Wald entropy of black holes: Logarithmic corrections and trace anomaly

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Quantum effects due to conformal matter in a black hole background result in universal logarithmic corrections to black-hole entropy. The universality resides in the connection of the log term coefficient with those of type-A and type-B Weyl anomalies, regularization-scheme independent quantities. We presently study the case of extremal black holes within Wald's Noether charge formalism. In the conformal class of flat metrics, we are again able to unveil the log term in the entropy from the horizon value of the solution to the Q-curvature uniformization problem. Beyond conformally flat backgrounds, type-B Weyl anomaly becomes an obstruction to considering flat space as the fiducial metric and the search for a metric of constant Q-curvature remains open. Notwithstanding, by a uniform scaling argument we show that the results based on heat kernel and Euclidean computations (namely entropy function and conical defect) can also be derived as Wald entropy, that is, as Noether charge of the integrated anomaly or conformal index. We finally comment on the relation with entanglement entropy.

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I. INTRODUCTION

Black holes in Einstein gravity are endowed with entropy given by the celebrated Bekenstein-Hawking (BH) *area law* [1,2]. Deviations from the area law are naturally expected to occur in more general classical gravitational theories and also due to quantum effects. This is certainly the case in string theory where the effective low-energy theory, in addition to Einstein-Hilbert action, contains higher curvature terms due to a finite size of strings (α') as well as quantum loop corrections (g_s). Here, a certain class of extremal black holes has a dual microscopic description and the statistical entropy resulting from the microscopic counting is found to be in remarkable agreement, for large charges, with the macroscopic counterpart that takes into account higher *local* curvature invariants from finite-size and quantum corrections (see, e.g., [3–5]). Crucial to this agreement is the fact that the corresponding macroscopic entropy, which deviates from the area law, is *Wald entropy*, that is, it satisfies the modified form of the first law derived by Wald [6] for a wide class of generally covariant actions; in Wald's formalism, the role of the entropy is played by the integral of a geometric density, the Noether potential, over a spatial cross-section of the horizon. In this respect, the "entropy function formalism" [7,8] has been developed as an efficient computational tool that exploits the attractor mechanism for extremal black holes; it can be shown that the entropy function is in fact Wald entropy.

Regarding quantum one-loop effects due to conformal matter, the *leading correction* to the black hole (bh) entropy in the semiclassical limit of large charges seems to be *universally* given by the logarithm of the horizon area (cf. [9,10])

$$S_{\rm bh} = S_{\rm BH} + {\rm const}\cdot \ln S_{\rm BH} + \mathcal{O}(1), \tag{1}$$

with a coefficient that involves those of the type-A and type-B trace anomalies [11]. This deep connection between trace anomaly and entropy had been established in the Euclidean formalism after renormalization of one-loop UV-divergent contribution of matter fields to the entropy [12,13], in the tunneling and exact differential formalisms [14,15], in the thermodynamics of certain black hole solutions stemming from an anomalous energy-momentum tensor [16] and, more recently, via the "quantum entropy function" [17–23]. Despite the many different approaches, there was little hope for a direct derivation based on Wald's formalism mainly due to the unavailability of the one-loop effective action, which in general contains nonlocal terms. Notwithstanding, for the conformally flat class of black hole backgrounds we were able to establish the connection between type-A trace anomaly and the logarithmic correction to the black hole entropy via Wald's Noether charge formalism [24]. The rationale behind our approach was to focus on the "anomaly-induced effective action" (${}^{A}S_{\rm anom}$), which suffices to correctly reproduce the anomalous trace of the energy-momentum tensor, and to render it local by the introduction of an auxiliary field ϕ to finally use Wald's formula to read off the contribution to the black hole entropy as Noether charge.

The purpose of the present note is to further elaborate on the Wald approach to log-corrections in two ways. We first study the extremal black hole case within the conformally flat class; second, we consider the general case where the type-B trace anomaly comes into play. We are able to obtain the log term, as was already done in nonextremal cases, from the solution the uniformization problem for Q-curvature. Beyond conformal flatness, the type-B Weyl

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anomaly enters the game and introduces ambiguities in the definition of the Q-curvature; it also becomes an obstruction for flat metric to be considered as the "fiducial" one and the search for a metric of constant Q-curvature remains open. Notwithstanding, by a uniform scaling argument we show that the results based on heat kernel and Euclidean computations (namely entropy function and conical defect) can also be derived as Wald entropy, that is, as Noether charge of the integrated anomaly or conformal index. We finally comment on the relation with entanglement entropy.

II. NOETHER CHARGE OF THE ANOMALY INDUCED ACTION

Let us first highlight the main steps of our previous work in establishing a connection between the type-A trace anomaly and the logarithmic correction to the black hole entropy via Wald's Noether charge formalism [24].

As said, the rationale behind our approach was to focus on the anomaly-induced effective action (${}^A S_{\rm anom}$), which suffices to correctly reproduce the anomalous trace of the energy-momentum tensor, and to render it local by the introduction of an auxiliary field ϕ. The type-A trace anomaly in four dimensions is given by the term involving the Euler density in the expectation value of the trace of the energy-momentum tensor

$$\langle T\rangle = -\frac{a}{16\pi^2}E_4, \tag{2}$$

with

$$E_4 = Riem^2 - 4Ric^2 + R^2. \tag{3}$$

The anomaly induced effective action is a conformal primitive of the trace anomaly that generalizes Polyakov's original computation in two dimensions [25]. In analogy with Liouville's local form of Polyakov's action, an auxiliary field can be introduced to cast the action in local form (see, e.g., [26,27])

$${}^A S_{\rm anom} = -\frac{a}{32\pi^2}\int dx^4\sqrt{-g}\quad\{-\phi\Delta_4\phi + Q\phi\}. \tag{4}$$

This four-dimensional analog involves the Q-curvature [28,29]

$$Q = E_4 - \frac{2}{3}\Delta R, \tag{5}$$

and the Paneitz's operator [31] (squared Laplacian plus lower curvature terms) [32]

$$\Delta_4 = \Delta^2 + 2\nabla_\mu\left(R^{\mu\nu} - \frac{1}{3}g^{\mu\nu}R\right)\nabla_\nu. \tag{6}$$

To correctly reproduce the anomaly from the metric variation of this action, the auxiliary field ϕ must satisfy

$$\Delta_4\phi = \frac{1}{2}\left(E_4 - \frac{2}{3}\Delta R\right), \tag{7}$$

that is, ϕ must solve the uniformization problem for the Q-curvature with the fiducial metric (flat one) having vanishing Q-curvature.

The corresponding Noether charge can be computed (see [24] for further details) and yields a leading correction to BH entropy

$$S_{\rm bh} = \frac{\mathcal{A}_{\mathcal{H}}}{4} - a\cdot\chi_{\mathcal{H}}\cdot\phi_{\mathcal{H}} + \cdots. \tag{8}$$

Here $\chi_{\mathcal{H}}$ is the (2 dimensional) Euler characteristic of the horizon, $\phi_{\mathcal{H}} = \phi(r_+)$ is the value at the horizon of the auxiliary field and the ellipsis stands for terms involving only derivatives of ϕ at the horizon. The derivatives of ϕ, and its powers, translate into inverse powers of the area, so that they do not talk to the logarithmic correction [35] and are absorbed in the $\mathcal{O}(1)$ term.

This neat formula for the quantum correction to *nonextremal* black hole entropy due to one-loop quantum field theoretic effects was the main outcome of our previous work [24].

III. EXTREMAL BLACK HOLES: NEAR-HORIZON GEOMETRY

We start by considering the near-horizon geometry of a black hole in the extremal limit. It is well known that this geometry factorizes in an AdS_2 times a transverse compact space [13,37] but the limiting procedure has its own subtleties. Let us briefly examine this limit in the following simple Euclidean spherically symmetric situation, described by the metric

$$ds^2 = f(r)d\tau^2 + f(r)^{-1}dr^2 + r^2 d\Sigma, \tag{9}$$

where $d\Sigma$ is the metric of the compact transverse part. If we expand around the outer horizon r_+

$$f(r) = a(r-r_+) + b(r-r_+)^2 + O((r-r_+)^3) \tag{10}$$

$$= (r-r_+)\left(r - r_+ + \frac{a}{b}\right)b + O((r-r_+)^3), \tag{11}$$

the extremal limit is then achieved by sending $a\to 0$ (i.e., $r_-\to r_+$) and rescaling the Euclidean time coordinate τ, where $\tau\sim\tau + 4\pi/a$. In suitable coordinates x, ϕ

$$r - r_+ = \frac{a}{b}\sinh^2\frac{x}{2}, \tag{12}$$

$$\frac{a\tau}{2} = \theta, \qquad \theta\sim\theta + 2\pi, \tag{13}$$

a two-dimensional hyperbolic H^2 factor shows up

$$ds^2 = \frac{1}{b}(dx^2 + \sinh^2 x d\theta^2) + r_+^2 d\Sigma^2. \tag{14}$$

The horizon is mapped to the center $x = 0$ of H^2 and the conformal boundary to $x\to\infty$.

A. Extremal Reissner-Nordström

As an illustration let us examine the conformally flat near-horizon geometry $H^2 \times S^2$ of the extremal Reissner-Nordström

$$ds^2 = r_+^2(dx^2 + \sinh^2 x d\theta^2 + d\Omega^2). \tag{15}$$

In order to solve the uniformization problem for the Q-curvature, we choose Fefferman-Graham coordinates and set $r_+ = 1$ for convenience,

$$\frac{\rho}{2} = \mathrm{e}^{-x}; \qquad x = 0 \mapsto \rho = 2; \qquad x \to \infty \mapsto \rho \to 0; \tag{16}$$

so that the H^2 metric becomes

$$ds_H^2 = \frac{d\rho^2 + (1 - \rho^2/4)^2 d\theta^2}{\rho^2}. \tag{17}$$

The Paneitz operator gets factorized in terms of the Laplacian as $\Delta_4 = \Delta^2 - 2\Delta$ and the solution, which is required to be regular at the horizon, has the following logarithmic term

$$\phi(\rho) \sim 2\ln\frac{(\rho + 2)^2}{\rho}. \tag{18}$$

Now we read off the logarithmic dependence on r_+, $\phi_{\mathcal{H}} \sim 2\ln r_+ \sim \ln A_{\mathcal{H}}$, and the logarithmic correction to the entropy in (8)

$$\triangle S_{\mathrm{bh}} = -2a \ln A_{\mathcal{H}}. \tag{19}$$

IV. BEYOND CONFORMAL FLATNESS: TYPE-B WEYL ANOMALY

When the background is not conformally flat, the type-B Weyl anomaly comes into play and things are more complicated

$$\langle T\rangle = -\frac{a}{16\pi^2}E_4 + \frac{c}{16\pi^2}W^2. \tag{20}$$

The anomaly induced effective action becomes ambiguous, essentially due to the ambiguity in the Q-curvature. One is free to shift any Q-curvature by the locally conformal invariant Weyl-squared term while preserving the linear transformation law under Weyl rescaling of the metric $g \to \hat{g} = e^{2w}g$

$$\sqrt{\hat{g}}\hat{Q} = \sqrt{g}(Q + \Delta_4 w) \quad \text{and} \quad \sqrt{\hat{g}}\hat{W}^2 = \sqrt{g}W^2. \tag{21}$$

This ambiguity carries over into the uniformization problem and we have, a priori, no preferred choice for the Q-curvature.

Now, as long as one is interested in the dependence on an overall scale factor, things are easier. Consider a uniform scaling of the metric $g \to \hat{g} = \Lambda^2 g$; the variation of the one-loop effective action (log of the functional determinant of the kinetic operator A of the free conformal field) is given by the associated zeta function at zero (see, e.g. [38])

$$\hat{S}_{\mathrm{eff}} - S_{\mathrm{eff}} = -\frac{1}{2}\log\frac{\det\hat{A}}{\det A} = \zeta_A(0)\log\Lambda. \tag{22}$$

Modulo zero mode contributions, this is essentially the integrated trace anomaly or conformal index

$$\zeta_A(0) = \int\sqrt{g}d^4x\langle T\rangle = \frac{1}{16\pi^2}\int\sqrt{g}d^4x(-aE_4 + cW^2). \tag{23}$$

We simply apply Wald's prescription to translate the dependence on the overall scale Λ from the effective action into the black hole entropy. That is, we compute the Noether charge of the above local quantity.

It is convenient to split the Weyl-square term as follows

$$W^2 = E_4 + 2\,\mathrm{Ric}^2 - \frac{2}{3}\mathrm{R}^2, \tag{24}$$

and compute the Noether charge of

$$\Delta S_{\mathrm{eff}} = -\frac{\log\Lambda}{16\pi^2}\int\sqrt{g}d^4x\left((a-c)E_4 + \frac{2c}{3}\mathrm{R}^2 - 2c\mathrm{Ric}^2\right). \tag{25}$$

The contribution of the first term in the right-hand side gives, upon integration, the Euler characteristic of the horizon ($\chi_{\mathcal{H}}$); the second, two times the Ricci scalar (R) of the four-dimensional space; and the third, the Ricci scalar of the four-dimensional space along the transverse directions to the horizon [36,39]

$$\Delta S_{\mathrm{bh}} = -\frac{\log\Lambda}{16\pi^2}4\pi\int_\Sigma\sqrt{h}d^2x\left(2(a-c)R_\Sigma + \frac{4c}{3}\mathrm{R} - 2c\mathrm{R}_{aa}\right). \tag{26}$$

where R_{aa} is the projection of the Ricci scalar perpendicular to the horizon.

In all, we end up with the following general relation between the correction to black hole entropy, logarithmic in the overall scale, and the type-A and type-B anomaly coefficients

$$\Delta S_{\mathrm{bh}} = \log\Lambda^2\left((c-a)\chi_{\mathcal{H}} - \frac{2c}{3}\hat{\Sigma}\mathrm{R}^{\mathcal{H}} + c\hat{\Sigma}\mathrm{R}^{\mathcal{H}}_{aa}\right), \tag{27}$$

where $\hat{\Sigma}$ stands for the horizon area over 4π and $R^{\mathcal{H}}$ and $R^{\mathcal{H}}_{aa}$ are the average values on the horizon of the corresponding quantities. In what follows, let us examine some illustrative examples.

A. Extremal Reissner-Nordström

This is just a crosscheck, the near-horizon geometry $H^2 \times S^2$ is conformally flat and therefore the type-B anomaly plays no role. The necessary inputs are

$$\chi_{\mathcal{H}} = 2, \qquad \hat{\Sigma} = 1, \qquad \mathrm{R}^{\mathcal{H}} = 0, \qquad \mathrm{R}^{\mathcal{H}}_{aa} = -2, \tag{28}$$

so that the log-correction is given by

$$\Delta S_{\rm bh} = -4a \log \Lambda, \tag{29}$$

in agreement with what we obtained before (19), noting that the area scales as Λ^2, and with the results via conical defect [13]. It is worthwhile to notice that the answer remains the same even if one considers the full geometry of the extremal Reissner-Nordström and not just the near-horizon one, very reminiscent of an underlying attractor mechanism.

B. Extremal topological black hole

We consider a family of extremal topological black holes [40]. The near-horizon geometries of these black holes have the form $H^2 \times T^2$ and $H^2 \times H^2/\Gamma$, the transverse compact space is a space of zero or constant negative curvature.

1. Torus

Starting with the torus, the inputs to the log-correction expression are

$$\chi_{\mathcal{H}} = 0, \qquad \hat{\Sigma} = 1, \qquad \mathrm{R}^{\mathcal{H}} = -2, \qquad \mathrm{R}^{\mathcal{H}}_{aa} = -2, \tag{30}$$

resulting in

$$\Delta S_{\rm bh} = -\frac{4c}{3} \log \Lambda. \tag{31}$$

This result is in conformity with [13] when one modifies the heat kernel computation to account for the conformal coupling of the scalar field.

2. Hyperbolic surface

For the case of a transverse hyperbolic surface of genus g

$$\begin{aligned} &\chi_{\mathcal{H}} = 2 - 2g, \qquad \hat{\Sigma} = g - 1, \\ &\mathrm{R}^{\mathcal{H}} = -4, \qquad \mathrm{R}^{\mathcal{H}}_{aa} = -2, \end{aligned} \tag{32}$$

resulting in

$$\Delta S_{\rm bh} = 4\left(a - \frac{2}{3}c\right)(g - 1) \log \Lambda, \tag{33}$$

in agreement with [13] adapted to the conformal scalar field.

3. Schwarzschild

This case is also amenable to treat, since there is only one dimensionful parameter, namely the radius of the horizon,

$$\chi_{\mathcal{H}} = 2, \qquad \hat{\Sigma} = 1, \qquad \mathrm{R}^{\mathcal{H}} = 0, \qquad \mathrm{R}^{\mathcal{H}}_{aa} = 0, \tag{34}$$

resulting in

$$\Delta S_{\rm bh} = 4(c - a) \log \Lambda, \tag{35}$$

which is easily compared with [12,41].

4. Reissner-Nordström

The nonextremal Reissner-Nordström can also be worked out

$$\chi_{\mathcal{H}} = 2, \qquad \hat{\Sigma} = 1, \qquad \mathrm{R}^{\mathcal{H}} = 0, \qquad \mathrm{R}^{\mathcal{H}}_{aa} = -\frac{2r_-}{r_+}. \tag{36}$$

The log term is then given by

$$\Delta S_{\rm bh} = 4\left(c - c\frac{r_-}{r_+} - a\right) \log \Lambda. \tag{37}$$

This is in concordance with [42], for the scalar field ($a = 1/360$, $c = 1/120$)

$$\Delta S_{\rm bh} = \frac{2r_+ - 3r_-}{90 r_+} \log \Lambda. \tag{38}$$

V. ENTANGLEMENT ENTROPY

It is a general belief that the proper interpretation of entanglement entropy in a black hole background is that of quantum correction to black hole entropy, with the event horizon playing the role of the entangling surface. We notice that recent computations of entanglement entropy [43] account for the connection between the logarithmic-in-the-cutoff term and trace anomaly coefficients. Also by means of a scaling argument, one can track down the trace anomaly dependence on the entanglement entropy; the computation requires the deformation of the background geometry by a conical defect. The final results (see also [44,45]) coincide with those derived here as Wald entropy; in fact, one can even show *a posteriori* that they are precisely given by Wald's formula [46]. The subtle difference with our approach is that we claim the correction to be Wald entropy from the outset.

It is worthwhile to stress that we get Wald entropy of the integrated conformal anomaly times the (logarithm of the) overall scale as a consistency requirement of our proposal to read off the entropy from the one-loop effective action. The scaling argument implies that we should get Wald entropy of the *local* conformal index. This is conceptually different from the observation (see, e.g., [45,47,48]) that the entropy obtained by the off-shell procedure (entanglement entropy) introducing a conical deformation is the same as the Wald entropy of the divergent piece of the off-shell action. This result is first derived and then recognized as the same as Wald entropy; in this formalism, there seems to be no a priori reason for this to happen. In fact, entanglement entropy for a general entangling surface is far more complicated, depending also on the extrinsic curvature [44,46]. We have recently learned that a "squashed-cone" regularization [49] in the off-shell approach has been proposed; in four dimensions it correctly reproduces the logarithmic terms in entanglement entropy that had been found invoking a holographic argument [44]. We believe that these quantum corrections to entanglement entropy are to be interpreted as quantum corrections to gravitational entropy (see, e.g., [50,51]), where the latter are not captured by Wald's formula in general but rather correspond to the "Wald-like" contributions recently proposed in a holographic setting [52].

Our scaling argument for stationary black holes, where Wald's formula applies, goes in the direction of the quantum entropy function, where calculations are done in the regular geometry (see, e.g., discussion in [19], pp. 7). In all, we get another instance where Wald's formula agrees with Euclidean and other approaches within their domains of applicability, this time including one-loop effects.

VI. CONCLUSION

The purpose of this short article was to further study the connection between trace anomaly and quantum corrections to black hole entropy within Wald's Noether charge formalism. The plan was to pursue the universality of this connection: even if in general it is not possible to know the full one-loop effective action, the part that is able to correctly reproduce the trace anomaly did suffice to track down the effect of the anomaly. This piece is given by the anomaly induced effective action which is rendered local by means of an auxiliary field. In conformally flat backgrounds, where only type-A trace anomaly plays a role, the on-shell condition for this auxiliary field corresponds to the mathematical problem of finding a Weyl scaling of the black hole metric to a fiducial one which in this case is nothing but the flat one where the Q-curvature vanishes. We showed that even in the extremal limit one is able to find a solution regular at the horizon and read off a term logarithmic in the horizon area. Beyond conformal flatness some difficulties are encountered, the nonvanishing Weyl tensor becomes an obstruction to considering flat metric as the fiducial one and the definition of the Q-curvature is plagued by ambiguities that, at this moment, we have been unable to tackle [53].

A more modest progress is achieved by considering the dependence on an overall scale. The change in the effective action is logarithmic in the scale factor and contains a local conformal invariant functional, namely, the integrated trace anomaly or conformal index. The effect of this term on the entropy can then be obtained à la Wald by computing its contribution to the Noether charge. We find agreement between this direct derivation via Wald entropy and the logarithmic terms in the entanglement entropy; the latter being derived by deformation of the effective action in the presence of a conical defect. In this respect, we close our initial program of providing a derivation à la Wald of the already known connection between trace anomaly (type A and type B) and the log-correction to (extremal and nonextremal) black hole entropy. For the case of nonconformally flat backgrounds, the dependence on the overall scale is correctly obtained; however, as already mentioned, the uniformization problem for the Q-curvature including type-B trace anomaly deserves further clarification.

Finally, let us comment that agreement is also found in the case of extremal black holes with the results via "quantum entropy function," at least the dependence of the logarithmic term on the trace anomaly [17–23,56]. At the level of the classical action, the entropy function is precisely Wald entropy; our present derivation lends support to considering that the quantum entropy function is also Wald entropy of the one-loop effective action.

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