Finite temperature Casimir effect for charged massless scalars in a magnetic field

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The zeta function regularization technique is used to study the finite temperature Casimir effect for a charged and massless scalar field confined between parallel plates and satisfying Dirichlet boundary conditions at the plates. A magnetic field perpendicular to the plates is included. Three equivalent expressions for the zeta function are obtained, which are exact to all orders in the magnetic field strength, temperature and plate distance. These expressions of the zeta function are used to calculate the Helmholtz free energy of the scalar field and the pressure on the plates, in the case of high temperature, small plate distance and strong magnetic field. In all cases, simple analytic expressions are obtained for the free energy and pressure which are accurate and valid for practically all values of temperature, plate distance and magnetic field.

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I. INTRODUCTION

The Casimir effect is a quantum phenomenon where an attractive or repulsive force is observed between electrically neutral conducting plates in vacuum, and can be regarded as a quantitative proof of the quantum fluctuations of the electromagnetic field. Casimir first predicted theoretically the effect, by calculating the attractive electromagnetic force between two parallel conducting plates [1]. The repulsive Casimir effect was discovered by Boyer some time later, when he showed that if the electromagnetic field is confined inside a perfectly conducting sphere, the wall of the sphere is subject to a repulsive force [2]. The first experimental evidence of the Casimir force was obtained more than 50 years ago by Sparnaay [3] and, since then, many greatly improved experimental observations have been reported. For a comprehensive review of these experiments, see the review article and the book by Bordag *et al.* [4,5].

Since Casimir forces have many applications—from nanotubes and nanotechnology [6–9], to branes and compactified extra dimensions [10–31], to string theory [32–35]—a large effort has gone into studying the Casimir effect and its generalization to quantum fields other than the electromagnetic field: fermions were first considered by Johnson [36] in connection with the bag model [37], then investigated by many others; for example [38,39], bosons and other scalar fields have also been investigated extensively [4].

It is well known that Casimir forces are very sensitive to the boundary conditions of the involved quantum fields on the plates. In the case of scalar fields, the most used boundary conditions are Dirichlet and Neumann; in the case of fermion fields or fields with spin, in general [40], bag boundary conditions are used. In this work we will use Dirichlet boundary conditions for a scalar field confined between two parallel plates.

Scalar fields, with or without charge or mass, appear in many different areas of physics. The Higgs field is responsible for spontaneous symmetry breaking in the Standard Model and is a charged massless scalar before the $SU(2)$ gauge symmetry is broken. Once the symmetry is broken, only a neutral massive scalar field remains in the unitary gauge. An ultralight or massless scalar is the dilaton field that breaks the conformal symmetry of strings in superstring theory [41,42]. Massless scalars called inflatons are used to solve the problem of a nonvanishing cosmological constant by causing the accelerated expansion of the Universe [43–45]. In condensed matter physics, scalar fields are important to describe spontaneous breaking of discrete symmetries. The Ginzburg-Landau scalar field is associated with type II superconductors, and it was shown that a description of quantum phase excitations in Ginzburg-Landau superconductors that uses a massless scalar phase field is equivalent to one that uses an antisymmetric Kalb-Ramond field [46]. Scalar fields are also used to explain Landau diamagnetism [47,48], etc. It is well known that the Casimir force between perfectly conducting parallel plates due to the electromagnetic field is obtained by multiplying by a factor of 2 the Casimir force due to a massless scalar field that satisfies Dirichlet boundary conditions on the plates, where the factor of 2 accounts for the two polarization states of the photon. Therefore the Casimir force between perfectly conducting parallel plane surfaces due to a massless, charged scalar field satisfying Dirichlet boundary conditions on the plates will be the same, apart from a multiplicative factor, as the force due to a massless, charged vector field satisfying bag boundary conditions on the plates. Vector fields of this type are the W field before symmetry breaking, or the gluon field in the presence of a chromomagnetic field [49,50].

The Casimir effect for charged scalar fields in a magnetic field has been studied in vacuum [51] and at finite

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temperature [52] using the Schwinger proper time method to calculate the effective action, but these authors are only able to obtain the free energy as an infinite sum of modified Bessel functions. In this paper we use a different method, the zeta function technique, to study the Casimir effect for massless scalar fields at finite temperature and in the presence of a magnetic field. This method allows us to obtain simple analytic forms for the free energy and Casimir pressure, valid for practically all values of the parameters involved. A similar investigation of the Casimir effect for massive scalar fields at finite temperature and in the presence of a magnetic field will be presented elsewhere.

In this paper we will calculate first the Casimir energy for two parallel plates, and then use it to calculate the Casimir force between the plates. While the Casimir force between distinct bodies, such as two parallel plates, is finite, their Casimir energy often needs to be regularized. In the simplest case of parallel plates with Dirichlet boundary conditions, the Casimir energy can be extracted without resorting to regularization. However, since in this work we add the complications of an external magnetic field and of finite temperature, a regularization technique will simplify our calculations, and we need to choose the most suitable regularization technique for our goal. Many regularization techniques are available nowadays, and many of them have been applied successfully to the Casimir effect, the cutoff method often used in various piston configurations [53,54], the world-line technique [55], the multiple-scattering method [56,57], the zeta function technique [58–60], and others. As we stated above, the choice for this paper is the zeta function technique, a powerful regularization technique used also in the computation of effective actions [61,62]. While this method is very powerful and convenient, it should be used with caution since it is the least physical of all methods, sweeping all but the logarithmic divergencies under the rug. We apply this regularization to obtain the free energy and Casimir pressure due to a scalar field confined between two parallel plates, at a distance a from each other. We assume Dirichlet boundary conditions on the plates and take our system to be in thermal equilibrium with a heat reservoir at finite temperature T, using the imaginary time formalism of finite temperature field theory, which is suitable for a system in thermal equilibrium. A uniform magnetic field $\vec{B}$ is present in the region between the plates and is perpendicular to the plates.

In Sec. II, we obtain three equivalent expressions of the zeta function for this system, exact to all orders in eB, T and a, where e is the scalar field charge. We also obtain simple analytic expressions for the zeta function in the case of high temperature ($T \gg a^{-1}, \sqrt{eB}$), small plate distance ($a^{-1} \gg T, \sqrt{eB}$), and strong magnetic field ($\sqrt{eB} \gg a^{-1}, T$). In Sec. III, we use the zeta function obtained in the previous section, to calculate the Helmholtz free energy of the scalar field and the pressure on the plates and to obtain simple analytic expressions for these quantities in the case of high temperature, small plate distance, and strong magnetic field. A discussion of our results is presented in Sec. IV.

II. ZETA FUNCTION EVALUATION

Using the imaginary time formalism of finite temperature field theory, we write the partition function $\mathcal{Z}$ for a bosonic system in thermal equilibrium at finite temperature T,

$$\mathcal{Z} = N \int_{\text{Periodic}} D\phi^{\star} D\phi \exp\left(\int_0^{\beta} d\tau \int d^3x \mathcal{L}\right), \quad (1)$$

where $\mathcal{L}$ is the Lagrangian density for the bosonic system, N is a constant, and "periodic" means that this functional integral is evaluated over field configurations satisfying

$$\phi(x, y, z, \tau) = \phi(x, y, z, \tau + \beta), \quad (2)$$

for any τ, where $\beta = 1/T$ is the periodic length in the Euclidean time axis. In addition to the finite temperature boundary conditions given by (2), we impose Dirichlet boundary conditions for scalar bosons between two square plates. In three-dimensional space with two large parallel plates perpendicular to the z axis and located at $z = 0$ and $z = a$, the Dirichlet boundary conditions constrain the scalar field to vanish at the plates,

$$\phi(x, y, 0, \tau) = \phi(x, y, a, \tau) = 0. \quad (3)$$

In the slab region there is also a uniform magnetic field pointing in the z direction, $\vec{B} = (0, 0, B)$. The scalar field has charge e and thus will interact with the magnetic field.

The scalar field Helmholtz free energy F and partition function $\mathcal{Z}$ are related by

$$F = -\beta^{-1} \log \mathcal{Z}. \quad (4)$$

A straightforward evaluation of the functional integral (1) yields

$$\log \mathcal{Z} = -\log \det(-D_{\mathrm{E}}|\mathcal{F}_a), \quad (5)$$

where the symbol $\mathcal{F}_a$ indicates the set of functions which satisfy boundary conditions (2) and (3), and the operator D_{E} is defined as

$$D_{\mathrm{E}} = \partial_\tau^2 + \partial_z^2 - (\vec{p} - e\vec{A})_\perp^2, \quad (6)$$

where $\vec{A}$ is the electromagnetic vector potential, the subscript E indicates Euclidean time, and we use the notation $\vec{p}_\perp = (p_x, p_y, 0)$.

The zeta function technique allows us to use the eigenvalues of D_{E} to evaluate $\log \mathcal{Z}$. The Dirichlet boundary conditions (3) are satisfied only if the allowed values for the momentum in the z direction are

$$p_z = \frac{\pi}{a} n, \quad (7)$$

where $n \in \{0, 1, 2, 3, \ldots\}$, and therefore the eigenvalues of $-\partial_\tau^2 - \partial_z^2$ whose eigenfunctions satisfy (2) and (3) are

$$\frac{\pi^2}{a^2} n^2 + \frac{4\pi^2}{\beta^2} m^2, \tag{8}$$

where $n \in \{0, 1, 2, 3, \ldots\}$ and $m \in \{0, \pm 1, \pm 2, \pm 3, \ldots\}$. The spectrum of the operator $(\vec{p} - e\vec{A})_\perp^2$ is well known from one-particle quantum mechanics, and its eigenvalues are the Landau levels

$$2eB\left(l + \frac{1}{2}\right), \tag{9}$$

with $l \in \{0, 1, 2, 3, \ldots\}$. Using the eigenvalues (8) and (9), we construct the zeta function $\zeta(s)$, which is given by

$$\zeta(s) = L^2\left(\frac{1}{2}\right) \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left(\frac{eB}{2\pi}\right) \times \sum_{l=0}^{\infty} \left[\frac{\pi^2}{a^2} n^2 + \frac{4\pi^2}{\beta^2} m^2 + eB(2l+1)\right]^{-s}, \tag{10}$$

where L^2 is the area of the plates and the factor $eB/2\pi$ takes into account the degeneracy per unit area of the Landau levels. In principle, summation in the index n should run from 0 to ∞. However, since n appears only squared, we run the summation from $-\infty$ to ∞ by including a factor of $1/2$. Note that with this procedure only half the $n = 0$ term is taken into account. This does not affect the physical result because the $n = 0$ term contributes to the Casimir energy a uniform energy density term, and such terms, as we will discuss in Sec. III, do not contribute to the Casimir pressure.

Once we put ζ in a suitable closed form, using the zeta function technique we will immediately obtain the partition function

$$\log Z = \zeta'(0), \tag{11}$$

and then the free energy using (4),

$$F = -\beta^{-1}\zeta'(0). \tag{12}$$

With the help of the following identity,

$$z^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} e^{-zt}, \tag{13}$$

where $\Gamma(s)$ is the Euler gamma function, we rewrite $\zeta(s)$ as

$$\zeta(s) = \frac{L^2}{8\pi\Gamma(s)} \int_0^\infty dt t^{s-2} \frac{eBt}{\sinh eBt} \left(\sum_{n=-\infty}^{\infty} e^{-\frac{\pi^2}{a^2} n^2 t}\right) \times \left(\sum_{m=-\infty}^{\infty} e^{-\frac{4\pi^2}{\beta^2} m^2 t}\right), \tag{14}$$

where we also used

$$\sum_{l=0}^{\infty} e^{-(2l+1)z} = \frac{1}{2\sinh z}. \tag{15}$$

The derivative of the zeta function is obtained easily by taking advantage of the useful fact that, for a well-behaved $G(s)$, the derivative of $G(s)/\Gamma(s)$ at $s = 0$ is simply $G(0)$, and therefore, using (12) and (14), we find the free energy

$$F = -\frac{L^2}{8\pi\beta} \int_0^\infty dt t^{-2} \frac{eBt}{\sinh eBt} \left(\sum_{n=-\infty}^{\infty} e^{-\frac{\pi^2}{a^2} n^2 t}\right) \times \left(\sum_{m=-\infty}^{\infty} e^{-\frac{4\pi^2}{\beta^2} m^2 t}\right). \tag{16}$$

Using the Poisson resummation formula [63], we are able to obtain three other expressions of the free energy, all equivalent to (16),

$$F = -\frac{L^2 a}{8\pi^{3/2}\beta} \int_0^\infty dt t^{-5/2} \frac{eBt}{\sinh eBt} \left(\sum_{n=-\infty}^{\infty} e^{-\frac{n^2 a^2}{t}}\right) \times \left(\sum_{m=-\infty}^{\infty} e^{-\frac{4\pi^2}{\beta^2} m^2 t}\right), \tag{17}$$

best suited for high temperature expansion ($2Ta \gg 1$ and $2T \gg \sqrt{eB}/\pi$),

$$F = -\frac{L^2}{16\pi^{3/2}} \int_0^\infty dt t^{-5/2} \frac{eBt}{\sinh eBt} \left(\sum_{n=-\infty}^{\infty} e^{-\frac{\pi^2}{a^2} n^2 t}\right) \times \left(\sum_{m=-\infty}^{\infty} e^{-\frac{m^2 \beta^2}{4t}}\right), \tag{18}$$

best suited for small plate distance expansion ($2Ta \ll 1$ and $a^{-1} \gg \sqrt{eB}/\pi$), and

$$F = -\frac{L^2 a}{16\pi^2} \int_0^\infty dt t^{-3} \frac{eBt}{\sinh eBt} \left(\sum_{n=-\infty}^{\infty} e^{-\frac{n^2 a^2}{t}}\right) \times \left(\sum_{m=-\infty}^{\infty} e^{-\frac{m^2 \beta^2}{4t}}\right), \tag{19}$$

best suited for strong magnetic field expansion. The last equation has been obtained by other authors [52], who used (19) to write the free energy as an infinite sum of modified Bessel functions.

It is not possible to evaluate (14) in closed form for arbitrary values of B, a and β, but it is possible to find simple expressions for $\zeta(s)$ when one or some of B, a and T are small or large. From these simple expressions of the zeta function, the free energy will be obtained immediately.

First we evaluate $\zeta(s)$ in the high temperature limit. To do so, we apply the Poisson resummation formula to the n sum in (14) and obtain

$$\zeta(s) = a[\zeta_B(s) + \zeta_{B,a}(s) + \tilde{\zeta}_{B,T}(s) + \zeta_{B,a,T}(s)], \tag{20}$$

where

$$\zeta_B(s) = \frac{L^2}{8\pi^{3/2}\Gamma(s)} \int_0^\infty dt t^{s-5/2} \frac{eBt}{\sinh eBt}, \tag{21}$$

$$\tilde{\zeta}_{B,T}(s) = \frac{L^2}{4\pi^{3/2}\Gamma(s)} \sum_{m=1}^{\infty} \int_0^{\infty} dt t^{s-5/2} \frac{eBt}{\sinh eBt} e^{-4\pi^2 m^2 t/\beta^2}, \tag{22}$$

$$\zeta_{B,a}(s) = \frac{L^2}{4\pi^{3/2}\Gamma(s)} \sum_{n=1}^{\infty} \int_0^{\infty} dt t^{s-5/2} \frac{eBt}{\sinh eBt} e^{-n^2 a^2/t}, \tag{23}$$

$$\zeta_{B,a,T}(s) = \frac{L^2}{2\pi^{3/2}\Gamma(s)} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \int_0^{\infty} dt t^{s-5/2} \frac{eBt}{\sinh eBt} \times e^{-(n^2 a^2/t + 4\pi^2 m^2 t/\beta^2)}. \tag{24}$$

After changing the integration variable from t to $tna\beta/2\pi m$ in (24), we obtain

$$\zeta_{B,a,T}(s) = \frac{L^2}{2\pi^{3/2}\Gamma(s)} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{na\beta}{2\pi m}\right)^{s-1/2} \int_0^{\infty} dt t^{s-5/2} \times \frac{eBt}{\sinh\left(\frac{eBtna\beta}{2\pi m}\right)} e^{-2\pi nma(t+1/t)/\beta}. \tag{25}$$

When $2aT \gg 1$, only the term with $n = m = 1$ contributes significantly to the double sum so, using the saddle point method, we evaluate the integral for $eB \ll 4\pi^2 T^2$ and obtain

$$\zeta_{B,a,T}(s) = \frac{L^2 eB}{2\pi a\Gamma(s)} \left(\frac{a\beta}{2\pi}\right)^s \frac{e^{-4\pi a/\beta}}{\sinh\left(\frac{eBa\beta}{2\pi}\right)}. \tag{26}$$

Next we evaluate (22) for $eB \ll 4\pi^2 T^2$. In this case, we can set

$$\frac{eBt}{\sinh eBt} \approx 1 - \frac{1}{6}(eBt)^2 \tag{27}$$

and, after substituting (27) into (22), we integrate to find

$$\tilde{\zeta}_{B,T}(s) = \frac{L^2}{\Gamma(s)} \left(\frac{\beta}{2\pi}\right)^{2s} \left[\frac{2\pi^{3/2}}{\beta^3} \Gamma\left(s - \frac{3}{2}\right) \zeta_R(2s - 3) - \frac{e^2 B^2 \beta}{48\pi^{5/2}} \Gamma\left(s + \frac{1}{2}\right) \zeta_R(2s + 1)\right], \tag{28}$$

where ζ_R is the Riemann zeta function of number theory. For calculating the free energy, we only need to know $\zeta(s)$ for $s \to 0$. For small s we have

$$z^{2s} \zeta_R(2s - 3) \frac{\Gamma(s - \frac{3}{2})}{\Gamma(s)} = \frac{\sqrt{\pi}}{90} s + \mathcal{O}(s^2), \tag{29}$$

and

$$z^{2s} \zeta_R(2s + 1) \frac{\Gamma(s + \frac{1}{2})}{\Gamma(s)} = \frac{\sqrt{\pi}}{2} + \sqrt{\pi}\left(\gamma_E + \ln\frac{z}{2}\right) s + \mathcal{O}(s^2), \tag{30}$$

where $\gamma_E = 0.5772$ is the Euler Mascheroni constant. Substituting (29) and (30) into (28), we obtain

$$\tilde{\zeta}_{B,T}(s) = L^2 \left[\frac{\pi^2}{45\beta^3} - \frac{e^2 B^2 \beta}{48\pi^2} \left(\frac{1}{2s} + \gamma_E + \ln\frac{\beta}{4\pi}\right)\right] s, \tag{31}$$

valid for $eB \ll 4\pi^2 T^2$ and small s. Notice that Eq. (22) is not valid for $s = 0$ but, after identifying the presence of Riemann zeta functions and Euler gamma functions in this equation, and assuming that an analytical continuation over the whole complex plane is subtended for these functions, expressions like Eq. (28) are well behaved for $s \to 0$. The same is true for Eqs. (21) and (23): they are not valid for $s = 0$ but, once Riemann zeta functions and Euler gamma functions are identified inside these equations and analytic continuation is subtended, they will become well behaved for $s \to 0$.

In the high temperature limit, both $eBa^2 \ll 1$ and $eBa^2 \gg 1$ are possible, and therefore we need to evaluate (23) for both scenarios. When $eBa^2 \ll 1$ we use (27) in (23), integrate, and find

$$\zeta_{B,a}(s) = \frac{L^2 a^{2s}}{4\pi^{3/2}\Gamma(s)} \left[\frac{1}{a^3} \Gamma\left(\frac{3}{2} - s\right) \zeta_R(3 - 2s) - \frac{e^2 B^2 a}{6} \Gamma\left(-\frac{1}{2} - s\right) \zeta_R(-1 - 2s)\right], \tag{32}$$

which, for small s, becomes

$$\zeta_{B,a}(s) = \frac{L^2}{8\pi} \left[\frac{\zeta_R(3)}{a^3} - \frac{e^2 B^2 a}{18}\right] s, \tag{33}$$

where $\zeta_R(3) = 1.2021$. When $eBa^2 \gg 1$ we use

$$\frac{1}{\sinh eBt} \approx 2e^{-eBt} \tag{34}$$

in (23), change the integration variable from t to $\frac{na}{\sqrt{eB}} t$, and find

$$\zeta_{B,a}(s) = \frac{L^2 eB}{2\pi^{3/2}\Gamma(s)} \sum_{n=1}^{\infty} \left(\frac{na}{\sqrt{eB}}\right)^{s-1/2} \times \int_0^{\infty} dt t^{s-3/2} e^{-\sqrt{eB} na(t+1/t)}. \tag{35}$$

Only the term with $n = 1$ contributes significantly to the sum when $eBa^2 \gg 1$ and, using the saddle point method, we evaluate the integral and find

$$\zeta_{B,a}(s) = \frac{L^2 eB}{2\pi a\Gamma(s)} \left(\frac{a}{\sqrt{eB}}\right)^s e^{-2\sqrt{eB} a}. \tag{36}$$

Finally, we calculate $\zeta_B(s)$, the only piece of the zeta function that can be evaluated exactly and, after integrating, we find

$$\zeta_B(s) = \frac{L^2 (eB)^{3/2-s}}{4\pi^{3/2}\Gamma(s)} (1 - 2^{1/2-s}) \Gamma\left(s - \frac{1}{2}\right) \zeta_R\left(s - \frac{1}{2}\right), \tag{37}$$

which, for small s, becomes

$$\zeta_B(s) = \frac{L^2(eB)^{3/2}}{2\pi}(\sqrt{2}-1)\zeta_R\left(-\frac{1}{2}\right)s, \tag{38}$$

where $\zeta_R(-\frac{1}{2}) = -0.2079$.

By adding (26), (31), (33), and (38), we find $\zeta(s)$ in the high temperature and very weak field limit, $2T \gg a^{-1}$, $2T \gg \sqrt{eB}/\pi$, and $eB \ll a^{-2}$,

$$\zeta(s) = L^2\left[\frac{\pi^2 a}{45\beta^3} + \frac{(eB)^{3/2}a}{2\pi}(\sqrt{2}-1)\zeta_R\left(-\frac{1}{2}\right) + \frac{\zeta_R(3)}{8\pi a^2} + \frac{eB}{2\pi}\frac{e^{-4\pi a/\beta}}{\sinh(\frac{eBa\beta}{2\pi})} - \frac{e^2B^2a^2}{144\pi} - \frac{e^2B^2\beta a}{48\pi^2}\left(\frac{1}{2s} + \gamma_E + \ln\frac{\beta}{4\pi}\right)\right]s, \tag{39}$$

where we took the small s limit. By adding (26), (31), (36), and (38), we find $\zeta(s)$ in the high temperature and very large plate distance limit, $2T \gg a^{-1}$, $2T \gg \sqrt{eB}/\pi$, and $eB \gg a^{-2}$,

$$\zeta(s) = L^2\left[\frac{\pi^2 a}{45\beta^3} + \frac{(eB)^{3/2}a}{2\pi}(\sqrt{2}-1)\zeta_R\left(-\frac{1}{2}\right) + \frac{eB}{2\pi}\frac{e^{-4\pi a/\beta}}{\sinh(\frac{eBa\beta}{2\pi})} + \frac{eB}{2\pi}e^{-2\sqrt{eB}a} - \frac{e^2B^2\beta a}{48\pi^2}\left(\frac{1}{2s} + \gamma_E + \ln\frac{\beta}{4\pi}\right)\right]s, \tag{40}$$

where we also took the small s limit.

Next we evaluate $\zeta(s)$ in the limit of small plate distance and apply the Poisson resummation formula to the m sum in (14) to obtain

$$\zeta(s) = \frac{\beta}{2}[\zeta_B(s) + \tilde{\zeta}_{B,a}(s) + \zeta_{B,T}(s) + \tilde{\zeta}_{B,a,T}(s)], \tag{41}$$

where $\zeta_B(s)$ is the same as in (21), and

$$\tilde{\zeta}_{B,a}(s) = \frac{L^2}{4\pi^{3/2}\Gamma(s)}\sum_{n=1}^{\infty}\int_0^\infty dt t^{s-5/2}\frac{eBt}{\sinh eBt}e^{-\pi^2n^2t/a^2}, \tag{42}$$

$$\zeta_{B,T}(s) = \frac{L^2}{4\pi^{3/2}\Gamma(s)}\sum_{m=1}^{\infty}\int_0^\infty dt t^{s-5/2}\frac{eBt}{\sinh eBt}e^{-m^2\beta^2/4t}, \tag{43}$$

$$\tilde{\zeta}_{B,a,T}(s) = \frac{L^2}{2\pi^{3/2}\Gamma(s)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\int_0^\infty dt t^{s-5/2} \times \frac{eBt}{\sinh eBt}e^{-(\pi^2n^2t/a^2 + m^2\beta^2/4t)}. \tag{44}$$

It is evident from (41)–(44) that $\zeta(s)$, in the limits $2aT \ll 1$ and $eB \ll \pi^2 a^{-2}$, is obtained from (20)–(24) by replacing a with $\beta/2$ and β with $2a$. For $eB(\frac{\beta}{2})^2 \ll 1$ and small s, we find

$$\zeta(s) = L^2\left[\frac{\pi^2\beta}{720a^3} + \frac{(eB)^{3/2}\beta}{4\pi}(\sqrt{2}-1)\zeta_R\left(-\frac{1}{2}\right) + \frac{\zeta_R(3)}{2\pi\beta^2} + \frac{eB}{2\pi}\frac{e^{-\pi\beta/a}}{\sinh(\frac{eBa\beta}{2\pi})} - \frac{e^2B^2\beta^2}{576\pi} - \frac{e^2B^2\beta a}{48\pi^2}\left(\frac{1}{2s} + \gamma_E + \ln\frac{a}{2\pi}\right)\right]s, \tag{45}$$

and for $eB(\frac{\beta}{2})^2 \gg 1$ and small s, we find

$$\zeta(s) = L^2\left[\frac{\pi^2\beta}{720a^3} + \frac{(eB)^{3/2}\beta}{4\pi}(\sqrt{2}-1)\zeta_R\left(-\frac{1}{2}\right) + \frac{eB}{2\pi}\frac{e^{-\pi\beta/a}}{\sinh(\frac{eBa\beta}{2\pi})} + \frac{eB}{2\pi}e^{-\sqrt{eB}\beta} - \frac{e^2B^2\beta a}{48\pi^2}\left(\frac{1}{2s} + \gamma_E + \ln\frac{a}{2\pi}\right)\right]s. \tag{46}$$

Last, we evaluate $\zeta(s)$ in the strong magnetic field limit, $eB \gg (\frac{\beta}{2})^{-2}$ and $eB \gg a^{-2}$. Under these conditions, after applying the Poisson resummation formula to both the n and m sums in (14), we find

$$\zeta(s) = a\beta[\zeta_W(s) + \tilde{\zeta}(s)] \tag{47}$$

where

$$\zeta_W(s) = \frac{L^2}{16\pi^2\Gamma(s)}\int_0^\infty dt t^{s-3}\frac{eBt}{\sinh eBt} \tag{48}$$

is the zeta function of the one-loop vacuum effective Lagrangian for massless scalar QED first calculated by Weisskopf, and

$$\tilde{\zeta}(s) = \frac{L^2}{16\pi^2\Gamma(s)}\int_0^\infty dt t^{s-3}\frac{eBt}{\sinh eBt} \times \left(\sum_{n,m=-\infty}^{\infty} e^{-a^2n^2/t}e^{-\beta^2m^2/4t} - 1\right). \tag{49}$$

The integral in (48) can be evaluated exactly, and we find

$$\zeta_W(s) = \frac{L^2(eB)^{2-s}}{8\pi^2\Gamma(s)}(1-2^{1-s})\Gamma(s-1)\zeta_R(s-1), \tag{50}$$

which, for small s, becomes

$$\zeta_W(s) = \frac{L^2e^2B^2}{96\pi^2}\left(\ln eB - \ln 3 - \frac{1}{2} - \frac{1}{s}\right)s, \tag{51}$$

where we used the interesting numerical fact [64]

$$\gamma_E + \ln\pi - \frac{6}{\pi^2}\zeta'(2) \simeq \ln 6 + \frac{1}{2}. \tag{52}$$

We evaluate $\tilde{\zeta}(s)$ by using (34), which is valid in the strong magnetic field limit; we then change the integration variable from t to $\sqrt{\frac{n^2a^2+m^2\beta^2/4}{eB}}t$, to find

$$\tilde{\zeta}(s) = \frac{L^2 eB}{8\pi^2\Gamma(s)} \sum_{n,m=-\infty}^{\infty} \left(\frac{n^2a^2 + m^2\beta^2/4}{eB}\right)^{\frac{s-1}{2}} \times \int_0^\infty dt t^{s-2} e^{-(t+1/t)\sqrt{eB}\sqrt{n^2a^2+m^2\beta^2/4}}, \tag{53}$$

where the term with $m = n = 0$ is excluded and only terms with $n = 0, \pm 1$ and $m = 0, \pm 1$ contribute significantly to the double sum. We integrate using the saddle point method and, for small s, obtain

$$\tilde{\zeta}(s) = \frac{L^2(eB)^{5/4}}{2\pi^{3/2}} \left[\frac{e^{-2a\sqrt{eB}}}{2a^{3/2}} + \frac{\sqrt{2}e^{-\beta\sqrt{eB}}}{\beta^{3/2}} + \frac{e^{-2\sqrt{eB}\sqrt{a^2+\beta^2/4}}}{(a^2 + \beta^2/4)^{3/4}}\right] s. \tag{54}$$

Adding (51) to (54) we find the zeta function in the limit of the strong magnetic field, $eB \gg a^{-2}$ and $eB \gg (\frac{\beta}{2})^{-2}$, and small s

$$\zeta(s) = L^2 a\beta\left[\frac{e^2B^2}{96\pi^2}\left(\ln eB - \ln 3 - \frac{1}{2} - \frac{1}{s}\right) + \frac{(eB)^{5/4}}{2\pi^{3/2}} \times \left(\frac{e^{-2a\sqrt{eB}}}{2a^{3/2}} + \frac{\sqrt{2}e^{-\beta\sqrt{eB}}}{\beta^{3/2}} + \frac{e^{-2\sqrt{eB}\sqrt{a^2+\beta^2/4}}}{(a^2 + \beta^2/4)^{3/4}}\right)\right] s. \tag{55}$$

III. FREE ENERGY AND CASIMIR PRESSURE

It is not possible to evaluate the free energy, (16)–(19), in closed form for arbitrary values of B, a and β but, using our results from Sec. II, we found simple analytic expressions for the free energy when one or some of those three quantities are small or large. The free energy in the high temperature limit, $2T \gg a^{-1}$ and $2T \gg \sqrt{eB}/\pi$, differs from the high temperature limit of the zeta function by a simple factor only; therefore, we divide (39) by $-\beta s$ and find F in the high temperature limit for $eB \ll a^{-2}$, and we divide (40) by the same quantity to find the high temperature limit of F for $eB \gg a^{-2}$. Notice that, once we divide (39) and (40) by $-\beta s$, the dominant term is the Stefan-Boltzmann term $-\frac{\pi^2}{45}VT^4$, where $V = L^2 a$ is the volume of the slab. Terms with a linear dependence on the plate distance, such as this one, are proportional to the volume of the slab and represent a uniform energy density. If the same magnetic field is present outside the slab and the medium outside the slab is also at temperature T, such terms do not contribute to the Casimir pressure. If there is vacuum outside the slab, i.e. no magnetic field and zero temperature, uniform energy density terms contribute a constant pressure which is very easily calculated. In this paper we assume that the same magnetic field is present inside and outside the slab, and that the medium outside the slab is at the same temperature as the one inside the slab, so we neglect contributions to the Casimir pressure from uniform energy density terms.

The pressure P on the plates is given by

$$P = -\frac{1}{L^2}\frac{\partial F}{\partial a}, \tag{56}$$

and therefore, for $2T \gg a^{-1} \gg \sqrt{eB}/\pi$, we find

$$P = -\frac{\zeta_R(3)}{4\pi\beta a^3} - \frac{2eB}{\beta^2}\frac{e^{-4\pi a/\beta}}{\sinh(\frac{eBa\beta}{2\pi})} - \frac{e^2B^2 e^{-4\pi a/\beta}}{4\pi^2}\frac{\coth(\frac{eBa\beta}{2\pi})}{\sinh(\frac{eBa\beta}{2\pi})} - \frac{e^2B^2 a}{72\pi\beta} \tag{57}$$

and

$$P = -\frac{2eB}{\beta^2}\frac{e^{-4\pi a/\beta}}{\sinh(\frac{eBa\beta}{2\pi})} - \frac{e^2B^2 e^{-4\pi a/\beta}}{4\pi^2}\frac{\coth(\frac{eBa\beta}{2\pi})}{\sinh(\frac{eBa\beta}{2\pi})} - \frac{(eB)^{3/2}}{\pi\beta}e^{-2\sqrt{eB}a}, \tag{58}$$

for $2T \gg \sqrt{eB}/\pi \gg a^{-1}$. Since the third term in (57) is negligible when compared to the other ones in (57) and (58), we write the high temperature Casimir pressure as

$$P = -\frac{\zeta_R(3)}{4\pi\beta a^3} - \frac{2eB}{\beta^2}\frac{e^{-4\pi a/\beta}}{\sinh(\frac{eBa\beta}{2\pi})} - \frac{e^2B^2 a}{72\pi\beta}, \tag{59}$$

for $2T \gg a^{-1} \gg \sqrt{eB}/\pi$, and

$$P = -\frac{2eB}{\beta^2}\frac{e^{-4\pi a/\beta}}{\sinh(\frac{eBa\beta}{2\pi})} - \frac{(eB)^{3/2}}{\pi\beta}e^{-2\sqrt{eB}a}, \tag{60}$$

for $2T \gg \sqrt{eB}/\pi \gg a^{-1}$.

To obtain the free energy in the small plate distance limit, we divide (45) and (46) by $-\beta s$, and find F for $a^{-1} \gg 2T \gg \sqrt{eB}/\pi$ and for $a^{-1} \gg \sqrt{eB}/\pi \gg 2T$, respectively. The dominant term here is $-\frac{\pi^2}{720}\frac{L^2}{a^3}$, which is the familiar vacuum Casimir energy for a complex scalar field and for the photon field [1]. The Casimir pressure for small plate distance is

$$P = -\frac{\pi^2}{240a^4} + \frac{eB}{2a^2}\frac{e^{-\pi\beta/a}}{\sinh(\frac{eBa\beta}{2\pi})} - \frac{e^2B^2 e^{-\pi\beta/a}}{4\pi^2}\frac{\coth(\frac{eBa\beta}{2\pi})}{\sinh(\frac{eBa\beta}{2\pi})} - \frac{e^2B^2}{48\pi^2}\left(\ln\frac{a}{2\pi} + 1\right) \tag{61}$$

in the case of a very weak magnetic field ($a^{-1} \gg 2T \gg \sqrt{eB}/\pi$), and it is identical in the case of very low temperature ($a^{-1} \gg \sqrt{eB}/\pi \gg 2T$). Since the third term in (61) is much smaller than the other ones, we can neglect it and write the pressure in the small plate distance limit as

$$P = -\frac{\pi^2}{240a^4} + \frac{eB}{2a^2}\frac{e^{-\pi\beta/a}}{\sinh(\frac{eBa\beta}{2\pi})} - \frac{e^2B^2}{48\pi^2}\left(\ln\frac{a}{2\pi} + 1\right). \tag{62}$$

Finally, for a strong magnetic field the free energy is found using (55),

$$F = -L^2 a\left[\frac{e^2B^2}{96\pi^2}\left(\ln eB - \ln 3 - \frac{1}{2}\right) + \frac{(eB)^{5/4}}{2\pi^{3/2}} \times \left(\frac{e^{-2a\sqrt{eB}}}{2a^{3/2}} + \frac{\sqrt{2}e^{-\beta\sqrt{eB}}}{\beta^{3/2}} + \frac{e^{-2\sqrt{eB}\sqrt{a^2+\beta^2/4}}}{(a^2+\beta^2/4)^{3/4}}\right)\right], \quad (63)$$

where the dominant term is the one-loop vacuum effective potential for massless scalar QED [64], and it is proportional to the volume of the slab, as expected. The effective potential is a uniform energy density term and therefore, under our assumptions, does not contribute to the Casimir pressure. The pressure, for $eB \gg (\frac{\beta}{2})^{-2}$ and $eB \gg a^{-2}$, is given by

$$P = -\frac{(eB)^{5/4}}{2\pi^{3/2}\sqrt{a}}e^{-2a\sqrt{eB}}\left(\sqrt{eB} + \frac{1}{4a}\right) + \frac{(eB)^{5/4}}{2\pi^{3/2}}\frac{e^{-2\sqrt{eB}\sqrt{a^2+\beta^2/4}}}{(a^2+\beta^2/4)^{3/4}} \times \left(1 - \frac{2a^2\sqrt{eB}}{\sqrt{a^2+\beta^2/4}} - \frac{3}{2}\frac{a^2}{a^2+\beta^2/4}\right), \quad (64)$$

and, neglecting the smaller terms, we obtain

$$P = -\frac{(eB)^{7/4}}{\pi^{3/2}}\left(\frac{e^{-2a\sqrt{eB}}}{2\sqrt{a}} + \frac{a^2 e^{-2\sqrt{eB}\sqrt{a^2+\beta^2/4}}}{(a^2+\beta^2/4)^{5/4}}\right). \quad (65)$$

IV. DISCUSSION AND CONCLUSIONS

In this paper we used the zeta function regularization technique to study the finite temperature Casimir effect of a massless charged scalar field confined between parallel plates and in the presence of a magnetic field perpendicular to the plates. We have obtained three expressions for the zeta function, (20), (41), and (47), which are exact to all orders in the magnetic field strength B, plate distance a and inverse temperature β, and we have used them to derive expressions for the Helmholtz free energy and for the Casimir pressure on the plates in the case of high temperature ($4T^2 \gg a^{-2}$, eB/π^2), small plate distance ($a^{-2} \gg 4T^2$, eB/π^2) and strong magnetic field ($eB \gg a^{-2}, 4T^2$).

We have been able to numerically evaluate the free energy with very high precision, using the three exact expressions (17)–(19), and we compared the values of the free energy obtained from our simple analytic expressions to the exact numerical values. In the high temperature case we found that, for $2aT = 4$, the high temperature limit of $\zeta(s)$ from Eq. (39) gives a value of F that is within 0.7 percent of the exact value of the free energy in the range $0 \leq eBa^2 \leq 1$, while $\zeta(s)$ from Eq. (40) gives a value of F that is within 0.7 percent of the exact value of the free energy in the range $1 \leq eBa^2 \leq \infty$. For $2aT = 10$, $\zeta(s)$ from Eq. (39) gives a value of F that is within 0.05 percent of the exact value of the free energy in the range $0 \leq eBa^2 \leq 1$, while $\zeta(s)$ from Eq. (40) gives a value of F that is within 0.05 percent of the exact value of the free energy in the range $1 \leq eBa^2 \leq \infty$, showing a very rapid decrease of the small discrepancy between F obtained from Eqs. (39) and (40) and the exact values of the free energy. We summarize this finding by stating that the free energy in the high temperature limit as obtained from Eqs. (39) and (40) is a simple analytic expression of F in the high temperature limit, valid for all values of the magnetic field B and the plate distance a, and with a discrepancy of no more than 0.7 percent from the exact value of F for $2aT \geq 4$. A similarly accurate expression of the Casimir pressure P, valid for $2aT \geq 4$ and all values of a and B, is obtained immediately from this simple expression of F, since $P = -\frac{1}{L^2}\frac{\partial F}{\partial a}$. To roughly indicate in what regimes of temperature, magnetic field, and plate separation our simple expression of F holds, we give two numerical examples for the high temperature limit, one with $T = 10^4$ K and the other with $T = 10^6$ K, and we take the charge e of the scalar field to equal the elementary charge. For $T = 10^4$ K, corresponding to 8.62×10^{-1} eV, our simple expression of F is valid for $a^{-1} \leq 4.31 \times 10^{-1}$ eV in natural units, corresponding to $a \geq 4.57 \times 10^{-7}$ m in SI units. Given a value of a within this range, for example $a = 10^{-5}$ m, the expression of F obtained from (39) should be used when $eB \leq 3.88 \times 10^{-4}$ eV^2 in natural units, which corresponds to $B \leq 6.55 \times 10^{-2}$ G in cgs units, while the expression of F from (40) should be used when $B \geq 6.55 \times 10^{-2}$ G. For $T = 10^6$ K, our expression of F is valid when $a \geq 4.57 \times 10^{-9}$ m. For a value of a within this range, for example $a = 10^{-6}$ m, the expression of F from (39) is valid when $B \leq 6.55$ G, and the other one is valid when $B \geq 6.55$ G.

In the small plate distance case we found that, for $2aT = \frac{1}{4}$, the small plate distance limit of ζ from Eq. (45) gives a value of F that is within 0.7 percent of the exact value of the free energy in the range $0 \leq eB(\frac{\beta}{2})^2 \leq 1$, while ζ from Eq. (46) gives a value of F that is within 0.7 percent of the exact value of the free energy in the range $1 \leq eB(\frac{\beta}{2})^2 \leq \infty$. For $2aT = \frac{1}{10}$, F from Eq. (45) is within 0.05 percent of the exact value of the free energy in the range $0 \leq eB(\frac{\beta}{2})^2 \leq 1$, while F from Eq. (46) is within 0.05 percent of the exact value of the free energy in the range $1 \leq eB(\frac{\beta}{2})^2 \leq \infty$, showing again a very rapid decrease of the small discrepancy between our analytical expressions and the exact values of the free energy. We summarize the small plate distance limit by stating that the free energy as obtained from (45) and (46) is a simple analytic expression of F, valid for all values of B and T, and with a discrepancy of no more than 0.7 percent from the exact value of F for $2aT \leq \frac{1}{4}$. The pressure in the case of small plate distance is obtained immediately from F for $2aT \leq \frac{1}{4}$ and all values of B and T. We now give two numerical examples for the small plate distance case, one with $T = 100$ K and the other with $T = 300$ K. For $T = 100$ K, our analytic expression of F is valid for $a \leq 2.86 \times 10^{-4}$ m. The form of F obtained from (45) should be used when $B \leq 5.01 \times 10^{-2}$ G, while the other form should be used when $B \geq 5.01 \times 10^{-2}$ G. For $T = 300$ K, our expression of F is valid for $a \leq 9.53 \times 10^{-5}$ m. F from (45) should be used when $B \leq 4.50 \times 10^{-1}$ G, while F from (46) should be used when $B \geq 4.50 \times 10^{-1}$ G. Notice that, if we set $T = 0$ in (45), we obtain the Casimir energy E_C for a massless and charged scalar field in a magnetic field,

$$\frac{E_C}{L^2} = -\frac{\pi^2}{720a^3} - \frac{(\sqrt{2}-1)\zeta_R(-\frac{1}{2})(eB)^{3/2}}{4\pi} + \frac{e^2B^2a}{48\pi^2}\left(\gamma_E + \ln\frac{a}{2\pi}\right), \quad (66)$$

where we see that the magnetic field, as it grows, inhibits the Casimir energy of the scalar field [51]. Our result, a simple analytic expression for E_C, is more explicit than that of [51], where the magnetic field correction to the Casimir energy is presented as an infinite sum of integrals.

In the case of a strong magnetic field, the free energy shown in Eq. (63) is valid for all values of a and T, and so is the pressure shown in Eq. (65). If we set $T = 0$ in (63), we can neglect the effective potential which is a uniform energy density term, and obtain E_C in the strong magnetic field case,

$$\frac{E_C}{L^2} = -\frac{1}{4\pi^{3/2}}\frac{(eB)^{5/4}e^{-2a\sqrt{eB}}}{\sqrt{a}}, \quad (67)$$

which agrees with [51] on the dependence of E_C from a and B, but is in disagreement for the overall sign since we obtain a negative value for E_C, not a positive one. We also obtain the numerical constant present in E_C, while the authors of Ref. [51] did not.

We conclude with a brief discussion of how observable this effect is. For a plate distance $a = 1$ μm and a magnetic field $B = 100$ G, eB is much larger than a^{-2} and, at low temperature, we use Eq. (67) to calculate the Casimir energy per unit area to find $\frac{E_C}{L^2} = -1.08 \times 10^8$ eV m^{-2}. We obtain the Casimir pressure using Eq. (65) with $T = 0$, and find $P = -1.35 \times 10^{-4}$ Pa. We compare these numbers to those of the electromagnetic Casimir effect for parallel plates at the same plate distance $a = 1$ μm, where we find that the Casimir energy per unit area is $\frac{E_C}{L^2} = -2.70 \times 10^9$ eV m^{-2}, and the Casimir pressure is $P = -1.30 \times 10^{-3}$ Pa, 1 to 2 orders of magnitude larger than what we obtain for the charged scalar field using Eqs. (67) and (65).

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