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C metric with a conformally coupled scalar field in a magnetic universe

Marco Astorino*

Centro de Estudios Científicos (CECs), Avenida Prat 514 Valdivia, Chile

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In Einstein-Maxwell gravity with a conformally coupled scalar field, the black hole found by Bocharova, Bronnikov, Melnikov, and Bekenstein (BBMB) breaks when embedded in the external magnetic field of the Melvin universe. The situation improves in the presence of acceleration, allowing one to build a magnetized and accelerating BBMB black hole with a thin membrane. But to overcome this and other disadvantages of BBMB spacetimes, a new class of black holes, including the rotating case, is proposed for the conformal matter coupling under consideration.

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I. INTRODUCTION

Solution generating techniques are a very powerful tool in general relativity. Taking advantage of the integrability properties of the system and its symmetries, they are not only a mere mechanism to build solutions hardly directly integrable from the (nonlinear system of partial differential) equations of motion, but their formalism is also useful to deepen conceptual problems in gravity, such as the Geroch conjecture[1] or black hole uniqueness.

Recently the Ernst solution generating technique, originally developed for axisymmetric spacetimes in Einstein general relativity without a cosmological constant [2], possibly coupled with Maxwell electromagnetism [3], was extended to the presence of a minimally or a conformally coupled scalar field in [4]. The latter theory admits a black hole discovered by Bocharova, Bronnikov, Melnikov, and Bekenstein (BBMB) in [5–7]. This was the first counterexample to the no-hair conjecture for black holes. Thanks to the generalized Ernst methods it was possible to extend the Harrison transformation, which allows one to embed an asymptotically flat and axisymmetric spacetime in the Melvin magnetic universe [8]. So the family of magnetized black holes, known as Ernst solutions, was widened to enclose also the BBMB black hole [4]. The presence of the scalar field, which for the BBMB black hole is divergent on the event horizon, makes the black hole break when immersed in an external magnetic field; that is, the magnetized solution displays curvature singularities on some points of the horizon. In the presence of the cosmological constant the divergence of the scalar field can be neutralized because it is hidden behind the event horizon, but unfortunately neither a solution generating technique nor a Harrison transformation is available at the moment for this system. Some attempts to adapt the Ernst method to the presence of the cosmological constant were done in [9,10], and small progresses were achieved there (for instance, the generalization of the Melvin magnetic universe in the presence of the cosmological constant), but basically the problem still remains open.

Recently also a *C* metric was discovered, in [11,12], for the Einstein-Maxwell theory with a conformally coupled scalar field, which is interpreted as a pair of accelerating BBMB black holes. A typical feature of these accelerating solutions is that the acceleration is provided by a conical singularity, physically interpreted as a string or a strut pulling or pushing, respectively, the two black holes. Ernst, in [13], using a Harrison transformation has shown how to regularize these accelerating (when intrinsically charged) solutions, removing the deficit or excess angle of the conical singularity by embedding the *C* metric (in the case without the scalar field) in an external magnetic field. Actually this regularization mechanism was invoked in [11], but the Harrison transformation in the presence of a scalar field was not known because the solution generating technique [4] was not available at that time. Furthermore, it is worthwhile to note that the accelerating BBMB solution has a better behaved scalar field than the nonaccelerating one, because the scalar field blows up only on one pole of the event horizon and not on the whole surface. Similar to the case with the cosmological constant (almost) all of the divergences are hidden inside the event horizon.

Since now we are in possession of the technology able to magnetize spacetimes in the Einstein-Maxwell theory with a conformally coupled scalar field, it would be interesting to explore the possibility of regularizing the *C* metric with a scalar hair by embedding it in an external magnetic field; this point is addressed in Sec. II. And since the accelerating BBMB has a more regular scalar field with respect to its static version, we hope also to be able to remove the naked singularities present in the magnetized BBMB black hole.

Furthermore one virtue of the Ernst solution generating technique, which at the beginning was probably the main motivation for its discovery, is to generate rotating solutions starting from a static seed, for instance, obtaining the Kerr spacetime from the Schwarzschild black hole. So it is

*marco.astorino@gmail.com

[1]The Geroch conjecture was proven by Hauser and Ernst in [1]. It states that any axially symmetric electrovacuum spacetime can be generated from the Minkowski one by Kinnersley-Chitre transformations.

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natural, with the help of the generalized solution technique, to explore the possibility of generating a rotating version of the BBMB black hole, which is still unknown. Unfortunately, the standard methods that work for the case without the scalar hair fails, so in Sec. III a rotating, scalar hairy black hole is considered to overcome this and other disadvantages typical of BBMB spacetimes.

While the existence of a (minimal or) conformally coupled scalar field is not proven in gravitational physics, it is theoretically widely used, especially in cosmology for studying dark energy and dark matter. On the other hand, the astrophysical interest in black holes embedded in an external magnetic source, such as the Melvin universe, comes from the fact that currents in the accretion disk around a massive black hole, especially the ones at the center of the galaxies, can presumably generate such kinds of magnetic fields.

II. ACCELERATING BBMB BLACK HOLE IN MELVIN MAGNETIC UNIVERSE

Thanks to the solution generating techniques developed in [4] for the Einstein-Maxwell gravity theory with a conformally (and minimally) coupled scalar field, it is now possible to embed the accelerating, scalar hairy black hole discovered in [11] into the Melvin magnetic universe.

A. *C* metric with a conformal scalar hair

Consider the action for general relativity coupled to the Maxwell electromagnetic field and to a conformally coupled self-interacting scalar field Ψ,

$$I[g_{\mu\nu}, A_\mu, \Psi] = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\Big[\mathrm{R} - F_{\mu\nu}F^{\mu\nu} - \kappa\Big(\nabla_\mu\Psi\nabla^\mu\Psi + \frac{\mathrm{R}}{6}\Psi^2\Big)\Big]. \tag{2.1}$$

The gravitational, electromagnetic, and scalar field equations are obtained by extremizing with respect to metric $g_{\mu\nu}$, the electromagnetic potential A_μ and the scalar field Ψ, respectively,

$$\mathrm{R}_{\mu\nu} - \frac{\mathrm{R}}{2}g_{\mu\nu} = \kappa(T^{(\mathrm{EM})}_{\mu\nu} + T^{(S)}_{\mu\nu}), \tag{2.2}$$

$$\partial_\mu(\sqrt{-g}F^{\mu\nu}) = 0, \tag{2.3}$$

$$\Box\Psi = \frac{1}{6}\mathrm{R}\Psi, \tag{2.4}$$

where

$$T^{(EM)}_{\mu\nu} = \frac{1}{4\pi\mu_0}\Big(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\Big), \tag{2.5}$$

$$T^{(S)}_{\mu\nu} = \partial_\mu\Psi\partial_\nu\Psi - \frac{1}{2}g_{\mu\nu}\partial_\sigma\Psi\partial^\sigma\Psi + \frac{1}{6}[g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu + G_{\mu\nu}]\Psi^2. \tag{2.6}$$

In this section we are interested in static and axisymmetric spacetimes characterized by two commuting killing vectors described by the Weyl metric

$$ds^2 = -f d\varphi^2 + f^{-1}[R^2dt^2 - e^{2\gamma}(dR^2 + dz^2)], \tag{2.7}$$

where the functions f, γ depend only on the coordinates (R, z) and $\kappa = 8\pi G$, while the electromagnetic potential will be taken of the form $A = A_t(R, z)dt + A_\varphi(R, z)d\varphi$.[2] An accelerating black hole solution for this model was found in [11] (see also [12]), for the null cosmological constant and electromagnetic charge, and it is

$$ds^2 = \frac{1}{(1 + Ar\cos\theta)^2}\Big[-\frac{Q(r)}{r^2}dt^2 + \frac{r^2}{Q(r)}dr^2 + \frac{r^2}{P(\theta)}d\theta^2 + r^2\sin^2\theta P(\theta)d\varphi^2\Big], \tag{2.8}$$

$$Q(r) = (1 - A^2r^2)(r - m)\Big(r - \frac{m}{1 + 2Am}\Big), \tag{2.9}$$

$$P(\theta) = (1 + Am\cos\theta)\Big(1 + \frac{Am}{1 + 2Am}\cos\theta\Big), \tag{2.10}$$

$$\Psi(r, \theta) = \sqrt{\frac{6}{\kappa}}\frac{m(Ar\cos\theta + 1)}{r(1 + Am) + m(Ar\cos\theta - 1)}. \tag{2.11}$$

A and m represent, respectively, the acceleration and the mass parameters of the black hole, and we will consider them positive. Actually this metric (2.8) is interpreted as a pair of black holes with a conformally coupled scalar hair uniformly accelerating apart along the axis $\theta = 0$. The inner r_-, outer r_+, and accelerating r_A horizons are given by

$$r_- = \frac{m}{1 + 2Am}, \qquad r_+ = m, \qquad r_A = \frac{1}{A}. \tag{2.12}$$

For the roots of the polynomial $Q(r)$ in (2.9) to be ordered according to the C-metric interpretation, the parameters have to satisfy the following relation:

$$0 \le Ar_- \le Ar_+ \le 1. \tag{2.13}$$

Moreover, as explained in [15], C metrics usually have a hidden parameter C in the range of azimuthal coordinate $\varphi \in (-C\pi, C\pi]$. When the acceleration parameter A goes

[2] It is shown by Carter in [14] (theorem 7) that this is the most generic circular electromagnetic field, compatible with the circular metric (2.7).

to zero, the black hole found by Bocharova, Bronnikov, and Melnikov in [5], and then studied by Bekenstein in [6,7], is recovered. In that case ($A = 0$) there is no accelerating horizon, and both the inner and outer horizons coincide: $r_\pm = m$.

It is worthwhile to examine the regularity of the axis of symmetry because in the literature this point is often not clear. This ambiguousness usually arises from a different choice of the radial coordinate r. As pointed out in [16], our radial coordinate choice is motivated by the facts that (i) the no accelerating limits are clearer, (ii) when the C metric is rotating, there are no torsion singularities (that is, rotating conical singularities, which generate closed timelike curves[3]), (iii) moreover, the interpretation of the extremal case is comprised in the standard case (the string and acceleration are not disappearing in the extremal case), and (iv) finally, the simpler algebra makes the position of the horizon clearer. To study the conicity of the metric (2.8) we consider a small circle around the half-axis $\theta = 0$ (with t, r constant). For the above range of φ, we obtain

$$\frac{\text{circumference}}{\text{radius}} = \lim_{\theta\to 0}\frac{2\pi C P(\theta)\sin\theta}{\theta} = 2\pi C\left(1 + Am + \frac{Am}{1+2Am} + \frac{A^2m^2}{1+2Am}\right). \tag{2.14}$$

When this value is different from 2π, the metric (2.8) has at least a conical singularity in $\theta = 0$. Similarly, around the other half axis $\theta = \pi$, we have

$$\frac{\text{circumference}}{\text{radius}} = \lim_{\theta\to \pi}\frac{2\pi C P(\theta)\sin\theta}{\pi - \theta} = 2\pi C\left(1 - Am - \frac{Am}{1+2Am} + \frac{A^2m^2}{1+2Am}\right). \tag{2.15}$$

A deficit angle is interpreted as a semi-infinite cosmic string pulling the BBMB black hole along the half axis with a force proportional to the tension of the string (i.e., a $T_{\mu\nu}$ localized on the string and proportional to the deficit angle), and conversely an excess angle is interpreted as a strut pushing the black hole.

Because the conicity of the conical singularities are different on the two half axes, in general it is not possible to remove them simultaneously, fixing the value of the constant C. Henceforward, to avoid the conical singularity for $\theta = 0$, we will set[4]

$$C = \left(1 + Am + \frac{Am}{1+2Am} + \frac{A^2m^2}{1+2Am}\right)^{-1}. \tag{2.16}$$

[3]This feature makes the two coordinate choices not physically equivalent in the presence of rotation.

[4]Alternatively a new axial angular coordinate, with canonical period 2π, can be defined dilatating the old one by a factor C^{-1}.

One cannot even remove the second conical singularity by a nontrivial fine-tuning between the parameters such that $Am + \frac{Am}{1+2Am} = 0$ because, apart from the trivial cases for $A = 0$ or $m = 0$, corresponding to Schwarzschild or Minkowski spacetimes in accelerating coordinates, respectively, the only remaining possibility is $Am = -1$; but unfortunately it is outside the range of permitted parameters (2.13). Note that the rotating solution of [12] lacks conical singularity, though it is accelerating, because it does not have a proper mass term.[5]

Usually, as was found by Ernst himself in [13] for the case of a vanishing scalar field, it is possible to introduce an external magnetic field to remove this residual conical singularity from the charged C metric. We will do the same with a non-null scalar field in Sec. II C.

We finally observe that, although the nonaccelerating case of solution (2.8) has a divergent scalar field $\Psi(r,\theta)$ behavior on the whole outer horizon $r = m$, when A is non-null the scalar field is well behaved except on one pole ($r = r_+$, $\theta = \pi$),

$$\Psi(r_+,\theta) = \frac{Am\cos\theta + 1}{Am(1+\cos\theta)}, \tag{2.17}$$

where it is divergent. The scalar field divergences were the origin of the problems in the magnetized BBMB solution in [4], and thus a better behaved scalar field on the horizon is favorable for magnetizing purposes.

B. *C* metric with a conformal scalar hair in the Melvin magnetic universe

Here we want to embed the metric (2.8), which we will consider our seed, in the external magnetic field of the Melvin magnetic universe. To do that it is necessary to have the Harrison transformation for the theory under consideration. Using the results of [4] we can write[6] such a kind of magnetizing transformation in the solution space of the Einstein-Maxwell theory of gravity with a conformally coupled scalar field. In terms of the Ernst potentials, for uncharged[7] and static seed spacetimes, the Harrison transformation is given by

$$\begin{aligned} \mathcal{E}_0 \to \mathcal{E} &= \frac{\mathcal{E}_0 - \frac{B^2}{4}(1-\frac{\kappa}{6}\Psi^2)^2\mathcal{E}_0^2}{[1-\frac{B^2}{4}(1-\frac{\kappa}{6}\Psi^2)\mathcal{E}_0]^2}, \\ \mathbf{\Phi}_0 \to \mathbf{\Phi} &= \frac{\frac{B}{2}(1-\frac{\kappa}{6}\Psi^2)\mathcal{E}_0}{1-\frac{B^2}{4}(1-\frac{\kappa}{6}\Psi^2)\mathcal{E}_0}. \end{aligned} \tag{2.18}$$

[5]As can be seen from the vanishing acceleration limit.

[6]One just has to pass to the Einstein frame with a conformal transformation, apply the desired transformation (in this case the Harrison one), and afterwards come back to the Jordan frame.

[7]A Harrison transformation preserving staticity is generalized in Sec. III C for a particular kind of charged seed.

We recall the definition of Ernst complex potentials that, just for this particular uncharged and static seed case, remain real,

$$\mathcal{E} := f - \mathbf{\Phi}^2, \qquad \mathbf{\Phi} := A_\varphi. \tag{2.19}$$

The Ernst potentials for the seed metric (2.8) are obtained by comparing it with the Weyl one (2.7),

$$\mathbf{\Phi}_0 = 0, \qquad \mathcal{E}_0 = f_0 = -\frac{P(\theta) r^2 \sin^2\theta}{(1 + Ar\cos\theta)^2}. \tag{2.20}$$

So, while the function $\gamma(r, \theta)$ remains unchanged as in the Weyl metric (2.7), the magnetized f is given by

$$f = \mathcal{E} + \mathbf{\Phi}^2 = \frac{f_0}{\Lambda^2(r,\theta)} \quad \text{where}$$
$$\Lambda(r,\theta) = 1 - \frac{B^2}{4}\left(1 - \frac{\kappa}{6}\Psi^2\right) f_0. \tag{2.21}$$

Finally the resulting magnetized version of the accelerating C metric (2.8) becomes

$$ds^2 = \frac{\Lambda^2(r,\theta)}{(1 + Ar\cos\theta)^2}\left[-\frac{Q(r)}{r^2}dt^2 + \frac{r^2}{Q(r)}dr^2 + \frac{r^2}{P(\theta)}d\theta^2 + \frac{r^2 \sin^2\theta P(\theta)}{\Lambda^4(r,\theta)}d\varphi^2\right], \tag{2.22}$$

supported by the magnetic field

$$A_\varphi = \frac{\frac{B}{2}(1 - \frac{\kappa}{6}\Psi^2) f_0}{1 - \frac{B^2}{4}(1 - \frac{\kappa}{6}\Psi^2) f_0}. \tag{2.23}$$

The conical singularity present at the point $\theta = \pi$ cannot be removed by the addition of an external magnetic field because, as shown in [4], the excess or deficit angle that stems from embedding conformal scalar hairy black holes in the Melvin universe is proportional to both the intensity of the external magnetic field B and the "intrinsic" electromagnetic charge e of the seed black hole. Since in this case the seed solution (2.8) is electromagnetically neutral, it is not possible to do a fine-tuning between the parameters (A, m, e, B) to elide the nodal singularity, exactly as in the case of the null scalar field [13]. In Sec. II C, an intrinsically charged solution will be considered so that we will be able to apply this Ernst trick.

The scalar curvature invariants of this metric (2.23), such as $R^{\mu\nu}R_{\mu\nu}$ and $R^{\mu\nu\sigma\lambda}R_{\mu\nu\sigma\lambda}$, are divergent only on the pole $(r = r_+, \theta = \pi)$. As expected, this is a reminiscence of the singular character of the field $\Psi(r, \theta)$ on that pole. So the magnetized C-metric solution is slightly better behaved than the nonaccelerating one of [4], but still it remains singular. Nevertheless, now it can be used, via the cut and paste procedure of [17], to build a regular black hole on the brane; this will be done in Appendix A.

C. Removing the conical singularity from the scalar hairy, charged C metric

The Ernst method [13] to remove the conical singularity typical of the C metric spacetime consists of embedding it in an external magnetic field. To achieve that, it is essential to have an interaction between the intrinsic charge of the black hole (which could be of electric or magnetic type) and the external field (which can be electric). The simpler implementation remains within a static framework, so it is necessary that the intrinsic electric charge and external charge are of the same type. For instance, we will consider an intrinsically magnetically charged accelerating BBMB black hole embedded in an external magnetic field. It is easy to see, via the electromagnetic duality in four dimensions, that the same result can be obtained by an electrically charged accelerating black hole embedded in an external electric field. On the other hand, when the intrinsic and external electromagnetic charges are of a different type, the metric becomes stationary due to the appearance of a $\vec{E} \times \vec{B}$ circulating momentum flux in the stress-energy tensor, which serves as a source for a twist potential. To be more precise in order to preserve the staticity of the seed spacetime, even when it is not electromagnetically neutral, the Ernst potential $\mathcal{E}$ must remain real.

Thus let us consider as a seed metric an accelerating BBMB black hole with intrinsic magnetic charge g. It has the same form as the uncharged one (2.8), but the matter fields are

$$A_\varphi = -g\cos(\theta),$$
$$\Psi(r,\theta) = \pm\sqrt{\frac{6}{\kappa}\frac{\sqrt{m^2 - g^2(1 + 2Am)}(Ar\cos\theta + 1)}{r(1 + Am) + m(Ar\cos\theta - 1)}}. \tag{2.24}$$

Because the metric is charged, we cannot use the Harrison transformation directly in the conformal frame, but we have to shift it in the minimal frame, magnetize the shifted metric, and then come back in the Jordan frame, as explained in [4]. The resulting magnetized metric remains formally the same as the uncharged case (2.22); also the scalar field remains the same as (2.24), but the magnetic potential becomes

$$A_\varphi = -\frac{g\cos\theta + \frac{B}{2}g^2\cos^2\theta + \frac{B}{2}\left\{1 - [m^2 - g^2(1 + 2mA)]\left[\frac{1 + Ar\cos\theta}{(1+Am)r + m(Ar\cos\theta - 1)}\right]^2\right\}\frac{P(\theta) r^2 \sin^2\theta}{(1 + Ar\cos\theta)^2}}{\Lambda(r,\theta)},$$

and Λ, for the charged case, modifies in

$$\Lambda(r,\theta) = 1 + gB\cos\theta + \frac{g^2B^2}{4}\cos^2\theta + \frac{B^2}{4}\left\{1 - [m^2 - g^2(1+2mA)] \times \left[\frac{1 + Ar\cos\theta}{(1+Am)r + m(Ar\cos\theta - 1)}\right]^2\right\} \times \frac{P(\theta)r^2\sin^2\theta}{(1+Ar\cos\theta)^2}.$$

Thus the introduction of the additional parameter B related to the external electromagnetic field makes possible the removal of the conical singularity from both the north and south poles. Expanding the metric around $\theta = 0$, as done in Sec. II A, it is possible to pull out the angular deficit or excess in $\theta = 0$ by just rescaling the φ coordinate

$$\varphi \rightsquigarrow \phi = \varphi\frac{(1+Am)[1 + Am/(1+2Am)]}{(1+gB/2)^4}. \qquad (2.25)$$

To eliminate also the conical singularity from $\theta = \pi$ one has to fix a particular relation between the parameters A, m, B, g,

$$mA = \frac{gB(4 + g^2B^2)}{4 - 4gB + 6g^2B^2 - g^3B^3 + g^4B^4/4}. \qquad (2.26)$$

From a physical point of view the removal of the conical singularity corresponds to removing the string (or strut) in charge to provide the acceleration to the C metric. It means that the acceleration of the black hole pair is entirely provided by the interaction force between the intrinsic electromagnetic charge of the black hole and the external magnetic field.

For small values of the electromagnetic field, $gB \ll 1$, the latter equation coincides with the Newtonian force felt by a massive magnetic monopole, of intensity g, in a uniform magnetic field whose strength is proportional to B (or alternatively, via electromagnetic duality, the weak electric field limit corresponds to an electric charge in a uniform electric field)

$$mA \approx gB.$$

In fact, this represents the nonrelativistic limit, i.e., $A \ll 1$, as can be seen inverting (2.26) and expanding for small acceleration parameter A,

$$gB = 2\frac{\left(\frac{1+3mA}{1-mA}\right)^{1/4} - 1}{\left(\frac{1+3mA}{1-mA}\right)^{1/4} + 1} \approx mA.$$

Usually these accelerating metrics, once regularized with the Ernst procedure, are of a certain interest because they provide a description of pair production of black holes in a magnetic field, as first pointed out in [18] (see also [19]). Unfortunately, this picture in the context of the BBMB black hole is ruined. In fact, despite the removal of the conical singularities and the strut/string interpretation related to that, not even the addition of the electromagnetic charge to the accelerating hairy metric is sufficient to make it regular because of the presence of a curvature singularity on the pole ($r = m$, $\theta = \pi$), hence the presence of a singularity not hidden behind an event horizon. This is due to the divergence of the scalar field, of the seed metric, at that point.

III. BLACK HOLES WITH A CONFORMALLY COUPLED CONSTANT SCALAR FIELD

As we have seen in the previous sections or as it is known from the literature, the BBMB solution, the actual model of a black hole with a conformally coupled scalar field, reveals several drawbacks or disadvantages, which are not present in ordinary black holes of the Kerr-Newman family. Let us list some:

(i) The scalar field is divergent on the horizon.
(ii) The spacetime is unstable under linear perturbations [20].
(iii) When embedded in an external magnetic field, it breaks down: it discloses curvature singularities on the horizon [4].
(iv) The introduction of the cosmological constant can hide the whole scalar field singularity behind the horizon, while the introduction of the acceleration cures just some divergences, but not all. These residual scalar field singularities, not hidden inside the event horizon, often cause naked singularities in the solution generating process, as seen in Sec. II B and in [4].[8]
(v) The BBMB black hole carries just a dichotomic, secondary hair, in the sense that there is not a continuous parameter associated with this scalar hair. There is no nonextremal extension, which might make the extra parameter continuous [21]. Moreover, because of the extremality its entropy is null.
(vi) It does not have a continuous limit to the Schwarzschild or Reissner-Nordstrom black hole. In [4] it is shown how, from a generalization of the BBMB solution, the Penney one, in the conformal frame it is possible to reach the Reissner-Nordstrom and the Schwarzschild black hole. But this is not an admissible physical process because, in order to do that, one has to pass through naked singularities.

[8]To be more precise, due to the scalar field divergence, some of the $SU(2,1)$ Kinnersley symmetry transformations, studied in [4], involve unbounded quantities when applied to the BBMB metric in the conformal frame. In this sense the BBMB black hole is not a "physically good" seed for the solution generating technique.

(vii) A stationary version of the BBMB black hole is not known. The Ernst generating algorithm fails to add rotation to the BBMB metric, and difficulties arise also in the slow rotating approximation [22]. The rotating metric of [12] does not have a proper mass term.

(viii) Higher dimensional flavors of the BBMB black hole are not known.

Thus, now, our purpose is to explore the possibility of a solution that is able to overcome these difficulties, or at least some. We restrict our research inside the most generic stationary axisymmetric Petrov type D class of metrics, which can be cast in the Plebanski-Demianski form. Recently this issue, in the presence of a scalar field coupling, was discussed in [23]. To begin with, we will focus on the conformal coupling for the scalar field without the cosmological constant. The most general nonstealth solution,[9] of the above form, admitting electromagnetic and Newman-Unti-Tamburino charges, acceleration, and in particular a standard mass[10] and rotation terms, of the Kerr type in the limit of vanishing acceleration, requires a constant scalar field,

$$ds^2 = \frac{1}{(y-x)^2}\left[\frac{F(y)(dt - x^2 d\varphi)^2}{1+x^2y^2} - \frac{1+x^2y^2}{F(y)}dy^2 + \frac{1+x^2y^2}{F(x)}dx^2 + \frac{F(x)(y^2dt + d\varphi)^2}{1+x^2y^2}\right], \tag{3.1}$$

$$F(\xi) = \sum_{i=0}^{4} f_i \xi^i, \qquad A = \frac{ey(dt - x^2 d\varphi)}{1+x^2y^2}, \qquad \Psi = \pm\sqrt{\frac{6}{\kappa}}\sqrt{1+\frac{e^2}{f_0+f_4}}. \tag{3.2}$$

Constant conformally coupled scalar black hole metrics are not a novelty; some static solutions were already discussed in [24,25] for a slightly different theory including the cosmological constant (and an extra conformal Ψ^4 potential term in the action, usually associated with the presence of the cosmological constant).

Even though the scalar field is constant, it contributes nontrivially to the equations of motion (2.2). In fact, for a constant scalar field Ψ_0, from (2.2) we have

$$\left(1 - \frac{\kappa}{6}\Psi_0^2\right)G_{\mu\nu} = \kappa T_{\mu\nu}^{(\mathrm{EM})}. \tag{3.3}$$

Hence (for $\Psi_0 \neq \pm\sqrt{6/\kappa}$) the presence of a constant conformally coupled scalar field has the property of rescaling the effective Newton coupling constant, thus rescaling the relative values of the electromagnetic charges. We will see hereinafter how the possibility of an arbitrary rescaling of the coupling constant, depending on the strength of the scalar field, has nontrivial physical effects. The basic difference with respect to the case with the cosmological constant [24,25] is that the value of the scalar field is not constrained by the coupling constants, as can be seen from (B6).

When the electromagnetic charges are vanishing, a new branch of solutions is allowed with $\Psi_0 = \pm\sqrt{6/\kappa}$, whose supporting spacetimes do not have to be Einstein manifold, but they have to obey the weaker condition coming from the scalar field equation (2.4): they simply are Ricci flat. It is possible to smoothly join these two branches in a unique family of metrics. To better clarify this point let us consider a specialization of (3.1) without the Newman-Unti-Tamburino term and in spherical coordinates ($r = -1/Ay$, $\cos\theta = x$),

$$ds^2 = \frac{\left[\frac{r^4 G(r)}{\rho^2}(dt + a\sin^2\theta d\varphi)^2 - \frac{\rho^2}{r^4 G(r)}dr^2 + \frac{\rho^2 \sin^2\theta}{G(\theta)}d\theta^2 + \frac{G(\theta)}{\rho^2}(adt + (r^2+a^2)d\varphi)^2\right]}{(1+Ar\cos\theta)^2}, \tag{3.4}$$

where

$$G(\xi) = (1-\xi^2)(1+r_+A\xi)(1+r_-A\xi), \qquad \xi = \{y = -1/Ar, x = \cos\theta\}, \tag{3.5}$$

$$\mathcal{A} = \frac{-erdt - ar e\sin^2\theta d\varphi}{r^2 + a^2\cos^2\theta}, \tag{3.6}$$

$$\Psi = \pm\sqrt{\frac{6}{\kappa}}\sqrt{\frac{s}{s+e^2}}, \tag{3.7}$$

$$r_\pm = m \pm \sqrt{m^2 - a^2 - e^2 - s}, \tag{3.8}$$

$$\rho = r^2 + a^2\cos^2\theta, \tag{3.9}$$

$r_\pm$ points the positions of the inner and outer horizons, while the accelerating horizon is located at $\xi^2 = 1$, that is, $r_A = \pm A^{-1}$. This metric clearly describes an accelerating Kerr-Newman black hole dressed with a conformally coupled scalar hair, which is represented by the continuous parameter s. In fact, when the scalar parameter s goes to zero, the standard accelerating Kerr-Newman black hole [15] is recovered. All subhierarchy of black holes until the Schwarzschild can be obtained by switching on and off the parameters (A, a, m, e, s). In this sense the hair can be classified as primary hair, contrary to the BBMB case.

No hair theorems [26] are avoided because the scalar field is often assumed to vanish asymptotically or because

[9]For some values of the parameters f_i there exist a matter configuration such that $T_{\mu\nu} = 0$, although the fields A_μ and Ψ are not null, so the matter does not have a backreaction with the background spacetime.

[10]In the notation of [23] the mass term is related to odd powers of the $F(\xi)$ function.

it is not possible to connect this family of black holes with the Einstein frame by a conformal transformation, in the case of null electromagnetic charge.

The spacetime (3.4)–(3.9) admits a further straightforward generalization, in the case of the cosmological constant (see Appendix B).

A. Scalar hairy Reissner-Nordstrom black hole

To have a clearer picture of the spacetime described by the metric (3.4), let us consider a simpler case. When the rotation a and acceleration A parameters are null in (3.4), we remain with the static Reissner-Nordstrom (RN) black hole enriched by the scalar hair s,

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{e^2+s}{r^2}\right)dt^2 + \left(1 - \frac{2m}{r} + \frac{e^2+s}{r^2}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2. \tag{3.10}$$

The scalar field remains the same as Eq. (3.7) while the electromagnetic potential reduces to the standard RN one: $A_t = -e/r$. The total energy momentum tensor

$$T^\mu{}_\nu = \frac{e^2+s}{r^4}\,\mathrm{diag}(-1,-1,1,1) \tag{3.11}$$

satisfies both dominant and strong energy conditions whenever $s \geq -e^2$. Therefore, without violating these overall energy conditions,[11] it is even possible to erase the contribution of the electromagnetic field by means of the constant scalar field, just setting $s = -e^2$, hence recovering the Schwarzschild spacetime, but in this borderline case the scalar field becomes divergent.

Following the Ernst magnetizing method for the accelerating version of this metric, it is possible to remove the conical singularity without constraining any of the physical parameters e, B, m, A, because of the presence of the scalar parameter s. Furthermore, this accelerating solution has not the curvature singularity of the BBMB C metric, and thus it is suitable to describe pair production of a scalar hairy black hole in the presence of an external magnetic field; these points are addressed in Sec. III C.

It is evident by the similarities to the RN metric that the spacetime (3.10) has the same causal structure of the static charged black hole. The only difference now is that the position of the horizons is shifted by the presence of the scalar field constant parameter s, as can be seen from (3.8), setting the rotation parameter $a = 0$. The electric charge remains the same as that of the RN spacetime,

$$Q = \frac{1}{4\pi}\int *F = e. \tag{3.12}$$

On the other hand, from a thermodynamical point of view, there are some dissimilarities with respect to the RN black hole, for instance, about local stability; this point is addressed in Sec. III B.

B. Thermodynamics of constant scalar hairy black hole

To analyze the thermodynamics of the charged black hole with a conformally coupled constant scalar field (3.10) we will use the Euclidean method, as done in [27]. The partition function for a thermodynamical ensemble is identified, around the Euclidean continuation of the classical solution, with the Euclidean path integral in the saddle point approximation [28]. Thus, first of all, we consider a minisuperspace of static Euclidean metrics given by

$$ds^2 = N(r)^2 f(r)^2 d\tau^2 + f(r)^{-2}dr^2 + r^2 d\Omega^2, \tag{3.13}$$

where the imaginary time τ, obtained by a wick rotation $t \to i\tau$, has period β, the inverse of the temperature T. It is obtained requiring regularity [no conical singularities in the (τ, r) section] on the horizon

$$T = \frac{1}{\beta} = \frac{N(r)}{4\pi}\frac{d}{dr}f(r)^2|_{r_+} = \frac{r_+ - r_-}{4\pi r_+^2}. \tag{3.14}$$

If the scalar field $\Psi(r)$ and the electromagnetic potential $A_\mu(r)$ are considered to depend at most on the radial coordinate r, then the reduced Euclidean action $\mathcal{I}$ becomes[12]

$$\mathcal{I} = \beta\int_{r_+}^{\infty}\left[N(r)\mathcal{H}(r) + A_t\left(\frac{r^2}{N(r)}A_t'(r)\right)'\right]dr + \mathcal{B}, \tag{3.15}$$

where $\mathcal{B}$ is the surface term and the prime denotes the d/dr derivative. The reduced Hamiltonian is given by

$$\mathcal{H} = \frac{r^2}{2G}\left\{\frac{\kappa}{6}\left[f^2(\Psi')^2 - \Psi\Psi'\left((f^2)' + \frac{4f^2}{r}\right) - 2\Psi f^2\Psi''\right] + \left(1 - \frac{\kappa\Psi^2}{6}\right)\left[\frac{(f^2)'}{r} - \frac{1-f^2}{r^2}\right] + G\frac{(A_t')^2}{N}\right\}.$$

In the grand canonical ensemble the variation of the action is implemented keeping the temperature fixed and the "injection voltage energy" $\Phi = A_t(\infty) - A_t(r_+)$. For the Euclidean solution under consideration $\Psi \propto$ const, $N = 1$, $\mathcal{H} = 0$, and $(r^2A_t')' = 0$, so the variation of the action evaluated on the classical solution is just given by the variation of the boundary term $\delta\mathcal{B}$.

$$\delta\mathcal{B} = -\frac{\beta}{2G}\left[r\left(1 - \frac{\kappa}{6}\Psi^2\right)\delta f^2 + 2GA_t\delta(r^2A_t')\right]_{r_+}^{\infty} \tag{3.16}$$

$$= \left(\frac{e^2}{e^2+s}\right)\frac{1}{G}\left(\beta\delta m - \frac{4\pi r_+\delta r_+}{2}\right) - \beta\Phi\delta e, \tag{3.17}$$

[11]Note that, when there is no electromagnetic field, the strong energy condition for the scalar field requires the positivity of s.

[12]Note that there is a sign discrepancy with respect to [27] because there the base manifold is hyperbolic.

where the following boundary variations of the fields at infinity and at the horizon r_+ were used:

$$\delta(r^2 A_t')|_\infty = \delta(r^2 A_t')|_{r_+} = \delta e, \tag{3.18}$$

$$\delta\Psi|_\infty = \delta\Psi|_{r_+}, \tag{3.19}$$

$$\delta f^2|_\infty = -\frac{2}{r}\delta m + O(r^{-2}), \tag{3.20}$$

$$\delta f^2|_{r_+} = -(f^2)'|_{r_+}. \tag{3.21}$$

Then defining an "effective Newton constant" $\tilde{G}$ as $\tilde{G}^{-1} = G^{-1}e^2/(e^2+s)$ and integrating (3.17) we obtain the finite Euclidean action, up to an arbitrary additive constant,

$$\mathcal{I} = \mathcal{B}(\infty) - \mathcal{B}(r_+) = \frac{\beta}{\tilde{G}}m - \frac{A_+}{4\tilde{G}} - \beta\Phi e. \tag{3.22}$$

In the grand canonical ensemble the Euclidean action is related (in the unit where Planck and Boltzmann constants are $\hbar = \kappa_B = 1$) to the free energy by $\mathcal{F} = \beta\mathcal{I}$. Thus the mass M, electric charge Q, and entropy S are obtained by the usual thermodynamical relations,

$$M = \partial_\beta \mathcal{I} - \beta^{-1}\Phi\partial_\Phi \mathcal{I} = \frac{m}{\tilde{G}}, \tag{3.23}$$

$$Q = -\beta^{-1}\partial_\Phi \mathcal{I} = e, \tag{3.24}$$

$$S = \beta\partial_\beta \mathcal{I} - \mathcal{I} = \frac{A_+}{4\tilde{G}}. \tag{3.25}$$

The first law of black hole thermodynamics is satisfied using only the effective Newton constant $\tilde{G}$; this is a typical feature of nonminimal coupling of the scalar field [27]. In the range of values of s respecting the dominant and strong energy conditions the entropy remains positive.

While, when the scalar field is vanishing, for $s = 0$, the standard results for the Reissner-Nordstrom black hole are retrieved. It is interesting to observe that for the uncharged case ($e = 0$) the total mass M and the entropy S become void. Thus the scalar hair can be considered to be primary since it does not depend on the presence of the electric charge; anyway some physical spacetime properties are better behaved for $e \neq 0$.

A natural question is now whether the charged constant hairy black hole (3.10) may decay into the Reissner-Nordstrom one, which is also a solution of the same action principle with a null scalar field, for a fixed temperature and electromagnetic potential injection. Evaluating the Euclidean action, for fixed β and Φ, for both RN and (3.10) spacetimes there is not a stable thermodynamical favored configuration, but it is possible to find numerically a critical point beyond which phase transitions can occur, for a certain range of parameters, that do not violate the strong and dominant energy conditions.

The local thermal stability with respect to the temperature fluctuation or electromagnetic fluctuation can be inferred by the analysis of the heat capacity at constant electric potential C_Φ and electrical permittivity at constant temperature ϵ_T, respectively, as done for the grand canonical ensemble in [29]

$$C_\Phi := T\left(\frac{\partial S}{\partial T}\right)_\Phi = T\left(\frac{\partial T}{\partial r_+}\right)_\Phi^{-1}\left(\frac{\partial S}{\partial r_+}\right)_\Phi = -\frac{2\pi r_+}{\tilde{G}}\frac{r_+^2 - e^2 - s}{r_+^2 - e^2 - 3s}\frac{e^2+2s}{e^2+s}. \tag{3.26}$$

The local thermodynamical stability is given by the positivity of the heat capacity; thus according to (3.26) and (3.8) in this case the presence of the scalar field improves the local stability since there is a parametric window for which $C_\Phi \geq 0$,

$$\frac{r_+^2 + e^2}{3} \leq s \leq m^2 - e^2.$$

The electrical permittivity is defined as

$$\epsilon_T := \left(\frac{\partial Q}{\partial \Phi}\right)_T = \left(\frac{\partial \Phi}{\partial r_+}\right)_T^{-1}\left(\frac{\partial Q}{\partial r_+}\right)_T. \tag{3.27}$$

But since the charge Q has dependence only on terms of the potential at the constant horizon, we have to decompose each factor in the previous equation as

$$\left(\frac{\partial Q}{\partial r_+}\right)_T = -\left(\frac{\partial T}{\partial Q}\right)_{r_+}^{-1}\left(\frac{\partial T}{\partial r_+}\right)_Q, \tag{3.28}$$

$$\left(\frac{\partial \Phi}{\partial r_+}\right)_T = -\left(\frac{\partial T}{\partial \Phi}\right)_{r_+}^{-1}\left(\frac{\partial T}{\partial r_+}\right)_\Phi. \tag{3.29}$$

So electrical permittivity (3.27) becomes

$$\epsilon_T = \left(\frac{\partial \Phi}{\partial e}\right)_{r_+}^{-1}\left(\frac{\partial T}{\partial r_+}\right)_\Phi^{-1}\left(\frac{\partial T}{\partial r_+}\right)_Q = r_+\frac{r_+^2 - 3e^2 - 3s}{r_+^2 - e^2 - 3s}. \tag{3.30}$$

Therefore even from just a naive[13] thermodynamical study, we can see how the presence of the scalar field affects the local thermal stability of the solution (3.10) with respect to the Reissner-Nordstrom black hole for $s = 0$.

In the next section we will present another application for which the presence of the scalar parameter s has nontrivial physical consequences.

C. Magnetized accelerating constant scalar hairy charged black hole pair

It could be of some interest to consider the magnetized version of the constant hairy charged and accelerating black hole, described by the metric (3.4), fixing, for

[13]Other thermodynamical settings may be considered, even including an extra chemical potential for the scalar field.

simplicity, the rotation parameter $a = 0$. This is because it possesses an extra free scalar parameter s, with respect to the not hairy one ($s = 0$), which allows us to achieve a regular equilibrium solution (with no conical singularity) without imposing any constraints on the mass m, charge g, external magnetic field B, and acceleration A parameters, as in the hairless case.

To keep the system as simple as possible we make use of the four-dimensional electromagnetic duality, in the metric (3.4) with $a = 0$, to obtain, as a seed, an intrinsic magnetically charged black hole instead of an electrically charged one,

$$ds^2 = \frac{1}{(1 + Ar\cos\theta)^2}\left[-\mathrm{g}(r)dt^2 - \frac{dr^2}{\mathrm{g}(r)} + \frac{r^2 d\theta^2}{p(\theta)} + r^2 p(\theta)\sin^2\theta d\varphi^2\right], \tag{3.31}$$

where

$$\mathrm{g}(r) = (1 - A^2r^2)\left(1 - \frac{2m}{r} + \frac{g^2 + s}{r^2}\right), \quad \Psi = \sqrt{\frac{6}{\kappa}}\sqrt{\frac{s}{s + g^2}}, \tag{3.32}$$

$$p(\theta) = 1 + 2mA\cos\theta + A^2\cos^2\theta(g^2 + s), \quad A_\varphi = -g\cos\theta. \tag{3.33}$$

This simplifies the discussion, because the solution after magnetization remains static. Otherwise, to guarantee staticity, we might have considered alternatively the intrinsic electric charge, but then, in that case, we should have embedded it in an external electric field. From a mathematical point of view this feature is portrayed by the fact that the Ernst potentials remain real (in the alternative case of an intrinsic electrically charged black hole in an external magnetic field, the electromagnetic Ernst potential $\mathbf{\Phi}$ remains purely imaginary; conversely the rotation is generated by fully complex potentials).

Using the results of [4], it is possible to write the Harrison magnetizing transformation for this class of static magnetically charged spacetimes, directly in the Jordan frame,

$$f = \frac{f_0}{\Lambda^2}, \qquad \mathbf{\Phi} = \frac{\mathbf{\Phi}_0 + \frac{B}{2}[(1 - \frac{k}{6}\Psi^2)f_0 - \mathbf{\Phi}_0^2]}{\Lambda}, \tag{3.34}$$

$$\text{where } \Lambda = 1 - B\mathbf{\Phi}_0 - \frac{B^2}{4}\left[\left(1 - \frac{k}{6}\Psi^2\right)f_0 - \mathbf{\Phi}_0^2\right]. \tag{3.35}$$

These are a generalization of (2.18) and (2.21) in the case of nonvanishing intrinsic magnetic charge g, simply expressed in terms of f and $\mathbf{\Phi}$.

From the comparison with the Weyl metric (2.7) we can identify the needed seed functions

$$f_0 = -\frac{p(\theta)r^2\sin^2\theta}{(1 + Ar\cos\theta)^2} \quad \text{and} \quad \mathbf{\Phi}_0 = -g\cos\theta. \tag{3.36}$$

Then after the action of the Harrison transform, according to (3.34), we get the magnetized ones. Plugging these latter ones again into the Weyl metric (2.7), we obtain the magnetized version of (3.31),

$$ds^2 = \frac{1}{(1 + Ar\cos\theta)^2}\left\{\Lambda^2\left[-\mathrm{g}(r)dt^2 - \frac{dr^2}{\mathrm{g}(r)} + \frac{r^2d\theta^2}{p(\theta)}\right] + \frac{r^2p(\theta)\sin^2\theta}{\Lambda^2}d\varphi^2\right\}, \tag{3.37}$$

where the functions $\mathrm{g}(r)$, $p(\theta)$, and Ψ remain the same as the nonmagnetized solution, while the electromagnetic potential

$$A_\varphi = -\frac{g\cos\theta + \frac{B}{2}[\frac{g^2}{g^2+s}\frac{p(\theta)r^2\sin^2\theta}{(1+Ar\cos\theta)^2} + g^2\cos^2\theta]}{1 + Bg\cos\theta + \frac{B^2}{4}[\frac{g^2}{g^2+s}\frac{p(\theta)r^2\sin^2\theta}{(1+Ar\cos\theta)^2} + g^2\cos^2\theta]},$$

which supports (3.37), includes both the intrinsic magnetic charge of the black hole and an external Melvin-like magnetic field. The metric (3.37) describes a pair of accelerating magnetically charged black holes in the presence of an external magnetic field and a conformally coupled scalar field. When the scalar field is null, that is $s = 0$, we recover the Ernst solution [13].

As usual, accelerating metrics (3.37) possess a couple of conical singularities on the poles, one (let us say around $\theta = 0$) is always easy to remove, following the same analysis of Sec. II A, by rescaling the angular coordinate φ, such that

$$\varphi \to \phi = \frac{1 + 2mA + A^2(g^2 + s)}{(1 + \frac{Bg}{2})^4}\varphi, \tag{3.38}$$

while the second singularity can be removed thanks to a constraint relation between the physical parameters m, g, B, A, s. An interesting feature of the conformally coupled constant scalar field solution is that it introduces a new parameter s with respect to the Reissner-Nordstrom spacetime, which, when expressed in terms of mass, acceleration, and intrinsic magnetic charge, allows us to remove the second conical singularity for $\theta = \pi$, without fine-tuning these latter charges as in the Ernst solution [13],

$$s = \frac{m(1 + \frac{2}{3}g^2B^2 + g^4B^4)}{A(gB + \frac{B^3g^3}{4})} - \frac{1}{A^2} - g^2. \tag{3.39}$$

Therefore, even though the effect of the constant scalar field is not dynamical and it reduces just to an effective rescaling of the Newton constant, it opens to the possibility of modeling less constrained magnetized charged black holes with respect to the null scalar field case. This feature has the effect that more general black holes in the pair creation process (in the spirit of [18,19]) can be admissible in a strong magnetic background, and also the pair creation rate[14] is affected by the extra parameter s. This is so because the pair creation probability depends on the position of the roots of the $g(r)$, which is modified with respect to RN spacetime whenever $s \neq 0$.

Another possibility to regularize the spacetime (3.31), without resorting to an external electromagnetic field, consists in directly fine-tuning the constant scalar field and rescaling the azimuthal coordinate in order to cancel the angular singularity of the C metric. But this can be done in a slightly different, not equivalent, radial coordinate, the one used for the C metric before [16].

IV. COMMENTS AND CONCLUSIONS

In this paper the Ernst solution generating technique, in the context of Einstein-Maxwell gravity conformally coupled to a scalar field, is applied to a C-metric solution, which describes a couple of accelerating BBMB black holes. Through a Harrison transformation we manage to embed the BBMB C metric into an external magnetic field. The resulting solution shows more regularity than the no accelerating one, but still it presents a curvature singularity on a pole of the event horizon, due to a divergence of the scalar field at that point. Thanks to this regularity enhancement we are able to build a fully regular black hole metric by a cut and paste procedure. The price to pay was the introduction of extra matter on the thin shell gluing surface.

Therefore a better behaved seed solution, which is able to overcome several disadvantages of the BBMB spacetime, is considered for the theory under consideration. The requirements of a proper mass term and rotation constrain the scalar field to be constant, at least in the realm of the Plebanski-Demianski spacetimes.[15] In that case it is possible to write a regular black hole family of solutions comprising the Kerr black hole and featuring acceleration, mass, rotation, intrinsic electromagnetic charge, and an extra scalar parameter. The thermodynamical properties of a simple black hole of this family (without acceleration and rotation) are studied and compared to the vanishing scalar field case, the Reissner-Nordstrom black hole. By a Harrison transformation we were able to embed some black holes of this family in an external magnetic field. It is interesting to note that the presence of the scalar field introduces an extra parameter s, which can be tuned (in terms of the other physical parameters) to cancel the string, encoded in the conical singularity, that is pulling the two black holes. This is the main difference compared to the case without the scalar field $s = 0$. A completely regular balanced solution can be obtained without constraining between themselves the mass, intrinsic charge, acceleration, and external magnetic strength. Possibly this is an astrophysically observable feature for the black hole family considered. Of course, another possible observable property is the correction to the standard Newton law due to the presence of the scalar field, which, for example, can be tested in galaxies' rotation curves. An upper limit constraint to the value of the scalar parameter s can also be found from solar system physics.

It may also be interesting, for a future perspective, to study whether this constant scalar field gives some contribution on a cosmological level, in particular concerning the open problems of the amount of dark energy and dark matter.

Eventually people interested in higher dimensions may find the four-dimensional C metrics presented in this paper of some utility in building novel, topological nontrivial solutions in five dimensions.

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APPENDIX A: THIN SHELL REGULARIZATION: MAGNETIZED AND ACCELERATING BBMB BLACK HOLE ON THE BRANE

It is possible to regularize at the same time both the conical singularity and the curvature singularity of the metric (2.22) localized at $\theta = \pi$. We will take advantage of the same procedure used in [17] to remove the conical singularity from the uncharged C metric. The basic idea is to consider the regular half part of the solution (2.22), that is the one with $0 \leq \theta \leq \pi/2$, to cut away the resting part for $\pi/2 \leq \theta \leq \pi$ and then gluing it into another copy of the regular one. While the continuity of the metric is assured, the price to pay is the introduction of an extra energy momentum tensor term $T^{\Sigma}_{\mu\nu} = \delta(\frac{\pi}{2} - \theta) S_{ij} e^i_\mu e^j_\nu$, localized on the $\bar{\theta} = \pi/2$ surface Σ, to regularize the discontinuity of the first fundamental form on the pasting surface $\theta = \pi/2$. Generalized junction conditions, for the theory we are considering, were discussed in [30]; the thin shell of extra matter content can be quantified as follows. First let us define h_{ij}, the metric on the three-surface characterized by constant θ, and the normalized outward orthogonal vector to the three-surface

[14] The pair creation rate is obtained (see [18,19] for details) as the difference of the action evaluated on the lukewarm instanton and the action evaluated on the Melvin magnetic background. The lukewarm instanton can be produced as the Wick rotated $t \to i\tau$ metric (3.37) regularized from conical singularities, in the Euclidean time, such that the temperature of the event and acceleration horizons coincides.

[15] Therefore an eventual stationary generalization of the BBMB black hole has to be searched for outside the Plebanski-Demianski ansatz.

$$n^\mu = \left[0, 0, \frac{\sqrt{P(\theta)}(1 + Ar\cos\theta)}{r\Lambda(r,\theta)}, 0\right]. \tag{A1}$$

So the extrinsic curvature on the three-surface is given by

$$K_{ij} = \nabla_i n_j = \frac{\sqrt{g(\theta)}(1 + Ar\cos\theta)}{2r\Lambda(r,\theta)} \frac{d}{d\theta} h_{ij}. \tag{A2}$$

The induced surface stress energy tensor is given by

$$\begin{aligned} S^{(S)}_{ij} &= -\frac{1}{8\pi G}\left\{[K_{ij}]^{\bar\theta+}_{\bar\theta-}\left(1 - \frac{\kappa}{6}\Psi^2\right) - h_{ij}[K]^{\bar\theta+}_{\bar\theta-}\left(1 - \frac{\kappa}{18}\Psi^2\right)\right\} \\ &= \frac{Ah_{ij}}{2\pi G\Lambda(r,\bar\theta)}\begin{bmatrix} 1 - \frac{m(1+Am)}{2r(1+2Am)} & 0 & 0 \\ 0 & 1 - \frac{m(1+Am)}{2r(1+2Am)} & 0 \\ 0 & 0 & 1 + \frac{\partial_\theta \log\Lambda(r,\theta)|_{\bar\theta}}{Ar} \end{bmatrix} \\ &\quad + \frac{A\Psi^2 h_{ij}}{36\pi G\Lambda(r,\bar\theta)}\begin{bmatrix} \frac{\partial_\theta \log\Lambda(r,\theta)|_{\bar\theta}}{Ar} + \frac{m(1+Am)}{2r(1+2Am)} & 0 & 0 \\ 0 & \frac{\partial_\theta \log\Lambda(r,\theta)|_{\bar\theta}}{Ar} + \frac{m(1+Am)}{2r(1+2Am)} & 0 \\ 0 & 0 & \frac{\partial_\theta \log\Lambda(r,\theta)|_{\bar\theta}}{-2Ar} - \frac{m(1+Am)}{r(1+2Am)} \end{bmatrix}. \end{aligned} \tag{A3}$$

Eventually also the electromagnetic field contribution may be taken into account, in the usual way,

$$S^{(\mathrm{EM})}_{ij} = \lim_{\bar\theta-\theta=\epsilon\to 0}\int_{-\epsilon}^{+\epsilon} T^{(\mathrm{EM})}_{ij}\, dn.$$

APPENDIX B: ACCELERATING, ROTATING, CHARGED CONSTANT HAIRY BLACK HOLE WITH COSMOLOGICAL CONSTANT

A further generalization of the metric (3.4), describing an accelerating, rotating, and intrinsically charged black hole with a conformally coupled, constant scalar hair can be found when we are in the presence of two additional terms in the action due to cosmological constant λ, and also due to an extra scalar conformally coupled potential $\alpha\Psi^4$. We present it here, but because the Harrison transformation in the presence of λ is not known at the moment, it will not be possible to embed it in an external electromagnetic field. The equations of motion are modified with respect to the null cosmological ones (2.2); in fact, the scalar energy momentum tensor $T^{(S)}_{\mu\nu}$ and scalar equation became

$$\begin{aligned} T^{(S)}_{\mu\nu} &= \partial_\mu\Psi\partial_\nu\Psi - \frac{1}{2}g_{\mu\nu}\partial_\sigma\Psi\partial^\sigma\Psi \\ &\quad + \frac{1}{6}[g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu + G_{\mu\nu}]\Psi^2 - \alpha g_{\mu\nu}\Psi^4, \end{aligned} \tag{B1}$$

$$\Box\Psi = \frac{1}{6}\mathrm{R}\Psi + 4\alpha\Psi^3. \tag{B2}$$

The electromagnetic equations remain the same as (2.6), so the potential A_μ also remains unchanged as in (3.7) (and also ρ, Ψ), while the metric in the presence of the cosmological constant becomes

$$ds^2 = \frac{\left[-\frac{F(r)}{\rho^2}(dt + a\sin^2\theta d\varphi)^2 + \frac{\rho^2}{F(r)}dr^2 + \frac{\rho^2}{G(\theta)}d\theta^2 + \frac{G(\theta)}{\rho^2}\sin^2\theta(adt + (r^2 + a^2)d\varphi)^2\right]}{(1 + Ar\cos\theta)^2}, \tag{B3}$$

where

$$F(r) = (1 - A^2r^2)\left[r^2 - 2mr + e^2 + s + a^2\left(1 + \frac{\lambda}{3A^2}\right)\right] - \frac{\lambda}{3}\left(r^4 + \frac{a^2}{A^2}\right), \tag{B4}$$

$$G(\theta) = 1 + 2Am\cos\theta + A^2\cos^2\theta\left[e^2 + s + a^2\left(1 + \frac{\lambda}{3A^2}\right)\right] \tag{B5}$$

$$\alpha = -\frac{\kappa\lambda}{36}\frac{s + e^2}{s}. \tag{B6}$$

The causal structure is the same as the standard accelerating, charged, and rotating C metric (which can be obtained in the smooth $s \longrightarrow 0$ limit). The basic difference with respect to this latter case, apart from the fact that the horizons are shifted in

$$r_{\pm} = m \pm \sqrt{m^2 - e^2 - s - a^2\left(1 + \frac{\lambda}{2A^2}\right)}, \qquad \text{(B7)}$$

is that the scalar hair parameter s allows one to set the strength of the scalar field and thus to arbitrarily tune the value of the coupling constants. As we have seen in Sec. III C, this feature can have relevant astrophysical consequences, at least in the balance between the string and external magnetic field strength of the magnetized C metric. The vanishing cosmological constant limit is well defined and gives the solution (3.4)–(3.9).

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