PHYSICAL REVIEW D **88,** 104039 (2013)

BRST-invariant boundary conditions and strong ellipticity

Ian G. Moss*

School of Mathematics and Statistics, Newcastle University, Newcastle Upon Tyne NE1 7RU, United Kingdom
(Received 1 March 2013; revised manuscript received 11 September 2013; published 26 November 2013)

The quantization of gauge theories usually proceeds through the introduction of ghost fields and Becchi-Rouet-Stora-Tyutin (BRST) symmetry. In the case of quantum gravity in the presence of boundaries, the BRST-invariant boundary value problem for the gauge field operators is nonelliptic, and consequently the definition of the effective action using heat-kernel techniques becomes problematic. This paper examines general classes of BRST-invariant boundary conditions and presents new boundary conditions for quantum gravity that fix the extrinsic curvature on the boundary and lead to a well-defined effective action. This prompts a discussion of the wider issue of nonellipticity in BRST-invariant boundary value problems and when the use of gauge-fixing terms on the boundary can resolve the issue.

DOI: 10.1103/PhysRevD.88.104039 PACS numbers: 04.60.−m

I. INTRODUCTION

A long-standing problem in quantum gravity has been the fact that the BRST-invariant boundary value problem is nonelliptic [1–6]. The BRST-invariant boundary value problem is the natural one in which the intrinsic metric on the boundary is fixed and the DeWitt gauge-fixing condition [7] is imposed at the boundary. Nonellipticity means that these boundary conditions do not allow the construction of functional determinants by the usual techniques [8]. As a consequence, physical quantities, such as the scaling behavior of coupling constants or coefficients of anomalies, are potentially ill-defined.

This paper explores BRST-invariant boundary value problems for gravity and other types of gauge field theories. In some cases nonellipticity arises from incomplete gauge fixing, and in these cases a simple remedy is to fix the residual gauge freedom by adding extra boundary gauge-fixing terms [9] or by projecting out the residual gauge modes. The boundary-value problem for quantum gravity that fixes the extrinsic curvature at the boundary is an example of this type of behavior.

Boundary value problems have important applications in quantum field theory. Possibly the earliest example was the derivation of the Casimir force between two parallel conducting plates in terms of quantum vacuum polarization [10]. In the 1980s, boundary value problems arose in quantum gravity in connection with the calculation of the Hartle-Hawking wave function of the universe [11], and later they appeared in the theory of strings and branes [9,12,13]. Recently, boundary value problems have featured in the theory of brane cosmology [14] and membranes [15].

Quantum effects are often described in terms of the one-loop effective action, (formally) given by a series of terms of the form log det P_m, where P_m are a set of second order differential operators with an appropriate set of boundary conditions. When the boundary value problem is strongly elliptic (as defined in Sec. IV), then the functional determinants can be defined by the heat kernels or the zeta functions of the operators. In most cases these functional determinants cannot be reduced to a simple analytic form, but useful quantities like the scaling behavior of coupling constants can often be determined by the geometrical invariants that appear in the asymptotic expansion on the heat kernel in the small time limit [16–21].

Early work on boundary value problems in quantum gravity [22–25] indicated that the boundary conditions would be combinations of Dirichlet and Neumann (or Robin) boundary conditions acting on different field components at the same point, which in the literature on quantum field theory are often called "mixed type." (This is different from the "mixed type" that refers to situations where Dirichlet and Neumann apply to different regions of the boundary.) In the general situation we have a field ϕ that is a section of a vector bundle V over a compact manifold with boundary. The vector bundle is decomposed at the boundary into $V = V_D \oplus V_N$ by projection matrices P_D and P_N. Given a normal derivative ∇_n and an endomorphism ψ, then mixed boundary conditions are

$$P_D\phi = 0, \qquad (\nabla_n + \psi)P_N\phi = 0. \tag{1}$$

If ψ is replaced by a differential operator acting on the boundary, then we call these boundary conditions "mixed type with tangential derivatives."

In BRST systems, there are ordinary fields ϕ and ghost fields c. Fields and ghosts are related by a BRST symmetry s, defined by a differential operator D,

$$s\phi = Dc. \tag{2}$$

We shall adopt the point of view that BRST invariance is an essential feature of quantum gauge theory and that, in particular, the boundary conditions should be invariant under (2) [26,27]. We also restrict attention to mixed boundary conditions, with or without tangential derivatives. BSRT invariant boundary conditions of mixed

*ian.moss@ncl.ac.uk

type without tangential derivatives have been found for Maxwell gauge theory and antisymmetric-tensor theory [28,29].

The BRST-invariant boundary conditions commonly used for quantum gravity are of mixed type with tangential derivatives [22,25,26,30]. These boundary conditions lead to a nonelliptic boundary value problem when applied to the graviton operator [1–6]. The first step toward extracting physical predictions would normally be to construct the heat kernel of the graviton operator, but this procedure is problematic when faced with a nonelliptic system. This paper introduces a new set of BRST-invariant boundary conditions that fix the extrinsic curvature. These boundary conditions are strongly elliptic if we allow two tangential derivatives. The heat kernels, functional determinants and effective action for these boundary conditions are all well-defined. (However, Barvinsky *et al.* [31] have pointed out that the small-time asymptotic expansion of the heat kernel is nonstandard when there are two or more tangential derivatives, so that the physical interpretation of the heat kernel coefficients is currently unresolved.)

We shall see in the following sections that nonelliptic boundary value problems are typical of BRST-invariant boundary conditions. The main result of this paper is that, in many cases, the nonellipticity can be explained as a residual gauge invariance in the system and it can be restored by allowing extra tangential derivatives in the boundary conditions. This has an interpretation in terms of adding extra gauge-fixing functions on the boundary. In certain limits, the heat kernel can be defined by the simpler procedure of projecting out the residual gauge modes.

Calculations on spheres have suggested that the generalized zeta function of the operators relevant to quantum gravity could be constructed despite the nonellipticity of the boundary value problem and divergences in the heat kernel [6,30,32]. It has also been suggested that the problems of non-ellipticity could be overcome by using non-Laplacian operators [3], but it has not been shown that this can be done in a way consistent with BRST symmetry. There are various techniques for dealing with specific nonelliptic boundary value problems in other areas of mathematical physics [33]. One strategy involves imposing additional boundary conditions and relating the problem of interest to an elliptic boundary value problem. The approach adopted here has some similarities, although we work all the time under the restrictions of BRST symmetry and have to take this into account when adding additional boundary conditions or gauge-fixing terms.

The plan of this paper is as follows. Section II gives a general account of BRST-invariant boundary conditions for a one-dimensional system. Section III extends this to field theory, with explicit results for Maxwell theory and linearized gravity. Section IV gets to grip with the issue of nonellipticity, its relation to residual gauge freedoms and its resolution. General remarks are made in Sec. V.

The conventions used in this paper are as follows. Spacetime is replaced by a manifold with boundary, on which there is a Riemannian metric and a Levi-Cività connection ∇_μ or $;\mu$. On the boundary, the unit normal vector n^μ is ingoing and the induced connection $|i$. Ordinary derivatives along the coordinate directions are denoted by ∂_μ or ∂_i.

II. BRST-INVARIANT BOUNDARY CONDITIONS

We begin with a general account of the BRST-invariant boundary conditions that will be used in subsequent sections. We shall follow the approach of Ref. [26], including details that were omitted in the earlier treatment. The relevant features of BRST invariance can be described in a simple one-dimensional system with an Abelian gauge symmetry. The boundary consists of just two points, representing the initial time and the final time. The time coordinate will eventually become the normal coordinate to the boundary when we deal with a field theory that has additional spatial dimensions.

Consider the one-dimensional system with coordinates $q = (q_i, q_a)$ and Lagrangian

$$L_q(q_i, \dot{q}_i, q_a). \tag{3}$$

A Lagrangian of this form has a set of primary constraints $\pi_q^a = 0$, where π_q^a are the momenta conjugate to q_a. The equations of motion $\dot{\pi}_q^a = 0$ lead to a set of secondary constraints $E_q^a = 0$, where

$$E_q^a = -\frac{\partial L_q}{\partial q_a}. \tag{4}$$

The gauge symmetry can be converted into a BRST symmetry s, with anticommuting ghosts c_a. We take the BRST symmetry to be linear, with transformations

$$sq_i = \alpha_i{}^a c_a, \qquad sq_a = \dot{c}_a + \beta_a{}^b c_b, \tag{5}$$

for some functions $\alpha_i{}^a$ and $\beta_a{}^b$ depending only on t. Gauge invariance of the action implies that the Lagrangian transforms by a total derivative, which we can write as a term linear in c_a with coefficient $\eta^a(q_i, q_a)$,

$$sL_q = (\eta^a c_a)^\cdot. \tag{6}$$

If we just examine the $\dot{c}_a$ part of the BRST transformed Lagrangian sL_q we find

$$E_q^a = \alpha_i{}^a \pi_q^i - \eta^a, \tag{7}$$

where π_q^i are the momenta conjugate to q_i. The BRST transformation of π_q^i can be found by differentiating sL_q with respect to $\dot{q}_i$,

$$s\pi_q^i = \frac{\partial \eta^a}{\partial q_i} c_a. \tag{8}$$

The gauge symmetry is fixed by introducing auxiliary fields b^a and gauge-fixing functions $f_a(q, \dot{q})$, which we take to be of the form

$$f_a = \dot{q}_a + \nu_a{}^i \dot{q}_i + \chi_a(q), \tag{9}$$

where $\nu_a{}^i$ are functions of t. The gauge-fixing and ghost Lagrangians are then

$$L_{gf} = b^a f_a - \frac{1}{2}\xi^{-1} G_{ab} b^a b^b, \tag{10}$$

$$L_{gh} = -\bar{c}^a s f_a + (\bar{c}^a s q_a)^{\cdot}, \tag{11}$$

where ξ is a constant and G_{ab} is a symmetric invertible matrix. The total derivative term is added to ensure that the Lagrangian $L = L_q + L_{gf} + L_{gh}$ is first order in time derivatives. BRST symmetry is imposed by requiring that

$$s\bar{c}^a = b^a, \qquad sc_a = sb^a = 0. \tag{12}$$

Note that, because of the addition of the total derivative terms, the total Lagrangian L transforms under BRST by a total derivative, $sL = dj/dt$, where

$$j = \eta^a c_a + s\bar{c}^a s q_a. \tag{13}$$

We are now ready to consider the BRST transformations of the full set of fields and their conjugate momenta. The momenta obtained from the Lagrangian L are

$$\pi^i = \pi_q^i + \nu_a{}^i b^a, \tag{14}$$

$$\pi^a = b^a, \tag{15}$$

$$\bar{p}_a = \dot{c}_a + \beta_a{}^b c_b, \tag{16}$$

$$p^a = -\dot{\bar{c}} + \bar{c}^b \left(\frac{\partial \chi_b}{\partial q_a} + \nu_b{}^i \alpha_i{}^a \right), \tag{17}$$

where the ghost momenta for c_a and $\bar{c}^a$ are denoted by p^a and $\bar{p}_a$, respectively. The field transformations we have seen already,

$$sq_i = \alpha_i{}^a c_a, \tag{18}$$

$$sq_a = \bar{p}_a, \tag{19}$$

$$s\bar{c}^a = b^a, \tag{20}$$

$$sc_a = sb^a = 0. \tag{21}$$

The momenta π_q^i transform by Eq. (8). The transformations of the remaining momenta can be obtained directly from the definitions (14)–(17),

$$s\pi^i = \frac{\partial \eta^a}{\partial q_i} c_a, \tag{22}$$

$$sp^a = E^a + \frac{\delta S}{\delta q_a}, \tag{23}$$

$$s\pi^a = s\bar{p}_a = 0, \tag{24}$$

where the constraint E_q^a of Eq. (7) has been modified to include a $\nu_a{}^i$ term,

$$E^a = \alpha_i{}^a \pi^i - \eta^a. \tag{25}$$

The transformations of the gauge-fixing function and the constraint are also useful,

$$sf_a = -\frac{\delta S}{\delta \bar{c}^a}, \tag{26}$$

$$sE^a = -\frac{\partial \eta^a}{\partial q_b} \bar{p}_b. \tag{27}$$

The first of these expressions is obtained by examination of the ghost Lagrangian (11), and the second follows from Eqs. (25), (22), and (19).

We look for boundary conditions in configuration space that are BRST invariant. The simplest possibility is a vanishing-ghost condition with q_i fixed,

$$c_a = \bar{c}^a = b^a = 0, \qquad q_i \text{ fixed}. \tag{28}$$

These boundary conditions are standard in the BRST formalism; see for example [34]. If we eliminate b^a using the field equation $b^a = \xi G^{ab} f_b$ and use Eq. (9) for the gauge-fixing functions, then we arrive at a set of boundary conditions that we denote by $\mathcal{B}_M$,

$$\mathcal{B}_M\colon\ c_a = \bar{c}^a = \dot{q}_a + \nu_a{}^i \dot{q}_i + \chi_a(q) = 0, \qquad q_i \text{ fixed}. \tag{29}$$

The BRST invariance of the boundary conditions in this form is restricted due to the elimination of b^a. If we define the ghost operator P_c by a functional derivative of the action S,

$$(P_c)_a{}^b = \frac{\delta^2 S}{\delta \bar{c}^a \delta c_b}, \tag{30}$$

then Eq. (26) shows that the BRST variation of the gauge-fixing functional is

$$sf_a = -(P_c)_a{}^b c_b. \tag{31}$$

Note that the boundary value problem for the eigenvalues of P_c is still fully BRST invariant, since $P_c c = \lambda c = 0$ on the boundary. We shall describe a set of boundary conditions as BRST invariant if they are BRST invariant for the eigenvalue problem. These boundary conditions can be used to evaluate the functional determinants that appear in one-loop quantum calculations.

The generalization of the boundary conditions (29) to electromagnetic field theory fixes the magnetic field on a spacelike boundary. There are other possibilities, however, one being a vanishing-ghost condition with the momenta π^i held fixed, which we denote by $\mathcal{B}_E$,

$$\mathcal{B}_E\colon\ c_a = \bar{c}^a = \dot{q}_a + \nu_a{}^i\dot{q}_i + \chi_a(q) = 0, \qquad \pi^i \text{ fixed.} \tag{32}$$

These correspond in electromagnetism to fixing the the electric field on a spacelike boundary. When we generalize to more than one dimension, these boundary conditions resemble the mixed type of boundary condition (1) but in general the $\chi_a(q)$ terms introduce derivatives tangential to the boundary.

Another important set of boundary conditions can be obtained by having the ghost momenta vanish on the boundary. It is then possible to require that the constraint function E^a vanishes on the boundary since its BRST variation vanishes by Eq. (27). In a field theory context, the constraint function usually has tangential derivatives. If we would like to remove as many tangential derivatives from the boundary conditions as possible, then it is useful to consider special cases where

$$\eta^a = \alpha_i{}^a h^i, \tag{33}$$

for some function $f^i(q)$. This includes *a fortiori* the situation where the Lagrangian is BRST invariant and $\eta^i = 0$. In these cases Eq. (25) reveals that $E^a = \alpha_i{}^a(\pi^i - h^i)$. Fixing E^a is then equivalent to the simpler condition of fixing $\pi^i - h^i$. Since $\alpha_i{}^a$ contain the tangential derivatives in the field theory case, this is an integration of the boundary conditions, and Eq. (33) can be regarded as an integrability condition. The corresponding set of BRST-invariant boundary conditions is denoted by $\mathcal{B}'_E$,

$$\mathcal{B}'_E\colon\ p^a = \bar{p}_a = q_a = 0, \qquad \pi^i - h^i \text{ fixed.} \tag{34}$$

The BRST invariance is restricted because the field equation appears in the variation of p^a [see Eq. (23)], but as before we have BRST symmetry in the eigenvalue problem. We will show later that for electromagnetism these boundary conditions are of mixed type with no tangential derivatives.

Many other sets of BRST-invariant boundary conditions can be constructed. The most general set of linear boundary conditions can be found by starting from the basic set (29) and applying canonical transformations that preserve the form of the BRST operator [26],

$$\Omega = \bar{p}_a\pi^a + c_a E^a. \tag{35}$$

This leads to a family of boundary conditions that depends on matrices $B_a{}^b$, $D_a{}^i$ and F_{ai},

$$B_b{}^a E^a + D_b{}^a f^b = 0, \tag{36}$$

$$B_b{}^a p^b + D_b{}^a \bar{c}^b = 0, \tag{37}$$

$$B_a{}^b \bar{p}_b - D_a{}^b c_b = 0, \tag{38}$$

$$B_a{}^b q_b + D_a{}^i q_i + F_{ai}\pi^i \qquad \text{fixed,} \tag{39}$$

where $D_a{}^b = D_a{}^i\alpha_i{}^a - F_{ai}\partial\eta^b/\partial q_i$. It is simple to check directly that these form a BRST-invariant set, but when generalized to field theory they usually have one or more tangential derivatives.

Up to now we have assumed that the Lagrangian L_q is presented in a form that has primary constraints $\pi_q^a = 0$, but this is often not the case. Suppose, instead, that a Lagrangian L'_q has primary constraints $\pi_q^{a\prime} = h^a(q)$. In this situation, we can introduce a canonical transformation generated by

$$L' = L + \dot{\epsilon}, \tag{40}$$

where $\epsilon \equiv \epsilon(q_i, q_a.\bar{c}^a, c_a)$. The momenta for L' and L are related by

$$\pi^{i\prime} = \pi^i + \frac{\partial\epsilon}{\partial q_i}, \tag{41}$$

$$\pi^{a\prime} = \pi^a + \frac{\partial\epsilon}{\partial q_a}, \tag{42}$$

$$p^{a\prime} = p^a + \frac{\partial\epsilon}{\partial c_a}, \tag{43}$$

$$\bar{p}'_a = \bar{p}_a + \frac{\partial\epsilon}{\partial\bar{c}^a}. \tag{44}$$

Note that substituting these into the formula (35) leaves the BRST operator unchanged. However, the boundary condition $\pi^{i\prime} = 0$ is not BRST invariant. What we will do in this case is determine the function ϵ that transforms the constraint into $\pi_q^a = 0$,

$$\frac{\partial\epsilon}{\partial q_a} = \frac{\partial L'_q}{\partial\dot{q}_a}. \tag{45}$$

The boundary condition (32) becomes

$$\mathcal{B}_E\colon\ c_a = \bar{c}^a = \dot{q}_a + \nu_a{}^i\dot{q}_i + \chi_a(q) = 0, \qquad \pi^{i\prime} - \partial\epsilon/\partial q_i \text{ fixed.} \tag{46}$$

A similar canonical transformation can be used to add additional gauge-fixing functions $F_a(q_i)$ on the boundary. This will prove useful later in the context of field theories. The new Lagrangian is defined as in Eq. (40) with extra terms

$$\epsilon = \frac{1}{2}\alpha G_{ab}F_aF^b. \tag{47}$$

Note that the field equations only depend on the value of F_a at the boundary because the Lagrangian is a function of $\dot{\epsilon}$. The new momentum is obtained from Eq. (41),

$$\pi^i = \pi^{i\prime} - \alpha G_{ab}\frac{\partial F^a}{\partial q_i}F^b. \tag{48}$$

The new set of BRST-invariant boundary conditions will be denoted by $\mathcal{B}_E(\alpha)$. From Eq. (32), these are

$$\mathcal{B}_E(\alpha):\ c_a = \bar{c}^a = \dot{q}_a + \nu_a{}^i \dot{q}_i + \chi_a(q) = 0,\qquad \pi^{i'} - \alpha G_{ab} F^a \partial F^b / \partial q_i \text{ fixed.} \tag{49}$$

The BRST symmetry can be shown directly using the transformations (18)–(24).

III. FIELD THEORIES

In this section we shall consider the application of BRST boundary conditions to linearized gauge field theories. The gauge-invariant action for a set of fields φ is of the form

$$S_\varphi = \int_{\mathcal{M}} d\mu\, \mathcal{L}_\varphi + \int_{\partial\mathcal{M}} d\mu\, \mathcal{L}_b, \tag{50}$$

where $\mathcal{M}$ is a Riemannian manifold with metric $g_{\mu\nu}$, volume measure $d\mu$, Levi-Cività connection ∇_μ and boundary $\partial\mathcal{M}$. The quadratic action for the linearized fields arises from expanding the fields about a background field configuration. This background can be chosen to satisfy an inhomogeneous boundary value problem for the classical field equations. We shall therefore focus on the homogeneous boundary value problems for the field fluctuations. The quadratic action defines a set of second order operators that depend on the background fields. The one-loop effective action is related to the functional determinants of these operators [35,36].

We can set up a normal coordinate system close to the boundary with the coordinate t along the unit normal direction and $t = 0$ on the boundary. The eigenvalue problem for the fluctuation operators will be required to be BRST invariant with boundary conditions of the mixed type described in the Introduction.

A. Maxwell theory

The simplest example is provided by vacuum electrodynamics in curved space with Maxwell field A_μ and field strength $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$. The Lagrangian density is the usual Maxwell form

$$\mathcal{L}_q = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{51}$$

with BRST symmetry

$$sA_\mu = c_{;\mu}, \tag{52}$$

and Lorentz gauge-fixing function

$$f = g^{\mu\nu} A_{\mu;\nu}. \tag{53}$$

The total Lagrangian density is therefore

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + bf - \frac{1}{2}\xi^{-1} b^2 + \bar{c}^{;\mu} c_{;\mu}. \tag{54}$$

Under the BRST symmetry, $s\mathcal{L}_q = 0$ with no boundary terms.

To set up the phase space near the boundary $\partial\mathcal{M}$ we need to decompose the Maxwell field into tangential and normal components (A_i, A) (see the Appendix). Decomposition of the Lagrangian density gives momenta

$$\pi^i = h^{ij}(\dot{A}_j - A_{|j} + K_j{}^k A_k), \qquad \pi = b. \tag{55}$$

A similar reduction of the the gauge-fixing function gives

$$f = \dot{A} + KA + A_i{}^{|i}. \tag{56}$$

The BRST generator is

$$\Omega = \bar{p}\pi + c\pi^i{}_{|i}. \tag{57}$$

The BRST boundary conditions are given in Table I. The sets $\mathcal{B}_M$ and $\mathcal{B}'_E$ consist of mixed Dirichlet and Robin boundary conditions with no tangential derivatives, while the set $\mathcal{B}_E$ is first order in normal derivatives and contains tangential derivatives. These BRST boundary conditions can be generalized to antisymmetric tensor fields [28,29], where the boundary conditions $\mathcal{B}_M$ and $\mathcal{B}'_E$ are equivalent to the relative and absolute boundary conditions used in the index theory of the de Rahm complex [16].

B. Linearized gravity

Linearized gravity forms the starting point for order $\hbar$ quantum gravity calculations based on Einstein gravity, as well as having wider applications to supergravity, low energy superstring theory and covariantly quantized strings and membranes. The Lagrangian density is obtained by decomposing the metric in the Einstein-Hilbert action into a background $g_{\mu\nu}$ and a field $\gamma_{\mu\nu}$,

$$g_{\mu\nu} + 2\kappa\gamma_{\mu\nu}, \tag{58}$$

where $\kappa^2 = 8\pi G$ and G is Newton's constant in m dimensions. We also use a dual field

$$\bar{\gamma}^{\mu\nu} = g^{(\mu\nu)(\rho\sigma)}\gamma_{\rho\sigma}, \tag{59}$$

defined by the DeWitt metric

$$g^{(\mu\nu)(\rho\sigma)} = \frac{1}{2}(g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\mu\nu}g^{\rho\sigma}). \tag{60}$$

Keeping only the quadratic terms and integrating by parts gives the Lagrangian [7,8]

TABLE I. Homogeneous BRST boundary conditions for Maxwell theory in curved space with gauge-fixing function $f = A_\mu{}^{;\mu}$. The field A_i is in the tangential direction, A is in the normal direction and the ghost is c.

Type	Fixes	Dirichlet	Non-Dirichlet
$\mathcal{B}_M$	F_{ij}, f	$A_i = c = 0$	$\dot{A} + KA = 0$
$\mathcal{B}_E$	F_{in}, f	$c = 0$	$\dot{A} + KA + A_i{}^{\vert i} = 0$ $\dot{A}_i - A_{\vert i} + K_i{}^j A_j = 0$
$\mathcal{B}'_E$	F_{in}, A	$A = 0$	$\dot{A}_i + K_i{}^j A_j = 0$ $\dot{c} = 0$

$$\mathcal{L}'_q = \frac{1}{2}\bar{\gamma}^{\mu\nu;\rho}\gamma_{\mu\nu;\rho} - g^{\mu\nu}\bar{\gamma}_{\mu\rho}{}^{;\rho}\bar{\gamma}_{\nu\sigma}{}^{;\sigma} + \frac{1}{2}Q_{\mu\nu}{}^{\rho\sigma}\bar{\gamma}^{\mu\nu}\gamma_{\rho\sigma}, \quad (61)$$

where

$$Q_{\mu\nu}{}^{\rho\sigma} = -2R_{(\mu}{}^{(\rho}{}_{\nu)}{}^{\sigma)} - 2\delta_{(\mu}{}^{(\rho}G_{\nu)}{}^{\sigma)} + \frac{2}{m-2}g_{\mu\nu}G^{\rho\sigma} + R_{\mu\nu}g^{\rho\sigma}, \quad (62)$$

and $R_{\mu\nu\rho\sigma}$ is the Riemann tensor, $R_{\mu\nu}$ is the Ricci tensor and $G_{\mu\nu}$ is the Einstein tensor. The prime is used in the same way as in the previous section to denote the fact that the primary constraint is not "$\pi^a = 0$," and we will have to perform a canonical transformation to simplify the constraint before we can apply the boundary conditions $\mathcal{B}_E$.

The BRST symmetry transforms the metric fluctuations into a ghost field c_μ,

$$s\gamma_{\mu\nu} = c_{\mu;\nu} + c_{\nu;\mu}. \quad (63)$$

The Lagrangian density transforms into a total derivative term and, in the notation of the previous section, the function η is nonvanishing. The commonly used gauge-fixing function is the DeWitt one,

$$f_\mu = \bar{\gamma}_{\mu\rho}{}^{;\rho}. \quad (64)$$

The extra terms in the Lagrangian density from the gauge fixing are

$$\mathcal{L}_{gf+gh} = b^\mu f_\mu - \frac{1}{4}\xi^{-1}b^\mu b_\mu + \bar{c}^{\mu;\nu}c_{\mu;\nu} - R_\mu{}^\nu\bar{c}^\mu c_\nu. \quad (65)$$

The Lagrangian has been put into first order form in order to apply the results of the previous section. The graviton and ghost operators are obtained from the Lagrangian densities by integration by parts after eliminating the auxiliary field b^μ. The graviton operator is

$$P = -\delta_{\mu\nu}{}^{\rho\sigma}\nabla^2 + 2(1-\xi)\delta_{\mu\nu}{}^{\alpha\delta}g^{(\beta\gamma)(\rho\sigma)}g_{\gamma\delta}\nabla_\alpha\nabla_\beta + Q_{\mu\nu}{}^{\rho\sigma}, \quad (66)$$

where $\delta_{\mu\nu}{}^{\rho\sigma}$ is the identity operator on symmetric tensors. The gauge parameter ξ has been scaled to make the graviton operator of Laplace type, i.e. with leading terms proportional to ∇^2, when $\xi = 1$.

Near the boundary $\partial\mathcal{M}$ we need to decompose the metric fluctuation into tangential and normal components $(\gamma_{ij}, \gamma_i, \gamma)$ (see the Appendix). We can identify γ_{ij} with the physical variables q_i of the previous section, γ_i and γ with the constrained variables q_a. The momenta conjugate to γ_i and γ are

$$\pi_q^{i\prime} = -2\gamma^{ij}{}_{|j} - 2(K^{ij} - Kh^{ij})\gamma_j + \gamma^k{}_k{}^{|i} + \gamma^{|i}, \quad (67)$$

$$\pi_q{}' = -\gamma_i{}^{|i} - K\gamma + K^{ij}\gamma_{ij}, \quad (68)$$

where indices are raised using the inverse boundary metric h^{ij}. These equations for the momenta become the primary constraints. The canonical transformation that trivializes these constraints corresponds to adding a term $\dot{\epsilon}$ to the Lagrangian density as in Eq (40). Solving Eq. (45) for ϵ gives

$$\epsilon = 2\gamma^i(2\gamma_{ik}{}^{|k} - \gamma^k{}_{k|i} - \gamma_{|i} + 2K_i{}^k\gamma_k + 2K\gamma_i) + \frac{1}{2}K\gamma^2 - K^{ij}\gamma_{ij}\gamma. \quad (69)$$

According to Eq. (41), the momentum $\pi^{ij} = \partial\mathcal{L}'/\partial\dot{\gamma}_{ij} - \partial\epsilon/\partial\gamma_{ij}$ that appears in the boundary condition $\mathcal{B}_E$ [see Eq. (46)] is then

$$\pi^{ij} = \dot{\gamma}^{ij} - h^{ij}\dot{\gamma}^k{}_k - 2\gamma^{(i|j)} - (K^{ij} - Kh^{ij})\gamma + 2h^{ij}\gamma_k{}^{|k}. \quad (70)$$

(Note that the convention here is to raise the indices after taking the normal derivatives.)

Unsurprisingly, the momentum π^{ij} has a physical interpretation in terms of the canonical decomposition of Einstein gravity. The Hamiltonian formalism gives the canonical momentum

$$p^{ij} = \frac{1}{2\kappa^2}(K^{ij} - h^{ij}K). \quad (71)$$

Let δp^{ij} be the perturbation in the canonical momentum corresponding to the metric perturbation $2\kappa\gamma_{\mu\nu}$, and then to first order one finds that

TABLE II. Homogeneous BRST boundary conditions for linearized gravity. The metric variation $\gamma_{\mu\nu}$ and ghost c_μ have been decomposed into tangential and normal components.

Type	Fixes	Dirichlet	Non-Dirichlet
$\mathcal{B}_M$	h_{ij}, f_μ	$\gamma_{ij} = c = c_i = 0$	$\dot{\gamma}_i - \frac{1}{2}\gamma_{\vert i} + K_i{}^j\gamma_j + K\gamma_i = 0$
			$\dot{\gamma} - \dot{\gamma}_i{}^i + 2\gamma_i{}^{\vert i} + 2K\gamma = 0$
$\mathcal{B}_E$	K_{ij}, f_μ	$c = c_i = 0$	$\dot{\gamma}_i - \frac{1}{2}\gamma_{\vert i} + \gamma_{ij}{}^{\vert j} - \frac{1}{2}\gamma^j{}_{j\vert i} + K_i{}^j\gamma_j + K\gamma_i = 0$
			$\dot{\gamma} + 2K\gamma - 2K^{ij}\gamma_{ij} = 0$
			$\dot{\gamma}_{ij} - 2\gamma_{(i\vert j)} - K_{ij}\gamma = 0$

$$\delta p^{ij} = \frac{1}{2\kappa}\pi^{ij}, \tag{72}$$

where π^{ij} is given by (70). Boundary conditions on π^{ij} correspond to fixing the canonical momentum on the boundary. This is analogous in the Maxwell case to fixing the electric field on the boundary.

There are two basic sets of BRST-invariant boundary conditions, $\mathcal{B}_M$ corresponding to fixing the intrinsic on the boundary [22], and the new boundary conditions $\mathcal{B}_E$ that fix the extrinsic curvature. Boundary conditions of type $\mathcal{B}'_E$ exist only when the integrability condition Eq. (33) is satisfied, We shall see later that this restricts the extrinsic curvature to the special form $K_{ij} = \kappa h_{ij}$, for some κ. Only the boundary conditions for linearized gravity that apply for a general background are listed in Table II.

IV. RESTORING STRONG ELLIPTICITY TO THE BOUNDARY VALUE PROBLEM

We now turn to the heat kernel of the BRST boundary value problem and the important issue of strong ellipticity. Avramidi *et al.* [1] have shown that the gravitational boundary value problem with a fixed boundary metric and a DeWitt gauge condition is not elliptic, and therefore the heat kernel and the propagator are ill-defined. We would like to examine whether this is also the case for the new BRST invariant gravitational boundary conditions.

Consider the eigenvalue problem

$$Pf = \lambda f \quad \text{on } \mathcal{M}, \tag{73}$$

$$Bf = 0 \quad \text{on } \partial\mathcal{M}, \tag{74}$$

where P is a second order operator and B is first order in normal derivatives. Ellipticity can be defined in terms of the leading symbols of the operators. The symbol of the operator P, denoted by $\sigma(P, x, \zeta)$, is constructed by replacing derivatives $\partial/\partial x^\mu$ by $i\zeta_\mu$. The leading symbol $\sigma_L(P, x, \zeta)$ is obtained by keeping only the leading terms in ζ. The operator is elliptic if $\det \sigma_L(P, x, \zeta) \neq 0$ for $\zeta \neq 0$.

For the boundary value problem, we replace the tangential components of ζ by k_i and the normal component by $-i\partial_t$ and consider the leading order system [37]

$$\sigma_L(P, x, k_i, -i\partial_t) f_L = \lambda f_L, \qquad t > 0, \tag{75}$$

$$\sigma_L(B, x, k_i, -i\partial_t) f_L = 0, \qquad t = 0. \tag{76}$$

The boundary value problem is said to be strongly elliptic in $C - R_+$ if the operator P is elliptic and there are no bounded exponential solutions f_L with $\lambda \in C - R_+$. The boundary value problem is elliptic if there are no bounded exponential solutions with $\lambda = 0$. Strong ellipticity implies the existence of a complete set of eigenfunctions to the original boundary value problem. Their eigenvalues are real and can be placed in an unbounded sequence $\lambda_1 \leq \lambda_2 \ldots$.

The heat kernel $K(x, x', \tau, P, B)$ for $x, x' \in \mathcal{M}$ and $\tau > 0$ is defined in terms of the normalized eigenfunctions f_n by

$$K(x, x', \tau, P, B) = \sum_{n=1}^{\infty} f_n(x) f_n(x')^\dagger e^{-\lambda_n \tau}. \tag{77}$$

The sum converges if the boundary value problem is strongly elliptic, but it might not exist if the operator is not strongly elliptic. In the strongly elliptic case, the integrated heat kernel can be written in the form

$$\mathrm{Tr}_{\mathcal{B}}(e^{-P\tau}) = \int_{\mathcal{M}} K(x, x, \tau, P, B) d\mu, \tag{78}$$

where $d\mu$ is the volume measure on the manifold. We also define the generalized zeta function by

$$\zeta(s) = \sum_{n=1}^{\infty'} \lambda_n^{-s}, \tag{79}$$

where the prime denotes the omission of any vanishing eigenvalues.

For the BRST boundary conditions we have considered so far, and with gauge parameter $\xi = 1$, the equations for ellipticity become

$$(-\partial_t^2 + k^2) f_L = \lambda f_L, \qquad t > 0, \tag{80}$$

$$(P_D + P_N \partial_t + \Gamma(k_i)) f_L = 0, \qquad t = 0, \tag{81}$$

where P_D and P_N are the projection matrices defined in the Introduction, and $\Gamma(k_i)$ is an Hermitian matrix acting on V_N. A bounded exponential solution would be a function of the form

$$f_L = \chi e^{-\mu t}, \tag{82}$$

where $\mathrm{Re}\mu > 0$ and $\lambda = k^2 - \mu^2$. Choose χ to be an eigenvector of Γ, and then Eq. (81) implies that μ is an eigenvalue of Γ. In particular, μ is real because Γ is Hermitian. If $\mu > k$, then $\lambda < 0$ and strong ellipticity is violated because there is a bounded exponential solution with $\lambda \in C - R_+$. Therefore strong ellipticity implies the spectrum $\mathrm{spec}(\Gamma) \subset (-\infty, k)$. Gilkey and Smith have shown that the converse is also true, and the boundary value problem is strongly elliptic if, and only if, the spectrum $\mathrm{spec}(\Gamma) \subset (-\infty, k)$ [38]. (It can be shown that the same condition applies for any gauge parameter $\xi > 0$.)

The matrices $\Gamma(k_i)$ can be obtained from the boundary conditions in Tables I and II. We keep only the derivative terms and replace the tangential derivatives with ik_i. It is convenient to use the spacetime components; for example the Maxwell $\mathcal{B}_E$ boundary condition becomes

$$\sigma_L(B, x, k_i, -i\partial_t) A = \dot{A}_\mu + i n_\mu k^\nu A_\nu - i k_\mu n^\nu A_\nu, \tag{83}$$

where $A_\mu = A_i e^i{}_\mu + A n_\mu$ and $k_\mu = k_i e^i{}_\mu$. (The tangent basis $e_i{}^\mu$ is defined in the Appendix.) We can see from Table III that in the cases with tangential derivatives, $\Gamma(k)$ always has an eigenvector with eigenvalue k. Therefore, of

TABLE III. The spectrum of the tangential term $\Gamma(k)$ in the boundary operator for different BRST boundary conditions. The eigenvector in the table is the eigenvector with eigenvalue k that makes the operator nonelliptic.

Field	Type	P_N	$\Gamma(k)$	Spectrum	k-eigenvector
Maxwell	$\mathcal{B}_E$	$\delta_\mu{}^\nu$	$i(n_\mu k^\nu - k_\mu n^\nu)$	$0, \pm k$	$kn_\mu - ik_\mu$
Gravity	$\mathcal{B}_M$	$2n_{(\mu}n^{(\rho}\delta_{\nu)}{}^{\sigma)} - n_\mu n_\nu n^\rho n^\sigma$	$i(n_\mu k_\nu n^\rho n^\sigma - 2n_\mu n_\nu k^\rho n^\sigma)$	$0, \pm k$	$kn_\mu n_\nu + ik_\mu n_\nu$
Gravity	$\mathcal{B}_E$	$\delta_{\mu\nu}{}^{\rho\sigma}$	$i(2k_\mu h_\nu{}^\rho n^\sigma - 2n_\mu h_\nu{}^\rho k^\sigma - n_\mu k_\nu \delta^{\rho\sigma})$	$0, \pm k$	$kn_\mu k_\nu - ik_\mu k_\nu$

the five sets of BRST boundary conditions considered so far, only those without tangential derivatives are strongly elliptic.

The explanation for this lack of strong ellipticity can be seen when we look at bounded exponential solutions to the leading order equation (76) when $\lambda = 0$,

$$f_L = u_L(k)e^{-kt}, \tag{84}$$

where u_L is the eigenvector with eigenvalue k given in the table. In each case, at leading order, these are gauge transformations of the form $f = Dv$, i.e.

$$f_L = \sigma_L(D, x, k, -i\partial_t)v_L(k)e^{-kt}, \tag{85}$$

for some v_L. We should therefore examine whether these correspond to an infinite set of zero modes $Pf = 0$. If so, there is a residual gauge invariance that has not been fixed by the gauge-fixing condition. In the next section, we shall see how adding a boundary term can fix the remaining gauge invariance and restore strong ellipticity. The idea of using boundary gauge-fixing terms is due to Barvinsky [9].

The extra gauge-fixing terms come with their own gauge parameter. In certain limits of the gauge parameter, the gauge choice is frozen and the effect on the heat kernel is equivalent to leaving out the the gauge zero modes. The heat kernel defined with only the nonzero modes will be denoted in the usual way by a prime,

$$K'(x, x', \tau, P, B) = \sum_{n=1}^{\infty'} f_n(x)f_n(x')^\dagger e^{-\lambda_n\tau}. \tag{86}$$

This can be used to define the trace Tr' and the generalized zeta function by

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty d\tau\, \tau^{s-1}\, \mathrm{Tr}'_{\mathcal{B}}(e^{-P\tau}). \tag{87}$$

The bounded exponential solutions might not correspond to gauge zero modes in every case, and then the additional gauge fixing will not help with the ellipticity problems.

A. Maxwell theory

The boundary conditions $\mathcal{B}_E$ for Maxwell theory are the ones that lead to a nonelliptic boundary value problem. We can try adding the boundary gauge-fixing function $F(A_i)$ given by

$$F(A_i) = A_i{}^{|i}. \tag{88}$$

The boundary conditions in Table II are modified into the new set $\mathcal{B}_E(\alpha)$ given by Eq. (49),

$$\dot{A}_i - A_{|i} + \alpha A_j{}^{|j}{}_i + K_i{}^j A_i = 0, \tag{89}$$

$$\dot{A} + A_i{}^{|i} + KA = 0. \tag{90}$$

Ellipticity depends on the matrix Γ, which can be read off (89) by replacing the tangential derivatives with ik_i and comparing with Eq. (81). The eigenvectors of Γ with a nonzero eigenvalue are given by two free parameters x and y and have the form $A_i = x\hat{k}_i$ and $A = y$, so that the eigenvalue problem for Γ reduces to

$$\begin{pmatrix} -\alpha k^2 & ik \\ -ik & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}. \tag{91}$$

The condition for strong ellipticity is that the largest eigenvalue is less than k, and this is the case for $\alpha > 0$. (This is also the correct sign choice for convergence of the functional integral in the quantum system.) We could therefore use the boundary conditions (89) and (90) to provide a well-defined heat kernel that can be used in one-loop quantum calculations. However, we shall see how it is possible to relate the boundary conditions $\mathcal{B}_E(\alpha)$ to a simpler set $\mathcal{B}'_E$.

The $\alpha \to \infty$ limit projects out (at least formally) the transverse modes with $F(A_i) = 0$. This can be examined further by decomposing the modes using the exterior derivative d and its conjugate $d^\dagger$. The Maxwell field operator is

$$P = d^\dagger d + \xi dd^\dagger. \tag{92}$$

Consider the pure gauge field

$$A = d\chi, \qquad P_c\chi = 0, \tag{93}$$

where $P_c = d^\dagger d$. Both dA and $d^\dagger A$ vanish, and so this field satisfies the boundary conditions $\mathcal{B}_E$. The Maxwell field (93) is also a zero mode of P. There are no restrictions on χ at the boundary and an infinite set of gauge zero modes like this can be constructed. Imposing the gauge condition $F(A_i) = 0$ removes these modes.

In the previous section we defined the heat kernel $K'(x, x', t, P, B_E)$ by taking a sum over the nonzero modes of P. Hodge decomposition gives two types of nonzero modes that satisfy the boundary conditions $\mathcal{B}_E$,

(1) Exact: $f_n = \lambda_n^{-1/2} d\chi_n$, where $P_c\chi_n = \lambda_n\chi_n$ and $\chi_n = 0$ on $\partial\mathcal{M}$.
(2) Coclosed: $d^\dagger f_n = 0$, where $F_{\rm in} = 0$ on $\partial\mathcal{M}$.

In the coclosed case, part of the boundary condition is redundant because it is enforced by the Hodge decomposition. It is possible to relate this boundary value problem to the boundary value problem $\mathcal{B}'_E$ (see Table I). Again, we use a Hodge decomposition of the modes, but with boundary conditions $\mathcal{B}'_E$,

(1) Exact: $f_n = \lambda_n^{-1/2} d\chi_n$, where $P_c\chi_n = \lambda_n\chi_n$ and $\dot\chi_n = 0$ on $\partial\mathcal{M}$.
(2) Coclosed: $d^\dagger f_n = 0$, where $F_{\rm in} = 0$ on $\partial\mathcal{M}$.

The coclosed modes for the two sets of boundary conditions are identical, whereas the exact modes come from Dirichlet scalars in the first case and Neumann scalars in the second. If we split the mode sums into exact and coexact mode sums, then $K'(x, x', t, P, B_E)$ is determined by the heat kernels of the strongly elliptic boundary value problems $\mathcal{B}'_E$, the scalar Dirichlet problem $\mathcal{B}_D$ and the scalar Neumann problem $\mathcal{B}_N$. For example, the integrated kernel

$$\mathrm{Tr}'_{\mathcal{B}_E}(e^{-P\tau}) = \mathrm{Tr}_{\mathcal{B}'_E}(e^{-P\tau}) - \mathrm{Tr}_{\mathcal{B}_N}(e^{-P_c\tau}) + \mathrm{Tr}_{\mathcal{B}_D}(e^{-P_c\tau}). \tag{94}$$

There is a nice physical interpretation of this result that arises when we combine the fields and the ghosts together into a supertrace. Let $\Delta = (P, P_c)$, and then define

$$\mathrm{STr}_{\mathcal{B}}(e^{-\Delta\tau}) = \mathrm{Tr}_{\mathcal{B}}(e^{-P_c\tau}) - \mathrm{Tr}_{\mathcal{B}}(e^{-P_c\tau}). \tag{95}$$

The supertrace determines the effective action of the quantum theory of gauge fields in the one-loop approximation. The ghost boundary conditions in $\mathcal{B}_E$ are Dirichlet, and the ghost boundary conditions in $\mathcal{B}'_E$ are Neumann. Using Eq. (94), we have

$$\mathrm{STr}'_{\mathcal{B}_E}(e^{-\Delta\tau}) = \mathrm{STr}_{\mathcal{B}'_E}(e^{-\Delta\tau}). \tag{96}$$

Therefore the one-loop effective action for the quantum field theory derived from the nonelliptic BRST boundary conditions $\mathcal{B}_E$ by omitting the zero modes is the same as the one-loop effective action derived from the strongly elliptic BRST boundary conditions $\mathcal{B}'_E$.

B. Linearized gravity

We shall repeat the preceding analysis for linearized gravity with the new BRST boundary conditions $\mathcal{B}_E$ that fix the extrinsic curvature at the boundary. The boundary condition $\pi^{ij} = 0$ can be replaced by

$$\pi^{ij} - 2\alpha(F^{(i|j)} - h^{ij}F^k{}_{|k}) = 0, \tag{97}$$

where

$$F_i = \gamma_{ij}{}^{|j} - \frac{1}{2}\gamma^j{}_{j|i}. \tag{98}$$

In terms of the metric components, using Eqs. (70), (A14), and (A15),

$$\dot\gamma_{ij} - 2\gamma_{(i|j)} - K_{ij}\gamma - 2\alpha F^{(i|j)} = 0, \tag{99}$$

$$\dot\gamma_i - \frac{1}{2}\gamma_{|i} + \gamma_{ij}{}^{|j} - \frac{1}{2}\gamma^j{}_{j|i} + K_i{}^j\gamma_j + K\gamma_i = 0, \tag{100}$$

$$\dot\gamma + K\gamma - 2K^{ij}\gamma_{ij} = 0. \tag{101}$$

We replace the tangential derivatives with ik_i and collect them together into the matrix Γ as in Eq. (81). The nonzero eigenvalues of Γ arise from eigenvectors of the form $\gamma_{ij} = x\hat k_i\hat k_j$ and $\gamma_i = y\hat k_i$. We have $F_i = k_i x/2$, and the eigenvalue problem becomes

$$\begin{pmatrix} -\alpha k^2 & ik \\ -ik & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}. \tag{102}$$

This is identical to the eigenvalue problem for the Maxwell theory, and again we have $\mathrm{spec}(\Gamma) \subset (-\infty, k)$; we conclude that Eqs. (99)–(101) provide a well-posed boundary value problem for the graviton operator when $\alpha > 0$.

To analyze the residual symmetry when $\alpha = 0$ it is convenient to use the operators D and $D^\dagger$ defined by

$$Dc_{\mu\nu} = c_{\mu;\nu} + c_{\nu;\mu}, \tag{103}$$

$$D^\dagger\gamma_\mu = \bar\gamma_{\mu\nu}{}^{;\nu}. \tag{104}$$

A pure gauge field $\gamma_{\mu\nu} = Dc_{\mu\nu}$ will satisfy the boundary conditions $\mathcal{B}_E$ if $s\pi_{ij} = 0$ and $D^\dagger\gamma = 0$. The BRST variation of π^{ij} using Eqs. (A16)–(A18) is

$$s\pi_{ij} = -c_{|ij} + h_{ij}c_{|k}{}^k - K^{(i}{}_k c^{j)|k} + K^{k(i}c_k{}^{|j)}. \tag{105}$$

A sufficient condition for $s\pi_{ij} = 0$ is that $c = 0$ and $c_{[i|j]} = 0$ on the boundary, leaving us free to pick one arbitrary function a on the boundary, where $c_i = a_{|i}$. The gauge mode will be a zero mode of gauge-fixed gravity operator P if it satisfies the gauge condition $D^\dagger\gamma = 0$, i.e.

$$D^\dagger Dc = 0. \tag{106}$$

This is an inhomogeneous Dirichlet problem, and we have a solution for each choice of the arbitrary function a on the boundary. These modes are responsible for the nonellipticity. They are eliminated by restricting the modes using the boundary gauge condition $F_a = 0$, or equivalently by using the boundary value system Eqs. (99)–(101).

In the Maxwell case it was possible to relate the boundary conditions $\mathcal{B}_E$ to a strongly elliptic BRST-invariant boundary value problem $\mathcal{B}'_E$ with no tangential derivatives. We can do this also for linearized gravity for a restricted class of backgrounds where the strongly elliptic BRST-invariant boundary value problem of this type exists. Consider, for example, the case where the extrinsic curvature is proportional to the surface metric,

$$K_{ij} = \kappa h_{ij}. \tag{107}$$

The BRST-invariant boundary value problem for these backgrounds was found in Ref. [26],

$$\mathcal{B}'_E\colon \dot{\gamma}_{ij} - K_{ij}\gamma = 0, \quad \dot{\gamma} + K\gamma - K\gamma_i{}^i = 0, \quad \gamma_i = 0,$$
$$\dot{c}_i - K_i{}^j c_j = 0, \quad c = 0. \tag{108}$$

As in the Maxwell case, the two sets of boundary conditions can be related by dividing the modes into the image of D and the kernel of $D^\dagger$, with the result that

$$\mathrm{STr}'_{\mathcal{B}_E}(e^{-\Delta\tau}) = \mathrm{STr}_{\mathcal{B}'_E}(e^{-\Delta\tau}). \tag{109}$$

In general, however, it seems that we have to resort to the more complicated system Eqs. (99)–(101).

We turn finally to the boundary conditions $\mathcal{B}_M$ that fix the field γ_{ij} on the boundary. A pure gauge mode would have to satisfy $s\gamma_{ij} = 0$,

$$c_{(i|j)} + K_{ij}c = 0. \tag{110}$$

In the special case $K_{ij} = 0$, we have solutions $c_i = 0$ for any function c on the boundary. As before, these correspond to the bounded exponential solutions to the leading order system of equations. However, for the more general case $K_{ij} = \kappa h_{ij}$, there are at most a finite number of solutions corresponding to conformal killing vectors of $\partial\mathcal{M}$, and the origin of the nonellipticity in this case does not lie in the gauge modes. This case has been investigated for spherical backgrounds [6,30,32], where it appears that the heat kernel diverges for points on the boundary, but nevertheless the generalized zeta function $\zeta(s)$ exists and can be analytically continued to $s = 0$.

V. CONCLUSION

We have seen that nonellipticity is a common feature in BRST-invariant boundary value problems, and it can be associated in some cases with a residual gauge invariance. In these cases a well-defined set of boundary conditions can be obtained by adding extra tangential derivatives, and this has an interpretation in terms of extra gauge-fixing terms on the boundary. The reduced heat kernel, constructed by leaving out the gauge zero modes, is also consistent with BRST invariance and can be used to calculate one-loop phenomena in quantum gauge theories with boundaries. We found examples where this procedure gives results that are equivalent to mixed BRST boundary value systems with no tangential derivatives.

A new set of BRST-invariant boundary conditions for quantum gravity that fix the extrinsic curvature at the boundary has been found. These boundary conditions give rise to a well-defined heat kernel and can be used to construct the effective action. Boundary conditions on the extrinsic curbature are consistent with local supersymmetry and arise in supergravity theories [39,40].

Although the addition of tangential derivatives formally restores strong ellipticity, it does necessarily allow for a nonproblematic asymptotic expansion of the heat kernel [31]. This can affect quantum field theory divergences and renormalization of couplings. The heat-kernel asymptotics for the boundary conditions discussed here is worth further investigation.

We have not been able to explain the origin of the nonellipticity for quantum gravity problems that fix the intrinsic metric of the boundary. These problems are important for applications of quantum gravity to quantum cosmology and the Hartle-Hawking state of the universe. It is a puzzling fact that the generalized zeta function exists for this problem when the background is a sphere with boundary [6,30,32]. An understanding of the origin and the role of nonellipticity in this case remains elusive.

APPENDIX A: TANGENTIAL DECOMPOSITIONS

Consider a manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$, tangential vectors $e_i{}^\mu$ and inward unit normal n^μ. Indices i and n will denote projection in the tangential and normal directions, respectively. The covariant derivative ";" on $\mathcal{M}$ induces a covariant derivative "|"on the boundary $\partial\mathcal{M}$ and the extrinsic curvature $K_{ij} = n_{i;j}$. The normal derivative will be denoted by a dot, and we choose an extension of the normal vector so that $\dot{n}^\mu = 0$ at the boundary.

1. Covectors

For a covector A_μ, let $A_i = e_i{}^\mu A_\mu$ and $A = n^\mu A_\mu$. The components of the covariant derivatives are

$$A_{i;j} = A_{i|j} + K_{ij}A, \tag{A1}$$

$$A_{n;j} = A_{|i} - K_i{}^j A_j, \tag{A2}$$

$$A_{i;n} = \dot{A}_i, \tag{A3}$$

$$A_{n;n} = \dot{A}. \tag{A4}$$

The gauge fixing function $f = A_\mu{}^{;\mu}$ is therefore

$$f = \dot{A} + A_i{}^{|i} + KA. \tag{A5}$$

For the Laplacian,

$$A_{i;\mu}{}^\mu = \ddot{A}_i + A_{i|j}{}^j + K\dot{A}_i - KK_i{}^j A_j - K_i{}^j K_j{}^k A_k$$
$$+ K_{ij}{}^{|j}A + K_i{}^j A_{|j} - \dot{K}_i{}^j A_j, \tag{A6}$$

$$A_{n;\mu}{}^\mu = \ddot{A} + A_{|i}{}^i + K\dot{A} - K^{ij}A_{i|j}. \tag{A7}$$

2. Symmetric tensors

For a symmetric tensor $\gamma_{\mu\nu}$, let $\gamma_{ij} = e_i{}^\mu e_j{}^\nu \gamma_{\mu\nu}$, $\gamma_i = e_i{}^\mu n^\nu \gamma_{\mu\nu}$ and $\gamma = n^\mu n^\nu \gamma_{\mu\nu}$. The components of the covariant derivatives are

$$\gamma_{ij;k} = \gamma_{ij|k} + 2K_{k(i}\gamma_{j)}, \tag{A8}$$

$$\gamma_{in;j} = \gamma_{i|j} + K_{ij}\gamma - K_i{}^k\gamma_{kj}, \tag{A9}$$

$$\gamma_{nn;k} = \gamma_i - 2K_i{}^j\gamma_j, \tag{A10}$$

$$\gamma_{ij;n} = \dot{\gamma}_{ij}, \tag{A11}$$

$$\gamma_{in;n} = \dot{\gamma}_i, \tag{A12}$$

$$\gamma_{nn;n} = \dot{\gamma}. \tag{A13}$$

The decomposition of the gauge-fixing function $f_\mu = \bar{\gamma}_{\mu\rho}{}^{;\rho}$ into $f_i = e_i{}^\mu f_\mu$ and $f = n^\nu f_\mu$ is

$$f_i = \dot{\gamma}_i + K_i{}^j\gamma_j + K\gamma_i + \gamma_{ij}{}^{|j} - \frac{1}{2}\gamma^j{}_{j|i} - \frac{1}{2}\gamma_{|i}, \tag{A14}$$

$$f = \frac{1}{2}\dot{\gamma} - \frac{1}{2}\dot{\gamma}_i{}^i + \gamma_i{}^{|i} + K\gamma - K^{ij}\gamma_{ij}, \tag{A15}$$

The BRST transformations become

$$s\gamma_{ij} = 2(c_{(i|j)} + K_{ij}c), \tag{A16}$$

$$s\gamma_i = \dot{c}_i - K_i{}^jc_j + c_{|i}, \tag{A17}$$

$$s\gamma = 2\dot{c}. \tag{A18}$$

When transforming a normal derivative, we use

$$(c_{i|j})^{\cdot} = \dot{c}_{i|j} + (K_{ij}{}^{|k} - K^k{}_{(i|j)})c_k. \tag{A19}$$

For example,

$$s\dot{\gamma}_{ij} = 2\dot{c}_{(i|j)} - 2K_{(i}{}^kc_{j)|k} + 2(K_{ij}{}^{|k} - K^k{}_{(i|j)})c_k + 2\dot{K}_{ij}c + 2K_{ij}\dot{c}. \tag{A20}$$

The tangential decomposition relates to the standard canonical decomposition of gravity in the following way. We write the metric in terms of an intrinsic metric h_{ij}, lapse N and shift N^i,

$$ds^2 = N^2dt^2 + h_{ij}(dx^i + N^idt)(dx^j + N^jdt). \tag{A21}$$

In the fixed basis $e_i = \partial_i$ and $n^\mu = N^{-1}(\partial_t - N^i\partial_i)$, the metric fluctuations are

$$\gamma_{ij} = (2\kappa)^{-1}\delta h_{ij}, \tag{A22}$$

$$\gamma_i = (2\kappa)^{-1}N^{-1}h_{ij}\delta N^j, \tag{A23}$$

$$\gamma = (2\kappa)^{-1}2N^{-1}\delta N. \tag{A24}$$

The canonical momentum is given by

$$p^{ij} = \frac{\partial\mathcal{L}}{\partial\dot{h}_{ij}} = \frac{1}{2\kappa}\frac{\partial\mathcal{L}}{\partial\dot{\gamma}_{ij}}, \tag{A25}$$

where $\mathcal{L}$ is the Lagrangian density with only first order time derivatives obtained from the Einstein-Hilbert action.

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