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Comparison between Jordan and Einstein frames of Brans-Dicke gravity a la loop quantum cosmology

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It is well known that the Jordan and Einstein frames are equivalent to each other in classical Brans-Dicke theory, provided that one and the same metric is employed for the physical space-time. Nevertheless, it is shown in this paper that by cosmological models the loop quantization in the two different frames leads to inequivalent effective theories. Analytical solutions are found in both frames for the effective loop quantum Brans-Dicke cosmology without potential in (i) the vacuum case and (ii) the additional massless scalar field case. In the Einstein frame, the analytical solution for the Brans-Dicke potential $\propto \varphi^2$ is found. In all of those solutions, the bouncing evolution of the scale factor is obtained around the Planck regime. The differences between the loop quantization of the two frames are reflected by (i) the evolution of the scale factor around the bounce and (ii) the scale of the bounce in the physical Jordan frame.

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I. INTRODUCTION

The Brans-Dicke theory of gravity [1] is one of the most popular modified gravity theories, whose various aspects have been investigated for over 50-years, especially in the recent decade. The scalar-tensor structure of this theory has been considered as a source of inflation and primordial inhomogeneities of space-time [2], dark energy [3–5], and in the context of the stability of stars [6]. The newest results on the cosmic microwave background [7] show that the R^2 inflation, which may be also expressed in terms of the Brans-Dicke field,[1] fits the data of the spectrum of perturbations. On the other hand, as the background-independent quantization of general relativity (GR), loop quantum gravity (LQG) [8,9] has been rather active in the recent two decades. The expectation that the singularity predicted by classical GR would be resolved by quantum gravity has been confirmed by the recent study of the loop quantum cosmology (LQC) [10,11], which is a simplified, symmetric model of LQG [12,13]. The big-bang singularity in the cosmological model of GR is replaced by the quantum bounce of LQC. Recently, the nonperturbative quantization scheme of LQG has been successfully extended to $f(R)$ theories [14,15] and Brans-Dicke theory [16–20]. The corresponding cosmological model for Brans-Dicke theory has been set up [20]. The purpose of this paper is to compare the Jordan frame with the Einstein frame of Brans-Dicke gravity by their loop quantum cosmology models. Note that the original formulation of Brans-Dicke theory was in the Jordan frame. If one and the same metric is employed to represent the physical space-time, then the Jordan and Einstein frames are equivalent to each other in classical Brans-Dicke theory. However, there is no guarantee for the equivalence of the quantization in two frames. As shown in Ref. [17], the quantization procedure of the Brans-Dicke theory distinguishes between two cases: $\omega = -\frac{3}{2}$ and $\omega \neq -\frac{3}{2}$. In this paper, we shall assume that $\omega \neq -\frac{3}{2}$. This assumption comes from the observational limitations of the Brans-Dicke theory, which prefers $\omega \gg 1$ [21,22].

To transform the Brans-Dicke theory into the Einstein frame, one has to redefine the metric tensor, which would cause the canonical form of the GR action. The LQC in the Einstein frame has been studied in Refs. [23–25]. In this paper, we take the original idea of Brans and Dicke that the Jordan frame is the physical one,[2] though in general this remains open. For instance, the quantization in different frames may give different results of the evolution of primordial gravitational waves, which in the future could help us to discriminate one frame and favor the other. Following the interpretation of the Jordan frame as the physical one, we compare the two methods of LQC quantization: in the Jordan and the Einstein frame. In the latter case, we shall transform the results into the Jordan frame for precise comparison.

In this paper, according to Refs. [20,27] and for simplicity, we only focus on the effective LQC of the two frames, where holonomy corrections are included, while neglecting inverse triad corrections. Therefore, by

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[1] In the rest of this paper, we will refer to $f(R)$ theories as their Brans-Dicke form in metric formalism.

[2] For arguments, see Ref. [26].

the LQC correction, we mean LQC holonomy corrections to the flat Friedmann-Robertson-Walker (FRW) space-time in different frames. This treatment is usually considered to be realistic and consistent with effective equations of LQC of GR. All calculations in this paper are performed in Planck units, i.e., for $8\pi G = M_{pl}^{-2} = 1$, $\hbar = 1$.

The structure of this paper is as follows: In Sec. II we introduce the classical Hamiltonian of Brans-Dicke theory in the Jordan frame in a flat FRW model. In Sec. III we calculate effective equations of motion in a semiclassical approach to LQC Brans-Dicke theory in the Jordan frame. In Sec. IV we present exact solutions of semiclassical equations of motion for the LQC Jordan frame quantization for the vacuum case and for the additional massless scalar field case. In Sec. V we introduce the Hamiltonian formalism of the same cosmological model of Brans-Dicke theory in the Einstein frame and its semiclassical equations of motion for the LQC Einstein frame quantization. In Sec. VI we solve the semiclassical equations for the Einstein frame analytically and compare the results of the LQC quantization in both frames. Finally, we conclude in Sec. VII.

II. CLASSICAL BRANS-DICKE THEORY

We start with the classical Brans-Dicke theory coupled with a scalar matter field. The Jordan frame action reads

$$S(g,\varphi,\chi) = \frac{1}{2}\int_\Sigma d^4x\sqrt{-g}\Big[\varphi\mathcal{R} - \frac{\omega}{\varphi}(\partial_\mu\varphi)\partial^\mu\varphi - 2V(\varphi) - (\partial_\mu\chi)\partial^\mu\chi - 2W(\chi)\Big], \tag{2.1}$$

where φ is the Brans-Dicke scalar field, χ is a scalar matter field, and $V(\varphi)$ and $W(\chi)$ are potentials. Now we consider an isotropic and homogenous $k = 0$ universe. We choose a fiducial Euclidean metric ${}^oq_{ab}$ on the spatial slice of the isotropic observers and introduce a pair of fiducial orthonormal triad and cotriad as $({}^oe^a_i, {}^o\omega^i_a)$, respectively, such that ${}^oq_{ab} = {}^o\omega^i_a {}^o\omega^i_b$. Then the physical spatial metric is related to the fiducial by $q_{ab} = a^{2o}q_{ab}$, and its line element can be described by the FRW form

$$ds^2 = -dt^2 + a^2(t)(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)),$$

where a is the scale factor. Then the classical action (2.1) reduces to

$$\mathcal{L} = -3\dot{a}^2a\varphi - 3\dot{a}a^2\dot{\varphi} + a^3\frac{\omega}{2}\frac{\dot{\varphi}^2}{\varphi} - a^3V(\varphi) + a^3\frac{1}{2}\dot{\chi}^2 - a^3W(\chi). \tag{2.2}$$

By the Legendre transformation, the canonical momenta read, respectively, as

$$\pi_\varphi = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = a^2\Big(-3\dot{a} + \frac{\omega a\dot{\varphi}}{\varphi}\Big), \quad \pi_\chi = \frac{\partial\mathcal{L}}{\partial\dot{\chi}} = a^3\dot{\chi}, \quad \pi_a = \frac{\partial\mathcal{L}}{\partial\dot{a}} = -3a(2\varphi\dot{a} + a\dot{\varphi}). \tag{2.3}$$

Therefore, from $\mathcal{H} = \pi_\varphi\dot{\varphi} + \pi_\chi\dot{\chi} + \pi_a\dot{a} - \mathcal{L}$, one obtains the classical Hamiltonian as a function of φ, χ, a, and their canonical momenta as

$$\mathcal{H}_{\text{class}}(a,\pi_a) = -\frac{1}{6\beta a^3}\Big(-3\beta(\pi_\chi^2 + 2a^6(V(\varphi) + W(\chi))) - 6\varphi\pi_\varphi^2 + 6a\pi_\varphi\pi_a + \frac{\omega}{\varphi}a^2\pi_a^2\Big), \tag{2.4}$$

where $\beta := 2\omega + 3$. While the spatial slice of our cosmological model is infinite, we may introduce an "elemental cell" $\mathcal{V}$ and restrict all integrals to $\mathcal{V}$. For simplicity, we let the elemental cell $\mathcal{V}$ be a cubic measured by our fiducial metric and denote its volume as V_o. Via fixing the degrees of freedom of local gauge and diffeomorphism transformations, we finally obtain the connection and densitized triad by symmetrical reduction as [28]

$$A^i_a = cV_0^{-\frac{1}{3}o}\omega^i_a, \qquad E^b_j = pV_0^{-\frac{2}{3}}\sqrt{\det({}^0q)}{}^oe^b_j, \tag{2.5}$$

where c, p are only functions of t. Note that the new variables are related to the old ones by

$$|p| = a^2V_0^{\frac{2}{3}}, \qquad c = -\gamma\,\mathrm{sgn}(p)\frac{\pi_a}{6a}V_0^{\frac{1}{3}}, \tag{2.6}$$

where γ is the so-called Barbero-Immirzi parameter. Now the gravitational part of the phase space of the cosmological model consists of conjugate pairs (c, p) and (φ, π_φ). The basic Poisson brackets between them can be simply read as

$$\{c,p\} = \frac{1}{3}\gamma, \qquad \{\varphi,\pi_\varphi\} = 1. \tag{2.7}$$

Thus, the classical Hamiltonian in terms of new variables is of the form

$$\mathcal{H}(c,p) = \frac{-6\frac{\omega}{\varphi}c^2p^2 + 6\gamma cp\pi_\varphi + \gamma^2\big(\frac{\beta}{2}\pi_\chi^2 + \beta|p|^3(V(\varphi) + W(\chi)) + \varphi\pi_\varphi^2\big)}{\gamma^2\beta|p|^{3/2}}. \tag{2.8}$$

From the Hamiltonian equations, one obtains the following classical equations of motion:

$$\dot{\xi} = \{\xi,\mathcal{H}\} = \frac{\gamma}{3}\Big(\frac{\partial\xi}{\partial c}\frac{\partial\mathcal{H}}{\partial p} - \frac{\partial\xi}{\partial p}\frac{\partial\mathcal{H}}{\partial c}\Big) + \frac{\partial\xi}{\partial\varphi}\frac{\partial\mathcal{H}}{\partial\pi_\varphi} - \frac{\partial\xi}{\partial\pi_\varphi}\frac{\partial\mathcal{H}}{\partial\varphi}, \tag{2.9}$$

where $\xi = \xi(c, p, \varphi, \pi_\varphi)$ is some function on the classical phase space.

Let us generalize the above equations to the Brans-Dicke theory coupled with any perfect fluid. Then, combining the Hamiltonian equations and the scalar constraint $\mathcal{H} = 0$, one obtains [29]

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{2}{\beta}(\varphi V_\varphi - 2V(\varphi)) = \frac{1}{\beta}(\rho_M - 3P_M), \quad (2.10)$$

$$3\left(H + \frac{\dot{\varphi}}{2\varphi}\right)^2 = \frac{\beta}{4}\left(\frac{\dot{\varphi}}{\varphi}\right)^2 + \frac{V(\varphi)}{\varphi} + \frac{\rho_M}{\varphi} = \frac{1}{\varphi^2}\rho_e, \quad (2.11)$$

where $V_\varphi \equiv \frac{dV}{d\varphi}$, $H := \frac{\dot{a}}{a}$ is the Hubble parameter, and $\rho_e := \frac{\beta}{4}\dot{\varphi}^2 + \varphi(V(\varphi) + \rho_M)$ is the effective energy density. For $V = \rho_M = P_M = 0$, Eqs. (2.10) and (2.11) have analytical solutions of the form

$$H = \frac{\dot{\varphi}}{2\varphi}\left(\pm\sqrt{\frac{\beta}{3}} - 1\right), \quad \dot{\varphi} = \frac{1}{2}\sqrt{\beta\rho_i}\left(\frac{\varphi}{\varphi_i}\right)^{\pm\frac{1}{2}(3-\sqrt{3\beta})}, \quad (2.12)$$

where φ_i is some initial value of φ. For $\beta = 3$ (i.e., $\omega = 0$) one obtains

$$H = 0, \quad \varphi = \text{const} \ \vee \ H = \frac{1}{2t}, \quad \varphi = \sqrt{\frac{2\alpha_1}{\alpha_2 - t}}. \quad (2.13)$$

III. LQC CORRECTIONS TO BRANS-DICKE THEORY IN THE JORDAN FRAME

Let us consider the LQC corrections to the Brans-Dicke theory we mentioned above. In this paper, we follow the hybrid approach: The connection and triad are quantized by the polymerlike quantization, while all other canonical variables are quantized by the Schrödinger quantization. The kinematic Hilbert space for the geometry part can be defined as $\mathcal{H}^{\text{gr}}_{\text{kin}} := L^2(R_{\text{Bohr}}, d\mu_H)$, where R_{Bohr} and $d\mu_H$ are, respectively, the Bohr compactification of the real line (the configuration space) and the Haar measure on it [28], while the kinematic Hilbert spaces for the scalar fields are defined as in usual quantum mechanics. The whole Hilbert space is their direct product. Let $|\mu\rangle$ be the eigenstates of $\hat{p}$ in the kinematic Hilbert space $\mathcal{H}^{\text{gr}}_{\text{kin}}$, such that

$$\hat{p}|\mu\rangle = \frac{\gamma}{6}\mu|\mu\rangle. \quad (3.1)$$

It turns out that those states satisfy the following orthonormal condition:

$$\langle\mu_i|\mu_j\rangle = \delta_{\mu_i,\mu_j}, \quad (3.2)$$

where δ_{μ_i,μ_j} is the Kronecker delta function rather than the Dirac distribution. Note that in the LQC framework, while there is no operator corresponding to the connection c, its holonomy $\exp(i\mu c/2)$ along a line with oriented length μ is a well-defined operator. In the improved dynamics setting [11], one employs the length $\bar{\mu} = \sqrt{\frac{\Delta}{|p|}}$, with $\Delta = 4\sqrt{3}\pi\gamma\ell_{\text{p}}^2$ being a minimum nonzero eigenvalue of the area operator [30,31] to construct the Hamiltonian constraint operator. In the semiclassical regime, as a basic variable, the holonomy will certainly lead to corrections to the classical equations. Here we only focus on the LQC holonomy correction, while neglecting inverse triad corrections. A heuristic and simple way to get the holonomy corrections is to replace the connection variable by its holonomy, i.e., $c \to \frac{\sin(\bar{\mu}c)}{\bar{\mu}}$, though its validity should be checked by detailed calculations.

A. Friedmann equation

By the following substitution,

$$\varphi \to \varphi, \quad \pi_\varphi \to \pi_\varphi, \quad \chi \to \chi,$$
$$\pi_\chi \to \pi_\chi, \quad p \to p, \quad c \to \sqrt{\frac{|p|}{\Delta}}\sin\left(c\sqrt{\frac{\Delta}{|p|}}\right), \quad (3.3)$$

an effective Hamiltonian constraint with holonomy corrections of loop quantum Brans-Dicke cosmology can be obtained from Eq. (2.8) as

$$\mathcal{H}_{\text{LQC}} = \frac{1}{|p|^{3/2}\beta\gamma\Delta}\left[\pi_\varphi\left(6p|p|^{1/2}\sqrt{\Delta}\sin\left(c\sqrt{\frac{\Delta}{|p|}}\right) + \gamma\Delta\varphi\pi_\varphi\right) + \frac{\beta}{2}\gamma\Delta\pi_\chi^2 + |p|^3\left(-6\frac{\omega}{\gamma\varphi}\sin^2\left(c\sqrt{\frac{\Delta}{|p|}}\right) + \beta\gamma\Delta(V(\varphi) + W(\chi))\right)\right]. \quad (3.4)$$

Note that this effective Hamiltonian constraint can also be derived by a systematic approach as in Ref. [20]. All semiclassical equations of motion can be obtained from Eq. (2.9) with $\mathcal{H}_{\text{LQC}}$ as a Hamiltonian. For instance, from $\dot{\varphi} = \{\varphi, \mathcal{H}_{\text{LQC}}\}$, one obtains

$$\dot{\varphi} = \frac{6p\sin\left(c\sqrt{\frac{\Delta}{|p|}}\right)}{\beta\sqrt{\Delta}\gamma|p|} + \frac{2\varphi\pi_\varphi}{\beta|p|^{3/2}} \Rightarrow \pi_\varphi = |p|^{3/2}\frac{-6\text{sgn}(p)\sin\left(c\sqrt{\frac{\Delta}{|p|}}\right) + \gamma\sqrt{\Delta}\beta\dot{\varphi}}{2\gamma\sqrt{\Delta}\varphi}. \quad (3.5)$$

From the equation $\dot{p} = 2pH = 2p\frac{\dot{a}}{a} = \{p, \mathcal{H}_{\text{LQC}}\}$, one finds

$$H = \frac{\cos\left(c\sqrt{\frac{\Delta}{|p|}}\right)\left(2\omega p|p|^{1/2}\sin\left(c\sqrt{\frac{\Delta}{|p|}}\right) - \gamma\sqrt{\Delta}\varphi\pi_\varphi\right)}{\gamma\beta\sqrt{\Delta}|p|^{3/2}\varphi}, \tag{3.6}$$

where H is a Hubble parameter. Substituting Eq. (3.5) into the scalar constraint $\mathcal{H}_{\rm LQC} = 0$, one obtains

$$\sin^2\left(c\sqrt{\frac{\Delta}{|p|}}\right) = \frac{\gamma^2\Delta}{3}\left(\frac{\beta}{4}\dot\varphi^2 + \varphi\frac{1}{2}\dot\chi^2 + \varphi V(\varphi) + \varphi W(\chi)\right). \tag{3.7}$$

From Eqs. (3.5), (3.6), and (3.7) and $\mathcal{H}_{\rm LQC} = 0$, one finds the semiclassical LQC version of the first Friedmann equation,

$$\left(H + \frac{\dot\varphi}{2\varphi}\right)^2 = \left(\frac{1}{\varphi}\sqrt{\frac{\rho_e}{3}}\sqrt{1 - \frac{\rho_e}{\rho_{\rm cr}}} + \frac{\dot\varphi}{2\varphi}\left(1 - \sqrt{1 - \frac{\rho_e}{\rho_{\rm cr}}}\right)\right)^2, \tag{3.8}$$

where $\rho_e = \frac{\beta}{4}\dot\varphi^2 + \varphi(\frac{1}{2}\dot\chi^2 + V(\varphi) + W(\chi))$ is the effective energy density and $\rho_{\rm cr} = \frac{3}{\gamma^2\Delta} \simeq 0.41 G^{-2} \simeq 260 M_{\rm pl}^4$ is the critical (maximal) energy density. Equation (3.8) coincides with the effective Friedmann equation in Ref. [20], where the potentials of scalar fields are not included.

B. Equation of motion of φ

From Eq. (3.5), we define

$$\tilde p_\varphi = |p|^{3/2}\dot\varphi = \frac{6p|p|^{1/2}\sin\left(c\sqrt{\frac{\Delta}{|p|}}\right)}{\beta\sqrt{\Delta}\gamma} + \frac{2}{\beta}\varphi\pi_\varphi. \tag{3.9}$$

Then we have

$$\dot{\tilde p}_\varphi = \frac{d}{dt}(a^3\dot\varphi) = a^3(\ddot\varphi + 3H\dot\varphi) = \{\tilde p_\varphi, \mathcal{H}_{\rm LQC}\}. \tag{3.10}$$

From Eqs. (3.5), (3.7), and (3.10), one obtains

$$\ddot\varphi + 3H\dot\varphi + \frac{2}{\beta}\varphi V_\varphi + \frac{2}{\beta}(V(\varphi) + W(\chi)) \times\left(1 - 3\sqrt{1 - \frac{\frac{\beta}{4}\dot\varphi^2 + \varphi(V(\varphi) + W(\chi) + \frac{1}{2}\dot\chi^2)}{\rho_{\rm cr}}}\right) = -\frac{\dot\chi^2}{\beta}. \tag{3.11}$$

This equation may be expressed in a more general way by expressing the scalar field χ with its energy density ρ_M and pressure P_M as

$$\ddot\varphi + 3H\dot\varphi + \frac{2}{\beta}\varphi V_\varphi + \frac{2}{\beta}(V(\varphi) + \rho_M - P_M) \times\left(1 - 3\sqrt{1 - \frac{\frac{\beta}{4}\dot\varphi^2 + \varphi(V(\varphi) + \rho_M)}{\rho_{\rm cr}}}\right) = -\frac{1}{\beta}(\rho_M + P_M). \tag{3.12}$$

Then, the χ field could be replaced by any other perfect fluid, e.g., by dust, radiation, cosmological constant, etc. From $\dot\pi_\chi = \{\pi_\chi, \mathcal{H}\}$, one obtains the equation of motion of χ as

$$\ddot\chi + 3H\dot\chi + W_\chi = 0, \tag{3.13}$$

where $W_\chi \equiv \frac{dW}{d\chi}$. The LQC corrections enter this equation due to the existence of potential terms of φ and χ. Even for $V(\varphi) = 0$ this correction may appear due to the existence of a nonzero $W(\chi)$. Therefore, the continuity equation is not modified by LQC quantization in the Jordan frame as long as $V(\varphi) = W(\chi) = 0$, which is the case considered in Ref. [16]. For $\rho_e \ll \rho_{\rm cr}$, from Eq. (3.12) one obtains

$$\ddot\varphi + 3H\dot\varphi + \frac{2}{\beta}(\varphi V_\varphi - 2V(\varphi)) \simeq \frac{1}{\beta}(4W(\chi) - \dot\chi^2) \simeq \frac{1}{\beta}(\rho_\chi - 3P_\chi), \tag{3.14}$$

which recovers the classical equation (2.10).

IV. EXACT SOLUTIONS OF EFFECTIVE LOOP QUANTUM BRANS-DICKE COSMOLOGY IN THE JORDAN FRAME

A. Vacuum solution

Let us consider a vacuum solution of the effective loop quantum Brans-Dicke cosmology in the Jordan frame with $V(\varphi) = 0$. Under the assumption $\rho_M = P_M = 0$, one obtains

$$\ddot\varphi + 3H\dot\varphi = 0 \Rightarrow \left(\frac{\dot\varphi}{2\varphi} - \frac{\ddot\varphi}{3\dot\varphi}\right)^2 = \left(\frac{\dot\varphi}{2\varphi}\right)^2\left(\left(\sqrt{\frac{\beta}{3}} - 1\right)\sqrt{1 - \frac{\beta}{4\rho_{\rm cr}}\dot\varphi^2} + 1\right)^2. \tag{4.1}$$

Since $\dot\varphi \propto a^{-3} > 0$, one can use φ as a time variable. Thus, let us consider $\dot\varphi$ as a function of φ, i.e., $\dot\varphi = f(\varphi)$. This implies $\ddot\varphi = f_\varphi f$, where $f_\varphi = \frac{df}{d\varphi}$. We already obtained the analytic solution of f and H in Ref. [20], where there exists a quantum bounce. In this paper, in order to see the singularity resolution more explicitly and also for a comparison with the Einstein frame quantization, we will plot the evolution of the volume of the elemental cell with respect to the scalar time. From Eq. (4.1) one obtains the analytical formula for the scale factor as a function of φ,

$$\frac{1}{a}\frac{da}{dt} = \frac{a_\varphi}{a} f = f \frac{d}{d\varphi}(\log a) = -\frac{1}{3} f_\varphi \Rightarrow a(\varphi) \propto f^{-1/3}. \tag{4.2}$$

The evolution of the volume of the elemental cell $V = a(\varphi)^3 V_o$ in this case, as well as its comparison with the classical evolution, is shown in Fig. 1.

The main motivation to consider the vacuum case without potential (as well as the additional massless scalar field scenario, which will be mentioned later) is due to the characteristic feature of theories with a quantum bounce: Around the bounce, kinetic terms of fields shall dominate over potentials. This means that the solutions obtained here shall also be a good approximation of the evolution around the bounce in more realistic theories with inflationary potentials.

B. Brans-Dicke cosmology with massless scalar field

Now we would like to extend the above results to incorporate a massless scalar field as an outside matter field. Let us consider Eq. (3.8) for $\rho_e = \frac{\beta}{4}\dot{\varphi}^2 + \frac{\varphi}{2}\dot{\chi}^2$, where χ is a massless scalar field. Since the LQC correction does not modify the conservation law for $V = W = 0$, one obtains

$$\ddot{\varphi} + 3H\dot{\varphi} = -\frac{1}{\beta}\dot{\chi}^2, \qquad \ddot{\chi} + 3H\dot{\chi} = 0 \Rightarrow H = -\frac{1}{3}\frac{\ddot{\chi}}{\dot{\chi}}. \tag{4.3}$$

From Eq. (4.3), one finds the analytical relation between φ and χ,

$$\begin{aligned} \varphi &= -\frac{1}{2\beta}\chi^2 + B\chi + C, \\ \chi &\in \left(B\beta - \sqrt{\beta}\sqrt{2C + B^2\beta}, B\beta + \sqrt{\beta}\sqrt{2C + B^2\beta}\right), \end{aligned} \tag{4.4}$$

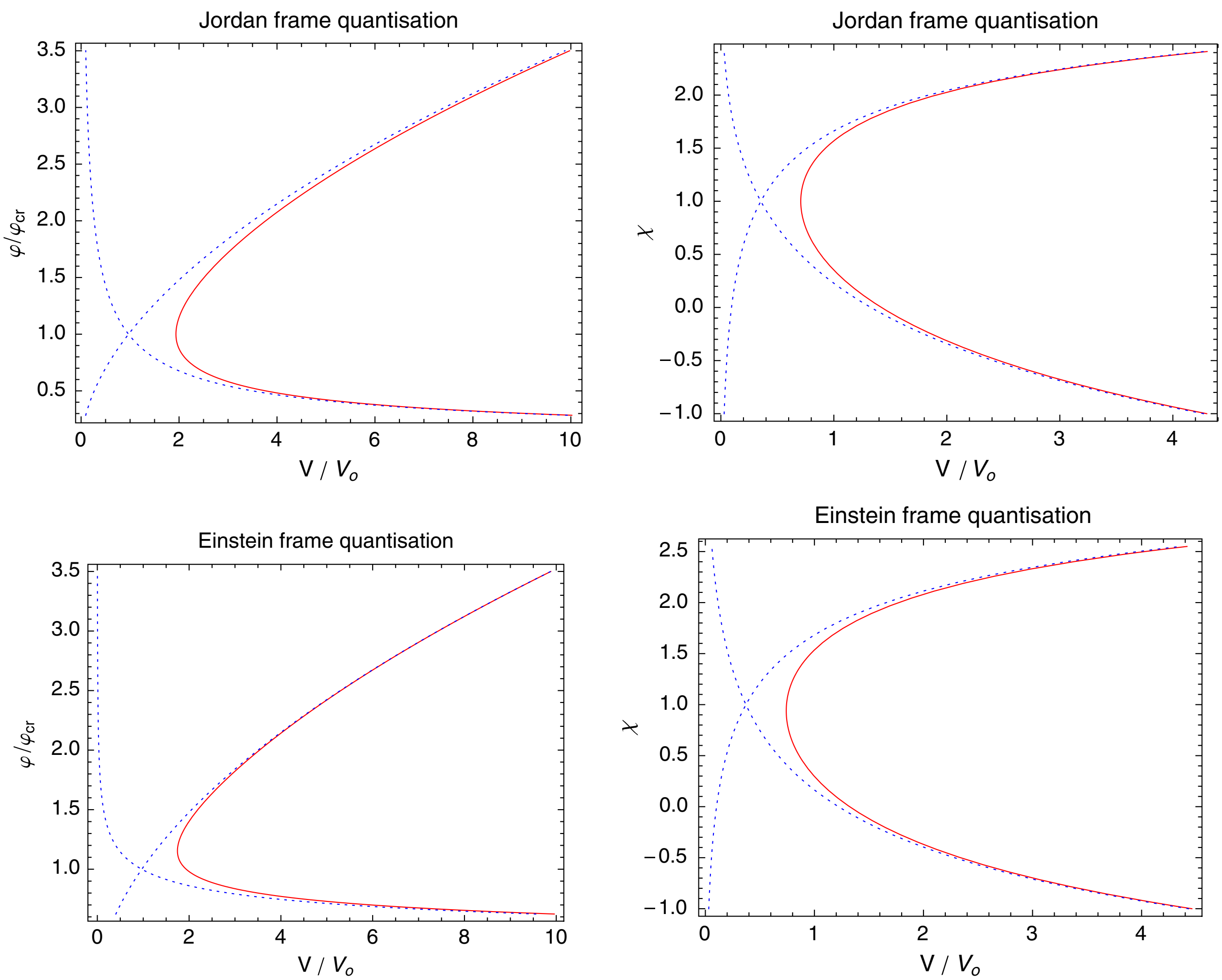

FIG. 1 (color online). All panels present the analytical results for the evolution of the volume of the elemental cell of the Universe as a function of φ (vacuum case, left panels) and χ (massless scalar field case, right panels, $\chi_{cr} = 1$) for various $\beta \sim O(10)$. Solid red lines represent the solution for effective LQC, while dotted blue lines represent classical solutions with singularity. Unlike the case of LQC GR, the LQC Brans-Dicke provides asymmetric evolution of V.

where B and C are constants satisfying $B := \frac{\dot\varphi_{\rm cr}}{\dot\chi_{\rm cr}} + \frac{\chi_{\rm cr}}{\beta}$ and $C := \varphi_{\rm cr} - \frac{\chi_{\rm cr}^2}{2\beta} - \frac{\dot\varphi_{\rm cr}}{\dot\chi_{\rm cr}}\chi_{\rm cr}$, respectively. Here the subscript cr denotes the value of the field at the moment of the bounce. Since $\dot\varphi$ may change its sign during the time evolution, one cannot use the Brans-Dicke field as a time variable. However, massless scalar field χ remains monotonic with respect to the cosmological time. Hence, in the following analysis, we shall use it to parametrize time flow. Note that the limit $\chi \to B\beta \pm \sqrt{\beta}\sqrt{2C + B^2\beta}$ corresponds to $t \to \pm\infty$. The relation (4.4) can be also obtained for the Einstein frame quantization as well as in the classical Brans-Dicke theory.

The GR limit of the Brans-Dicke theory is obtained for $\varphi \to 1$. In the considered scenario, χ is always growing with time because $\dot\chi$ is always positive. Since for late time (big χ) $\varphi \to 0$, one does not recover the GR limit. However, this shall not be a problem in this analysis, since the physical evolution of φ at late time depends on the other matter fields which would fill the Universe. We have assumed that around the bounce, the Universe is dominated by χ. But this assumption is not realistic from the point of view of the observable Universe. With $V = W = 0$, one does not obtain inflation, reheating, baryogenesis, etc. Therefore, for energy much smaller than $\rho_{\rm cr}$, one needs to consider the existence of additional fields or at least potential terms of φ or χ. This due to Eq. (3.12) shall modify the evolution of φ, which could obtain the desired limit $\varphi \to 1$.

To obtain the analytical solution for the Hubble parameter, let us note that Eq. (3.8) may be rewritten as a second-order differential equation of the χ field by taking account of Eqs. (3.8), (4.3), and (4.4) as

$$\ddot\chi + \frac{\dot\chi^2\sqrt{1 - \frac{(2C+B^2\beta)\dot\chi^2}{4\rho_{\rm cr}}}\left(3(\chi - B\beta) + \beta\sqrt{3(2C + B^2\beta)}\right)}{2C\beta + 2B\beta\chi - \chi^2} = 0. \tag{4.5}$$

The effective energy density is now equal to

$$\rho_e = \frac{1}{4}(2C + B^2\beta)\dot\chi^2. \tag{4.6}$$

This comes from Eq. (4.4) and the fact that $\rho_\chi \propto \rho_e \propto a^{-3}$. Since Eq. (4.5) does not contain any explicit dependence of t, one can substitute $\dot\chi$ by $g(\chi) = \dot\chi$. Then one obtains

$$g_\chi + g\frac{\left(\sqrt{3(2C + B^2\beta)} - 3B + 3\frac{\chi}{\beta}\right)}{(2C + 2B\chi - \frac{\chi^2}{\beta})}\sqrt{1 - \frac{(2C + B^2\beta)g^2}{4\rho_{\rm cr}}} = 0, \tag{4.7}$$

where $g_\chi = \frac{dg}{d\chi}$. The exact solution of this equation is

$$g(\chi) = \frac{4\sqrt{\rho_{\rm cr}}}{\sqrt{2C + B^2\beta}}\,\frac{\left(\frac{B\beta+\sqrt{\beta}\sqrt{2C+B^2\beta}-\chi}{B\beta+\sqrt{\beta}\sqrt{2C+B^2\beta}-\chi_{\rm cr}}\right)^{\frac{1}{2}(\sqrt{3\beta}+3)}\left(\frac{-B\beta+\sqrt{\beta}\sqrt{2C+B^2\beta}+\chi}{-B\beta+\sqrt{\beta}\sqrt{2C+B^2\beta}+\chi_{\rm cr}}\right)^{\frac{1}{2}(\sqrt{3\beta}-3)}}{\left(\frac{B\beta+\sqrt{\beta}\sqrt{2C+B^2\beta}-\chi}{B\beta+\sqrt{\beta}\sqrt{2C+B^2\beta}-\chi_{\rm cr}}\right)^{\sqrt{3\beta}+3} + \left(\frac{-B\beta+\sqrt{\beta}\sqrt{2C+B^2\beta}+\chi}{-B\beta+\sqrt{\beta}\sqrt{2C+B^2\beta}+\chi_{\rm cr}}\right)^{\sqrt{3\beta}-3}}. \tag{4.8}$$

The Hubble parameter is equal to $H = -\frac{1}{3}\frac{\ddot\chi}{\dot\chi} = -\frac{1}{3}g_\chi$. Analogously to the vacuum scenario, one obtains $a(\chi) \propto g^{-1/3}$. The evolution of the elementary cell in this case, as well as its comparison with the classical evolution, is also shown in Fig. 1.

V. LQC CORRECTIONS IN THE EINSTEIN FRAME

The nonminimal coupling between a scalar field and the Ricci scalar may be replaced by the minimally coupled system with a redefined metric tensor, which leads to the GR form of the action. This approach (the so-called Einstein frame) is equivalent to the Jordan frame analysis at the classical level. It is often more convenient to perform calculations in the Einstein frame and (under the assumption that the Jordan frame is the physical one) to express results in terms of physical variables. Let us define $\tilde g_{\mu\nu}$ of the form of

$$\tilde g_{\mu\nu} = \varphi g_{\mu\nu}. \tag{5.1}$$

In the FRW model, we let

$$d\tilde t = \sqrt{\varphi}dt, \qquad \tilde a = \sqrt{\varphi}a, \tag{5.2}$$

where t and a are the cosmological time and scale factor in the Jordan frame. Then the action (2.1) in the cosmological model may be expressed as

$$\tilde S = \int d\tilde t\left(-3\tilde a'^2\tilde a + \tilde a^3\frac{\beta}{4}\left(\frac{\varphi'}{\varphi}\right)^2 - \tilde a^3\frac{V}{\varphi^2}\right) + S_M(\tilde g_{\mu\nu}, \varphi), \tag{5.3}$$

where $'$ denotes the derivative with respect to $\tilde t$. The action of the matter fields depends on φ due to the transformation to the Einstein frame. The action (5.3) may be simplified with a new scalar field defined by

$$\phi = \sqrt{\frac{\beta}{2}}\ln(\varphi) \Rightarrow \varphi = \exp\left(\sqrt{\frac{2}{\beta}}\phi\right). \tag{5.4}$$

Then the the action (5.3) takes the form of a scalar field ϕ minimally coupled to the gravity, as in the case of matter coupling in general relativity. Thus, the Lagrangian looks as follows:

$$\mathcal{L} = -3\tilde{a}'^2\tilde{a} + \tilde{a}^3\frac{1}{2}\phi'^2 - \tilde{a}^3\tilde{V}(\phi) + \mathcal{L}_M, \quad (5.5)$$

where $\tilde{V}(\phi) = \frac{V}{\varphi^2}$ (with φ taken as a function of ϕ), and $\mathcal{L}_M$ is the Lagrangian of matter fields in the Einstein frame. Similar to Eq. (2.5), the coefficient $\tilde{c}$ of the connection and the coefficient $\tilde{p}$ of the densitized triad in the FRW model can be also isolated by the symmetric reduction. Since the connection and densitized triad come from the Einstein frame, one gets their relation to those of the Jordan frame as

$$|\tilde{p}| = \tilde{a}^2 V_o^{2/3} = \varphi|p|,$$
$$\tilde{c} = \gamma\tilde{a}' = \gamma\left(\dot{a} + a\frac{\dot{\varphi}}{2\varphi}\right) = c + \gamma a\frac{\dot{\varphi}}{2\varphi}, \quad (5.6)$$

with the Poisson bracket $\{\tilde{c}, \tilde{p}\} = \frac{\gamma}{3}$. The Lagrangian density (5.5) also implies $\pi_\phi = \tilde{a}^3\phi'$. Under the assumption that the only matter field is a scalar field χ with potential $W(\chi)$, one obtains the following classical Hamiltonian in terms of new variables as

$$\tilde{\mathcal{H}} = \frac{\pi_\phi^2 + e^{\sqrt{\frac{2}{\beta}}\phi}\pi_\chi^2}{2\tilde{p}^{3/2}} - \frac{3\tilde{c}^2\sqrt{\tilde{p}}}{\gamma^2} + \tilde{p}^{3/2}(\tilde{V} + e^{-2\sqrt{\frac{2}{\beta}}\phi}W), \quad (5.7)$$

where $\pi_\chi \equiv \tilde{a}^3 e^{-\sqrt{\frac{2}{\beta}}\phi}\chi'$ is the canonical momentum of χ. All classical equations of motion may be obtained from the Hamiltonian equations. This means that for a given function ξ on the classical phase space, one obtains $\xi' = \{\xi, \tilde{\mathcal{H}}\}$.

The main motivation to implement LQC corrections in the Einstein frame is that the issue of the physical interpretation of both frames is still open. We need to know how to distinguish on the experimental level between the Jordan and Einstein frames' LQC quantization. This is a strong suggestion to analyze and compare both of them. In this paper, we follow the assumption that the Jordan frame is the physical one. So, to compare the quantization in both frames, we shall express the results of the Einstein frame quantization as a function of the Jordan frame variables and fields. One can still treat the Jordan frame as an underlying frame for quantization of all degrees of freedom, from which the evolution in the physical (Jordan) frame emerges. The other reason to consider the Einstein frame quantization is that the Einstein frame Hamiltonian obtains its canonical form, which is easy to use methods of LQC quantization discussed in detail in the literature (e.g., in Ref. [12]). The procedure of the LQC quantization of the Brans-Dicke theory in the Einstein frame is similar to that in the Jordan frame, but now the kinematical Hilbert space is defined over the Bohr compactification of the configuration space of $\tilde{c}$. The momentum operator $\tilde{p}$ acts on its orthonormal eigenstate $|\tilde{\mu}\rangle$ in the same way as Eq. (3.1). In the construction of the Hamiltonian operator, one employs the holonomy $\exp(i\tilde{\bar{\mu}}\,\tilde{c}/2)$ with $\tilde{\bar{\mu}} = \sqrt{\Delta/|\tilde{p}|}$. Again, we limit ourselves to the LQC holonomy correction. Thus, in the semiclassical regime of LQC quantization, one shall transform $\tilde{c}$ into $\sqrt{|\tilde{p}|/\Delta}\sin(\tilde{c}\sqrt{\Delta/|\tilde{p}|})$, while $\tilde{p}$ remains unchanged [23–25]. This gives the effective Friedmann equation and equation of motion of the form

$$3\tilde{H}^2 = \tilde{\rho}\left(1 - \frac{\tilde{\rho}}{\rho_{\rm cr}}\right),$$
$$\frac{d^2\phi}{d\tilde{t}^2} + 3\tilde{H}\frac{d\phi}{d\tilde{t}} + \frac{dV}{d\phi} = e^{-2\sqrt{\frac{2}{\beta}}\phi}\sqrt{\frac{1}{2\beta}}\left(4W - e^{\sqrt{\frac{2}{\beta}}\phi}\chi'^2\right), \quad (5.8)$$

where $\tilde{H} = \frac{\tilde{a}'}{\tilde{a}}$ and $\tilde{\rho} = \frac{\rho_e}{\varphi^3}$. The effective Friedmann equation may be rewritten in terms of the Jordan frame metric as

$$3\left(H + \frac{\dot{\varphi}}{2\varphi}\right)^2 = \frac{\rho_e}{\varphi^2}\left(1 - \frac{\rho_e}{\varphi^3\rho_{\rm cr}}\right). \quad (5.9)$$

Thus, Eqs. (3.8) and (5.9) give sufficiently different forms of the Friedmann equation. However, they both obtain the same classical limit for $\rho_e \ll \rho_{\rm cr}$, which is described by Eq. (2.11). Note that the Einstein frame quantization does not change the effective equations of motion of matter fields as well as those of the Brans-Dicke field. It seems natural because the LQC quantization is performed for the scalar field ϕ minimally coupled to the metric $\tilde{g}_{\mu\nu}$. The situation changes when the frame changes, since for Jordan frame, any potential term would imply the LQC correction to the equation of motion of φ. For the Einstein frame quantization, the bounce in the Einstein frame defined by $\tilde{H} = 0$ appears for $\tilde{\rho} = \rho_{\rm cr}$, while the bounce in the Jordan frame ($H = 0$) requires

$$\frac{4\rho_e^2}{\varphi_{\rm cr}^3(4\rho_e - 3\dot{\varphi}_{\rm cr}^2)} = \rho_{\rm cr}, \quad (5.10)$$

where the index cr denotes the value of the field at the moment of the bounce in the Jordan frame. In general, scales of bounces which originate from quantization in the Jordan frame (3.8) and Einstein frame (5.10), respectively, may be sufficiently different, since $\varphi_{\rm cr}^3(1 - 3\dot{\varphi}_{\rm cr}^2/4\rho_e)$ does not need to be equal to 1. Thus, not only the exact evolution, but even the scales of the bounces (understood as the value of the effective energy density at the moment, in which $H = 0$) are different in the different frame quantizations.

An interesting scenario of the Einstein frame quantization is the vacuum case with $\beta = 3$ and nonzero potential. Then at the moment of the Jordan frame bounce, one obtains $\rho_e = \varphi_{\rm cr}^2\sqrt{\rho_{\rm cr}V(\varphi_{\rm cr})}$. If the φ_{cr} is close to the minimum of the potential V (which in the most realistic scenario would be in $\varphi = 1$), one obtains a very low scale of a Jordan frame bounce. In particular, the scale of the bounce may be close to the inflationary scale, and the LQC effects may be visible for the biggest scales of the power spectrum of primordial curvature perturbations.

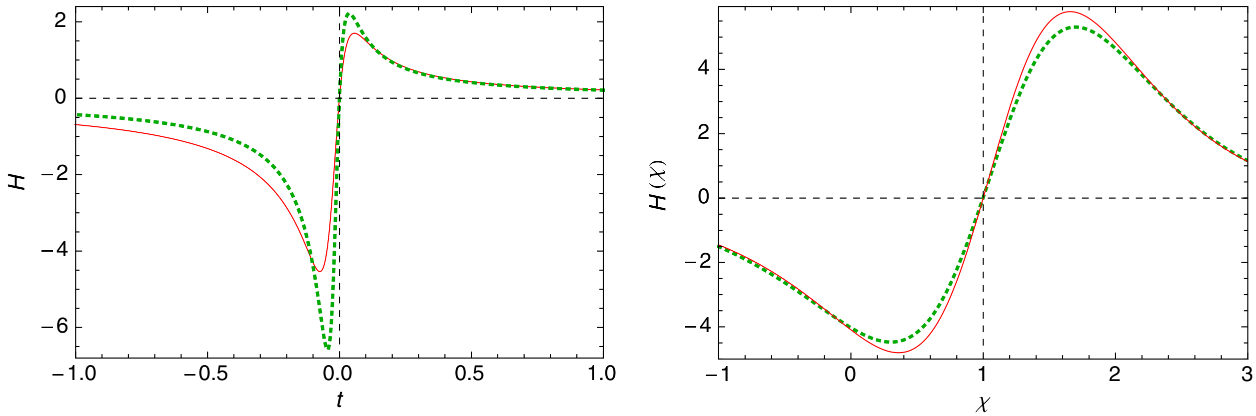

FIG. 2 (color online). Both panels' lines represent the analytical solutions for the physical Hubble parameter H as a function of cosmological time t (left panel, vacuum solution, $\beta = 5$, φcr = 1) or χ (right panel, massless scalar field solution, $\beta = 15$, $\chi_{cr} = 1$, $B = 0$, $C = 1$) for the LQC quantization in the Jordan (red line) and Einstein (dashed green line) frames. The $t = 0$ or $\chi = 1$ correspond to the moment of the bounce in the Jordan frame. For $t > t_{pl}$, both methods of quantization approach the same evolution trajectory. The comparison takes place in the Jordan frame due to our assumption that this frame is the physical one.

VI. EXACT SOLUTIONS OF EFFECTIVE LOOP QUANTUM BRANS-DICKE COSMOLOGY IN THE EINSTEIN FRAME

A. Vacuum solution

The LQC quantization in the Einstein frame provides several analytical solutions in the semiclassical regime. For instance, the case $\rho_M = P_M = V = 0$ corresponds to the domination of a massless scalar field in the Universe with the scale factor $\tilde{a}$. Then Eq. (5.8) has the exact solution of the form[3]

$$\frac{\varphi'}{\varphi} = \phi' \propto \tilde{a}^{-3} \Rightarrow \tilde{a} = \tilde{a}_{cr}(1 + 3\rho_{cr}\tilde{t}^2)^{1/6}, \quad (6.1)$$

where $\tilde{a}_{cr}$ is a value of $\tilde{a}$ at the moment of the bounce in the Einstein frame. For this scenario, the evolution of the Hubble parameters for the Brans-Dicke theory with LQC corrections in the Jordan and Einstein frames are presented, respectively, in Fig. 2.

Let us note that the analytical solution in the vacuum case with $V(\varphi) = 0$ may be also obtained as a function of the Brans-Dicke field. From Eqs. (2.10) and (5.9), one finds

$$\ddot{\varphi} + \frac{3\dot{\varphi}^2}{2\varphi}\left(\sqrt{\frac{\beta}{3}\left(1 - \frac{\beta\dot{\varphi}^2}{4\varphi^3\rho_{cr}}\right)} - 1\right) = 0$$
$$\Rightarrow j' + \frac{3j}{2\varphi}\left(\sqrt{\frac{\beta}{3}\left(1 - \frac{\beta j^2}{4\varphi^3\rho_{cr}}\right)} - 1\right) = 0, \quad (6.2)$$

where $j = j(\varphi) = \dot{\varphi}$ and $j' = \frac{dj}{d\varphi}$. The solution of this equation is the following:

$$j = 4\sqrt{\frac{\rho_{cr}}{\beta}}\frac{\varphi^{3/2}(\varphi/\tilde{\varphi}_{cr})^{\frac{\sqrt{3\beta}}{2}}}{1 + (\varphi/\tilde{\varphi}_{cr})^{\sqrt{3\beta}}},$$
$$H = -\frac{1}{3}j', \qquad a(\varphi) \propto a^{-1/3}, \quad (6.3)$$

where $\tilde{\varphi}_{cr} \neq \varphi_{cr}$ is the value of φ at the moment of a bounce in the Einstein frame. The bounce in the Jordan frame happens for $\varphi = \varphi_{cr} = \tilde{\varphi}_{cr}(\frac{\sqrt{\beta}+\sqrt{3}}{\sqrt{\beta}-\sqrt{3}})^{1/\sqrt{3\beta}}$ and for energy density $\tilde{\rho}(\varphi_{cr}) = (1 - 3/\beta)\rho_{cr}$. In the limit of $\beta \gg 1$, one obtains $\tilde{\varphi}_{cr} \simeq \varphi_{cr}$ and $\tilde{\rho}(\varphi_{cr}) \simeq \rho_{cr}$.

B. The $V(\varphi) = \Lambda\varphi^2$ case

The other analytical solution comes from the case $V(\varphi) = \Lambda\varphi^2$ and $\rho_M = 0$. Then, in the Einstein frame one obtains $\tilde{V} = \Lambda$, so if the φ field has a square potential in the Jordan frame, it gives the Universe filled with massless scalar field ϕ and cosmological constant Λ in the Einstein frame. Let us assume that at the moment of a bounce in the Einstein frame, when $\tilde{H} = 0$, one obtains $\Lambda = (1 - \alpha)\rho_{cr}$ and $\frac{1}{2}\dot{\phi}^2_{cr} = \alpha\rho_{cr}$. This comes from the fact that $\Lambda + \frac{1}{2}\dot{\phi}^2_{cr} = \rho_{cr}$. The α parameter has an interpretation of percentage contribution of ρ_m to the critical energy density. Usually, one expects a cosmological constant to be subdominant around the big bounce. Thus, it is natural to consider $|1 - \alpha| \ll 1$. One shall note that $\alpha < 1$ and $\alpha > 1$ correspond to $\Lambda > 0$ and $\Lambda < 0$, respectively. The exact solution of Eq. (5.8) looks as follows:

$$\tilde{a} = \tilde{a}_{cr}\left(\frac{\cosh(6\sqrt{\theta}\tilde{t}) - 2\alpha + 1}{2(1 - \alpha)}\right)^{\frac{1}{6}} \quad \text{for } \Lambda > 0, \quad (6.4)$$

$$\tilde{a} = \tilde{a}_{cr}\left(\frac{-\cos(6\sqrt{\theta}\tilde{t}) + 2\alpha - 1}{2(\alpha - 1)}\right)^{\frac{1}{6}} \quad \text{for } \Lambda < 0, \quad (6.5)$$

[3]A more general solution of Eq. (5.8) is given in Ref. [32].

where $\theta = |1-\alpha|\alpha\rho_{\rm cr}/3$. It is easy to show that for $\alpha \leq 1/2$, one obtains $\frac{d}{d\tilde{t}}\tilde{H} > 0$ at any time. In such a case, $\tilde{H}$ grows to its finite maximal value $\tilde{H}_{\rm max} = \sqrt{\theta}$. For $\alpha > 1/2$, $\tilde{H}$ grows initially to reach its global maximum $\tilde{H}_{\rm max} = \sqrt{\rho_{\rm cr}/12}$ at $\rho = \rho_{\rm cr}/2$. Later on, $\tilde{H}$ decreases together with $\tilde{\rho}$. For $\sqrt{\theta}t \ll 1$, one finds $\cosh(6\sqrt{\theta}\tilde{t}) \simeq 1 + 6\alpha\rho_{\rm cr}(1-\alpha)\tilde{t}^2$ and $\cos(6\sqrt{\theta}\tilde{t}) \simeq 1 - 6\alpha\rho_{\rm cr}(1-\alpha)\tilde{t}^2$. Thus, for both positive and negative cosmological constants, around a bounce, one recovers the solution (6.1). Let us note that in the case of $V = \Lambda\varphi^2$, it is possible to solve analytically only for the quantization in the Einstein frame. An interesting feature of this model is that, although φ has a potential term, one may still use it as a time variable because $\dot{\varphi} > 0$ is always valid. However, this is the case only in the classical limit or for the LQC quantization in the Einstein frame. For $\varphi \gg 1$, the considered potential is a good approximation of the potential of the Starobinsky inflation [33], for which $V \propto (\varphi - 1)^2$. However, $V = \Lambda\varphi^2$ does not provide the graceful exit, and it generates a too flat power spectrum of initial curvature perturbations.

C. With massless scalar field

For the Einstein frame quantization, one may also obtain the analytical solutions of the effective theory in the case of $V = W = 0$. Since classical equations of motion of matter fields are still valid for the effective theory of the Einstein frame quantization, one may use the relation (4.4) and the semiclassical Friedmann equation (5.8) to obtain an equation of motion of only one degree of freedom, which is χ. Again, we define $h(\chi) = \dot{\chi}$, which gives following equation of motion:

$$\frac{dh}{d\chi} - \frac{3(B\beta - \chi)h}{2C\beta + (2B\beta - \chi)\chi} + \sqrt{\frac{3}{2\beta}(2C\beta + (2B\beta - \chi)\chi)}\sqrt{\tilde{\rho}\left(1 - \frac{\tilde{\rho}}{\rho_{\rm cr}}\right)}, \tag{6.6}$$

where

$$\tilde{\rho} = \frac{2\beta^3(2C + B^2\beta)h^2}{(2C\beta + 2B\beta\chi - \chi^2)^3}. \tag{6.7}$$

This equation has the following exact solution:

$$h = \sqrt{\frac{2\rho_{\rm cr}}{\beta^3(2C+B^2\beta)}}\frac{\left(\frac{B\beta+\sqrt{\beta}\sqrt{2C+B^2\beta}-\chi}{B\beta+\sqrt{\beta}\sqrt{2C+B^2\beta}-\tilde{\chi}_{\rm cr}}\right)^{\frac{1}{2}\sqrt{3\beta}}\left(\frac{-B\beta+\sqrt{\beta}\sqrt{2C+B^2\beta}+\chi}{-B\beta+\sqrt{\beta}\sqrt{2C+B^2\beta}+\tilde{\chi}_{\rm cr}}\right)^{\frac{1}{2}\sqrt{3\beta}}}{\left(\frac{B\beta+\sqrt{\beta}\sqrt{2C+B^2\beta}-\chi}{B\beta+\sqrt{\beta}\sqrt{2C+B^2\beta}-\tilde{\chi}_{\rm cr}}\right)^{\sqrt{3\beta}} + \left(\frac{-B\beta+\sqrt{\beta}\sqrt{2C+B^2\beta}+\chi}{-B\beta+\sqrt{\beta}\sqrt{2C+B^2\beta}+\tilde{\chi}_{\rm cr}}\right)^{\sqrt{3\beta}}}(2C\beta + (2B\beta - \chi)\chi)^{3/2}, \tag{6.8}$$

where $\tilde{\chi}_{\rm cr} \neq \chi_{\rm cr}$ represents the moment in which $\tilde{H} = 0$. Similar to the Jordan frame quantization, the Hubble parameter in the Jordan frame is of the form $H(\chi) = -\frac{1}{3}\frac{dh}{d\chi}$. This result is also compared with the Jordan frame quantization in Fig. 2.

VII. CONCLUSIONS

In this paper, we consider the issue of the semiclassical evolution of the Universe in both the Jordan and Einstein frames of the LQC Brans-Dicke theory with scalar potential $V(\varphi)$ coupled with an additional scalar field χ with a potential $W(\chi)$. The Hamiltonian formalism of the corresponding cosmological models is presented in terms of geometrical variables as well as connection variables.

To compare the Jordan frame with the Einstein frame of Brans-Dicke theory, the same cosmological model is quantized by the LQC method in both frames separately. We then consider the effective equations with the LQC holonomy corrections that result from the different frames' quantization. In particular, the effective Friedmann equations and equations of motion for the scalar fields are obtained in both frames. In the Jordan frame quantization, it is shown that not only the potential term of the Brans-Dicke field φ but also $W(\chi)$ can lead to corrections to the effective equation of motion of φ. In the $\rho_e \ll \rho_{\rm cr}$ limit, the classical Brans-Dicke theory can be recovered from the effective theory. In the $W = 0$ case, equations of motion for scalar fields and the semiclassical Friedmann equation are of the the same form as the Brans-Dicke theory with and without a potential.

Analytical solutions are found for the effective equations that result from both frames' quantization without potential in the vacuum case and in the additional massless scalar field case separately. In the vacuum case, the Brans-Dicke scalar field can be employed as an internal time. In the matter coupled case, the matter scalar field χ is used as a time variable, and the analytical relation (valid in the effective theory of both frames and in the classical theory) between the two scalar fields is obtained. In all those solutions, the bouncing evolution of the scale factor is obtained around the Planck regime. However, the quantization of different frames leads to different scales of the bounces of the scale factor $a(t)$. The Jordan frame quantization and the Einstein frame quantization require $\rho_e = \rho_{\rm cr}$ and $\frac{4\rho_e^2}{\varphi_{\rm cr}^3(4\rho_e - 3\dot{\varphi}_{\rm cr}^2)} = \rho_{\rm cr}$, respectively, for the occurrence of the bounce. Hence, different frame quantization gives different physics. Moreover, as shown in Fig. 1, the evolutional trajectories of the Jordan frame volume of the elementary cell of the Universe are different for the different frames' quantization.

However, as shown in Fig. 2, in the vacuum case, the difference of the evolutional trajectories of the Jordan frame Hubble parameter disappears for time $t > t_{pl}$ after the bounces. Therefore, it may be difficult to distinguish between the frames' quantization by observations of the primordial inhomogeneities. The situation may change by the fine-tuning of the initial conditions of the Einstein frame quantization, which (comparing to the Jordan frame quantization) may sufficiently decrease the scale of the Jordan frame bounce. In particular, this could lead to the Jordan frame bounce around the grand unification theory scale with the period of superinflation visible in the power spectrum of primordial inhomogeneities. This issue analyzed in the context of Starobinsky inflation shall be the goal of our future work.

Another interesting feature of the bounce in loop quantum Brans-Dicke cosmology explored in Fig. 1 is that, unlike the case of LQC of GR, the evolutional trajectories of the two sides of the bounce point are obviously asymmetric. As a by-product, we also find an analytical solution with Jordan frame potential $V = \Lambda\varphi^2$ in the theory of Einstein frame quantization.

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