PHYSICAL REVIEW D **88,** 105027 (2013)

Gauge-invariant functional measure for gauge fields on $\mathbb{CP}^2$

V. P. Nair*

Physics Department, City College of the CUNY, New York, New York 10031, USA and Niels Bohr International Academy, Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen, Denmark

(Received 9 August 2013; published 26 November 2013)

We introduce a general parametrization for non-Abelian gauge fields on the four-dimensional space $\mathbb{CP}^2$. The volume element for the gauge-orbit space or the space of physical configurations is then investigated. The leading divergence in this volume element is obtained in terms of a higher dimensional Wess-Zumino-Witten action, which has previously been studied in the context of Kähler-Chern-Simons theories. This term, it is argued, implies that one needs to introduce a dimensional parameter to specify the integration measure, a step which is a nonperturbative version of the well-known dimensional transmutation in four-dimensional gauge theories.

DOI: 10.1103/PhysRevD.88.105027 PACS numbers: 11.15.−q, 11.10.Ef, 12.38.−t

I. INTRODUCTION

The importance of the gauge-orbit space needs no emphasis given that Yang-Mills theories are the foundational paradigm for the interactions of fundamental particles. The relevant space over which the functional integration for such theories is carried out is the space of gauge potentials ($\mathcal{A}$) modulo the space of all gauge transformations which are fixed to be identity at one point on the spacetime manifold ($\mathcal{G}_*$). In particular, the measure of integration is the volume element of this gauge-orbit space $\mathcal{C} = \mathcal{A}/\mathcal{G}_*$, equivalently, the space of physical field configurations [1].

This volume element can be calculated exactly for gauge fields in two dimensions in terms of a Wess-Zumino-Witten (WZW) action [2]. It plays a role in the Chern-Simons-WZW relationship [3] and, albeit indirectly, in the solution of Yang-Mills theory on Riemann surfaces [4]. It can be incorporated into a Hamiltonian formalism for $(2+1)$-dimensional Yang-Mills theories leading to string tension calculations and insight into the mass gap [5,6], including supersymmetric cases [7].

The situation for four-dimensional gauge theories has been much less clear. Gauge-fixing and the Faddeev-Popov procedure construct this volume element for a local section of $\mathcal{A}$ viewed as a $\mathcal{G}_*$-bundle over $\mathcal{C}$; this is adequate for the perturbative calculations, but does not really give any insight into anything beyond that. The volume element for the gauge-orbit space is the subject of this paper. The calculations in lower dimensions utilized the possibility that one could view two-dimensional space as a complex manifold, which then led to a parametrization of fields which was very suitable for the calculation of the volume element for $\mathcal{C}$. There is no natural complex structure for $\mathbb{R}^4$ since any choice of complex coordinates would not be 4d-rotationally invariant (or Lorentz invariant with Minkowski signature). One could consider a twistor space version which would include the set of all local complex structures. However, a simpler situation is obtained with $\mathbb{CP}^2$, which is a complex Kähler manifold. The standard metric for this space is the Fubini-Study metric given, in local coordinates z^a, $\bar{z}^{\bar{a}}$, $a = 1, 2$, $\bar{a} = \bar{1}, \bar{2}$, by

$$ds^2 = \frac{dz^a d\bar{z}^{\bar{a}}}{(1 + z\cdot\bar{z}/R^2)} - \frac{\bar{z}\cdot dz z\cdot d\bar{z}}{R^2(1+z\cdot\bar{z}/R^2)} \tag{1}$$

where we have also included a scale parameter for the coordinates. As the parameter $R \to \infty$, the metric becomes that of flat space (although there are some global issues which will not be important for us). This is, therefore, an interesting space to consider, being endowed with a complex structure and with a suitable limit to the flat four-dimensional space. So, in this paper, we will consider gauge fields on $\mathbb{CP}^2$.

In the next section we will introduce a suitable parametrization for gauge fields on $\mathbb{CP}^2$ and identify the gauge-invariant variables of the problem. We will then proceed to the evaluation of the leading divergent term in the functional integration measure. This is shown to be given by a higher dimensional generalization of the WZW action. The functional integration for gauge fields in four dimensions, it is well known, should show dimensional transmutation with a freely specifiable dimensional parameter characterizing the theory. In the last section, we argue that the leading divergence in the calculation of the volume element for $\mathcal{C}$ introduces just such a parameter, which is, effectively, a nonperturbatively defined version of the Λ-parameter of QCD. The main result of the paper is then contained in Eqs. (52) and (53), which give the definition of the functional integral with the measure defined in terms of the gauge-invariant variables. The computation of the subleading and finite terms in the Jacobian of the transformation to the gauge-invariant variables and the extensions of the result to supersymmetric theories are briefly alluded to in the discussion section; they are interesting directions to explore in future.

*vpn@sci.ccny.cuny.edu

II. THE VOLUME ELEMENT FOR THE GAUGE ORBIT SPACE

A. Parametrization of fields

We will begin with a suitable parametrization of the gauge fields on $\mathbb{CP}^2$. This space may be thought of as the group coset $SU(3)/U(2)$. Thus functions, vectors, etc. on this space may be realized in terms of the Wigner functions $\langle R, A|\hat{g}|R, B\rangle$ which are the representation of a $SU(3)$ element g in a general irreducible representation labeled as R. For the defining fundamental representation, we take g to be a 3×3 unitary matrix of unit determinant and it can be parametrized as $g = \exp(it_a\varphi^a)$, where t_a form a basis for Hermitian traceless 3×3 matrices, with $\mathrm{Tr}(t_at_b) = \frac{1}{2}\delta_{ab}$, and φ^a are the coordinates for $SU(3)$. In terms of the functions $\langle R, A|\hat{g}|R, B\rangle$, the states on the right, namely, $|R, B\rangle$ must be so chosen as to give the correct transformation property under $U(2)\in SU(3)$. Notice that, for $\mathbb{CP}^2$, $SU(3)$ plays the role of the Poincaré group and $U(2)$ plays the role of the Lorentz group; so vectors, tensors, etc., must be characterized by their transformation property under $U(2)$. We will refer to the $SU(2)$ part of $U(2)$ as isospin (denoted by I) and the $U(1)$ part of $U(2)$ as hypercharge (denoted by Y). Specifically, we take the $SU(2)$ to be generated by t_1, t_2, t_3 and the hypercharge to correspond to t_8. For functions on $\mathbb{CP}^2$, which must be invariant under $U(2)$, we need states with $Y = 0$ and $I = 0$. For vectors, we need a $SU(2)$ doublet ($I = \frac{1}{2}$ representation). A general $SU(3)$ representation is of the form $T^{a_1a_2\cdots a_p}_{b_1b_2\cdots b_q}$, a_i, $b_j = 1$, 2, 3, which may be labeled as (p, q). These are totally symmetric in all a_i's and totally symmetric in all b_j's with the trace (or contraction between any two upper and lower indices) vanishing. The value of hypercharge is given by

$$Y = \begin{cases} 1/3 & a_i = 1, 2 \\ -2/3 & a_i = 3 \\ -1/3 & b_i = 1, 2 \\ 2/3 & b_i = 3. \end{cases} \tag{2}$$

For the derivative operators on $\mathbb{CP}^2$ we can use a subset of the right translation operators R_a defined by

$$R_a g = g t_a. \tag{3}$$

More explicitly, we can write

$$g^{-1}dg = -it_aE^a_i d\varphi^i, \qquad R_a = i(E^{-1})^i_a\frac{\partial}{\partial\varphi^i}. \tag{4}$$

For $\mathbb{CP}^2$, the derivatives will be taken as $\nabla_1 = R_4 + iR_5$ and $\nabla_2 = R_6 + iR_7$, and $\nabla_{\bar{1}} = R_4 - iR_5$, $\nabla_{\bar{2}} = R_6 - iR_7$. The ∇'s correspond to derivatives in the tangent frame, $\nabla_i = (e_i^{-1})^\mu(\partial/\partial x^\mu)$ in terms of the usual local coordinates, e's being the frame fields. (The group theoretic approach we use, with derivatives given by R_a, for $\mathbb{CP}^k$ spaces is essentially along the lines of [8]; it is also similar to what was done for gauge fields on $\mathbb{CP}^1$ in [9].)

The operators ∇_i form a $SU(2)$ doublet with $Y = 1$, $\nabla_{\bar{i}}$ are again a $SU(2)$ doublet with $Y = -1$. A gauge field A is to be added to these operators, so we need a $SU(2)$ doublet with $Y = 1$ for A_i, and a $SU(2)$ doublet with $Y = -1$ for $A_{\bar{i}}$. This corresponds to the states of the form $T^{i33\cdots 3}_{33\cdots 3}$ which give $Y = 1$ for $p = q$, and $Y = -1$ for $p = q + 3$. Likewise, $T^{33\cdots 3}_{i33\cdots 3}$ would give a $SU(2)$ doublet with $Y = -1$ for $p = q$ and $Y = 1$ for $q = p + 3$. Thus for a vector field, we need three types of representations:

(1) $R_1 \equiv (p, p)$-type, $p \neq 0$: These contribute to both A_i and $A_{\bar{i}}$.
(2) $R_2 \equiv (p, p + 3)$-type: These contribute to A_i.
(3) $R_3 \equiv (p + 3, p)$-type: These contribute to $A_{\bar{i}}$.

The general expression for an Abelian vector field is thus

$$\begin{aligned} A_i &= \sum_{A,R_1} C^{R_1}_A\langle R_1, A|\hat{g}|R_1, i\rangle + \sum_{A,R_3} B^{R_3}_A\langle R_3, A|\hat{g}|R_3, i\rangle \\ A_{\bar{j}} &= \sum_{A,R_1} \bar{C}^{R_1}_A\langle R_1, A|\hat{g}|R_1, \bar{j}\rangle + \sum_{A,R_2} \bar{B}^{R_2}_A\langle R_2, A|\hat{g}|R_2, \bar{j}\rangle, \end{aligned} \tag{5}$$

where $C^{R_1}_A$ and $B^{R_3}_A$ are arbitrary complex numbers. The representations R_2 and R_3 are conjugates of each other; R_1 is invariant under conjugation. The state on the right for R_1, namely, $T^{i3\ldots 3}_{3\ldots 3}$ can be obtained by the action of $t_4 + it_5$ and $t_6 + it_7$ on a state $|w\rangle$ of the form $T^{3\cdots 3}_{3\ldots 3}$, which is $SU(2)$ invariant with zero hypercharge. In other words it can be obtained by the action of ∇_i on a function. Thus the first terms in (5) are of the form of derivatives acting on a function. In a similar way, the relevant state $|R_3, i\rangle$ can be obtained by the action of $\epsilon_{ij}\nabla_{\bar{j}}$ on a state $|z\rangle$ which is $SU(2)$ invariant with $Y = 2$. [A point of clarification: Even though there is only one irreducible doublet representation for $SU(2)$, it is only pseudoreal. Thus to convert doublet with upper indices to ones with lower indices, we have to use the ϵ_{ij} symbol.] In a similar way, we can obtain the relevant state for $|R_2, \bar{j}\rangle$ by the action of $\epsilon_{ij}\nabla_j$ on a state which is $SU(2)$ invariant with $Y = -2$. Combining these results, we see that the parametrization given above reduces to

$$A_i = -\nabla_i\theta + \epsilon_{ij}\nabla_{\bar{j}}\chi, \qquad A_{\bar{j}} = \nabla_{\bar{i}}\bar{\theta} - \epsilon_{ij}\nabla_j\bar{\chi}. \tag{6}$$

(In preparation for the non-Abelian case where we use anti-Hermitian basis for the gauge fields, we have changed over, compared to (5), to the conjugation property $A_i^\dagger = -A_{\bar{i}}$. In other words, A_i corresponds to $-i(A_4 + iA_5)$, $-i(A_6 + iA_7)$.) In (6), φ is a complex function on $\mathbb{CP}^2$ and hence is expandable in terms of $\langle R_1, A|\hat{g}|w\rangle$. The quantity χ is expandable in terms of $\langle R_3, A|\hat{g}|z\rangle$; it does not define a function on $\mathbb{CP}^2$ since $|z\rangle$ has $Y = 2$. The term $\epsilon_{ij}\nabla_{\bar{j}}\chi$ may be thought of as the divergence of a twoform. The four real independent components for general a vector field in four dimensions are captured by the θ, $\bar{\theta}$, χ and $\bar{\chi}$.

The generalization to the non-Abelian case is straightforward. Notice that, the product of a state of the form $|w\rangle$ with another state of the form $|w\rangle$ still gives a state of the same type. Thus functions can be multiplied to form other functions. Also, the product of a state of the type $|w\rangle$ with $|z\rangle$ still gives a state of the form $|z\rangle$. Thus multiplying χ by functions is also possible. We may combine this to write a parametrization for A_i as

$$A_i = -\nabla_i M M^{-1} - M a_i M^{-1}, \tag{7}$$

where M is a complex matrix in the complexification of the gauge group. We will take the gauge group as $SU(N)$ for the rest of this paper, so $M \in SL(N, \mathbb{C})$. Gauge transformations act on M as $M \to M^U = UM$; the term $M a_i M^{-1}$ transforms covariantly under this, so that A_i in (7) has the expected transformation property

$$A_i \to A_i^U = U A_i U^{-1} - \nabla_i U U^{-1}. \tag{8}$$

The conjugate components are given by

$$A_{\bar{i}} = M^{\dagger -1} \nabla_{\bar{i}} M^\dagger + M^{\dagger -1} \bar{a}_{\bar{i}} M^\dagger. \tag{9}$$

Since the inhomogeneous parts in the gauge transformation are generated from $-\nabla_i M M^{-1}$ and $M^{\dagger -1}\nabla_{\bar{i}} M^\dagger$, $D_i f \equiv \nabla_i f + [-\nabla_i M M^{-1}, f]$ and $\bar{D}_{\bar{i}} \equiv \nabla_{\bar{i}} f + [M^{\dagger -1}\nabla_{\bar{i}} M^\dagger, f]$ are gauge-covariant derivatives, for f's which transform as $f \to f^U = U f U^{-1}$. Thus another way to generalize (6) is

$$\begin{aligned} A_i &= -\nabla_i M M^{-1} + \epsilon_{ij} \bar{D}_{\bar{j}} \phi \\ A_{\bar{i}} &= M^{\dagger -1} \nabla_{\bar{i}} M^\dagger - \epsilon_{ij} D_j \phi^\dagger \end{aligned} \tag{10}$$

Here ϕ transforms covariantly under gauge transformations. Primarily, the parametrization of the gauge fields we use will be (10). But we may also view it as equivalent to (7) and (9), defining

$$a_i = -M^{-1} \epsilon_{ij} \bar{D}_{\bar{j}} \phi M, \quad \bar{a}_{\bar{i}} = -M^\dagger \epsilon_{ij} D_j \phi^\dagger M^{\dagger -1}. \tag{11}$$

Both these ways of viewing the parametrization of the gauge fields will be useful later. In terms of the matrix structure, the gauge fields are of the form $A_1 = (-iT^a)A_1^a = (-iT^a)(A_4^a + iA_5^a)$, $A_{\bar{1}} = (-iT^a)A_{\bar{1}}^a = (-iT^a)(A_4^a - iA_5^a)$, etc., where $\{T^a\}$ form a basis for the Lie algebra of the gauge group, say, $SU(N)$.

The gauge transformation properties show that the gauge-invariant degrees of freedom are described by $H = M^\dagger M$ and $\chi = M^{-1}\phi M$, $\bar{\chi} = M^\dagger \phi^\dagger M^{\dagger -1}$ (or $M^{ba}\phi^b$ and $(M^\dagger)^{ab}\phi^{\dagger b}$). (We use the same letter χ, although these are matrices and parametrize the non-Abelian fields now.) These fields constitute the coordinates for the space of gauge-invariant configurations or the gauge-orbit space $\mathcal{C}$.

B. The metric and volume

We now turn to the metric on the space of these gauge potentials. It is given by

$$ds^2 = -2\int d\mu \, \mathrm{Tr}(\delta A_{\bar{i}} \delta A_i) = \int d\mu \, \delta A_{\bar{i}}^a \delta A_i^a, \tag{12}$$

where $d\mu$ is the volume element for $\mathbb{CP}^2$. Taking the variations of (10) we find

$$\begin{aligned} \delta A_i^a &= -(D_i\theta)^a + \epsilon_{ij}(\bar{D}_{\bar{j}}\delta\phi)^a + \epsilon_{ij} f^{abc}(\bar{D}_{\bar{j}}\theta^\dagger)^b \phi^c, \\ \delta A_{\bar{i}}^a &= -(\bar{D}_{\bar{i}}\theta^\dagger)^a + \epsilon_{ij}(D_j\delta\phi^\dagger)^a + \epsilon_{ij} f^{abc}(D_j\theta)^b \phi^{\dagger c}, \end{aligned} \tag{13}$$

where $\theta = \delta M M^{-1} = (-iT^a)\theta^a$ and f^{abc} are the structure constants of the Lie algebra defined by $[T^a, T^b] = i f^{abc} T^c$. Using these variations in (12), we obtain

$$\begin{aligned} ds^2 &= (ds^2)_0 + (ds^2)_1 + (ds^2)_2, \\ (ds^2)_0 &= \int d\mu [(\bar{D}_{\bar{i}}\theta^\dagger - \epsilon_{ij} D_j \delta\phi^\dagger)^a (D_i\theta - \epsilon_{ik}\bar{D}_{\bar{k}}\delta\phi)^a], \\ (ds^2)_1 &= \int d\mu [-\epsilon_{ij} f^{abc}(D_j\theta)^b \phi^{\dagger c}(D_i\theta - \epsilon_{ik}\bar{D}_{\bar{k}}\delta\phi)^a \\ &\quad - (\bar{D}_{\bar{i}}\theta^\dagger - \epsilon_{ij} D_j\delta\phi^\dagger)^a \epsilon_{ik} f^{abc}(\bar{D}_{\bar{k}}\theta^\dagger)^b \phi^c], \\ (ds^2)_2 &= \int d\mu [\epsilon_{ij}\epsilon_{ik} f^{abc} f^{amn}(D_j\theta)^b \phi^{\dagger c}(\bar{D}_{\bar{k}}\theta^\dagger)^m \phi^n]. \end{aligned} \tag{14}$$

We have separated the metric into terms with no power of ϕ or $\phi^\dagger$, with one power of the same, or two powers. It is worth emphasizing that the connections in the covariant derivatives D_i and $\bar{D}_{\bar{i}}$ are $-\nabla_i M M^{-1}$ and $M^{\dagger -1}\nabla_{\bar{i}} M^\dagger$, respectively. As a result, we can further simplify $(ds^2)_0$ by partial integration, by noting that

$$\begin{aligned} &\int d\mu [\epsilon_{ij}(D_j\delta\phi^\dagger)^a (D_i\theta)^a] \\ &= -\int d\mu [\delta\phi^{\dagger a} \epsilon_{ij}(D_j D_i \theta)^a] = 0. \end{aligned} \tag{15}$$

Since D_i only involves $\nabla_i M M^{-1}$, the holomorphic covariant derivatives commute and so $\epsilon_{ij} D_i D_j = 0$. Thus

$$\begin{aligned} (ds^2)_0 = \int d\mu [&\theta^{\dagger a}(-\bar{D}_{\bar{i}} D_i)^{ab}\theta^b \\ &+ \delta\phi^{\dagger a}(-D_i \bar{D}_{\bar{i}})^{ab}\delta\phi^b]. \end{aligned} \tag{16}$$

We can simplify the other term in ds^2 in a similar way to get

$$\begin{aligned} (ds^2)_1 &= \int d\mu [\theta^a(\epsilon_{ik} D_i \Phi^\dagger D_k)^{ab}\theta^b \\ &\quad + \theta^a(D_k \Phi^\dagger \bar{D}_{\bar{k}})^{ab}\delta\phi^b + \theta^{\dagger a}(\epsilon_{ik}\bar{D}_{\bar{i}}\Phi\bar{D}_{\bar{k}})^{ab}\theta^{\dagger b} \\ &\quad + \delta\phi^{\dagger a}(-D_k \Phi \bar{D}_{\bar{k}})^{ab}\theta^{\dagger b}], \\ (ds^2)_2 &= \int d\mu [\theta^{\dagger a}(\bar{D}_{\bar{k}}\Phi\Phi^\dagger D_k)^{ab}\theta^b], \end{aligned} \tag{17}$$

where Φ is a matrix, $(\Phi)^{ab} = \phi^c f^{abc}$ and $(\Phi^\dagger)^{ab} = \phi^{\dagger c} f^{abc}$. Define a 4×4 matrix of operators $\mathcal{M}$ by

$$\mathcal{M}_{11} = \mathcal{M}_{33} = (-\bar{D}_{\bar{i}} D_i + \bar{D}_{\bar{k}} \Phi \Phi^\dagger D_k),$$
$$\mathcal{M}_{22} = \mathcal{M}_{44} = (-D_i \bar{D}_{\bar{i}}), \qquad \mathcal{M}_{13} = 2(\epsilon_{ik} \bar{D}_{\bar{i}} \Phi \bar{D}_{\bar{k}}),$$
$$\mathcal{M}_{23} = 2(-D_k \Phi \bar{D}_{\bar{k}}), \qquad \mathcal{M}_{31} = 2(\epsilon_{ik} D_i \Phi^\dagger D_k),$$
$$\mathcal{M}_{32} = 2(D_k \Phi^\dagger \bar{D}_{\bar{k}}), \tag{18}$$

with all other elements being zero. Then the metric is

$$ds^2 = \frac{1}{2} \int d\mu \, \xi_A^\dagger \mathcal{M}_{AB} \xi_B, \tag{19}$$
$$(\xi_1, \xi_2, \xi_3, \xi_4) = (\theta, \delta\phi, \theta^\dagger, \delta\phi^\dagger).$$

The volume element corresponding to this is given, up to an overall normalization factor, by $\sqrt{\det \mathcal{M}}$ times the volume defined by the differentials ξ. For the latter, θ and $\theta^\dagger$ give the standard Cartan-Killing volume element of $SL(N, \mathbb{C})$ (at each spacetime point); the differentials $\delta\phi$, $\delta\phi^\dagger$ give the standard functional integration measure $[d\phi d\phi^\dagger]$. Thus the volume element corresponding to (19) becomes

$$dV = \sqrt{\det \mathcal{M}} d\mu_{SL(N,\mathbb{C})} [d\phi d\phi^\dagger]. \tag{20}$$

(In $d\mu_{SL(N,\mathbb{C})}$, we have a product of the volume of $SL(N, \mathbb{C})$ over all spacetime points; this is not explicitly displayed, but left as understood.) As discussed in [5], by doing a polar decomposition of M into a unitary matrix and a Hermitian matrix, we can factor out the volume of gauge transformations from $d\mu_{SL(N,\mathbb{C})}$,

$$d\mu_{SL(N,\mathbb{C})} = d\mu(H) d\mu(SU(N)). \tag{21}$$

The integration measure for the ϕ's is gauge invariant since these fields transform covariantly, in much the same way matter fields have a gauge-invariant measure in standard functional integration. Factoring out the volume of gauge transformations, we get the volume element for the gauge-orbit space $\mathcal{C}$ (or the space of physical configurations) as

$$d\mu(\mathcal{C}) = \sqrt{\det \mathcal{M}} d\mu(H) [d\phi d\phi^\dagger]. \tag{22}$$

Also, as noted in [5], parametrizing the matrix H in terms of a set of real fields λ^a, $H^{-1} dH = d\lambda^a r_{ak}(\lambda) T^k$ and $d\mu(H) = \prod_x (\det r)[d\lambda]$.

C. Calculating the Jacobian factor

The problem is now reduced to the computation of the determinant of $\mathcal{M}$. For this we note that the off-diagonal terms in the matrix $\mathcal{M}$ depend on Φ or $\Phi^\dagger$, so in the neighborhood of the subspace $\Phi = 0$, $\mathcal{M}$ has only diagonal elements given by the operators $(-\bar{D}_{\bar{k}} D_k)$ and $(-D_k \bar{D}_{\bar{k}})$. Our strategy will be to calculate the volume around this subspace. The terms which depend on Φ, $\Phi^\dagger$ can then be included in a series expansion. The gauge transformation of the potentials A_i, $A_{\bar{i}}$ is fully captured by M and $M^\dagger$ in the parametrization we have used; thus setting Φ and $\Phi^\dagger$ to zero (in $\sqrt{\det \mathcal{M}}$) is consistent with gauge invariance requirements. Taking a Hamiltonian point of view for a moment, the two polarization states which would normally be eliminated by the Gauss law—which being a first class constraint eliminates two degrees of freedom—are contained in M, $M^\dagger$. The fields Φ and $\Phi^\dagger$ act almost as matter fields describing the surviving two polarizations. We may therefore expect to gain some insight into many of the issues of low energy physics from the analysis of the measure near the subspace with $\Phi = \Phi^\dagger = 0$.

The quantity to be calculated is thus $\log[\det(-\bar{D}_{\bar{k}} D_k) \times \det(-D_k \bar{D}_{\bar{k}})]$. In two dimensions, the analogous quantity would be $\log \det(-\bar{D} D)$. Formally we can factorize this, calculating $\log \det \bar{D}$ and $\log \det D$ separately and putting them together with the standard Schwinger-Quillen counterterm to obtain the gauge invariant result. In four dimensions, such a factorization is obviously not possible since we have a sum over the two complex indices in $\bar{D}_{\bar{k}} D_k$. So we will first recalculate the two-dimensional case in a way that will help us generalize to the four dimensions. In two dimensions, we need to calculate $\Gamma = \log \det(-\bar{D} D) = \mathrm{Tr} \log(-\bar{D} D)$. Consider the variation of this with respect to A. We get

$$\delta\Gamma = \mathrm{Tr}\Big[-\delta(\bar{D} D)\Big(\frac{1}{-\bar{D} D}\Big)\Big]$$
$$= \int d^2x \, \mathrm{Tr}[-\bar{D}_x(\delta A_x G(x,y))]_{y\to x}, \tag{23}$$

where $G(x,y) = (-\bar{D} D)^{-1}_{x,y}$. The operator $\bar{D}$ acts on both δA and $G(x,y)$. When it acts on δA, we have $G(x,y)]_{y\to x}$. This is proportional to the identity in any regularized version of G and hence this contribution vanishes by the matrix trace. The surviving term is

$$\delta\Gamma = \int d^2x \, \mathrm{Tr}[-\delta A_x \bar{D}_x G(x,y)]_{y\to x}. \tag{24}$$

(We have written out the functional trace; the remaining trace is just over the matrices.) This shows that we need a regularized version of the short-distance behavior of $\bar{D}_x G(x,y)$. We know that $(-\bar{\partial}\partial)^{-1}$ behaves as $\log[(x-y)\times(\bar{x}-\bar{y})]$ at short separations, so that $\bar{\partial} G(x,y)$ behaves as $1/(\bar{x}-\bar{y})$. However, we need to put in phase factors which ensure the correct gauge transformation properties. It can then be seen that the short-distance behavior should be given by

$$\bar{D}_x G(x,y) \simeq -\frac{M(x) M^{-1}(y) W(y,x)}{\pi(\bar{x}-\bar{y})},$$
$$W(y,x) = P \exp\Big(-\int_x^y A\Big). \tag{25}$$

$W(y,x)$ is the Wilson line matrix which transforms under gauge transformations as

$$W(y,x) \longrightarrow W(y,x,A^U) = U(y)W(y,x)U^{-1}(x) \quad (26)$$

so that $\bar{D}_x G(x,y) \to U(x)\bar{D}_x G(x,y)U^{-1}(x)$. Since δA transforms covariantly, $\delta A_x \to U(x)\delta A_x U^{-1}(x)$, this makes the trace in (24) gauge invariant. Further, since the numerator of $\bar{D}_x G(x,y)$ in (25) transforms covariantly, we have

$$\begin{aligned} &D(M(x)M^{-1}(y)W(y,x)) \\ &\quad = \partial(M(x)M^{-1}(y)W(y,x)) + [A, M(x)M^{-1}(y)W(y,x)] \\ &\quad = 0. \end{aligned} \quad (27)$$

The action of ∂ on $1/\pi(\bar{x}-\bar{y})$ leads to a delta function, verifying

$$-D_x[\bar{D}_x G(x,y)] = \delta^{(2)}(x-y). \quad (28)$$

This verifies the correctness of the short-distance behavior of the $\bar{D}_x G(x,y)$ given in (25).

It is now straightforward to expand (25) to first order in $x-y$, $\bar{x}-\bar{y}$ and find

$$\begin{aligned} \delta\Gamma &= \frac{1}{\pi}\int d^2x\, \mathrm{Tr}[\delta A(\bar{A} + \bar{\partial}MM^{-1})], \\ &= \frac{1}{\pi}\int d^2x [\mathrm{Tr}(\delta A\bar{A}) - \mathrm{Tr}(\bar{\partial}\theta\partial MM^{-1})], \\ &= \frac{1}{\pi}\int d^2x\, \mathrm{Tr}(\delta A\bar{A}) + \delta S_{wzw}(M), \end{aligned} \quad (29)$$

where the WZW action is given by

$$\begin{aligned} S_{wzw}(M) = &\frac{1}{2\pi}\int d^2x\, \mathrm{Tr}(\partial M\bar{\partial}M^{-1}) \\ &+ \frac{i}{12\pi}\int \mathrm{Tr}(M^{-1}dM)^3. \end{aligned} \quad (30)$$

There is a similar result for the variation of $M^\dagger$ or $\bar{A}$ and the combined result is

$$\Gamma = S_{wzw}(H). \quad (31)$$

In arriving at (29), we have used the symmetric way of taking the limit $y \to x$, so that $(x-y)/(\bar{x}-\bar{y})$ gives zero.

Before going to the four-dimensional case, there is one other point worth emphasizing. In parametrizing the fields as $A = -\partial MM^{-1}$, $\bar{A} = M^{\dagger -1}\bar{\partial}M^\dagger$ there is an ambiguity since M and $MV(\bar{x})$ where $V(\bar{x})$ is antiholomorphic give the same A. The use of M's in (25) carry this ambiguity over to the short-distance behavior. However, it is immaterial, as the corresponding correction to $\delta\Gamma$ vanishes,

$$\begin{aligned} \delta\Gamma]_{M\to MV} &= \frac{1}{\pi}\int \mathrm{Tr}[\delta AM\bar{\partial}VV^{-1}M^{-1}] \\ &= \frac{1}{\pi}\int \mathrm{Tr}[-D\theta M\bar{\partial}VV^{-1}M^{-1}] \\ &= \frac{1}{\pi}\int \mathrm{Tr}[\theta D(M\bar{\partial}VV^{-1}M^{-1})] \\ &= \frac{1}{\pi}\int \mathrm{Tr}[\theta M\partial(\bar{\partial}VV^{-1})M^{-1}] = 0. \end{aligned} \quad (32)$$

In four dimensions, the connections in the covariant derivatives are $-\nabla_k MM^{-1}$ and $M^{\dagger -1}\nabla_{\bar{k}}M^\dagger$. The degree of divergence for the short-distance behavior is worse since $G(x,y) \sim (-\nabla_{\bar{k}}\nabla_k)^{-1}_{x,y} \sim (x-y)^{-2}$. We will introduce a Pauli-Villars type regulator which corresponds to the replacement

$$\left(\frac{1}{-\bar{D}_{\bar{k}}D_k}\right) \to \left(\frac{1}{-\bar{D}_{\bar{k}}D_k}\right)\left(\frac{\Lambda^2}{-\bar{D}_{\bar{j}}D_j + \Lambda^2}\right) \equiv G_{\mathrm{reg}}(x,y). \quad (33)$$

The parameter Λ^2 [with the dimension of (mass)^2] is the ultraviolet cutoff. The short-distance behavior of this function is given by $G_{\mathrm{reg}}(x,y) \sim \Lambda^2(-\bar{\partial}\partial)^{-2} \sim \Lambda^2\log(x-y)^2$, just as in two dimensions. Thus we get the short-distance behavior

$$\bar{D}_{\bar{k}}G(x,y)]_{\mathrm{reg}} \simeq -\frac{\Lambda^2}{\pi}\frac{(x-y)_k}{|x-y|^2}(M(x)M^{-1}(y)W(y,x)). \quad (34)$$

(The numerical factors are not quite precise; it is immaterial since they can all be absorbed into Λ^2.) As before, defining $\Gamma = \mathrm{Tr}\log(-\bar{D}_{\bar{k}}D_k)$ we find

$$\begin{aligned} \delta\Gamma &= \int d\mu\, \mathrm{Tr}[\delta(-\nabla_k MM^{-1})(-\bar{D}_{\bar{k}}G(x,y))_{\mathrm{reg}}]_{y\to x} \\ &= \frac{\Lambda^2}{\pi}\int d\mu\, \mathrm{Tr}[\delta(-\nabla_k MM^{-1}) \\ &\quad \times (M^{\dagger -1}\nabla_{\bar{k}}M^\dagger + \nabla_{\bar{k}}MM^{-1})]. \end{aligned} \quad (35)$$

We have taken the angular symmetric limit as $y \to x$, so that

$$\left[\frac{(x-y)_k(\bar{x}-\bar{y})_{\bar{a}}}{|x-y|^2}\right]_{y\to x} = c\,\delta_{ka} \quad (36)$$

for some constant c, which has been absorbed into the cutoff Λ^2. We have a similar result for the variation with respect to $M^\dagger$ and the results can be combined to obtain

$$\Gamma = \Lambda^2 S_{4d}(H), \quad (37)$$

where S_{4d} is the four-dimensional WZW action appropriate to a four-dimensional Kähler manifold. This action is basically contained in Donaldson's paper [10], but was independently derived as the boundary action for the Kähler-Chern-Simons theory in [11] in an attempt to generalize conformal field theories to four dimensions. It has since been studied by a number of authors, most notably starting with the work of Losev *et al.* [12]. For an arbitrary matrix N, it is explicitly given by

$$\begin{aligned} S_{4d}(N) &= \frac{1}{2\pi}\int d\mu\, \mathrm{Tr}(\nabla_k N \nabla_{\bar{k}} N^{-1}) \\ &\quad + \frac{i}{12\pi}\int \omega \wedge \mathrm{Tr}(N^{-1}dN)^3, \\ &= \frac{1}{2\pi}\int d\mu g^{a\bar{a}}\, \mathrm{Tr}(\partial_a N \bar{\partial}_{\bar{a}} N^{-1}) \\ &\quad + \frac{i}{12\pi}\int \omega \wedge \mathrm{Tr}(N^{-1}dN)^3, \end{aligned} \tag{38}$$

where ω is the Kähler form for $\mathbb{CP}^2$. For Γ, we need $S_{4d}(M^\dagger M) = S_{4d}(H)$. In the first term of the first line of (38), we are still using the derivatives in the tangent frame (given by the right translation operators on the group element which coordinatizes the manifold). In the second line, we show the expression in terms of the derivatives in the local coordinate description, with $g^{a\bar{a}}$ as the inverse to the Kähler metric $g_{a\bar{a}}$. In local coordinates, the metric and the Kähler form are given by

$$\begin{aligned} ds^2 &= \left[\frac{d\bar{z}\cdot dz}{(1+\bar{z}\cdot z)} - \frac{z\cdot d\bar{z}\,\bar{z}\cdot dz}{(1+\bar{z}\cdot z)^2}\right] \equiv g_{a\bar{a}}dz^a d\bar{z}^{\bar{a}}, \\ \omega &= \frac{i}{2} g_{a\bar{a}} dz^a \wedge d\bar{z}^{\bar{a}}. \end{aligned} \tag{39}$$

In this convention,

$$d\mu = \frac{1}{4}(\det g_{a\bar{a}}) dz^1 d\bar{z}^{\bar{1}} dz^2 d\bar{z}^{\bar{2}} = (\det g) d^4x. \tag{40}$$

This higher dimensional WZW action also satisfies a Polyakov-Wiegmann identity of the form [11,13]

$$\begin{aligned} S_{4d}(NM) &= S_{4d}(N) + S_{4d}(M) \\ &\quad - \frac{1}{\pi}\int d\mu\, \mathrm{Tr}(N^{-1}\nabla_{\bar{a}} N \nabla_a M M^{-1}). \end{aligned} \tag{41}$$

This is easily verified by direct substitution and simplification in (38). This identity shows that Γ in (37) satisfies (35), thereby justifying (37) as the integrated version of (35).

There are a number of refinements to be considered. First of all, so far we have only calculated the leading term proportional to Λ^2; there can be subleading terms and finite terms, which are not captured because of the way we have taken the short distance limit. So the result (37) should, more accurately, be expressed as

$$\mathrm{Tr}\log(-\bar{D}_{\bar{k}} D_k) = \Lambda^2 S_{4d}(H) + \text{subleading} + \text{finite terms}. \tag{42}$$

Second, for the measure calculation, we also need $\det(-D_k \bar{D}_{\bar{k}})$. Notice that if we make the transformation $z^a \leftrightarrow \bar{z}^{\bar{a}}$ and $\nabla_k \leftrightarrow \nabla_{\bar{k}}$ and $M \leftrightarrow M^{\dagger -1}$, then $D_k \leftrightarrow \bar{D}_{\bar{k}}$. So this second determinant is the same as the first with $z^a \leftrightarrow \bar{z}^{\bar{a}}$ and $H \leftrightarrow H^{-1}$. The first term of S_{4d} is obviously unchanged; the second term changes sign under $H \leftrightarrow H^{-1}$ and there is another minus sign from $z^a \leftrightarrow \bar{z}^{\bar{a}}$. So it is unchanged as well as we find

$$\mathrm{Tr}\log(-D_k \bar{D}_{\bar{k}}) = \Lambda^2 S_{4d}(H) + \text{subleading} + \text{finite terms}. \tag{43}$$

Going back to (22), we can now write our result so far as

$$\begin{aligned} d\mu(\mathcal{C}) &= \sqrt{\det \mathcal{M}}\, d\mu(H)[d\phi d\phi^\dagger] \\ &\approx \det(-\bar{D}_{\bar{k}} D_k)\det(-D_m \bar{D}_{\bar{m}}) d\mu(H)[d\phi d\phi^\dagger] \\ &\approx e^{2\Lambda^2 S_{4d}(H)} d\mu(H)[d\phi d\phi^\dagger]. \end{aligned} \tag{44}$$

We will now look at how this result can be improved by some of the Φ, $\Phi^\dagger$-dependent terms. Separating off the $\mathcal{M}_{13}$, $\mathcal{M}_{23}$, etc., we can write $\log\sqrt{\det \mathcal{M}}$ as

$$\begin{aligned} \log\sqrt{\det \mathcal{M}} &= \frac{1}{2}\mathrm{Tr}\log \mathcal{M} \\ &= \mathrm{Tr}\log \mathcal{M}_{11} + \mathrm{Tr}\log \mathcal{M}_{22} + \frac{1}{2}\mathrm{Tr}\log(1+\mathbf{X}) \\ &= \mathrm{Tr}\log(-\bar{D}_{\bar{k}} D_k + \bar{D}_{\bar{k}}\Phi\Phi^\dagger D_k) \\ &\quad + \mathrm{Tr}\log(-D_m \bar{D}_{\bar{m}}) + \frac{1}{2}\mathrm{Tr}\mathbf{X} - \frac{1}{4}\mathrm{Tr}(\mathbf{X}\mathbf{X}) \\ &\quad + \cdots, \\ \mathbf{X} &= \begin{bmatrix} 0 & 0 & \mathcal{M}_{11}^{-1}\mathcal{M}_{13} \\ 0 & 0 & \mathcal{M}_{22}^{-1}\mathcal{M}_{23} \\ \mathcal{M}_{11}^{-1}\mathcal{M}_{31} & \mathcal{M}_{11}^{-1}\mathcal{M}_{32} & 0 \end{bmatrix}. \end{aligned} \tag{45}$$

The term in $\log\sqrt{\det \mathcal{M}}$ which is second order in Φ, $\Phi^\dagger$ is then

$$\begin{aligned} &(\log\sqrt{\det \mathcal{M}})_2 \\ &= \int d\mu_x \mathrm{Tr}[\bar{D}_{\bar{k}}\Phi\Phi^\dagger D_k G(x,y)]_{y\to x} \\ &\quad - 2\int d\mu_x d\mu_y \epsilon_{ik}\epsilon_{mn} \mathrm{Tr}[D_i \Phi^\dagger D_k G(x,y) \bar{D}_{\bar{m}} \Phi \bar{D}_{\bar{n}} G(y,x)] \\ &\quad + 2\int d\mu_x d\mu_y \mathrm{Tr}[D_k \Phi \bar{D}_{\bar{k}} G(x,y) D_m \Phi^\dagger \bar{D}_{\bar{m}} \tilde{G}(y,x)], \end{aligned} \tag{46}$$

$$G(x,y) = (-\bar{D}_{\bar{k}} D_k)^{-1}_{x,y}, \qquad \tilde{G}(x,y) = (-D_k \bar{D}_{\bar{k}})^{-1}_{x,y}. \tag{47}$$

This term can be calculated with a suitable regulator and clearly one can go to higher powers as there is a systematic expansion of (45) in powers of Φ, $\Phi^\dagger$. Nevertheless, it is an involved process and we will postpone further discussion of this. It will not be needed for the arguments presented in the next section. Instead, what we will do here is determine some of the quadratic terms in Φ, $\Phi^\dagger$ using a symmetry argument. For this we go back to the parametrization (7) and (9). Notice that the components of a_i and $\bar{a}_{\bar{i}}$ are all related to the single complex quantity ϕ, as in (11). However, consider evaluating the Jacobian factor for the θ, $\theta^\dagger$ part of $d\mu(\mathcal{C})$ for arbitrary a_i and $\bar{a}_{\bar{i}}$, setting them to the values given in (11) at the end. This can be done by taking variations of (7) and (9) at fixed a_i, $\bar{a}_{\bar{i}}$. Notice that (7) and (9) have something of a "fake gauge symmetry,"

$$M \to M^S = MS, \qquad a_i \to a_i^S = S^{-1} a_i S - S^{-1}\nabla_i S,$$
$$M^\dagger \to M^{\dagger S} = S^{-1} M^\dagger, \qquad \bar{a}_{\bar{i}} \to \bar{a}_{\bar{i}}^S = S^{-1}\bar{a}_{\bar{i}} S + S^{-1}\nabla_{\bar{i}} S. \tag{48}$$

The calculation of the Jacobian must have this symmetry implying that we must consider the gauged version of the WZW action. This is given by

$$S_{4d}(H, a, \bar{a}) = S_{4d}(H) - \frac{1}{\pi}\int d\mu \,\mathrm{Tr}[H^{-1}\nabla_{\bar{i}} H a_i + \bar{a}_{\bar{i}}\nabla_i H H^{-1} + \bar{a}_{\bar{i}} H a_i H^{-1} - \bar{a}_{\bar{i}} a_i]. \tag{49}$$

We can now substitute for a_i, $\bar{a}_{\bar{i}}$ from (11) and simplify to get

$$S_{4d}(H, \chi, \bar{\chi}) = S_{4d}(H) - \frac{1}{\pi}\int d\mu \,\mathrm{Tr}[H^{-1}(\mathcal{D}_a\bar{\chi}) H \mathcal{D}_a \chi - (\mathcal{D}_a\bar{\chi})\mathcal{D}_a\chi - \epsilon^{\bar{a}\bar{b}} H^{-1}\nabla_{\bar{a}} H \mathcal{D}_{\bar{b}}\chi - \epsilon^{ab}\mathcal{D}_b\bar{\chi}\nabla_a H H^{-1}], \tag{50}$$

where, as stated before, $\chi = M^{-1}\phi M$, $\bar{\chi} = M^\dagger \phi^\dagger M^{\dagger -1}$. Since we can consider $[d\phi d\phi^\dagger] = [d\chi d\bar{\chi}]$ as well, we can now summarize our results so far as follows.

$$d\mu(\mathcal{C}) = d\mu(H)[d\chi d\bar{\chi}]\exp[2\Lambda^2 S_{4d}(H, \chi, \bar{\chi}) + \cdots], \tag{51}$$

where the ellipsis denotes terms which are subleading in the divergence, or finite, or involve higher powers of χ, $\bar{\chi}$.

III. DISCUSSION

We are now in a position to discuss the relevance of this result (51) for the functional integration in a gauge theory.

First of all, note that the term S_{4d} is only obtained for the non-Abelian theory. The variables θ^a and ϕ^a transform in the adjoint representation and the trace in S_{4d} is in the same representation, hence vanishing for the Abelian theory. Second, we note that S_{4d} has the properties of a mass term for the gauge fields in the sense of being a gauge-invariant completion of $A_i\bar{A}_{\bar{i}}$. In fact, it is well known that the WZW action in two dimensions is a mass term for the gauge fields [13]; this even goes back to Schwinger's original calculation in the Abelian case. It is also known that such a term defined on a light cone (with a suitable integration over the orientations of the light cone) can describe the screening mass in four-dimensional Yang-Mills theory at finite temperature [14].

Usually, when we integrate over fermions in a four-dimensional gauge theory, there is a quadratic divergence proportional to A^2, but this is generally rejected on the grounds that there is no such term which is both gauge and Lorentz invariant. In other words, there is no such term consistent with gauge invariance and the isometries of the underlying space. (And, indeed, with a gauge- and Lorentz-invariant regularization, no such term is generated.) However, in our case, the term we find is gauge invariant and invariant under the isometries of the space $\mathbb{CP}^2$. Therefore the conclusion is that we must define the gauge theory by including such a term from the beginning with a bare parameter m_0^2, so that

$$d\mu(\mathcal{C}) = d\mu(H)[d\chi d\bar{\chi}]\sqrt{\det \mathcal{M}}\exp[m_0^2 S_{4d}(H, \chi, \bar{\chi})], \\ = d\mu(H)[d\chi d\bar{\chi}]\exp[m_R^2 S_{4d}(H, \chi, \bar{\chi}) + \cdots]. \tag{52}$$

The renormalized value of this parameter, namely m_R^2, then defines a mass scale for the theory. Thus the functional integral for Yang-Mills theory would be defined as

$$Z = \int d\mu(\mathcal{C}) e^{-S_{\rm YM}(A)}, \\ = \int d\mu(H)[d\chi d\bar{\chi}]\sqrt{\det\mathcal{M}} \times \exp[m_0^2 S_{4d}(H, \chi, \bar{\chi})] e^{-S_{\rm YM}(H,\chi,\bar{\chi})}. \tag{53}$$

This is the main conclusion of this paper. For the term $\exp[m_0^2 S_{4d}(H, \chi, \bar{\chi})]$ which we need to have for a well-defined definition of the integration measure, it is sufficient to understand the divergence structure of $\sqrt{\det \mathcal{M}}$. To identify the nature of the terms needed, it is sufficient to calculate the divergent terms in $\sqrt{\det \mathcal{M}}$. This is why we concentrated on such terms in this paper. Eventually, in calculating physical processes, the higher terms with χ, $\bar{\chi}$ will be needed as well.

It is straightforward to take the $R \to \infty$ for the term $m_0^2 S_{4d}(H, \chi, \bar{\chi})$ to obtain the flat space limit. With the metric scaled as indicated in (1), $S_{4d}(H, \chi, \bar{\chi})$ has the dimension of $(\mathrm{mass})^{-2}$, so that m_0^2 is retained as such. But the coordinates of the $\mathbb{R}^4$ would still be organized into two complex coordinates. Viewing this as one choice of local complex structure for $\mathbb{R}^4$, it may be possible to use twistor space and obtain a more symmetric form as $R \to \infty$; this is one of the issues for future work. However, we do emphasize that for any finite value of R, no matter how large, the term $S_{4d}(H, \chi, \bar{\chi})$ is obtained, and hence it will remain relevant to the question of the mass gap. For this question, it is sufficient to consider the case $R \gg m_R^{-1}$, but finite.

The need for a dimensional parameter to define non-Abelian gauge theories in four dimensions is certainly not a surprise. We may in fact view this as a nonperturbative version of the standard dimensional transmutation. It should further be possible to carry out perturbation theory starting from (53)—we will not need gauge-fixing and ghosts—and relate m_R^2 to the Λ-parameter of QCD.

Going back to the role of the volume element, the two-dimensional version of $d\mu(\mathcal{C})$, set into a Hamiltonian formulation, has also been very useful for understanding many features of Yang-Mills theory in three dimensions. The WZW action $S_{wzw}(H)$ from the measure is again crucial for the mass gap in the theory, although it is not directly a 3d-covariant mass term. In fact, generalizing to the extended supersymmetric cases, one can show a complete concordance between such terms (or lack thereof) in the functional measure and the results regarding mass gap expected from other independent considerations. It would be interesting to generalize these considerations to supersymmetric theories in four dimensions by analyzing the measure along the lines of this paper. (There is a small caveat though: Since $\mathbb{CP}^2$ is not a spin-manifold, a spin-$\mathbb{C}$ structure will have to be used.) We will postpone such an analysis to a future work.

The importance of a masslike term for Yang-Mills theory in four dimensions was first emphasized by Cornwall [15] and there have been many attempts to elucidate its origin and implications [16]. Our calculation shows a clear and specific realization of this suggestion.

The masslike term in the functional measure, whether for the wave functions at equal time (as is relevant for a Hamiltonian formulation) or for the Euclidean spacetime functional integral, provides a cutoff on fluctuations of the low momentum modes of the fields and this is the key to the mass gap. It is worth emphasizing that this is a general property of the geometry of the gauge-orbit space and not reliant on any special configurations or matter content.

Some of the properties of $S_{4d}(H)$ as an action in its own right are are also worthy of a few remarks. The equations of motion for this action give anti–self-dual instantons, which are also obviously related to holomorphic vector bundles. It was in this context that Donaldson originally considered this action. The action S_{4d} was obtained in [11] as an attempt to generalize the WZW theory to four dimensions and relate it to the Kähler-Chern-Simons theory as a replay of the CS-CFT correspondence in two-three dimensions [3]. As shown in [11], and elaborated in [12,17], the 4-d theory S_{4d} admits a holomorphically factorized current algebra very similar to the 2-d case. Such theories have also been found in higher dimensional quantum Hall systems [8], and are also realized as the target space dynamics of (world sheet) $N = 2$ heterotic superstrings [18]. Finally, as a small addendum to the remark on the instanton connection, the action $S_{4d}(H)$ evaluated on instantons is a function of the instanton moduli and it would be interesting to see how the integration over the moduli is controlled by the measure in (53).

ACKNOWLEDGMENTS

This work was initiated during a visit to the Niels Bohr International Academy in April-May, 2012. I thank Poul Damgaard and the members of the academy for hospitality and D. O'Connell and R. Monteiro for discussions. I also thank Abhishek Agarwal for useful comments. This research was also supported by the U.S. National Science Foundation Grant No. PHY-1213380 and by PSC-CUNY.

[1] Some of the general issues concerning the volume element for the gauge orbit space are discussed in: I. M. Singer, Phys. Scr. **24**, 817 (1981); Commun. Math. Phys. **60**, 7 (1978); P. K. Mitter and C. M. Viallet, Commun. Math. Phys. **79**, 457 (1981); Phys. Lett. **85B**, 246 (1979); M. Asorey and P. K. Mitter, Commun. Math. Phys. **80**, 43 (1981); O. Babelon and C. M. Viallet, Commun. Math. Phys. **81**, 515 (1981); Phys. Lett. **103B**, 45 (1981); P. Orland, arXiv:hep-th/9607134; Phys. Rev. D **70**, 045014 (2004).

[2] K. Gawedzki and A. Kupiainen, Phys. Lett. **215**, 119 (1988); Nucl. Phys. **B320**, 625 (1989); M. Bos and V. P. Nair, Int. J. Mod. Phys. A **05**, 959 (1990).

[3] E. Witten, Commun. Math. Phys. **121**, 351 (1989).

[4] B. Rusakov, Mod. Phys. Lett. A **05**, 693 (1990); E. Witten, Commun. Math. Phys. **141**, 153 (1991); D. Fine, Commun. Math. Phys. **134**, 273 (1990); M. Blau and G. Thompson, Int. J. Mod. Phys. A **07**, 3781 (1992); D. Gross, Nucl. Phys. **B400**, 161 (1993); D. Gross and W. Taylor, Nucl. Phys. **B400**, 181 (1993).

[5] D. Karabali and V. P. Nair, Nucl. Phys. **B464**, 135 (1996); Phys. Lett. **379**, 141 (1996); D. Karabali, Chanju Kim, and V. P. Nair, Nucl. Phys. **B524**, 661 (1998); Phys. Lett. **434**, 103 (1998); D. Karabali and V. P. Nair, Phys. Rev. D **77**, 025014 (2008); D. Karabali, V. P. Nair, and A. Yelnikov, Nucl. Phys. **B824**, 387 (2010).

[6] For a recent review, see V. P. Nair, Proc. Sci., QCD-TNT09 030.

[7] A. Agarwal and V. P. Nair, Nucl. Phys. **B816**, 117 (2009).

[8] D. Karabali and V. P. Nair, Nucl. Phys. **B641**, 533 (2002); **B679**, 427 (2004); **B697**, 513 (2004); V. P. Nair and S. Randjbar-Daemi, Nucl. Phys. **B679**, 447 (2004); D. Karabali, V. P. Nair and R. Randjbar-Daemi, in *From Fields to Strings: Circumnavigating Theoretical Physics*, Ian Kogan Memorial Collection, edited by M. Shifman, A. Vainshtein, and J. Wheater (World Scientific, Singapore, 2004), pp. 831–876.

[9] A. Agarwal and V. P. Nair, Phys. Rev. D **85**, 085011 (2012).

[10] S. K. Donaldson, Proc. London Math. Soc. **s3-50**, 1 (1985).

[11] V. P. Nair and J. Schiff, Phys. Lett. **246**, 423 (1990); Nucl. Phys. **B371**, 329 (1992).

[12] A. Losev, Gr. Moore, N. Nekrasov, and S. Shatashvili, Nucl. Phys. B, Proc. Suppl. **46**, 130 (1996); Nucl. Phys. **B484**, 196 (1997).

[13] A. M. Polyakov and P. B. Wiegmann, Phys. Lett. **141B**, 223 (1984).

[14] The relationship between the screening mass and the WZW action is given in R. Efraty and V. P. Nair, Phys. Rev. Lett. **68**, 2891 (1992); Phys. Rev. D **47**, 5601 (1993); for earlier work on hard thermal loops, which are the relevant Feynman diagrams for this, see R. Pisarski, Phys. Rev. Lett. **63**, 1129 (1989); E. Braaten and R. Pisarski, Phys. Rev. D **42**, 2156 (1990); Nucl. Phys. **B337**, 569 (1990); Phys. Rev. D **45**, R1827 (1992); J. C. Taylor and S. M. H. Wong, Nucl. Phys. **B346**, 115 (1990).

[15] J. M. Cornwall, Phys. Rev. D **10**, 500 (1974); **26**, 1453 (1982).

[16] The literature on this is enormous, but a good status report and conduit to earlier work is J. M. Cornwall, J. Papavassiliou, and D. Binosi, *The Pinch Technique and its Application to Nonabelian Gauge Theories* (Cambridge University Press, Cambridge, England, 2011); A complementary review is N. Vandersickel and D. Zwanziger, Phys. Rep. **520**, 175 (2012).

[17] T. Inami, H. Kanno, T. Ueno, and C-S. Xiong, Phys. Lett. **399**, 97 (1997); T. Inami, H. Kanno, and T. Ueno, Mod. Phys. Lett. A **12**, 2757 (1997).

[18] H. Ooguri and C. Vafa, Nucl. Phys. **B367**, 83 (1991).