.edu

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potential. According to the holographic principle this is translated into a modulated boundary value for the electrostatic potential in the AdS black hole gravity background. The corresponding Einstein-Maxwell-scalar systems can be obtained which below a critical temperature undergo a phase transition to a condensate with a nonvanishing modulation. Depending on what symmetries are broken, the modulated condensate gives rise to ordered states like CDW or SDW in the boundary theory [13,25].

To explore the properties of spatial inhomogeneities in holographic superfluids, gravitational backgrounds which are not spatially homogeneous were introduced in [26–29]. In [30] the breaking of the translational invariance is sourced by a scalar field with a nontrivial profile in the x direction. Upon perturbing the one-dimensional "lattice," the Einstein-Maxwell-scalar field equations were numerically solved at first order and the optical conductivity was calculated. Further properties of this construction were studied in [31].

In this work, we study a holographic superfluid in which a spatially inhomogeneous phase is spontaneously generated. The gravity sector consists of an AdS-Reissner-Nordström black hole, an electromagnetic field, and a scalar field. We introduce high-derivative interaction terms between the electromagnetic field and the scalar field. These higher-order terms are essential in spontaneously generating the inhomogeneous phase in the boundary theory. Alternative approaches for spontaneously breaking translational symmetries have been found by use of an interaction with the Einstein tensor [32,33], a Chern-Simons interaction [34,35], and more recently with a dilaton [17].

We put the gravitational background on a one-dimensional "lattice" generated by an x-dependent profile of the scalar field. At the onset of the condensation of the scalar field, we calculate the transition temperature. We find that as the wave number of the scalar field increases starting from zero (homogeneous profile), the transition temperature increases, showing that inhomogeneous configurations dominate at higher temperatures. We find a maximum transition temperature corresponding to a certain finite wave number. This is the critical temperature (T_c) of our system. Below T_c the system undergoes a second-order phase transition to an inhomogeneous phase. This occurs in a range of parameters of the system that we discuss.

We then solve the equations of motion below the critical temperature. We use perturbation theory to expand the bulk fields right below T_c, thus obtaining an analytic solution to the coupled system of Einstein-Maxwell-scalar field equations at first order. We find that a spatially inhomogeneous charge density is spontaneously generated in the boundary theory.

The paper is organized as follows. In Sec. II, we present the basic setup of the holographic model, and introduce the higher-derivative couplings. In Sec. III, we discuss the instability to a spatially inhomogeneous phase. We calculate numerically the critical temperature of the system, and analyze its dependence on the various parameters of the system. In Sec. IV, we use perturbation theory to obtain an analytic solution below the critical temperature, and show that the charge density in the boundary theory is spatially inhomogeneous. Finally, in Sec. V, we present our conclusions.

II. THE SETUP

In this section we introduce a holographic model whose main feature is the spontaneous generation of spatially inhomogeneous phases in the boundary theory. This cures the main deficiency of an earlier proposal [25]. This is achieved by introducing higher-derivative coupling of the electromagnetic field to the scalar field.

Consider a system consisting of a $U(1)$ gauge field, A_μ, with corresponding field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and a scalar field ϕ with charge q under the $U(1)$ group. The fields live in a spacetime of negative cosmological constant $\Lambda = -6/L^2$.

The action is given by

$$S = \int d^4x\sqrt{-g}\mathcal{L},$$
$$\mathcal{L} = \frac{R + 6/L^2}{16\pi G} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - (D_\mu\phi)^* D^\mu\phi - m^2|\phi|^2, \tag{1}$$

where $D_\mu\phi = \partial_\mu\phi - iqA_\mu\phi$. For simplicity, we shall set $16\pi G = L = 1$.

Our main concern is to generate spatially inhomogeneous phases in the boundary theory. To this end, we may introduce higher-derivative interaction terms of the form

$$\mathcal{L}_{\text{int}} = \phi^*[\eta\mathcal{G}^{\mu\nu}D_\mu D_\nu + \eta'\mathcal{H}^{\mu\nu\rho\sigma}D_\mu D_\nu D_\rho D_\sigma + \cdots]\phi + \text{c.c.}, \tag{2}$$

which may arise from quantum corrections. The possible operators in the above expression and their emergence from string theory are worth exploring. Candidates for $\mathcal{G}_{\mu\nu}$ include contributions to the stress-energy tensor form the electromagnetic field and the scalar field, the Einstein tensor [32,33], etc., and similarly for $\mathcal{H}_{\mu\nu\rho\sigma}$, etc. Here we shall be content with a special choice which leads to inhomogeneities,

$$\mathcal{L}_{\text{int}} = \eta\mathcal{G}^{\mu\nu}(D_\mu\phi)^* D_\nu\phi - \eta'|D_\mu\mathcal{G}^{\mu\nu}D_\nu\phi|^2, \tag{3}$$

where

$$\mathcal{G}_{\mu\nu} = T^{(\text{EM})}_{\mu\nu} + g_{\mu\nu}\mathcal{L}^{(\text{EM})} = F_{\mu\rho}F_\nu{}^\rho - \frac{1}{2}g_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}, \tag{4}$$

coupling the scalar field ϕ to the gauge field. This coupling is the essential tool for the generation of spatial inhomogeneities.

From the action (1), together with the interaction term (3), we obtain the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - 3g_{\mu\nu} = \frac{1}{2}T_{\mu\nu}, \tag{5}$$

where $T_{\mu\nu}$ is the stress-energy tensor,

$$T_{\mu\nu} = T^{(\mathrm{EM})}_{\mu\nu} + T^{(\phi)}_{\mu\nu} + \eta\Theta_{\mu\nu} + \eta'\Theta'_{\mu\nu}, \tag{6}$$

containing gauge, scalar, and interaction term contributions, respectively,

$$\begin{aligned} T^{(\mathrm{EM})}_{\mu\nu} &= F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}g_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}, \\ T^{(\phi)}_{\mu\nu} &= (D_\mu\phi)^*D_\nu\phi + D_\mu\phi(D_\nu\phi)^* \\ &\quad - g_{\mu\nu}(D_\alpha\phi)^*D^\alpha\phi - m^2g_{\mu\nu}|\phi|^2, \\ \Theta_{\mu\nu} &= \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int d^4x\sqrt{-g}\mathcal{G}^{\mu\nu}(D_\mu\phi)^*D_\nu\phi, \\ \Theta'_{\mu\nu} &= -\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int d^4x\sqrt{-g}|D_\mu\mathcal{G}^{\mu\nu}D_\nu\phi|^2. \end{aligned} \tag{7}$$

Varying the Lagrangian with respect to A_μ we find the Maxwell equations

$$\nabla_\mu F^{\mu\nu} = J^\nu, \tag{8}$$

where J_μ is the current,

$$J_\mu = qJ^{(\phi)}_\mu + \eta\mathcal{J}_\mu + \eta'\mathcal{J}'_\mu, \tag{9}$$

containing scalar and interaction term contributions, respectively,

$$\begin{aligned} J_\mu &= i[\phi^*D_\mu\phi - (D_\mu\phi)^*\phi], \\ \mathcal{J}_\mu &= \frac{1}{\sqrt{-g}}\frac{\delta}{\delta A^\mu}\int d^4x\sqrt{-g}\mathcal{G}^{\mu\nu}(D_\mu\phi)^*D_\nu\phi, \\ \mathcal{J}'_\mu &= -\frac{1}{\sqrt{-g}}\frac{\delta}{\delta A^\mu}\int d^4x\sqrt{-g}|D_\mu\mathcal{G}^{\mu\nu}D_\nu\phi|^2. \end{aligned} \tag{10}$$

Finally, the equation of motion for the scalar field is

$$\begin{aligned} D_\mu D^\mu\phi - m^2\phi &= \eta D_\mu(\mathcal{G}^{\mu\nu}D_\nu\phi) \\ &\quad + \eta' D_\rho(\mathcal{G}^{\mu\rho}D_\mu(D_\nu(\mathcal{G}^{\nu\sigma}D_\sigma\phi))). \end{aligned} \tag{11}$$

Our aim is to study the Einstein-Maxwell-scalar system of equations first at the critical temperature, and then below the critical temperature using perturbation theory.

III. THE CRITICAL TEMPERATURE

At the critical temperature, we have $\phi = 0$. The Einstein-Maxwell system has a static solution with metric of the form

$$ds^2 = \frac{1}{z^2}\left[-h(z)dt^2 + \frac{dz^2}{h(z)} + dx^2 + dy^2\right]. \tag{12}$$

The system possesses a scaling symmetry. The arbitrary scale is often taken to be the radius of the horizon. It is convenient to fix the scale by using a radial coordinate z so the horizon is at $z = 1$. Since the scale has been fixed, we should only be reporting on scale-invariant quantities.

The Maxwell equations admit the solution

$$A_t = \mu(1 - z), \tag{13}$$

so that the $U(1)$ gauge field has an electric field in the z direction equal to the chemical potential, $E_z = \mu$.

The Einstein equations are then solved by

$$h(z) = 1 - \left(1 + \frac{\mu^2}{4}\right)z^3 + \frac{\mu^2}{4}z^4. \tag{14}$$

The temperature is given as

$$\frac{T_c}{\mu} = -\frac{h'(1)}{4\pi\mu} = \frac{3}{4\pi\mu}\left(1 - \frac{\mu^2}{12}\right), \tag{15}$$

where we divided by μ to create a scale-invariant quantity.

Additionally, at the critical temperature the scalar field satisfies the wave equation,

$$\begin{aligned} &\partial_z^2\phi + \left[\frac{h'}{h} - \frac{2}{z}\right]\partial_z\phi + \frac{1}{h}(1 - \eta\mu^2z^4 - \eta'\mu^4z^{10}\nabla_2^2)\nabla_2^2\phi \\ &\quad - \frac{1}{h}\left[\frac{m^2}{z^2} - q^2\frac{A_t^2}{h}\right]\phi = 0, \end{aligned} \tag{16}$$

where $\nabla_2^2 = \partial_x^2 + \partial_y^2$, and we fixed the gauge so that ϕ is real.

The wave equation (16) can be solved by separating variables,

$$\phi(z, x, y) = \Phi(z)Y(x, y), \tag{17}$$

where Y is an eigenfunction of the two-dimensional Laplacian,

$$\nabla_2^2 Y = -\tau Y. \tag{18}$$

We will keep translation invariance in the y direction, and concentrate on the one-dimensional "lattice" defined by

$$Y = \cos(kx), \tag{19}$$

with $\tau = k^2$, and leave the two-dimensional lattices for future study.

The radial function $\Phi(z)$ satisfies the wave equation

$$\begin{aligned} &\Phi'' + \left[\frac{h'}{h} - \frac{2}{z}\right]\Phi' + \frac{\tau}{h}[1 - \eta\mu^2z^4 - \eta'\tau\mu^4z^{10}]\Phi \\ &\quad - \frac{1}{h}\left[\frac{m^2}{z^2} - q^2\frac{A_t^2}{h}\right]\Phi = 0. \end{aligned} \tag{20}$$

The asymptotic behavior (as $z \to 0$) is $\Phi \sim z^{\Delta}$, where $\Delta(\Delta - 3) = m^2$. It is convenient to write

$$\Phi(z) = \frac{\langle \mathcal{O}_\Delta \rangle}{\sqrt{2}} z^{\Delta} F(z), \quad F(0) = 1. \tag{21}$$

For general Δ, we obtain

$$F'' + \left[\frac{2(\Delta - 1)}{z} + \frac{h'}{h}\right]F' + \left[-\frac{\tau(1 - \eta\mu^2 z^4 - \eta'\tau\mu^4 z^{10})}{h} + \frac{q^2 A_t^2}{h^2} + \frac{m^2}{z^2}\left(1 - \frac{1}{h}\right) + \frac{\Delta h'}{zh}\right]F = 0. \tag{22}$$

The maximum transition temperature of the system can be calculated by solving (22) numerically, and using the expression (15) for the temperature. The maximum transition temperature is the critical temperature T_c of the system. Figure 1 shows the numerically calculated transition temperature without the higher-derivative couplings ($\frac{\eta}{\mu^2} = \frac{\eta'}{\mu^4} = 0$) dependent on the wave number k. Both quantities are divided by the chemical potential μ to render them dimensionless. The maximum value is found at $k = 0$, which shows that the homogeneous solution is dominant. When the higher-derivative interaction terms are turned on, the critical temperature T_c of the system also depends on the coupling constants η, η', and the homogeneous solution no longer dominates.

In the limit of vanishing second higher-derivative coupling, $\eta' \to 0$, we may analytically calculate the asymptotic critical temperature. The latter is found in the limit in which the wave number diverges ($\tau \to \infty$). In this limit, the wave equation (22) is dominated by the term proportional to τ near the horizon ($z \to 1$), thus giving the critical value for the chemical potential as $\mu_c^4 = \frac{\mu^2}{\eta}$, and corresponding temperature (15)

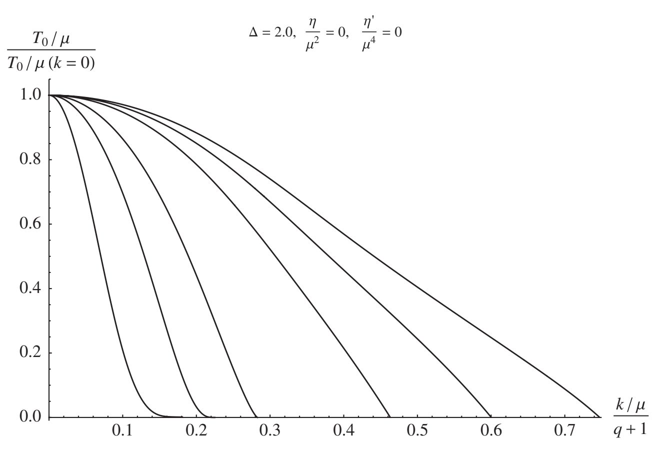

FIG. 1. Dependence of the transition temperature T_0/μ on the wave number k/μ for $\Delta = 2$ in the absence of higher-derivative interactions ($\frac{\eta}{\mu^2} = \frac{\eta'}{\mu^4} = 0$). From left to right the lines correspond to $q = 0$, 1, 1.5, 3, 5, 10. The maximum transition temperature (i.e., the critical temperature) occurs at $k = 0$ and the homogeneous configuration is dominant.

$$\lim_{\tau/\mu^2 \to \infty} \frac{T_0}{\mu} = \frac{3}{4\pi}\left(\frac{\eta}{\mu^2}\right)^{1/4}\left(1 - \frac{1}{12\sqrt{\frac{\eta}{\mu^2}}}\right). \tag{23}$$

The critical temperature for a standard Einstein-Maxwell-scalar system, i.e., $\frac{\eta}{\mu^2} = \frac{\eta'}{\mu^4} = 0$, with a neutral scalar was calculated in [36]. For $\Delta = 2$, it was found that $\frac{T_c}{\mu} \approx 0.00009$. For η large enough, the asymptotic ($\tau/\mu^2 \to \infty$) transition temperature will be higher than that of the homogeneous solution. In this case, the transition temperature monotonically increases as we increase the wave number k and asymptotes to (23). Hence the higher-derivative coupling's encoding of the electric field's back-reaction near the horizon is the cause of spontaneous generation of spatial modulation.

As we switch on $\eta' > 0$, the transition temperature is bounded from above by (23). It attains a maximum value at a finite k. Thus the second higher-derivative coupling acts as a UV cutoff on the wave number, which in turn determines the size of the "lattice" through $k = \frac{2\pi}{a}$, where a is the lattice spacing. As $\eta' \to 0$, the lattice spacing also vanishes ($a \to 0$) and the wave number diverges ($k \to \infty$). We will focus on small-to-zero charge, realizing a transition temperature at zero wave number *below* (23), which guarantees that $k \neq 0$ at the maximum transition temperature, hence the dominance of inhomogeneous modes. For our purposes, we do not expect any quantitative differences between small and zero charge, even with a neutral scalar leaving the boundary $U(1)$ symmetry intact.

In Fig. 2, we show the transition temperature T_0 as a function of the wave number k with $q = 1$, $\frac{\eta}{\mu^2} = 1$, $\frac{\eta'}{\mu^4} = 0.005$, for various values of the conformal dimension of the scalar field Δ. The critical temperature of the system is the maximum transition temperature which occurs at finite $k/\mu \approx 3.98$. The effect of the charge q is shown in Fig. 3. It can be seen that for finite coupling constants η and η', and small enough charge q, the system produces a *critical* temperature at nonvanishing finite values of k/μ. For large enough q, the homogeneous solution ($k = 0$) remains the dominant solution.

Figure 4 shows the dependence of the critical temperature T_c and wave number k on the second higher-derivative coupling constant $\frac{\eta'}{\mu^4}$. There is little to no dependence of T_c on the conformal dimension Δ. We see numerical confirmation that in the absence of the second coupling ($\eta' = 0$), the maximum transition temperature corresponds to $k \to \infty$. Thus η' acts as an effective UV cutoff, determining the wave number k at the critical temperature, and therefore the size of the "lattice" of the system (if $k = \frac{2\pi}{a}$, where a is the lattice spacing). The value of the wave number decreases with increasing coupling η', as shown on the right panel of Fig. 4. The UV cutoff η', or effective lattice spacing, can be understood as stabilizing the inhomogeneous modes introduced by the first higher-derivative

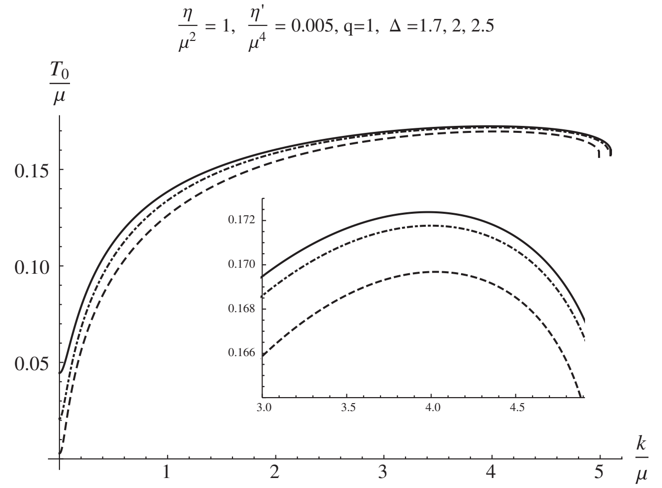

FIG. 2. Dependence of the transition temperature $\frac{T_0}{\mu}$ on the wave number $\frac{k}{\mu}$ for $\frac{\eta}{\mu^2} = 1$, $\frac{\eta'}{\mu^4} = 0.005$, $q = 1$, and $\Delta = 1.7$ (solid line), 2 (dash-dotted line), and 2.5 (dotted line).

coupling η. Looking forward, we trust our linearization below the critical temperature because it will not rely on a gradient expansion but on an order parameter proportional to $(T - T_c)^{1/2}$.

In summary, at the critical temperature the electric field backreacts on the system, the Einstein-Maxwell field equations admit solutions with a spatially dependent scalar field while the electric field attains a constant value equal to the chemical potential. In the next section we will perturb around the critical temperature and show the system develops a spatially inhomogeneous phase in the boundary theory.

IV. BELOW THE CRITICAL TEMPERATURE

In this section we study the system below the critical temperature. The equations of motion resulting from the considered action (1) together with (3) may be perturbed near the critical temperature with spatially dependent

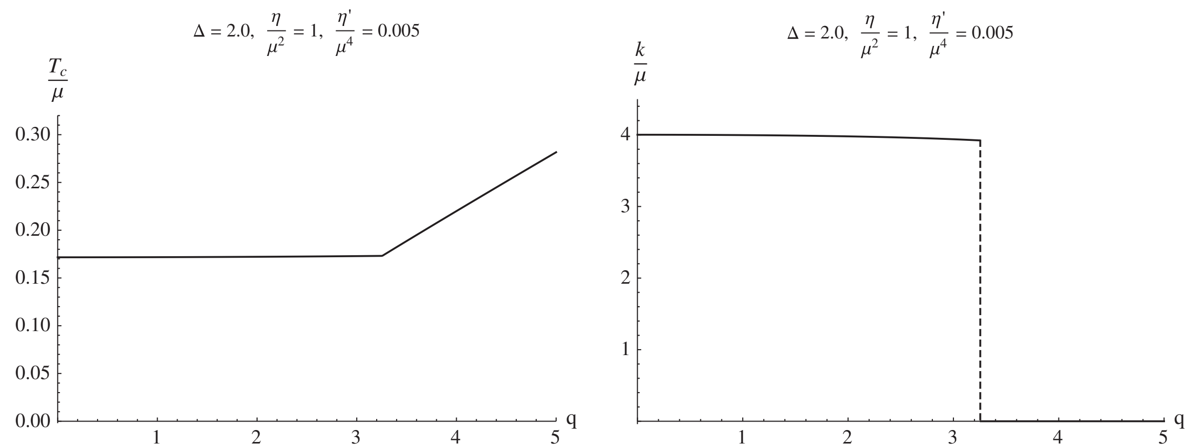

FIG. 3. The critical temperature $\frac{T_c}{\mu}$ (left panel) and corresponding wave number $\frac{k}{\mu}$ (right panel) as functions of q with $\frac{\eta}{\mu^2} = 1$, $\frac{\eta'}{\mu^4} = 0.005$, and $\Delta = 2$. When $q \lesssim 3.2$, the charge is small enough that the higher-derivative couplings spontaneously generate inhomogeneity. Above that range, the homogeneous scalar is dominant.

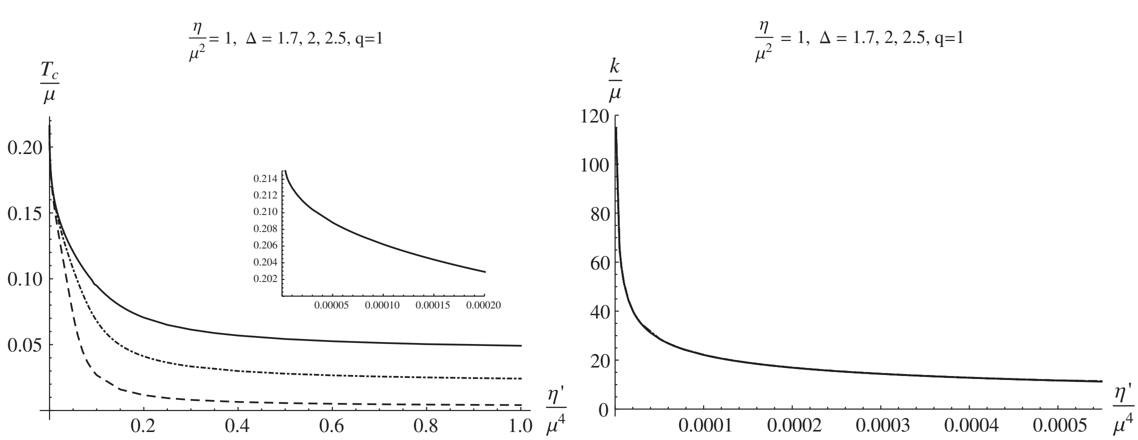

FIG. 4. Dependence of the critical temperature $\frac{T_c}{\mu}$ (left panel) and corresponding wave number $\frac{k}{\mu}$ (right panel) on the second higher-derivative coupling constant $\frac{\eta'}{\mu^4}$, for $\frac{\eta}{\mu^2} = 1$, $q = 1$, $\Delta = 1.7$ (solid), 2 (dash-dotted), and 2.5 (dashed). The inset is shown only for $\Delta = 2$ because, at the scale shown, no difference can be seen among the three different conformal dimensions of the full graph.

solutions. We will study the behavior of the system analytically, leaving a full numerical study for the future. To simplify the discussion somewhat, we shall assume that the effects of the cutoff are negligible near the critical temperature ($T \approx T_c$), and set $\eta' = 0$. We will build a perturbative expansion on the departure below T_c and not in terms of gradients or momentum of the scalar mode. It is straightforward, albeit tedious, to include the effects of the cutoff below T_c.

Below the critical temperature the scalar field backreacts on the metric. Consider the following ansatz:

$$ds^2 = \frac{1}{z^2}\left[-h(z,x)e^{-\alpha(z,x)}dt^2 + \frac{dz^2}{h(z,x)} + e^{\beta(z,x)}dx^2 + e^{-\beta(z,x)}dy^2\right]. \tag{24}$$

To solve the equations of motion (5), (8), and (11) below the critical temperature T_c, we expand in the order parameter

$$\xi = \frac{\langle \mathcal{O}_\Delta\rangle}{\sqrt{2}}, \tag{25}$$

and write

$$\begin{aligned} h(z,x) &= h_0(z) + \xi^2 h_1(z,x) + \mathcal{O}(\xi^4),\\ \alpha(z,x) &= \xi^2\alpha_1(z,x) + \mathcal{O}(\xi^4),\\ \beta(z,x) &= \xi^2\beta_1(z,x) + \mathcal{O}(\xi^4),\\ \phi(z,x) &= \xi\phi_0(z,x) + \xi^3\phi_1(z,x) + \mathcal{O}(\xi^5),\\ A_t(z,x) &= A_{t0}(z) + \xi^2 A_{t1}(z,x) + \mathcal{O}(\xi^4), \end{aligned} \tag{26}$$

where A_{t0}, h_0, and $\xi\phi_0$ are defined at the critical temperature T_c by Eqs. (13), (14), and (16), respectively. The chemical potential is given as

$$\begin{aligned} \mu \equiv A_t(0,x) &= \mu_0 + \xi^2\mu_1 + \mathcal{O}(\xi^2),\\ \mu_0 = A_{t0}(0), &\qquad \mu_1 = A_{t1}(0,x). \end{aligned} \tag{27}$$

It should be noted that we are working with an ensemble of fixed chemical potential, which seems to contradict Eq. (27) in which the chemical potential appears to receive corrections below the critical temperature. However, the reported chemical potential is measured in units in which the radius of the horizon is 1 and a change in μ, in these units, is due to a change in our scale as we lower the temperature.

At each given order of the parameter ξ, only a finite number of modes of the various fields are generated. At $\mathcal{O}(\xi^2)$, we have only 0 and $2k$ Fourier modes,

$$\begin{aligned} h_1(z,x) &= z^3(h_{10}(z) + h_0^2(z)h_{11}(z)\cos 2kx),\\ \alpha_1(z,x) &= \alpha_{10}(z) + z^3 h_0(z)\alpha_{11}(z)\cos 2kx,\\ \beta_1(z,x) &= \beta_{10}(z) + z^3\beta_{11}(z)\cos 2kx,\\ A_{t1}(z,x) &= A_{t10}(z) + zh_0(z)A_{t11}(z)\cos 2kx, \end{aligned} \tag{28}$$

where we included explicit factors of z and $h_0(z)$ for convenience. From (27), we obtain the boundary condition

$$A_{t10}(0) = \mu_1. \tag{29}$$

Then from the Maxwell equation (8), and the boundary condition (29), we find

$$A_{t10}(z) = C(1-z) + \frac{\mu_0}{4}\int_1^z dw \times \int_1^w dw' w'^{2\Delta-2}h_0(w')\mathcal{A}(w'), \tag{30}$$

where

$$\begin{aligned} C &= \mu_1 + \frac{\mu_0}{4}\int_0^1 dz\int_1^z dw w^{2\Delta-2}h_0(w)\mathcal{A}(w),\\ \mathcal{A}(z) &= \left[q^2\frac{\mu_0^2(1-z)^2z^3 + 4q^2(1-z)h_0(z)}{h_0^2(z)} + z(\Delta^2 + 8k^2\eta(1+\Delta)z^2)\right]F^2(z)\\ &\quad + 2z^2[\Delta + 4k^2\eta z^2]F(z)F'(z) + z^3[F'(z)]^2. \end{aligned} \tag{31}$$

Thus the integration constant $\mathcal{C}$ is expressed in terms of the chemical potential parameters μ_0 and μ_1. While μ_0 is determined at the critical temperature, μ_1 still needs to be determined. Subsequently, we will determine $\mathcal{C}$ using the scalar equation and use that value in Eq. (30) to find μ_1.

After some algebra, from the Einstein equations we deduce that the mode function $\alpha_{10}(z)$ is given by

$$\alpha_{10}(z) = \frac{1}{2}\int_0^z dw w^{2\Delta-1}\left[\left(q^2\mu_0^2\frac{(1-w)^2w^2}{h_0^2(w)} + \Delta^2\right)F^2(w) + 2\Delta wF(w)F'(w) + w^2[F'(w)]^2\right]. \tag{32}$$

Notice that the mode function α_{10} contributes to α_1 at an order higher than $\mathcal{O}(z^3)$ near the boundary for $\Delta > \frac{3}{2}$.

The mode function $\beta_{10}(z)$ is given by

$$\beta_{10}(z) = \frac{k^2}{2}\int_0^z \frac{dw w^2}{h_0(w)}\int_w^1 dw' w'^{2\Delta-2}(1-\eta\mu_0^2 w'^4)F^2(w'). \tag{33}$$

The mode function β_{10} also contributes at $\mathcal{O}(z^3)$ near the boundary, because $\beta_{10}\sim z^3$ at the boundary.

Finally, the mode function $h_{10}(z)$ is given by

$$h_{10}(z) = -\frac{\mu_0[2C + \mu_0\alpha_{10}(1)]}{4}(1-z) - \frac{1}{4}\int_z^1 dw w^{2\Delta-4}\mathcal{H}(w), \tag{34}$$

where

$$\mathcal{H}(z) = \left[m^2 + \frac{q^2\mu_0^2 z^2(1-z)^2}{h_0(z)} + k^2 z^2(1+\eta z^4\mu_0^2) + \Delta^2 h_0(z)\right]F^2(z) + 2z\Delta h_0(z)F(z)F'(z) + z^2 h_0(z)[F'(z)]^2$$
$$- z^{4-2\Delta}\mu_0^2\int_1^z dw w^{2\Delta}F(w)\left[\left(\frac{2q^2(1-w)}{w^2 h_0(w)} + 4\tau\eta(\Delta+1)w\right)F(w) + 4\tau\eta w^2 F'(w)\right]. \quad (35)$$

The mode function h_{10} contributes at $\mathcal{O}(z^3)$ to the metric (24) near the boundary because $h_{10}(0)$ is finite, and we removed a factor of z^3 in the definition (28). We fix one of the integration constants by setting $h_{10}(1) = 0$, so that the horizon remains at $z = 1$. C [Eq. (31)] is the remaining integration constant to be determined.

The remaining first-order modes α_{11}, β_{11}, h_{11}, A_{t11} are determined by a system of coupled linear ordinary differential equations,

$$\alpha'_{11} + \frac{zh'_0 + 3h_0 + 2k^2z^2}{zh_0}\alpha_{11} - \frac{4k^2 z}{h_0}h_{11} - \frac{z^{2\Delta-4}}{2h_0}\mathcal{A}_1 = 0,$$
$$\beta'_{11} - \frac{3}{2}z\mu_0 A'_{t11} - 4h_0 h'_{11} + 3\frac{2k^2z^2 + h_0}{zh_0}\beta_{11} - \frac{\mu_0(5h_0 + 3zh'_0)}{2h_0}A_{t11} + \frac{1}{4}(-8k^2 z + 3\mu_0^2 z^3 + 2h'_0)\alpha_{11}$$
$$+ \frac{(10k^2z^2 - h_0 - 8zh'_0)}{z}h_{11} + \frac{z^{2\Delta-4}}{4h_0^2}\mathcal{A}_2 = 0,$$
$$h'_{11} - \alpha'_{11} - \frac{\beta'_{11}}{h_0} + \left[\frac{1}{z} + \frac{2h'_0}{h_0}\right]h_{11} + \frac{\mu_0 A_{t11}}{h_0} - \frac{3}{2}\left[\frac{2}{z} + \frac{h'_0}{h_0}\right]\alpha_{11} - \frac{3\beta_{11}}{zh_0} + \frac{z^{2\Delta-4}}{2h_0}\mathcal{A}_3 = 0,$$
$$A''_{t11} + 2\left[\frac{1}{z} + \frac{h'_0}{h_0}\right]A'_{t11} + \frac{2h'_0 + z(-4k^2 + h''_0)}{zh_0}A_{t11} - \frac{\mu_0}{2}z^2\alpha'_{11} - \frac{\mu_0}{2}z^2\left[\frac{3}{z} + \frac{h'_0}{h_0}\right]\alpha_{11} - \frac{z^{2\Delta-3}\mu_0 F^2[q^2(1-z) - 2\tau\eta z^3 h_0]}{h_0^2} = 0,$$
$$(36)$$

where

$$\mathcal{A}_1(z) = z^2h_0^2F'^2 + 2z\Delta h_0^2 FF' + [q^2(1-z)^2\mu_0^2 + \Delta^2 h_0^2]F^2,$$
$$\mathcal{A}_2(z) = 5z^2h_0^2F'^2 + 2z(5\Delta - 1)h_0^2FF' + [5q^2\mu_0^2(1-z)^2z^2 + 3(m^2 - k^2z^2(1+2\eta z^4\mu_0^2))h_0 + \Delta(5\Delta-2)h_0^2]F^2, \quad (37)$$
$$\mathcal{A}_3(z) = F(\Delta F + zF'),$$

with F defined as in (21). The system of equation (36) can be seen to possess a unique solution by requiring finiteness of all functions in the entire domain $z \in [0, 1]$. Notice that the unknown parameter $\mathcal{C}$ is absent, which is due to the fact that at first order the 10 modes decouple from the 11 modes [see Eq. (28) for the definition of the Fourier modes]. However, explicit solutions can only be obtained numerically. A complete numerical analysis will be presented elsewhere.

To complete the determination of the first-order modes, we need to calculate the integration constant C [or, equivalently, the chemical potential parameter μ_1; see Eq. (31)]. To this end, we turn to the scalar wave equation. At zeroth order, the chemical potential parameter μ_0 was obtained as an eigenvalue of the scalar wave equation. The first-order correction, μ_1, is determined by the first-order equation of the scalar wave equation.

Considering (11) below the critical temperature, the scalar field at first order has two Fourier modes,

$$\phi_1(z,x) = \Phi_{10}(z)\cos kx + \Phi_{11}(z)\cos 3kx. \quad (38)$$

The first (Φ_{10}) mode satisfies the equation

$$\Phi''_{10} + \left[\frac{h'_0}{h_0} - \frac{2}{z}\right]\Phi'_{10} + \frac{\tau}{h_0}(1 - \eta\mu_0^2 z^4)\Phi_{10}$$
$$- \frac{1}{h_0}\left[\frac{m^2}{z^2} - q^2\frac{A_{t0}}{h_0}\right]\Phi_{10} + z^{\Delta+1}\mathcal{B} + Cz^{\Delta+2}\mathcal{C} = 0, \quad (39)$$

where

$$\mathcal{B} = \mathcal{B}_2F'' + \mathcal{B}_1F' + \mathcal{B}_0F,$$
$$\mathcal{C} = \mathcal{C}_2F'' + \mathcal{C}_1F' + \mathcal{C}_0F, \quad (40)$$

and the coefficients $\mathcal{B}_i$ and $\mathcal{C}_i$ ($i = 0, 1, 2$) are

$$\mathcal{B}_0 = \frac{z\mu_0^2[q^2(1-z)^2 + k^2z^4\eta h_0]}{h_0}\alpha_{10} - \frac{1}{2}\Delta h_0\alpha'_{10} + \tau z(1-\eta z^4\mu_0^2)\beta_{10} + \frac{z^2[-q^2(1-z)^2z^2\mu_0^2 + \Delta^2 h_0^2]}{h_0^2}\mathbf{h}$$
$$+ z^3\Delta\mathbf{h}' - \frac{2q^2(-1+z)z\mu_0}{h_0}\mathbf{A} - 2\tau\eta\mu_0 z^5\mathbf{A}' + \frac{1}{4}z^2(2q^2(1-z)^2z^2\mu_0^2 - 3\Delta h_0^2 - zh_0(2k^2z + \Delta h_0'))\alpha_{11}$$
$$-\frac{1}{4}z^3\Delta h_0^2\alpha'_{11} - \frac{1}{2}\tau z^4(1-\eta z^4\mu_0^2)\beta_{11} - z^2\mu_0(q^2(-1+z) + 2k^2z^3\eta h_0)A_{t11} + \frac{1}{2}z^2(-q^2(1-z)^2z^2\mu_0^2$$
$$+\Delta^2h_0^2 + 2z\Delta h_0h_0')h_{11} + \frac{1}{2}z^3\Delta h_0^2h'_{11},$$
$$\mathcal{B}_1 = z^3(1+2\Delta)\mathbf{h} + z^4\mathbf{h}' - \frac{1}{2}zh_0\alpha'_{10} - \frac{1}{4}z^3h_0(3h_0 + zh_0')\alpha_{11} - \frac{1}{4}z^4h_0^2\alpha'_{11} + \frac{1}{2}z^3h_0(h_0 + 2\Delta h_0 + 2zh_0')h_{11} + \frac{1}{2}z^4h_0^2h'_{11},$$
$$\mathcal{B}_2 = z^4\mathbf{h} + \frac{1}{2}z^4h_0^2h_{11}, \tag{41}$$

$$\mathcal{C}_0 = \frac{\mu_0}{2h_0^2}[q^2(1-z)^2((z-1)z^3\mu_0^2 + 4h_0) + z(-\Delta^2 + \Delta(\Delta+1)z + 4\tau\eta z^3)h_0^2],$$
$$\mathcal{C}_1 = \frac{\mu_0}{2}z^2[-1-2\Delta + 2z(1+\Delta)], \qquad \mathcal{C}_2 = \frac{\mu_0}{2}(z-1)z^3. \tag{42}$$

We defined [see Eqs. (31) and (35)]

$$\mathbf{h}(z) = -\frac{1}{4}\int_z^1 dww^{2\Delta-4}\mathcal{H}(w),$$
$$\mathbf{A}(z) = \frac{\mu_0}{4}\int_1^z dw\int_1^w dw'w'^{2\Delta-2}h_0(w')\mathcal{A}(w'). \tag{43}$$

By using the zeroth-order wave equation (16), we obtain

$$C = -\frac{\int_0^1 dzz^{2\Delta+1}F[\mathcal{B}_2F'' + \mathcal{B}_1F' + \mathcal{B}_0F]}{\int_0^1 dzz^{2\Delta+2}F[\mathcal{C}_2F'' + \mathcal{C}_1F' + \mathcal{C}_0F]}. \tag{44}$$

Having obtained the integration constant C, the remaining unknown parameter μ_1 is calculated using Eq. (30).

The temperature of our system below the critical temperature T_c can be calculated using

$$\frac{T}{\mu} = -\frac{h'(1)e^{-\alpha(1)}}{4\pi\mu}. \tag{45}$$

We obtain

$$\frac{T}{T_c} = 1 - \xi^2\left(\alpha_{10}(1) + \frac{\mu_1}{\mu_0}\right) - \frac{\xi^2}{3-\frac{\mu_0^2}{4}}h'_{10}(1), \tag{46}$$

where ξ is given by Eq. (25).

Equation (46) can be inverted to find the energy gap (25) as a function of temperature near the critical temperature,

$$\frac{\langle\mathcal{O}_\Delta\rangle^{1/\Delta}}{T_c} \approx \gamma\left(1-\frac{T}{T_c}\right)^{\frac{1}{2\Delta}},$$
$$\gamma = \frac{4\pi}{3-\frac{\mu_0^2}{4}}\left(\frac{\alpha_{10}(1)}{2} + \frac{\mu_1}{2\mu_0} + \frac{h'_{10}(1)}{2(3-\frac{\mu_0^2}{4})}\right)^{-\frac{1}{2\Delta}}. \tag{47}$$

Thus, as the temperature of the system is lowered below the critical temperature T_c the condensate is spontaneously generated. The dependence of the condensate on the temperature is of the same form as in conventional holographic superconductors.

Finally, the charge density of the system is determined by using

$$\frac{\rho}{\mu^2} = -\frac{\partial_z A_t(0,x)}{[A_t(0,x)]^2} = \frac{\rho_0 + \xi^2\rho_1(x)}{\mu_0^2}, \tag{48}$$

where $\rho_0 = \mu_0$ is the charge density at or above the critical temperature, and

$$\rho_1(x) = -2\mu_1 - A'_{t10}(0) - A_{t11}(0)\cos 2kx, \tag{49}$$

from Eq. (32). This is an important result showing the generation of a spatially inhomogeneous charge density below the critical temperature in the presence of a *spatially homogeneous* (constant) chemical potential. This is the case provided $A_{t11}(0) \neq 0$, which is guaranteed analytically from the system of equation (36) for the 11 modes. Indeed, from the last equation in (36), we obtain $A'_{t11}(0) = 0$. Moreover, there is a boundary condition at the horizon $z = 1$ where we demand finiteness of A_{t11} [$A_{t11}(1) < \infty$]. If additionally $A_{t11}(0) = 0$, then the second-order differential equation is overdetermined and has no solution. Thus, a general solution has $A_{t11}(0) \neq 0$.

V. CONCLUSIONS

We have discussed a holographic model in which the gravity sector consists of a $U(1)$ gauge field and a scalar field coupled to an AdS charged black hole under a *constant* chemical potential. We introduced higher-derivative interaction terms between the $U(1)$ gauge field and the scalar field. A gravitational lattice was generated spontaneously by a spatially dependent profile of the scalar field. The transition temperature was calculated as a function of the wave number. The critical temperature was determined as the maximum transition temperature. This occurred at finite nonvanishing wave number, showing that the inhomogeneous solution was dominant over a range of the parameters of the system which we discussed.

The system was then studied below the critical temperature. We obtained analytic expressions for the various fields using perturbation theory in the (small) order parameter. It was found that a spatial inhomogeneous phase is generated at the boundary. In particular, we showed analytically that a spatially inhomogeneous charge density is spontaneously generated in the system while it is held at constant chemical potential.

It will be illuminating to compare features between different mechanisms for generating spontaneous translation symmetry breaking seen with the Einstein tensor-scalar coupling [32], Chern-Simons interaction [34,35], and dilaton [17]. Additionally, this work only considered a unidirectional lattice. It would be interesting to extend our discussion to a more general two-dimensional lattice and determine which configuration is energetically favorable. Finally, one would like to understand the origin of the higher-derivative couplings we introduced in terms of quantum corrections within string theory. Work in these directions is in progress.

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