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Point multimonopoles in SU(3) gauge theory

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It is known that the spherically symmetric singular Wu-Yang monopole solution of the SU(2) gauge field equations is equivalent to the Dirac U(1) monopole (with one gauge group embedded into the other). We consider a multicenter configuration of k Dirac monopoles and its embedding into SU(3) gauge theory. Using this embedding, we construct an explicit multimonopole solution of the SU(3) Yang-Mills equations which generalizes the SU(2) Wu-Yang solution and the known spherically symmetric point SU(3) monopole solutions.

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I. INTRODUCTION

Magnetic monopoles [1–5] play an important role in the $(3+1)$-dimensional Yang-Mills-Higgs theory [6,7]. In particular, it is believed that quark confinement can be explained by the condensation of monopoles and the dual Meissner effect [8–10]. This dual superconductor mechanism of confinement is discussed in terms of the Dirac monopoles [1], whose embedding into SU(2) Yang-Mills are point (singular) Wu-Yang monopoles [2,3]. That is why it is important to understand better how Abelian monopoles arise in the non-Abelian pure gauge theory. Construction of spherically symmetric point monopoles in the SU(3) gauge theory was considered in [11–15]. In these papers, some *Ansätze* for the gauge potential reducing monopole equations to nonlinear ordinary differential equations were explored. However, even after this reduction, the construction of explicit solutions remains problematic. In this paper, we construct an SU(3) point multimonopole configuration generalizing the known point monopole solutions for the SU(2) and SU(3) gauge theories. This explicit solution to the Yang-Mills equations can also be considered as an approximation at large distances $r\longrightarrow\infty$ of unknown smooth finite-energy multimonopole configuration.

II. DIRAC MULTIMONOPOLE

We consider the configuration of k Dirac monopoles at points $\vec{a}_i=\{a_i^1,a_i^2,a_i^3\}$ with $i=1,\dots,k$. We introduce two regions in $\mathbb{R}^3$:

$$\begin{aligned}\mathbb{R}^3_N &:= \mathbb{R}^3\Big\backslash\bigcup_{i=1}^k\{(a_i^1,a_i^2,x^3)|x^3\le a_i^3\},\\ \mathbb{R}^3_S &:= \mathbb{R}^3\Big\backslash\bigcup_{i=1}^k\{(a_i^1,a_i^2,x^3)|x^3\ge a_i^3\},\end{aligned}\tag{1}$$

assuming[1] that $a_i^{1,2}\neq a_j^{1,2}$ for $i\neq j$,

$$\mathbb{R}^3_N\cup\mathbb{R}^3_S=\mathbb{R}^3\setminus\{\vec{a}_1,\dots,\vec{a}_k\}.\tag{2}$$

We consider a principal U(1) bundle L over the topologically nontrivial space (2) and a connection A on this bundle. The Dirac monopole located at the point $\vec{a}_j$ is described by the gauge potentials $A^{N,j}$ and $A^{S,j}$ defined on $\mathbb{R}^3_N$ and $\mathbb{R}^3_S$, respectively, as

$$\begin{aligned}&A^{N,j}=A^{N,j}_a\mathrm{d}x^a\quad\text{with}\quad A^{N,j}_1=\frac{\mathrm{i}x_j^2}{2r_j(r_j+x_j^3)},\\ &A^{N,j}_2=-\frac{\mathrm{i}x_j^1}{2r_j(r_j+x_j^3)},\qquad A^{N,j}_3=0,\end{aligned}\tag{3}$$

$$\begin{aligned}&A^{S,j}=A^{S,j}_a\mathrm{d}x^a\quad\text{with}\quad A^{S,j}_1=-\frac{\mathrm{i}x_j^2}{2r_j(r_j-x_j^3)},\\ &A^{S,j}_2=\frac{\mathrm{i}x_j^1}{2r_j(r_j-x_j^3)},\qquad A^{S,j}_3=0,\end{aligned}\tag{4}$$

where

$$x_j^c=x^c-a_j^c,\quad r_j^2=\delta_{ab}x_j^ax_j^b,\quad a,b,c=1,2,3.\tag{5}$$

On the overlap region $\mathbb{R}^3_N\cap\mathbb{R}^3_S$, the gauge potentials are related via transition functions $f^j_{NS}:=(y_j/\bar{y}_j)^{1/2}$:

$$A^{N,j}=A^{S,j}+\mathrm{d}\ln\left(\frac{\bar{y}_j}{y_j}\right)^{\frac{1}{2}},\tag{6}$$

where $y_j=x_j^1+\mathrm{i}x_j^2$ and the bar denotes the complex conjugation. The configuration of k Dirac monopoles is described by the gauge potentials[2]

[1]In the case $a_i^{1,2}=a_j^{1,2}$ for some $i\neq j$, one has to introduce more than two open sets covering the space $\mathbb{R}^3\setminus\{\vec{a}_1,\dots,\vec{a}_k\}$. For simplicity we will not consider this case.

[2]Here $A|_N$ is the restriction of the connection A to the region $\mathbb{R}^3_N$, and $A|_S$ is the restriction of A to $\mathbb{R}^3_S$.

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$$A|_N := A^{N,k} = \sum_{j=1}^{k} A^{N,j} \quad \text{and} \quad A|_S := A^{S,k} = \sum_{j=1}^{k} A^{S,j}, \tag{7}$$

such that on the intersection $\mathbb{R}^3_N \cap \mathbb{R}^3_S$ we have

$$A^{N,k} = A^{S,k} + \mathrm{d}\ln\left(\prod_{j=1}^{k}\left(\frac{\bar{y}_j}{y_j}\right)^{\frac{1}{2}}\right). \tag{8}$$

III. SPHERICAL COORDINATES

Let us introduce the following functions of the coordinates:

$$\begin{aligned} v_j &:= \frac{\bar{y}_j}{r_j + x_j^3} = \mathrm{e}^{-\mathrm{i}\varphi_j}\tan\frac{\vartheta_j}{2} \quad \text{and} \\ w_j &:= \frac{1}{v_j} = \frac{y_j}{r_j - x_j^3} = \mathrm{e}^{\mathrm{i}\varphi_j}\cot\frac{\vartheta_j}{2}. \end{aligned} \tag{9}$$

Here the angle variables φ_j and ϑ_j are introduced via formulas

$$\begin{aligned} x_j^1 = r_j\cos\varphi_j\sin\vartheta_j, \quad x_j^2 = r_j\sin\varphi_j\sin\vartheta_j, \quad \text{and} \\ x_j^3 = r_j\cos\vartheta_j. \end{aligned} \tag{10}$$

Note that v_j and w_j are well defined on $\mathbb{R}^3_N$ and $\mathbb{R}^3_S$, respectively. It is not difficult to see that

$$A^{N,k} = \sum_{i=1}^{k}\frac{1}{2(1+v_i\bar{v}_i)}(\bar{v}_i\mathrm{d}v_i - v_i\mathrm{d}\bar{v}_i), \tag{11}$$

$$A^{S,k} = \sum_{i=1}^{k}\frac{1}{2(1+w_i\bar{w}_i)}(\bar{w}_i\mathrm{d}w_i - w_i\mathrm{d}\bar{w}_i), \tag{12}$$

and on the overlap $\mathbb{R}^3_N \cap \mathbb{R}^3_S$ we have

$$A^{N,k} = A^{S,k} + \mathrm{d}\ln\left(\prod_{i=1}^{k}\left(\frac{v_i}{\bar{v}_i}\right)^{\frac{1}{2}}\right), \tag{13}$$

since $\bar{y}_i/y_i = \bar{w}_i/w_i = v_i/\bar{v}_i$.

For the gauge field strength describing k Dirac monopoles, we have

$$F^{D,k} = \mathrm{d}A^{N,k} = -\sum_{i=1}^{k}\frac{\mathrm{d}v_i\wedge\mathrm{d}\bar{v}_i}{(1+v_i\bar{v}_i)^2} = -\sum_{i=1}^{k}\frac{\mathrm{d}w_i\wedge\mathrm{d}\bar{w}_i}{(1+w_i\bar{w}_i)^2} = \mathrm{d}A^{S,k}. \tag{14}$$

It is not difficult to see that $F^{D,k} = \frac{1}{2}F^{D,k}_{ab}\mathrm{d}x^a\wedge\mathrm{d}x^b$ is singular[3] only at points $\{\vec{a}_1,\ldots,\vec{a}_k\}$, where monopoles are located, and satisfies the Maxwell equations

$$\partial_a F^{D,k}_{ab} = 0 \tag{15}$$

on $\mathbb{R}^3\setminus\{\vec{a}_1,\ldots,\vec{a}_k\}$. Here we denote $\partial_a := \frac{\partial}{\partial x^a}$.

[3] Notice that $v_i\to\infty$ for $x^{1,2}\to a_i^{1,2}$, $x^3\le a_i^3$ and $w_i\to\infty$ for $x^{1,2}\to a_i^{1,2}$, $x^3\ge a_i^3$.

IV. SU(2) POINT MONOPOLES

After 't Hooft analysis [4], it was realized that the SU(2) Wu-Yang singular monopole solution of gauge field equations is nothing but the Abelian Dirac monopole in disguise (see e.g., [3,11,14]). One can consider this solution as an approximation of a nonsingular monopole, since the gauge potential of the finite-energy spherically symmetric SU(2) monopole [4,5] approaches the Wu-Yang monopole gauge potential for large r. A multimonopole generalization of the SU(2) Wu-Yang solution was described in Ref. [16]. Here we construct the multimonopole generalization of solutions from [11–15] for the SU(3) gauge theory.

Recall that SU(3) has two U(1) subgroups, generators of which can be taken as matrices

$$I_3 = -\mathrm{i}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix} \quad \text{and} \quad I_8 = -\mathrm{i}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}. \tag{16}$$

Let us split $k = m + n$ with $0\le m\le k$ and introduce the gauge potentials

$$A^{N,m} := \mathrm{i}I_3\sum_{i=1}^{m}A^{N,i}, \qquad A^{S,m} := \mathrm{i}I_3\sum_{i=1}^{m}A^{S,i}, \tag{17}$$

$$A^{N,n} := \mathrm{i}I_8\sum_{i=m+1}^{k}A^{N,i}, \qquad A^{S,n} := \mathrm{i}I_8\sum_{i=m+1}^{k}A^{S,i}, \tag{18}$$

as well as

$$A^{N,m,n} := A^{N,m} + A^{N,n} \quad \text{and} \quad A^{S,m,n} := A^{S,m} + A^{S,n}, \tag{19}$$

where $A^{N,i}$ and $A^{S,i}$ are given in (3)–(5).

Let us multiply Eq. (6) by the matrix $\mathrm{i}I_3$, sum over j from 1 till m, and rewrite it as

$$A^{N,m} = f^{(m)}_{NS}A^{S,m}(f^{(m)}_{NS})^{-1} + f^{(m)}_{NS}\mathrm{d}(f^{(m)}_{NS})^{-1}, \tag{20}$$

where

$$f^{(m)}_{NS} = \begin{pmatrix}\prod_{j=1}^{m}\left(\frac{w_j}{\bar{w}_j}\right)^{\frac{1}{2}} & 0 & 0\\ 0 & \prod_{j=1}^{m}\left(\frac{\bar{w}_j}{w_j}\right)^{\frac{1}{2}} & 0\\ 0 & 0 & 1\end{pmatrix}. \tag{21}$$

This matrix can be split as

$$f^{(m)}_{NS} = (g^{(m)}_N)^{-1}g^{(m)}_S, \tag{22}$$

where the 3×3 unitary matrices

$$g_N^{(m)} = \frac{1}{(1+\prod_{i=1}^m v_i\bar{v}_i)^{\frac{1}{2}}} \times \begin{pmatrix} \prod_{j=1}^m v_j & 1 & 0 \\ -1 & \prod_{j=1}^m \bar{v}_j & 0 \\ 0 & 0 & \left(1+\prod_{j=1}^m v_j\bar{v}_j\right)^{\frac{1}{2}} \end{pmatrix} \quad (23)$$

and

$$g_S^{(m)} = \frac{1}{(1+\prod_{i=1}^m w_i\bar{w}_i)^{\frac{1}{2}}} \times \begin{pmatrix} 1 & \prod_{j=1}^m \bar{w}_j & 0 \\ -\prod_{j=1}^m w_j & 1 & 0 \\ 0 & 0 & \left(1+\prod_{j=1}^m w_j\bar{w}_j\right)^{\frac{1}{2}} \end{pmatrix} \quad (24)$$

are well defined on $\mathbb{R}^3_N$ and $\mathbb{R}^3_S$, respectively. Substituting (9) into (23) and (24), one can write these matrices in x_i^a for $i = 1, \ldots, m$, $a = 1, 2, 3$. Note that the matrices (21)–(24) belong to the subgroup SU(2) in SU(3). Substituting (22) into (20), one obtains

$$g_N^{(m)} A^{N,m} (g_N^{(m)})^\dagger + g_N^{(m)} \mathrm{d}(g_N^{(m)})^\dagger = g_S^{(m)} A^{S,m} (g_S^{(m)})^\dagger + g_S^{(m)} \mathrm{d}(g_S^{(m)})^\dagger =: A^{(m)}_{\mathrm{su}(2)}, \quad (25)$$

where † denotes the Hermitian conjugation. The su(2)-valued gauge potential (25) is well defined everywhere on $\mathbb{R}^3 \setminus \{\vec{a}_1, \ldots, \vec{a}_m\}$. It realizes the embedding of the solution [16] into the Lie algebra su(3).

V. SU(3) POINT MONOPOLES

Let us now multiply Eq. (6) by the matrix $\mathrm{i}I_8$, sum over j from $m+1$ till $k = m+n$, and rewrite it as

$$A^{N,n} = h_{NS}^{(n)} A^{S,n} (h_{NS}^{(n)})^{-1} + h_{NS}^{(n)} \mathrm{d}(h_{NS}^{(n)})^{-1}, \quad (26)$$

where

$$h_{NS}^{(n)} = \begin{pmatrix} \prod_{j=m+1}^k \left(\frac{w_j}{\bar{w}_j}\right)^{\frac{1}{2}} & 0 & 0 \\ 0 & \prod_{j=m+1}^k \left(\frac{w_j}{\bar{w}_j}\right)^{\frac{1}{2}} & 0 \\ 0 & 0 & \prod_{j=m+1}^k \frac{\bar{w}_j}{w_j} \end{pmatrix}. \quad (27)$$

This matrix can be split as

$$h_{NS}^{(n)} = (h_N^{(n)})^{-1} h_S^{(n)}, \quad (28)$$

where the unitary 3×3 matrices

$$h_N^{(n)} = (1+u\bar{u})^{-\frac{3}{2}} \begin{pmatrix} 1 & -\sqrt{2}\bar{u} & -\bar{u}^2 \\ \sqrt{2}u & \frac{2+u\bar{u}+(u\bar{u})^2}{2+u\bar{u}} & -\frac{\sqrt{2}u\bar{u}^2}{2+u\bar{u}} \\ u^2 & -\frac{\sqrt{2}\bar{u}u^2}{2+\bar{u}u} & \frac{2+3u\bar{u}}{2+u\bar{u}} \end{pmatrix} \times \begin{pmatrix} \bar{u} & -1 & 0 \\ 1 & u & 0 \\ 0 & 0 & (1+u\bar{u})^{\frac{1}{2}} \end{pmatrix}, \quad (29)$$

with

$$u = \prod_{i=m+1}^k v_i, \quad (30)$$

and

$$h_S^{(n)} = (1+t\bar{t})^{-\frac{3}{2}} \begin{pmatrix} t^2 & -\sqrt{2}t & -1 \\ \sqrt{2}t & \frac{1+t\bar{t}+2(t\bar{t})^2}{1+2t\bar{t}} & -\frac{\sqrt{2}\bar{t}}{1+2t\bar{t}} \\ 1 & -\frac{\sqrt{2}\bar{t}}{1+2t\bar{t}} & \frac{\bar{t}^2(3+2t\bar{t})}{1+2t\bar{t}} \end{pmatrix} \times \begin{pmatrix} 1 & -t & 0 \\ \bar{t} & 1 & 0 \\ 0 & 0 & (1+t\bar{t})^{\frac{1}{2}} \end{pmatrix}, \quad (31)$$

with

$$t = \prod_{i=m+1}^k w_i, \quad (32)$$

are well defined on $\mathbb{R}^3_N$ and $\mathbb{R}^3_S$, respectively. One can rewrite these matrices in terms of x_i^a for $i = m+1, \ldots, k = m+n$ by using Eqs. (9). Substituting (28) into (26), we obtain the formula

$$h_N^{(n)} A^{N,n} (h_N^{(n)})^\dagger + h_N^{(n)} \mathrm{d}(h_N^{(n)})^\dagger = h_S^{(n)} A^{S,n} (h_S^{(n)})^\dagger + h_S^{(n)} \mathrm{d}(h_S^{(n)})^\dagger =: A^{(n)}_{\mathrm{su}(3)}, \quad (33)$$

where the so-defined su(3)-valued gauge potential $A^{(n)}_{\mathrm{su}(3)}$ is well defined everywhere on $\mathbb{R}^3 \setminus \{\vec{a}_{m+1}, \ldots, \vec{a}_k\}$. Note that the existence of splittings (22) and (28) means that Dirac's nontrivial U(1) bundle L over $\mathbb{R}^3 \setminus \{\vec{a}_1, \ldots, \vec{a}_k\}$ trivializes after embedding into an SU(3) bundle. The matrices (23), (24), (29), and (31) define such trivializations, when $f_{NS}^{(m)} \to g_N^{(m)} f_{NS}^{(m)} (g_S^{(m)})^{-1} = \mathbf{1}_3$ and $h_{NS}^{(n)} \to h_N^{(n)} h_{NS}^{(n)} (h_S^{(n)})^{-1} = \mathbf{1}_3$.

VI. GAUGE FIELD STRENGTH

The field strengths for the configurations (25) and (33) are

$$F^{(m)}_{\mathrm{su}(2)} = \mathrm{d}A^{(m)}_{\mathrm{su}(2)} + A^{(m)}_{\mathrm{su}(2)} \wedge A^{(m)}_{\mathrm{su}(2)} = \mathrm{i}F^{D,m} Q_{(m)} \quad (34)$$

and

$$F^{(n)}_{\mathrm{su}(3)} = \mathrm{d}A^{(n)}_{\mathrm{su}(3)} + A^{(n)}_{\mathrm{su}(3)} \wedge A^{(n)}_{\mathrm{su}(3)} = \mathrm{i}F^{D,n}\tilde{Q}_{(n)}, \quad (35)$$

where the matrices

$$Q_{(m)} := g^{(m)}_N I_3 (g^{(m)}_N)^\dagger = g^{(m)}_S I_3 (g^{(m)}_S)^\dagger \in \mathrm{su}(2) \subset \mathrm{su}(3), \quad (36)$$

$$\tilde{Q}_{(n)} := h^{(n)}_N I_8 (h^{(n)}_N)^\dagger = h^{(n)}_S I_8 (h^{(n)}_S)^\dagger \in \mathrm{su}(3) \quad (37)$$

are well defined on $\mathbb{R}^3_N \cup \mathbb{R}^3_S$. Here $F^{D,m}$ and $F^{D,n}$ are Abelian multimonopole field strengths from (14). Both $Q_{(m)}$ and $\tilde{Q}_{(n)}$ may be considered as generators of two groups U(1) embedded into SU(3). This clarifies the Abelian nature of the configurations (25), (34), (33), and (35). Furthermore, it is not difficult to show that

$$\partial_a F^{(m)}_{ab} + [A^{(m)}_a, F^{(m)}_{ab}] = \mathrm{i}(\partial_a F^{D,m}_{ab}) Q_{(m)}, \quad (38)$$

$$\partial_a F^{(n)}_{ab} + [A^{(n)}_a, F^{(n)}_{ab}] = \mathrm{i}(\partial_a F^{D,n}_{ab}) \tilde{Q}_{(n)}, \quad (39)$$

and therefore on $\mathbb{R}^3 \setminus \{\vec{a}_1, \ldots, \vec{a}_k\}$ both multimonopole fields satisfy the Yang-Mills equations

$$\partial_a F_{ab} + [A_a, F_{ab}] = 0. \quad (40)$$

In (38) and (39) we used

$$A^{(m)}_{\mathrm{su}(2)} = A^{(m)}_a \mathrm{d}x^a \quad \text{and} \quad F^{(m)}_{\mathrm{su}(2)} = \frac{1}{2} F^{(m)}_{ab} \mathrm{d}x^a \wedge \mathrm{d}x^b, \quad (41)$$

$$A^{(n)}_{\mathrm{su}(3)} = A^{(n)}_a \mathrm{d}x^a \quad \text{and} \quad F^{(n)}_{\mathrm{su}(3)} = \frac{1}{2} F^{(n)}_{ab} \mathrm{d}x^a \wedge \mathrm{d}x^b. \quad (42)$$

VII. SUPERPOSITION OF $m + n$ MONOPOLES

We have constructed two explicit multimonopole solutions of the SU(3) gauge theory starting from the Abelian solutions (17) and (18) embedded into the Lie algebra su(3). Now we consider the similar construction for their sum (19) which after non-Abelian "dressing" will not be the sum of (25) and (33).

Note that the transition matrices (21) and (27) commute with one another as well as g_N and h_{NS}. That is why we have the equality

$$h^{(n)}_N g^{(m)}_N (f^{(m)}_{NS} h^{(n)}_{NS}) = h^{(n)}_S g^{(m)}_S, \quad (43)$$

where we used (22) and (28). Thus, introducing

$$f^{(m,n)}_{NS} := f^{(m)}_{NS} h^{(n)}_{NS}, \qquad g^{(m,n)}_N := h^{(n)}_N g^{(m)}_N, \quad \text{and} \quad g^{(m,n)}_S := h^{(n)}_S g^{(m)}_S, \quad (44)$$

we get[4]

$$A^{N,m,n} = f^{(m,n)}_{NS} A^{S,m,n} (f^{(m,n)}_{NS})^{-1} + f^{(m,n)}_{NS} \mathrm{d}(f^{(m,n)}_{NS})^{-1}. \quad (45)$$

After splitting

$$f^{(m,n)}_{NS} = (g^{(m,n)}_N)^{-1} g^{(m,n)}_S, \quad (46)$$

following from (43) and (44), we obtain

$$g^{(m,n)}_N A^{N,m,n} (g^{(m,n)}_N)^\dagger + g^{(m,n)}_N \mathrm{d}(g^{(m,n)}_N)^\dagger = g^{(m,n)}_S A^{S,m,n} (g^{(m,n)}_S)^\dagger + g^{(m,n)}_S \mathrm{d}(g^{(m,n)}_S)^\dagger := A^{(m,n)}_{\mathrm{su}(3)}, \quad (47)$$

where $A^{(m,n)}_{\mathrm{su}(3)}$ is the su(3)-valued gauge potential which is well defined on $\mathbb{R}^3 \setminus \{\vec{a}_1, \ldots, \vec{a}_k\}$.

For the field strength we have

$$F^{(m,n)}_{\mathrm{su}(3)} = \mathrm{d}A^{(m,n)}_{\mathrm{su}(3)} + A^{(m,n)}_{\mathrm{su}(3)} \wedge A^{(m,n)}_{\mathrm{su}(3)} = \mathrm{i}F^{D,m} Q_{(m,n)} + \mathrm{i}F^{D,n} \tilde{Q}_{(m,n)}, \quad (48)$$

where $F^{D,m}$ and $F^{D,n}$ are given in (14) and

$$Q_{(m,n)} := g^{(m,n)}_N I_3 (g^{(m,n)}_N)^\dagger = g^{(m,n)}_S I_3 (g^{(m,n)}_S)^\dagger, \qquad \tilde{Q}_{(m,n)} := g^{(m,n)}_N I_8 (g^{(m,n)}_N)^\dagger = g^{(m,n)}_S I_8 (g^{(m,n)}_S)^\dagger \quad (49)$$

are well defined on $\mathbb{R}^3_N \cup \mathbb{R}^3_S$. Again one can show that the gauge field

$$A^{(m,n)}_{\mathrm{su}(3)} = A^{(m,n)}_a \mathrm{d}x^a, \qquad F^{(m,n)}_{\mathrm{su}(3)} = \frac{1}{2} F^{(m,n)}_{ab} \mathrm{d}x^a \wedge \mathrm{d}x^b \quad (50)$$

of the constructed multimonopole configuration satisfies the Yang-Mills equations (40) on $\mathbb{R}^3 \setminus \{\vec{a}_1, \ldots, \vec{a}_k\}$. From (23), (29), and (44) it follows that

$$g^{(m,n)}_N = (1 + u\bar{u})^{-\frac{3}{2}} (1 + v\bar{v})^{-\frac{1}{2}} \times \begin{pmatrix} 1 & -\sqrt{2}\bar{u} & -\bar{u}^2 \\ \sqrt{2}u & \frac{2+u\bar{u}+(u\bar{u})^2}{2+u\bar{u}} & -\frac{\sqrt{2}u\bar{u}^2}{2+u\bar{u}} \\ u^2 & -\frac{\sqrt{2}\bar{u}u^2}{2+\bar{u}u} & \frac{2+3u\bar{u}}{2+u\bar{u}} \end{pmatrix} \times \begin{pmatrix} 1+\bar{u}v & \bar{u}-\bar{v} & 0 \\ v-u & 1+u\bar{v} & 0 \\ 0 & 0 & (1+u\bar{u})^{\frac{1}{2}}(1+v\bar{v})^{\frac{1}{2}} \end{pmatrix}, \quad (51)$$

where

$$v = \prod_{j=1}^{m} v_j, \qquad u = \prod_{j=m+1}^{k} v_j, \qquad v_j = \frac{x^1_j - \mathrm{i}x^2_j}{r_j + x^3_j}, \quad \text{and} \quad r^2_j = (x^1_j)^2 + (x^2_j)^2 + (x^3_j)^2. \quad (52)$$

[4]Note also that $h^{(n)}_{NS} I_3 (h^{(n)}_{NS})^{-1} = I_3$ and $h^{(n)}_{NS} I_8 (h^{(n)}_{NS})^{-1} = I_8$ and similarly for $f^{(m)}_{NS}$.

Equation (51) gives the explicit expression of the matrix $g_N^{(m,n)}$ via the coordinates x^1, x^2, x^3 and parameters $\vec{a}_j$ of location of monopoles. The explicit expression of the gauge potential $A_{\mathrm{su}(3)}^{(m,n)}$ and the field strength $F_{\mathrm{su}(3)}^{(m,n)}$ via $g_N^{(m,n)}$ are given in (47)–(49). Further simplification of these formulas via multiplications of matrices, summations, etc., is not possible, since expressions produced in this way are very cumbersome. Notice that the matrices $g_N^{(m)} = g_N^{(m,0)}$ and $h_N^{(n)} = g_N^{(0,n)}$, used in the potentials (25) and (33), can be obtained from (51) by putting $u = 0$ and $v = 0$, respectively. Note again that our multimonopole solutions can be considered as guides to the asymptotic behavior at large radii to be satisfied by smoothed-out finite-energy solutions. Also, they can be used in constructing monopole wall-type solutions considered e.g., in Refs. [17,18].

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[1] P. A. M. Dirac, Proc. R. Soc. A **133**, 60 (1931).

[2] T. T. Wu and C. N. Yang, in *Properties of Matter under Unusual Conditions*, edited by H. Mark and S. Fernbach (Wiley, New York, 1969), p. 349.

[3] T. T. Wu and C. N. Yang, Phys. Rev. D **12**, 3843 (1975).

[4] G. 't Hooft, Nucl. Phys. **B79**, 276 (1974).

[5] A. M. Polyakov, JETP Lett. **20**, 194 (1974).

[6] R. Rajaraman, *Solitons and Instantons. An Introduction to Solitons and Instantons in Quantum Field Theory* (North-Holland, Amsterdam, 1982).

[7] N. S. Manton and P. Sutcliffe, *Topological Solitons* (Cambridge University Press, Cambridge, England, 2004).

[8] G. 't Hooft, in *Proceedings of the European Physical Society International Conference on High Energy Physics, Palermo, Italy, 1975*, edited by A. Zichichi (Editrice Compositori, Bologna, 1976).

[9] S. Mandelstam, Phys. Rep. **23**, 245 (1976).

[10] G. 't Hooft, in *Proceedings of the International Workshop "Large N(c) QCD 2004," Trento, Italy, 2004*, edited by J. L. Goity *et al.* (World Scientific, Hackensack, NJ, 2005).

[11] A. C. T. Wu and T. T. Wu, J. Math. Phys. (N.Y.) **15**, 53 (1974).

[12] A. Chakrabarti, Ann. Inst. Henri Poincaré **23**, 235 (1975).

[13] W. J. Marciano and H. Pagels, Phys. Rev. D **12**, 1093 (1975).

[14] E. Corrigan, D. I. Olive, D. B. Fairlie, and J. Nuyts, Nucl. Phys. **B106**, 475 (1976).

[15] P. Goddard, J. Nuyts, and D. I. Olive, Nucl. Phys. **B125**, 1 (1977).

[16] A. D. Popov, J. Math. Phys. (N.Y.) **46**, 073506 (2005).

[17] R. S. Ward, Phys. Rev. D **75**, 021701 (2007).

[18] S. A. Cherkis and R. S. Ward, J. High Energy Phys. 05 (2012) 090.