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Noether symmetries and analytical solutions in $f(T)$ cosmology: A complete study

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We investigate the main features of the flat Friedmann-Lemaître-Robertson-Walker cosmological models in the $f(T)$ modified gravity regime. In particular, a general approach to find out exact cosmological solutions in $f(T)$ gravity is discussed. Instead of taking into account phenomenological models, we consider as a selection criterion, the existence of Noether symmetries in the cosmological $f(T)$ pointlike Lagrangian. We find that only the $f(T) = f_0 T^n$ model admits extra Noether symmetries. The existence of extra Noether integrals can be used in order to simplify the system of differential equations (equations of motion) as well as to determine the integrability of the $f(T) = f_0 T^n$ cosmological model. Within this context, we can solve the problem analytically and thus we provide the evolution of the main cosmological functions such as the scale factor of the Universe, the Hubble expansion rate, the deceleration parameter, and the linear matter perturbations. We show that the $f(T) = f_0 T^n$ cosmological model suffers from two basic problems. The first problem is related to the fact that the deceleration parameter is constant which means that it never changes sign, and therefore the Universe always accelerates or always decelerates depending on the value of n. Second, we find that the clustering growth rate remains always equal to unity implying that the recent growth data disfavor the $f(T) = f_0 T^n$ gravity. Finally, we prove that the $f(T) = f_0 T^n$ gravity can be cosmologically equivalent with the $f(R) = R^n$ gravity model and the time varying vacuum model $\Lambda(H) = 3\gamma H^2$ (for $n^{-1} = 1 - \gamma$) because the above cosmological scenarios share exactly the same Hubble expansion, despite the fact that the three models have a different geometrical origin. Finally, some important differences with power-law $f(R)$ gravity are pointed out.

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I. INTRODUCTION

Nonstandard gravity models provide an alternative possibility towards understanding the accelerated expansion of the Universe (see [1] and references therein). The physical mechanism which is responsible for the present accelerating stage of the Universe can be driven by a modification of the Einstein-Hilbert action, while the matter content of the Universe remains the same (relativistic and cold dark matter). In the literature there are plenty of modified gravity models proposed by different authors, such as the braneworld Dvali, Gabadadze and Porrati [2] model, $f(R)$ gravity [3], scalar-tensor theories [4], Gauss-Bonnet gravity [5], Hořava-Lifshitz gravity [6], nonlinear massive gravity [7], etc.

Another gravitational scenario which has recently gained a lot of attention is the so-called $f(T)$ gravity. The intrinsic properties of this scenario are based on the rather old formulation of the teleparallel equivalent of general relativity (TEGR) [8–10]. Specifically, instead of using the torsionless Levi-Civita connection of the classical general relativity (GR), one utilizes the curvatureless Weitzenböck connection in which the corresponding dynamical fields are the four linearly independent *vierbeins.* Therefore, all the information concerning the gravitational field is included in the torsion tensor. Within this framework, considering invariance under general coordinate transformations, global Lorentz-parity transformations, and requiring up to second order terms of the torsion tensor, one can write down the corresponding Lagrangian density T [9] by using some suitable contractions. A natural generalization of TEGR gravity is $f(T)$ gravity which is based on the fact that we allow the Lagrangian to be a function of T [11–13], inspired, of course, by the well-known extension of $f(R)$ Einstein-Hilbert action. However, $f(T)$ gravity does not coincide with $f(R)$ extension, but it rather consists of a different class of modified gravity. It is interesting to mention that the torsion tensor includes only products of first derivatives of the vierbeins, giving rise to second-order field differential equations in contrast with the $f(R)$ gravity that provides fourth-order equations which potentially may lead to some problems, for example in the well position and well formulation of the Cauchy problem [14].

Despite the fact that TEGR coincides completely with GR, both at the background and perturbation levels, it has been shown that $f(T)$ gravity provides different structural properties with respect to GR as well as different black-hole solutions and cosmological features [11–13,15–19]. An important question here is what classes of $f(T)$ extensions

are allowed. From the phenomenological viewpoint, the aforementioned cosmological and spherical analysis lead to a variety of such expressions. Using cosmological [16,17,20] and Solar System [18] observations, one can show that the deviations from TEGR must be small.

In this paper, we use a model-independent selection rule based on first integrals, due to Noether symmetries of the equations of motion, in order to identify the viability of $f(T)$ gravity in the context of flat Friedmann-Lemaître-Robertson-Walker (FLRW) cosmologies. Actually, the idea to use Noether symmetries in cosmology is not new and indeed there is a lot of work in the literature (see [21–33]) along this line. In this context, recently we have shown (see Basilakos *et al.* [34]; Paliathanasis *et al.* [35]) that the existence of Noether symmetries can be used as a selection criterion in order to distinguish the scalar dark energy models [34] as well as the $f(R)$ gravity models [35]. Inspired by the above works in the current article, we would like to estimate the Noether symmetries of the $f(T)$ gravity. The aim here is (a) to identify the $f(T)$ functional forms which accommodate extra Noether symmetries, and (b) for these models, to solve the system of the resulting field equations and derive analytically the main cosmological functions (the scale factor, the Hubble expansion rate, deceleration parameter and growth factor) and finally to compare with other cosmological patterns which are outside and inside GR.

The structure of the article is as follows: In Sec. II, we discuss the issue of torsion in GR and its connection with unholonomic frames. This discussion is useful in order to clarify some misunderstandings on the role of torsion that are present in literature. In particular, we shall discuss its dependence on the frame where observations are made.

In Sec. III, we give the basic FLRW cosmological equations in the framework of $f(T)$ gravity. The main properties and theorems of the Noether symmetry approach are summarized in Sec. IV. Noether symmetries for $f(T)$ cosmology are discussed in Sec. V. In Sec VI we provide analytical solutions for $f(T)$ models that admit nontrivial Noether symmetries. A comparison with analogous $f(R)$ cosmology is pursued putting in evidence similarities and differences. We draw conclusions in Sec. VI.

II. THE ROLE OF TORSION IN GENERAL RELATIVITY

Before starting our considerations on $f(T)$ gravity and its cosmological realization, it is useful to discuss in detail the role of torsion in GR considering, in particular, how it behaves with respect to holonomic and unholonomic frames.

Let us start with some definitions. In an n-dimensional manifold $\mathcal{M}$ consider a coordinate neighborhood $\mathcal{U}$ with a coordinate system $\{x^\mu\}$. At each point $P \in \mathcal{U}$, we have the resulting holonomic frame $\{\partial_\mu\}$. We define in $\mathcal{U}$ a new frame $\{e_a(x^\mu)\}$ which is related to the holonomic frame $\{\partial_a\}$ as follows:

$$e_a(x^\mu) = h_a^\mu \partial_\mu \qquad a, \mu = 1, 2, \ldots, n, \tag{1}$$

where the quantities $h_a^\mu(x)$ are in general functions of the coordinates (i.e. depend on the point P). Notice that Latin indices count vectors, while Greek indexes are tensor indices. We assume that $\det h_a^\mu \neq 0$ which guarantees that the vectors $\{e_a(x^\mu)\}$ form a set of linearly independent vectors. We define the "inverse" quantities h_a^μ by means of the following "orthogonality" relations:

$$h_a^\mu h_\nu^a = \delta_\nu^\mu, \qquad h_b^\mu h_\mu^c = \delta_b^c. \tag{2}$$

The commutators of the vectors $\{e_a\}$ are not in general all zero. If they are zero, then there exists a new coordinate system in $\mathcal{U}$, $\{y^b\}$ so that $e_b = \frac{\partial}{\partial y^b}$, i.e. the new frame is holonomic. If there are commutators $[e_a, e_b] \neq 0$ then the new frame $\{e_b\}$ is called unholonomic and at least a number of vectors e_b cannot be written in the form $e_b = \partial_b$. The quantities which characterize an unholonomic frame are the objects of unholonomicity or Ricci rotation coefficients $\Omega^a{}_{bc}$ defined by the relation

$$[e_a, e_b] = \Omega^c{}_{ab} e_c. \tag{3}$$

Let us compute:

$$[e_a, e_b] = [h_a^\mu \partial_\mu, h_b^\nu \partial_\nu] = [h_a^\mu h^\nu_{b,\mu} h_\nu^c - h_b^\nu h^\mu_{a,\nu} h_\mu^c] e_c$$

from which follows that the Ricci rotation coefficients of the frame $\{e_a\}$ are

$$\Omega^a{}_{bc} = 2 h^\mu_{[b} h^\nu_{c],\mu} h^a_\nu. \tag{4}$$

The condition for $\{e_a\}$ to be a holonomic basis is $\Omega^a{}_{bc} = 0$ at all points $P \in \mathcal{U}$. This is a set of linear partial differential equations whose solution defines all holonomic frames and all coordinate systems in $\mathcal{U}$. One obvious solution is $h_b^c = \delta_b^c$. The set of all coordinate systems in $\mathcal{U}$, equipped with the operation of composition of transformations, has the structure of an infinite dimensional Lie group which is called the *Manifold Mapping Group* [36].

Let us consider now the special unholonomic frames which satisfy the Jacobi identity:

$$[[e_a, e_b], e_c] + [[e_b, e_c], e_a] + [[e_c, e_a], e_b] = 0. \tag{5}$$

These frames are the generators of a Lie algebra, therefore they have an extra role to play. Replacing the commutator in terms of the unholonomicity objects, we find the following identity:

$$\Omega^d{}_{ab,c} + \Omega^d{}_{ba,a} + \Omega^d{}_{ca,b} - \Omega^l{}_{ab}\Omega^d_{cl} - \Omega^l{}_{bc}\Omega^d_{al} - \Omega^l{}_{ca}\Omega^d_{bl} = 0. \tag{6}$$

Using the definition of the covariant derivative we write

$$\nabla_{e_i} e_j = \Gamma^k_{ij} e_k, \tag{7}$$

where Γ^k_{ij} are the connection coefficients in the frame $\{e_i\}$. If we compute the Γ^k_{ij} assuming

$$[e_i, e_j] = C^k_{.ij} e_k$$

it follows that

$$C^k_{.ij} = \Omega^k_{.jk}.$$

Let us consider now three vector fields X, Y, Z and the covariant derivative of the metric vector X. Then we have

$$\nabla_X g(Y,Z) = X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_Y Z) \quad (8)$$

and by interchanging the role of X, Y, Z

$$\nabla_Y g(Z,X) = Y(g(Z,X)) - g(\nabla_Y Z, X) - g(Z, \nabla_Z X), \quad (9)$$

$$\nabla_Z g(X,Y) = Z(g(X,Y)) - g(\nabla_Z X, Y) - g(X, \nabla_X Y). \quad (10)$$

Adding Eqs. (8) and (9) and subtracting (10), one obtains

$$\begin{aligned}\nabla_X g(Y,Z) &+ \nabla_Y g(Z,X) - \nabla_Z g(X,Y)\\ &= X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y))\\ &\quad - [g(\nabla_X Y, Z) + g(\nabla_Y Z, X) - g(\nabla_Z X, Y)]\\ &\quad - [g(Y, \nabla_X Z) + g(Z, \nabla_Y X) - g(X, \nabla_Z Y)]\end{aligned}$$

then

$$\begin{aligned}\nabla_X g(Y,Z) &+ \nabla_Y g(Z,X) - \nabla_Z g(X,Y)\\ &= X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y))\\ &\quad - [g(\nabla_X Y, Z) + g(Z, \nabla_Y X)]\\ &\quad - [g(\nabla_Y Z, X) - g(X, \nabla_Z Y)]\\ &\quad - [g(Y, \nabla_X Z) - g(\nabla_Z X, Y)]\end{aligned}$$

that is

$$\begin{aligned}\nabla_X g(Y,Z) &+ \nabla_Y g(Z,X) - \nabla_Z g(X,Y)\\ &= X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y))\\ &\quad - [g(Z, \nabla_X Y + \nabla_Y X) + g(X, \nabla_Y Z - \nabla_Z Y)\\ &\quad + g(Y, \nabla_X Z - \nabla_Z X)],\end{aligned}$$

where

$$g(Z, \nabla_X Y + \nabla_Y X) = 2g(Z, \nabla_X Y) + g(Z, \nabla_Y X - \nabla_X Y).$$

Replacing in the last relation and solving for $2g(Z, \nabla_X Y)$, we find

$$\begin{aligned}2g(Z, \nabla_X Y) = &[X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y))]\\ &- [\nabla_X g(Y,Z) + \nabla_Y g(Z,X) - \nabla_Z g(X,Y)]\\ &- [g(Z, \nabla_Y X - \nabla_X Y) + g(X, \nabla_Y Z - \nabla_Z Y)\\ &+ g(Y, \nabla_X Z - \nabla_Z X)]\end{aligned}$$

or

$$\begin{aligned}2g(Z, \nabla_X Y) = &[X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y))]\\ &- [\nabla_X g(Y,Z) + \nabla_Y g(Z,X) - \nabla_Z g(X,Y)]\\ &- [g(Z, \nabla_X Y - \nabla_Y X - [Y,X])\\ &+ g(X, \nabla_Y Z - \nabla_X Y - [Y,X])\\ &+ g(Y, \nabla_X Z - \nabla_Z X - [X,Z])]\\ &- [g(Z,[Y,X]) + g(X,[Y,Z]) + g(Y,[X,Z])].\end{aligned}$$

At this point, we can define the quantities

$$T_\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \quad A_\nabla(X,Y,Z) = \nabla_X g(Y,Z).$$

The tensors T_∇ and A_∇ are called the *torsion* ($T_\nabla \equiv T$) and the *metricity* of the connection ∇ respectively. The last relation in terms of the fields T_∇ and A_∇ is written as follows:

$$\begin{aligned}2g(Z, \nabla_X Y) = &[X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y))]\\ &- [A_\nabla(X,Y,Z) + A_\nabla(Y,Z,X)\\ &- A_\nabla(Z,X,Y)] + -[g(Z, T_\nabla(Y,X))\\ &+ g(X, T_\nabla(Y,Z)) + g(Y, T_\nabla(X,Z))]\\ &- [g(Z,[Y,X]) + g(X,[Y,Z]) + g(Y,[X,Z])].\end{aligned} \quad (11)$$

Let $X = e_l$, $Y = e_j$ and $Z = e_k$. Contracting with $\frac{1}{2} g^{il}$, we have

$$\begin{aligned}2g(Z, \nabla_X Y) &\to \Gamma^i_{jk},\\ [X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y))] &\to \left\{ {i \atop jk} \right\},\\ g(X, T_\nabla(Y,Z)) &\to Q^i_{.kj},\\ g(Z, T_\nabla(Y,X)) + g(Y, T_\nabla(X,Z)) &\to g^{il}(g_{tj} Q^t_{kl} + g_{tk} Q^t_{jl})\\ &= -\bar{S}^i_{.kj},\\ g(X,[Y,Z]) &\to \frac{1}{2} C^i_{.jk},\\ g(Z,[Y,X]) + g(Y,[X,Z]) &= \frac{1}{2} g^{il}(g_{tj} C^t_{lk} + g_{tk} C^t_{jl})\\ &= -S^i_{.kj},\end{aligned}$$

and

$$A_\nabla(X,Y,Z) + A_\nabla(Y,Z,X) - A_\nabla(Z,X,Y) \to \frac{1}{2} g^{il} \Delta_{jkl}.$$

Replacing in Eq. (11), we find the connection coefficients in the frame $\{e_i\}$, that is

$$\Gamma^i_{jk} = \left\{ {i \atop jk} \right\} + \bar{S}^i_{.kj} + S^i_{.kj} - \frac{1}{2} g^{il} \Delta_{jkl} + Q^i_{jk} - \frac{1}{2} C^i_{.jk}, \quad (12)$$

where $\{ {i \atop jk} \}$ are the standard Levi-Civita connection coefficients (i.e. the Christofell symbols). This is the most general expression for the connection coefficients in terms of the fields $\{ {i \atop jk} \}$, T_∇, A_∇ and C^i_{jk}. Concerning the symmetric and antisymmetric part, we have

$$\Gamma^i_{.(jk)} = \left\{ {i \atop jk} \right\} + \bar{S}^i_{.jk} + S^i_{.jk} - \frac{1}{2} g^{il} \Delta_{jkl}, \tag{13}$$

$$\Gamma^i_{.[jk]} = Q^i_{.jk} - \frac{1}{2} C^i_{.jk}, \tag{14}$$

and then we can draw the following conclusions:

(1) The connection coefficients in a frame $\{e_i\}$ are determined from the metric, the torsion, the metricity and the unholonomicity objects (equivalently the commutators) of the frame.
(2) The symmetric part $\Gamma^i_{.(jk)}$ of Γ^i_{jk} depends on all fields. This means that the geodesics and the autoparallels in a given frame depend on the geometric properties of the underlying manifold (fields g_{ij}, $Q^i_{.kj}$, $g_{ij|k}$) and the unholonomicity of the frame (field $C^i_{.jk}$).
(3) The antisymmetric part $\Gamma^i_{.[jk]}$ of Γ^i_{jk} depends only on all fields $Q^i_{.kj}$ and $C^i_{.jk}$.
(4) The objects of unholonomicity $C^i_{.jk}$ behave in the same way as the components of torsion. This means that even in a Riemannian space where $Q^i_{.kj} = 0$, $g_{ij|k} = 0$ in an unholonomic basis the antisymmetric part $\Gamma^i_{.[jk]} = -\frac{1}{2} C^i_{.jk} \neq 0$.

This result has lead to the misunderstanding that when one works in an unholonomic frame then the torsion is introduced. This statement is clearly not correct. This misunderstanding has important consequences because the effects one will observe in an unholonomic frame will be frame dependent and not covariant effects. Therefore all conclusions made in a specific unholonomic frame must be restricted to that frame only.

III. $f(T)$ GRAVITY AND COSMOLOGY

With the above considerations in mind, let us consider TEGR and its straightforward extension $f(T)$. Teleparallelism uses as dynamical objects the *vierbiens* as unholonomic frames in spacetime. Following the definitions in the previous section, they are defined by the requirement $g(e_i, e_j) = e_i.e_j = \eta_{ij}$, where $\eta_{ij} = \text{diag}(-1, +1, +1, +1)$ is the Lorentz metric in canonical form. Obviously $g_{\mu\nu}(x) = \eta_{ij} h^i_\mu(x) h^j_\nu(x)$ where $e^i(x) = h^i_\mu(x) dx^i$ is the dual basis. Differing from GR, which uses the torsionless Levi-Civita connection, Teleparallelism utilizes the curvatureless *Weitzenböck connection*, whose non-null torsion tensor is defined as

$$T^\beta_{\mu\nu} = \hat{\Gamma}^\beta_{\nu\mu} - \hat{\Gamma}^\beta_{\mu\nu} = h^\beta_i (\partial_\mu h^i_\nu - \partial_\nu h^i_\mu). \tag{15}$$

Notice the Ricci rotation coefficients are $\Omega^i_{jk} = T^i_{jk}$ and encompass all the information concerning the gravitational field. The TEGR Lagrangian for the gravitational field equations (Einstein equations) is assumed to be

$$T = S_\beta{}^{\mu\nu} T^\beta{}_{\mu\nu}, \tag{16}$$

where

$$S_\beta{}^{\mu\nu} = \frac{1}{2} (K^{\mu\nu}{}_\beta + \delta^\mu_\beta T^{\theta\nu}{}_\theta - \delta^\nu_\beta T^{\theta\mu}{}_\theta) \tag{17}$$

and $K^{\mu\nu}{}_\beta$ is the *contorsion* tensor

$$K^{\mu\nu}{}_\beta = -\frac{1}{2} (T^{\mu\nu}{}_\beta - T^{\nu\mu}{}_\beta - T_\beta{}^{\mu\nu}), \tag{18}$$

which equals the difference of the Levi-Civita connection in the holonomic and the unholonomic frame (see Sec. II for details).

Here, the gravitational field will be driven by a Lagrangian density which is a function of the trace T. Therefore, the corresponding action of $f(T)$ gravity reads as

$$\mathcal{A}_T = \frac{1}{16\pi G} \int d^4x e f(T), \tag{19}$$

where $e = \det(e^i_\mu \cdot e^i_\nu) = \sqrt{-g}$. Obviously, TEGR and thus GR, are restored for $f(T) = T$.

In order to construct a realistic theory of gravity, we have to incorporate the matter and radiation fields too. Therefore, the total action is written as

$$A_{\text{tot}} = \mathcal{A}_T + \frac{1}{16\pi G} \int d^4x e (L_m + L_r), \tag{20}$$

where the matter and radiation Lagrangians are assumed to correspond to perfect fluids with energy densities ρ_m, ρ_r and pressures p_m, p_r respectively. If matter couples to the metric in the standard form then the variation of the action with respect to the vierbein leads to the equations [11]

$$e^{-1} \partial_\mu (e S_i^{\mu\nu}) f'(T) - h^\lambda_i T^\beta_{\mu\lambda} S_\beta^{\nu\mu} f'(T) + S_i^{\mu\nu} \partial_\mu(T) f''(T) + \frac{1}{4} h^\nu_i f(T) = 4\pi G h^\beta_i T^{(m)\nu}{}_\beta, \tag{21}$$

where a prime denotes differentiation with respect to T, $S_i{}^{\mu\nu} = h_i{}^\beta S_\beta^{\mu\nu}$ and $T^{(m)}_{\mu\nu}$ is the matter energy-momentum tensor. It is easy to show that, for $f(T) = T$, Eq. (21) reduces to the standard Einstein equations [37].

In order to consider the related $f(T)$ cosmology, let us assume a spatially flat FLRW metric which, in the holonomic (comoving) frame $\{\partial t, \partial x, \partial y, \partial z\}$, assumes the form

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2),$$

where $a(t)$ is the cosmological scale factor. In this space we define the vierbein (unholonomic frame) $\{e_i\}$ which becomes

$$h^i_\mu(t) = \text{diag}(1, a(t), a(t), a(t)). \tag{22}$$

In order to derive the cosmological equations in a FLRW metric, we need to deduce a pointlike Lagrangian from the action (19). As a consequence, the infinite degrees of freedom of the original field theory will be reduced to a finite number as in mechanical systems. This fact allows

one to deal with minisuperspaces of finite dimensions (see [38] for details).

In this framework, considering $\{a, T\}$ as the canonical variables of the configuration space the $f(T)$ action becomes formally

$$\mathcal{A}_T = \int \mathcal{L}(a, \dot{a}, T, \dot{T})dt.$$

Due to the fact that T, in GR, reduces to

$$T = -6\left(\frac{\dot{a}}{a}\right)^2 = -6H^2, \tag{23}$$

where H is the Hubble parameter [15], one can rewrite the $f(T)$ action using a Lagrange multiplier $\lambda_{\mathcal{L}}$ as follows:

$$\mathcal{A}_T = 2\pi^2 \int dt\left\{f(T)a^3 - \lambda_{\mathcal{L}}\left[T + 6\left(\frac{\dot{a}^2}{a^2}\right)\right]\right\}. \tag{24}$$

In order to determine $\lambda_{\mathcal{L}}$, we need to vary the $f(T)$ action with respect to T, that is

$$a^3\frac{df(T)}{dT}\delta T - \lambda_{\mathcal{L}}\delta T = 0$$

from which follows

$$\lambda_{\mathcal{L}} = a^3 f'(T).$$

Replacing in the Lagrangian we find

$$\mathcal{L} = a^3[f(T) - Tf'(T)] - 6\dot{a}^2 a f'(T), \tag{25}$$

which is canonical in the variables $\{a, T\}$.

Also, the substitution of the vierbein (22) in Eq. (21) for $i = \nu = 0$ (as well as the energy condition) yields

$$12H^2 f'(T) + f(T) = 16\pi G\rho. \tag{26}$$

Besides, for $i = \nu = 1$ Eq. (21) gives

$$48H^2\dot{H}f''(T) - 4(\dot{H} + 3H^2)f'(T) - f(T) = 16\pi G p, \tag{27}$$

where $\rho = \rho_m + \rho_r$ and $p = p_m + p_r$ are the total energy density and pressure respectively which they have been measured in the unholonomic frame. It is important to stress that Eqs. (26) and (27) can be derived by the Euler-Lagrange equations

$$E_{\mathcal{L}} = \frac{\partial\mathcal{L}}{\partial\dot{a}}\dot{a} + \frac{\partial\mathcal{L}}{\partial\dot{T}}\dot{T} - \mathcal{L} \tag{28}$$

and

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{a}} = \frac{\partial\mathcal{L}}{\partial a}, \tag{29}$$

respectively. The Euler-Lagrange equation,

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{T}} = \frac{\partial\mathcal{L}}{\partial T}, \tag{30}$$

gives the constraint (23). In this sense, the pointlike Lagrangian (25) completely defines the related dynamical system in the minisuperspace $\{a, T\}$.

It is interesting to mention that using the conservation equation $\dot{\rho} + 3H(\rho + p) = 0$ one can rewrite Eqs. (26) and (27) in the Friedmann-Einstein form

$$H^2 = \frac{8\pi G}{3}(\rho + \rho_T), \tag{31}$$

$$2\dot{H} + 3H^2 = -8\pi G(p + p_T), \tag{32}$$

where

$$\rho_T = \frac{1}{16\pi G}[2Tf'(T) - f(T) - T], \tag{33}$$

$$p_T = \frac{1}{16\pi G}\{4\dot{H}[2Tf''(T) + f'(T) - 1]\} - \rho_T \tag{34}$$

are the unholonomicity contributions to the energy density and pressure that disappears as soon as $f(T) = T$. Finally, $f(T)$ gravity can mimic, under specific circumstances, the scalar field for dark energy [15]. In order to address this crucial question, we need to derive an effective equation-of-state parameter $w(a)$ for the $f(T)$ cosmology. Indeed, utilizing Eqs. (33) and (34), we can easily obtain the effective unholonomicity equation of state as

$$\omega_T \equiv \frac{p_T}{\rho_T} = -1 + \frac{4\dot{H}[2Tf''(T) + f'(T) - 1]}{2Tf'(T) - f(T) - T}. \tag{35}$$

It is easy to see that possible deviations from the ΛCDM model can be addressed by the second term in such an equation.

IV. NOETHER SYMMETRIES

Generally, Noether symmetries play an important role in physics because they can be used to simplify a given system of differential equations as well as to determine the integrability of the system. In general, the existence of a Noether symmetry can be related to a conserved quantity bringing a physical meaning. The so-called *Noether Symmetry Approach* results extremely useful in cosmology in order to find out exact solutions (see [21] for a comprehensive review of the method). We would like to remind the reader that a fundamental approach to derive the Noether symmetries for a given dynamical problem (in a Riemannian space) has been published recently by Tsamparlis and Paliathanasis [39] (a similar analysis can be found in [36,40–44]).

Let us consider the Hamiltonian $\mathcal{H}$ which depends on one independent variable $\{t\}$ and n dependent variables $\{x^i(t)\colon i = 1, \ldots, n\}$, i.e. $\mathcal{H} = \mathcal{H}(t, x^k, \dot{x}^k, \ldots, x^{[n]k})$, where a dot over a symbol means differentiation with respect to t. We perform the one parameter point transformation

$$\bar{t} = \Xi(t, x^k, \varepsilon), \qquad \bar{x}^A = \Phi(t, x^k, \varepsilon). \tag{36}$$

In that case, the generating vector of the one parameter point transformation is

$$X = \xi(t, x^k, \varepsilon)\partial_t + \eta^i(t, x^k, \varepsilon)\partial_i, \tag{37}$$

where

$$\xi(t, x^k) = \frac{\partial \Xi^i(t, x^k, \varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon\to 0},$$
$$\eta^i(t, x^k) = \frac{\partial \Phi(t, x^k, \varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon\to 0}.$$

The extension of the generator vector in the jet space $B_M = \{t, x^k, \dot{x}^k, \ddot{x}^k, \ldots, x^{[n]k}\}$ is [36]

$$X^{[n]} = X + \eta^A_i \partial_{u_i} + \cdots + \eta^A_{ij,\ldots,i_n}\partial_{u_{ij,\ldots,i_n}},$$

where

$$\eta^{[1]i} = \frac{d}{dt}\eta^i - \dot{x}^i\frac{d}{dt}\xi, \tag{38}$$

$$\eta^{[n]i} = \frac{d}{dt}\eta^{[n-1]i} - x^{[n]i}\frac{d}{dt}\xi. \tag{39}$$

$X^{[n]}$ is called the nth prolongation of the generator (37).

We say that the function $\mathcal{H}(t, x^k, \dot{x}^k, \ddot{x}^k, \ldots, x^{[n]k}) = 0$ is invariant under the transformation of Eq. (36) if and only if there is a function $\lambda_{\mathcal{L}}$ such as the following condition holds:

$$X^{[n]}(\mathcal{H}) = \lambda_{\mathcal{L}}\mathcal{H}, \qquad \mathrm{mod}\mathcal{H} = 0, \tag{40}$$

where $\lambda_{\mathcal{L}}$ is a function to be determined [45]. Moreover, the generating vector (37) is a Lie symmetry of the function $\mathcal{H}(t, x^k, \dot{x}^k, \ddot{x}^k, \ldots, x^{[n]k})$. In the following sections we are interested in systems of second order which implies that the Hamiltonian becomes $\mathcal{H} = \mathcal{H}(t, x^k, \dot{x}^k, \ddot{x}^k)$.

A. Noether theorems

Let $\mathcal{L}(t, x^k, \dot{x}^k)$ be a function which describes the dynamics of a system. The equations of motion of the dynamical system follow from the action of the Euler Lagrange vector E_i on the function $\mathcal{L}$, i.e.,

$$E_i(L) = 0, \tag{41}$$

where the Euler Lagrange vector is

$$E_i = \frac{d}{dt}\frac{\partial}{\partial \dot{x}^i} - \frac{\partial}{\partial x^i}. \tag{42}$$

If the Lagrangian is invariant under the action of the transformation (36), namely $X^{[1]}\mathcal{L} = 0$, then it is easy to see that the Euler Lagrange equations (41) are also invariant under the transformation (36). In general we have the following theorem [36].

Theorem 1.—Let

$$X = \xi(t, x^k)\partial_t + \eta^i(t, x^k)\partial_i \tag{43}$$

be the infinitesimal generator of the transformation (36) and

$$\mathcal{L} = \mathcal{L}(t, x^k, \dot{x}^k) \tag{44}$$

be a Lagrangian describing the dynamical system (41). The action of the transformation (36) on (44) leaves the Euler Lagrange equations (41) invariant, if and only if there exists a function $g = g(t, x^k)$ such that the following condition holds:

$$X^{[1]}L + L\frac{d\xi}{dt} = \frac{dg}{dt}, \tag{45}$$

where $X^{[1]}$ is the first prolongation of (43).

If the generator of Eq. (43) satisfies Eq. (45) then the generator (43) is a Noether symmetry of the dynamical system described by the Lagrangian (44). Noether symmetries form a Lie algebra called the Noether algebra. We also have the following result.

Theorem 2.—For any Noether symmetry (43) of the Lagrangian (44) there corresponds a function $I(t, x^k, \dot{x}^k)$

$$I = \xi\left(\dot{x}^i\frac{\partial L}{\partial \dot{x}^i} - L\right) - \eta^i\frac{\partial L}{\partial x^i} + g \tag{46}$$

which is a first integral i.e. $\frac{dI}{dt} = 0$. The function (46) is called a Noether integral (first integral) of the dynamical system (41).

V. NOETHER SYMMETRIES FOR $f(T)$ COSMOLOGY

In this section we apply the Noether symmetries approach to $f(T)$ cosmology in which the corresponding Lagrangian of the field equations is given by Eq. (25). Here we consider a one parameter point transformation in the space $\{t, a, T\}$ and the generator is written as

$$X = \xi(t, a, T)\partial_t + \eta_1(t, a, T)\partial_a + \eta_2(t, a, T)\partial_T.$$

Notice that the Lagrangian (25) is a singular Lagrangian (the Hessian vanishes), hence the jet space is $\bar{B}_M = \{t, a, T, \dot{a}\}$ and thus the first prolongation of X in the jet space $\bar{B}_M$ is

$$X^{[1]} = \xi\partial_t + \eta_1\partial_a + \eta_2\partial_T + \eta^{[1]}_1\partial_{\dot{a}}, \tag{47}$$

where $\eta^{[1]}_1 = \dot{\eta}_1 - \dot{a}\,\dot{\xi}$ [45–48]. Now we compute each term in the symmetry condition (45).

The term $X^{[1]}L$ gives

$$\begin{aligned} X^{[1]}L = {} & [3a^2\eta_1(f_T T - f) + a^3 f_{TT}T\eta_2] + [12 f_T a\eta_{1,t}]\dot{a} \\ & + [12 f_T a\xi_{,a}]\dot{a}^3 + 6[f_T\eta_1 + f_{TT}a\eta_2 \\ & + 2f_T a\eta_{1,a} - 2f_T a\xi_{,t}]\dot{a}^2 \\ & + [12 f_T a\eta_{1,T}]\dot{a}\,\dot{T} + [12 f_T a\xi_{,T}]\dot{a}^2\dot{T}. \end{aligned}$$

The second term $L\dot{\xi}$ gives

$$\begin{aligned} L\dot{\xi} = {} & [a^3(f_T T - f)\xi_{,t}] + [a^3(f_T T - f)\xi_{,a}]\dot{a} \\ & + [a^3(f_T T - f)\xi_{,T}]\dot{T} + [6 f_T a\xi_{,t}]\dot{a}^2 \\ & + [6 f_T a\xi_{,a}]\dot{a}^3 + [6 f_T a\xi_{,T}]\dot{a}^2\dot{T}. \end{aligned}$$

Finally the right-hand side of Eq. (45) is

$$\dot{g} = g_{,t} + g_{,a}\dot{a} + g_{,T}\dot{T}.$$

Replacing the results in Eq. (45) and setting the terms with the powers of $\dot{a}$ and $\dot{T}$ equal to zero in order to select the Lie vector (see [21] for details), we find the following set of Noether symmetry conditions:

$$\xi_{,a} = 0, \qquad \xi_{,T} = 0, \qquad \eta_{1,T} = 0, \tag{48}$$

$$a^3(f_T T - f)\xi_{,T} = g_{,T}, \tag{49}$$

$$3a^2\eta_1(f_T T - f) + a^3 f_{TT} T\eta_2 + a^3(f_T T - f)\xi_{,t} = g_{,t}, \tag{50}$$

$$12 f_T a\eta_{1,t} + a^3(f_T T - f)\xi_{,a} = g_{,a}, \tag{51}$$

$$f_T\eta_1 + f_{TT} T a\eta_2 + 2f_T a\eta_{1,a} - f_T a\xi_{,t} = 0. \tag{52}$$

From Eqs. (48) and (49) it follows

$$\xi = \xi(t), \qquad \eta_1 = \eta_1(t, a), \qquad g = g(t, a).$$

Then Eq. (51) becomes $12 f_T a\eta_{1,t} = g_{,a}$ Because η_1, g are independent of T which follows that

$$\eta_1 = \eta_1(a), \qquad g = g(t).$$

Dividing Eq. (52) with $a f_T$ we find

$$2\eta_{1,a} + \frac{\eta_1}{a} + \frac{f_{TT}}{f_T}\eta_2 - \xi_{,t} = 0 \tag{53}$$

from which follows that

$$\eta_2 = \frac{f_T}{f_{TT}} S(a, t),$$

where S is an arbitrary function of its arguments. Taking this result into consideration the conditions (50') and (53) become respectively

$$2\eta_{1,a} + \frac{\eta_1}{a} + S(a, t) - \xi_{,t} = 0, \tag{54}$$

$$3a^2\eta_1(f_T T - f) + a^3 f_{TT} T S + a^3(f_T T - f)\xi_{,t} = g_{,t}. \tag{55}$$

From Eq. (54) follows that $S(a, t) = M(a) + N(t)$ hence we have the final symmetry conditions (where $f \neq e^{kT}$ $k =$ constant):

$$2\eta_{1,a} + \frac{\eta_1}{a} + M + N - \xi_{,t} = 0, \tag{56}$$

$$3\frac{\eta_1}{a} + \frac{f_T T}{f_T T - f} M + \frac{f_T T}{f_T T - f} N + \xi_{,t} = \frac{1}{a^3(f_T T - T)} g_{,t}. \tag{57}$$

It is obvious that Eqs. (54) and (55) hold for arbitrary $f(T)$ as long as $\xi = c_0$ and $\eta_1 = \eta_2 = 0$ (i.e. $S = 0$). In this case the corresponding Noether integral is the Hamiltonian $\mathcal{H}$, implying that the dynamical system is autonomous.

Moreover, the conditions (56) and (57) give the following system of equations:

$$\frac{f_T T}{f_T T - f} = \frac{n}{n - 1} \tag{58}$$

and

$$g_{,t} = 0, \qquad N = c + \xi_{,t}, \qquad 2\eta_{1,a} + \frac{\eta_1}{a} + M = c,$$
$$3\frac{\eta_1}{a} + \frac{n}{n - 1} M = m, \qquad \frac{n}{1 - n} N - \xi_{,t} = m.$$

Solving the first equation of the system (58) we find that

$$f(T) = f_0 T^n, \tag{59}$$

where f_0 is the integration constant. In this context we can obtain the Noether symmetries. Specifically, in the case of $n \neq \frac{1}{2}, \frac{3}{2}$, the Noether symmetry vector is

$$X_1 = \left(\frac{3C}{2n - 1} t\right)\partial_t + (Ca + c_3 a^{1-\frac{3}{2n}})\partial_a + \left[\frac{1}{n}((c - m)n + 3c_3 a^{-\frac{3}{2n}}) + \frac{3C}{2n - 1} + c\right] T\partial_T$$

as well as the corresponding Noether integral is

$$I_1 = \left(\frac{3C}{2n - 1} t\right)\mathcal{H} - 12 f_0 n(Ca^2 + c_3 a^{2-\frac{3}{2n}}) T^{n-1}\dot{a},$$

where $C = \frac{m(1-n)+nc}{3}$.

For $n = \frac{3}{2}$, the Noether symmetry is given by

$$X_2 = \frac{1}{5}(3c - 2m) t\partial_t + \left[\left(\frac{c}{2} - \frac{m}{6}\right)a + c_4\right]\partial_a + \left[(m + 11c) - \frac{c_4}{a} + \frac{2}{5}(8c - 2m)\right] T\partial_T \tag{60}$$

with corresponding Noether integral

$$I_2 = \frac{1}{5}(3c - 2m) t\mathcal{H} - 18 f_0\left[\left(\frac{c}{2} - \frac{m}{6}\right)a^2 + c_4 a\right] T^{\frac{1}{2}}\dot{a}.$$

Finally for $n = \frac{1}{2}$, the Noether symmetry becomes

$$X_3 = c_1 t\partial_t + (-2c_1 + c_3 a^{\frac{1}{4}})\partial_a + \left(4c_1 + c_2 + \frac{3c_3}{2} a^{-\frac{3}{4}}\right) T\partial_T$$

and the Noether integral is

$$I_3 = c_1 t\mathcal{H} - 6 f_0(-2c_1 a + c_3 a^{\frac{3}{4}}) T^{-\frac{1}{2}}\dot{a}.$$

We would like to stress that our results are in agreement with those of [33] but they are richer because we have considered the term $\xi\partial_t$ in the generator which is not done in [33]. To this end it becomes evident that $f(T) = f_0 T^n$ is the only form that admits extra Noether symmetries implying the existence of exact analytical solutions (see the next section).

VI. EXACT COSMOLOGICAL SOLUTIONS

In this section we proceed in an attempt to analytically solve the basic cosmological equations of the $f(T) = f_0 T^n$ gravity model. In particular from the Lagrangian (25), we obtain the main field equation

$$\ddot{a} + \frac{1}{2a}\dot{a}^2 + \frac{f''}{f'}\dot{a}\,\dot{T} - \frac{1}{4}a\frac{f'T - f}{f'} = 0. \tag{61}$$

Also differentiating Eq. (23) we find

$$\dot{T} = 12\left[\left(\frac{\dot{a}}{a}\right)^3 - \frac{\dot{a}\,\ddot{a}}{a^2}\right]. \tag{62}$$

Finally, inserting $f(T) = f_0 T^n$, Eq. (23) and (62) into Eq. (61) we derive, after some algebra, that

$$(2n-1)\left[\ddot{a} - \frac{\dot{a}^2}{2a}\frac{(2n-3)}{n}\right] = 0, \tag{63}$$

a solution of which is

$$a(t) = a_0 t^{2n/3} \qquad H(t) = \frac{\dot{a}}{a} = \frac{2n}{3t} \tag{64}$$

or

$$H = H_0 a^{-3/2n} = H_0(1+z)^{3/2n}, \tag{65}$$

where $n \in \mathcal{R}_+^\star - \{\frac{1}{2}\}$, $a(z) = (1+z)^{-1}$ and H_0 is the Hubble constant in agreement with [33]. Also using Eq. (65) the deceleration parameter is given by

$$q = -1 - \frac{\mathrm{d}\ln H}{\mathrm{d}\ln a} = -1 + \frac{3}{2n}. \tag{66}$$

From Eq. (64) it is evident that this cosmological model has no inflection point. Therefore, the main drawback of the $f(T) = f_0 T^n$ gravity model is that the deceleration parameter preserves sign, and therefore the Universe always accelerates or always decelerates depending on the value of n. Indeed, if we consider $n = 1$ (TEGR) then the above solution boils down to the Einstein–de Sitter model as it should. On the other hand, the accelerated expansion of the Universe ($q < 0$) is recovered for $n > \frac{3}{2}$. The latter points that even if we would admit $n > \frac{3}{2}$ as a mere phenomenological possibility, we would be also admitting that the Universe has been accelerating forever, which is of course difficult to accept.

Now, we proceed to provide the growth factor of the $f(T) = f_0 T^n$. In general, the basic equation which governs the evolution of the matter fluctuations in the linear regime is given by

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G_{\rm eff}\rho_m\delta_m = 0, \tag{67}$$

where ρ_m is the matter density and $G_{\rm eff}$ is the effective Newton's parameter which is written as [49]

$$G_{\rm eff} = \frac{G}{f'(T)}. \tag{68}$$

Note that G denotes Newton's gravitational constant. On the other hand, using Eqs. (31) and (33) one can easily write

$$4\pi G\rho_m = \frac{3H^2}{2} - 4\pi G\rho_T = \frac{3H^2}{2} - \frac{2Tf'(T) - f(T) - T}{4}. \tag{69}$$

Therefore, inserting Eqs. (23), (68), and (69) into Eq. (67) we have the following general equation:

$$\ddot{\delta}_m + 2H\dot{\delta}_m + \frac{2Tf'(T) - f(T)}{4f'(T)}\delta_m = 0. \tag{70}$$

We focus now on the $f(T) = f_0 T^n$ gravity model. First of all for GR ($n = 1$) we have $G_{\rm eff} = G$ and thus, without losing the generality, we can set $f_0 = 1$.[1] Therefore, Eq. (70) becomes

$$\ddot{\delta}_m + \frac{4n}{3t}\dot{\delta}_m - \frac{2n(2n-1)}{3t^2}\delta_m = 0. \tag{71}$$

Notice that in order to derive Eq. (71) we have utilized Eqs. (23) and (64). Interestingly, the above differential equation modifies that of the Einstein–de Sitter model in which $n = 1$ (GR). From the mathematical point of view, Eq. (71) is of Euler type whose general solution is

$$\delta_m(t) = C_1 t^{2n/3} + C_2 t^{1-2n} \tag{72}$$

or

$$\delta_m(a) = \tilde{C}_1 a + \tilde{C}_2 a^{3(1-2n)/2n}, \tag{73}$$

where $\tilde{C}_1 = C_1/a_0^{3/2n}$ and $\tilde{C}_2 = C_2/a_0^{3(1-2n)/2n}$. In the case of $0 < n < \frac{1}{2}$ we have two growth factors while for $n > \frac{1}{2}$ the only growth factor is $D_+ = a \propto t^{2n/3}$. It is interesting to mention that if we write the growth factor as a function of the scale factor then mathematically it coincides with that of the Einstein–de Sitter model [50]. This result means that the growth rate of clustering $f_+(a) = d\ln D_+/d\ln a$ remains constant and equal to unity for every scale factor, implying that the present growth data disfavor the $f(T) = f_0 T^n$ gravity. Indeed, in Fig. 1 we plot the growth data as collected by Basilakos *et al.* (see [51] and references therein) with the estimated growth rate function, $f_+(z)\sigma_8(z)$ [see $f(T)$ (solid line) and ΛCDM (dashed line)]. Notice that the theoretical $\sigma_8(z)$ is given by $\sigma_8(z) = \sigma_8 D_+(z)$, where σ_8 is the rms mass fluctuation on $R_8 = 8h^{-1}$ Mpc scales at redshift $z = 0$.

A. Cosmological analogue to other models

In this section (assuming flatness) we present the cosmological equivalence at the background level between the current $f(T)$ gravity with $f(R)$ modified gravity and dark

[1] If $f(T) = f_0 T$ then the Newton's constant is just rescaled to be $G_{\rm eff} = G/f_0$ which is also constant in time. This result comes directly from the action (19) (see also [49]).

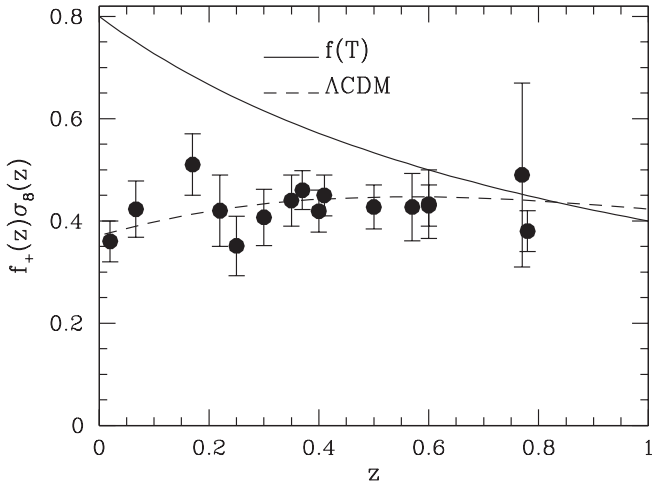

FIG. 1. Comparison of the observed (solid points) and theoretical evolution of the growth rate $f_+(z)\sigma_8(z)$. The solid and dashed lines correspond to $f(T) = f_0 T^n$ and ΛCDM. As in Basilakos *et al.* [51] we use $\sigma_8 = 0.8$ while for the ΛCDM case we set $\Omega_{m0} = 0.272$.

energy, through a specific reconstruction of the $f(R)$ and vacuum energy density, namely, $f(R) = R^n$ and $\Lambda(H) = 3\gamma H^2$. In the case of $f(R) = R^n$ it has been found by Paliathanasis (see the Appendix in [52]) that the corresponding scale factor obeys Eq. (64), where $n \in \mathcal{R}_+^\star - \{2, \frac{3}{2}, \frac{7}{8}\}$.[2] In [53,54], it has been shown that the particular model $f(R) \propto R^{3/2}$ has the cosmological solution $a(t) = \sqrt{a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t}$ capable of addressing both dark-energy and dark-matter dominated phases. However, despite the analogies, we have to point out that $f(R)$ gravity is a fourth-order theory while $f(T)$ gravity remains of second order.

On the other hand, considering a spatially flat FLRW metric in the context of GR, the combination of the Friedmann equations with the total (matter + vacuum) energy conservation in the matter dominated era provides (for more details see [55])

$$\dot{H} + \frac{3}{2}H^2 = \frac{\Lambda}{2}. \tag{74}$$

Solving Eq. (74) for $\Lambda(H) = 3\gamma H^2$ (see Refs. [56–58]) we end up with

$$H = H_0 a^{-3(1-\gamma)/2} = H_0(1+z)^{3(1-\gamma)/2}. \tag{75}$$

Now, comparing Eqs. (65) and (75) and connecting the above coefficients such as $n^{-1} = 1 - \gamma$, we find that the $f(T) = f_0 T^n$ and the flat $\Lambda(H) = 3\gamma H^2$ models can be viewed as equivalent cosmologies as far as the Hubble expansion is concerned, despite the fact that the current time varying vacuum model adheres to GR. However, if the $\Lambda(H) = 3\gamma H^2$ cosmological model is confronted with the current observations provides a poor fit [55]. Since the current time varying vacuum model shares exactly the same Hubble parameter with the $f(T) = f_0 T^n$ gravity model, this fact implies that the latter is also under observational pressure when we compare against the background cosmological data (SnIa, BAOs and CMB data). The same observational situation holds also for $f(R) = R^n$ modified gravity.

[2]The Lagrangian here is $\mathcal{L}_R = 6naR^{n-1}\dot{a}^2 + 6n(n-1)\times a^2 R^{n-2}\dot{a}\dot{R} + (n-1)a^3 R^n$, where R is the Ricci scalar. For $n = 1$ the solution of the Euler-Lagrange equations is the Einstein de-Sitter model [$a(t) \propto t^{2/3}$] as it has to be. Note, that for $n = 2$ one can find a de-Sitter solution ($a(t) \propto e^{H_0 t}$, see [52]).

VII. CONCLUSIONS

In this paper, we present a general study of Noether symmetries for $f(T)$ gravity and discuss the role of torsion and unholonomic frames in the context of teleparallel gravity and its straightforward extension. In particular, we point out the misunderstanding that, when one works in an unholonomic frame, the torsion is introduced showing that this statement is not correct. The misunderstanding consists in the fact that the effects one observes in an unholonomic frame are frame dependent and not covariant effects. Therefore all conclusions made in a specific unholonomic frame must be restricted to that frame only.

Coming to the specific *Noether Symmetry Approach*, this article extends the works by Basilakos *et al.* [34], Paliathanasis *et al.* [35], and Wei *et al.* [33]. We confirm the result of [33] that amongst the variety of $f(T)$ modified gravity theories, $f(T) = f_0 T^n$ gravity admits Noether symmetries (integrals of motion). However, we provide here a more general family of Noether integrals with respect to that of [33]. From the mathematical viewpoint the existence of extra integrals of motion points out the existence of further analytical solutions.

Based on the $f(T) = f_0 T^n$ models, we derive analytical solutions and thus we find the evolution of the main cosmological functions, namely the scale factor of the Universe, the Hubble parameter, the deceleration parameter, and for the first time to our knowledge the growth of matter fluctuations in the linear regime. Furthermore, we discuss the linear matter fluctuations from these background solutions. The analysis of the deceleration parameter points out that the $f(T) = f_0 T^n$ gravity models include an intrinsic problem, namely, the fact that the expansion of the Universe always accelerates or always decelerates without spanning the different trends of cosmic evolution. Another basic problem is related to the fact that the growth rate of clustering is constant and always equal to unity which means that the present growth data cannot accommodate the $f(T) = f_0 T^n$ gravity. As shown in [59], a robust cosmographic reconstruction of $f(T)$ cosmology needs more complicated models to address data.

Finally, we find that flat $f(T) = f_0 T^n$ cosmologically models are perfectly equivalent to the cosmic expansion history of the flat $f(R) = R^n$ modified gravity and the flat

time varying vacuum model $\Lambda(H) = 3\gamma H^2$ (where $n^{-1} = 1 - \gamma$), despite the fact that the three models live in a completely different geometrical background. This fact is a further indication of the high degeneracy problem affecting cosmological models capable of addressing the dark energy issue.

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