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Quantum instability of the emergent universe

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We perform a semiclassical analysis of the emergent universe scenario for inflation. Fixing the background, and taking the inflaton to be homogenous, we cast the inflaton's evolution as a one-dimensional quantum mechanics problem. The potential is taken to be flat or linear, as an approximation to the monotonic and slowly varying asymptotic behavior of the emergent universe potential. We find that the tuning required over a long time scale for this inflationary scenario is unstable quantum mechanically. Considering the inflaton field value as a wave packet, the spreading of the wave packet destroys any chance of both starting and ending with a well-formed state. Thus, one cannot have an Einstein static universe to begin with that evolves into a well-defined beginning to inflation a long time later.

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I. INTRODUCTION

The ancient question of whether the Universe had a beginning, or has existed eternally, has in recent decades been brought into focus using the tools and knowledge of general relativity (GR) and modern cosmology. The question's resolution, however, is still far from clear. The standard Friedmann–Lemaître–Robertson–Walker (FLRW) big bang cosmology includes an initial singularity, and there is a number of theorems purporting that such singularities are generic [1], suggesting that *classical* spacetime, at least, has a "beginning" in the sense of a global spacelike surface at which classical GR breaks down. However, these theorems all brook various exceptions and loopholes, and several scenarios have been developed that circumvent these theorems and form the basis for classical or semiclassical "past eternal" cosmologies (see Ref. [2] for review of some of these involving inflation).

One scenario that has garnered significant attention is the "emergent universe" of Ellis and Maartens [3]. This cosmology has several attractive features: there is no initial singularity or "beginning of time," no horizon problem, and (it is claimed) no quantum gravity era because the curvature scale always greatly exceeds the Planck scale. In this scenario at some early time, the Universe is a closed FLRW cosmology that as time $t \to -\infty$ asymptotes to an Einstein static universe, with a negative-pressure energy component that stabilizes the universe against gravitational collapse. Thus, the closed universe exists "eternally" but then at some point begins inflation. That is, the first e-folding of inflation takes an unbounded amount of time, but the second and subsequent e-foldings proceed essentially as usual, ending in reheating and ordinary cosmological evolution.

A key question about this scenario is whether the initial state can self-consistently exist for eternity. Classically, the Einstein static universe is unstable to homogeneous perturbations but stable to inhomogeneous perturbations if the fluid sound speed is sufficiently high [4]. This indicates that eternality is possible in principle but only if the homogeneous mode of the positive-pressure content *precisely* balances the negative energy repulsive component of the energy density. This appears problematic, however, in that we might expect that quantum fluctuations can destabilize this careful balance, causing the universe to collapse or expand uncontrollably.

In this paper, we examine the simplest version of the emergent universe, based on a single rolling scalar field in GR. Our analysis will treat the metric classically (and in fact fixed) and assume the scalar field is homogeneous but quantized. Treating the metric quantum mechanically (e.g., using the Wheeler–de Witt formalism) or including anisotropic perturbations seems very unlikely to increase the cosmology's stability. These assumptions allow us to cast the problem as a simple one-dimensional quantum mechanics problem, following the method of Ref. [5], where the degree of freedom is the value of the scalar field. This reveals a result that is intuitively perhaps unsurprising: due to the spreading of any wave packet, it is inconsistent to have a well-defined state (Gaussian wave packet) in the asymptotic past as an Einstein static universe as well as a well-defined state at the beginning of inflation (i.e., at the end of the first e-folding). We show this by analyzing both a flat and nearly flat (linear) potential for the wave packet evolution. Computing the probability of the initial state evolving into a well-defined preinflationary state, we see

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that it is severely suppressed (and effectively zero in the past eternal limit). We discuss these calculations and also comment on the emergent universe in alternative gravitational theories.

II. THE EMERGENT UNIVERSE

The emergent universe evades relevant singularity theorems [1] by explicitly violating the assumptions that $K = -1$, 0 and $H \equiv \dot{a}/a > 0$. The emergent universe is closed, $K = +1$, and has $H = 0$ initially. This scenario does not bounce but starts as an Einstein static universe with finite size in the infinite past, inflates, and then reheats in the usual manner. In this way it avoids an initial singularity and horizon problem, and the initial size can be large enough for this scenario to avoid a quantum gravity era. Although there is an infinite time for inflation, the amount of inflation is finite (and can be made large).

The simplest setup for the emergent universe is an Einstein static universe with a cosmological constant [absorbed into the constant term of the scalar field's potential, $V(\phi)$] and minimally coupled scalar field, ϕ. The Friedmann equations are

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3}(\dot{\phi}^2 - V(\phi)), \tag{1}$$

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) - \frac{K}{a^2}, \tag{2}$$

where a dot represents a time derivative, G is Newton's constant, and $a(t)$ is the scale factor using the conventions of Refs. [3,5] (it has units of length[1]).

A positive minimum for the (initial) scale factor, a_0,[2] has $H_0 = 0$ and

$$\frac{3K}{8\pi G a_0^2} = \frac{1}{2}\dot{\phi}_0{}^2 + V_0, \tag{3}$$

where the zero subscript denotes the initial value (the same time as a_0 is defined). Furthermore, Eq. (1), since $\ddot{a} = 0$, tells us that

$$\dot{\phi}_0{}^2 = V_0. \tag{4}$$

Then the value of the potential at this minimum, or the initial vacuum energy, is

$$V_0 = \frac{K}{4\pi G a_0^2} (= \dot{\phi}_0^2). \tag{5}$$

Here we see the beginning of a potential problem for the emergent universe: the Einstein static universe requires a precise balancing of the kinetic energy of the scalar field with the vacuum energy. This balancing must persist if the universe is to be considered static and thus will be sensitive to quantum fluctuations in the scalar field. In this note we attempt to analyze this problem via a "semiclassical"-like analysis.

One might also worry about the classical stability of the Einstein static universe [4]. For the simplest case, we are considering, where there is no matter, the static universe is neutrally stable for inhomogeneous linear perturbations. Homogeneous perturbations will break the balance of the curvature to vacuum energy, leading to inflation; thus, to perdure for an indefinite amount of time, this balance in the zero-mode must be mathematically perfect. We do not address concerns about this here.[3] Rather, we assume that such a perfect balance is maintainable classically and investigate the same problem when quantized.

III. INFLATION AS ONE-DIMENSIONAL QUANTUM MECHANICS

First, let us define our setup and conventions, which follows closely from Ref. [3,5]. We will take the simplest case of the universe filled with just a scalar field, ϕ, in a FLRW background with $K = +1$, scale factor $a(t)$, and Hubble expansion rate $H \equiv \dot{a}/a$. The scalar field obeys

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \tag{6}$$

where the dots denote time derivatives, $V(\phi)$ is the inflaton potential, and the prime denotes a derivative with respect to the argument.

We can cast the evolution of the inflaton as a one-dimensional quantum mechanics problem by making the following simplifications:

(i) Treat the background geometry classically (and fixed); the scale factor is not treated as a field.
(ii) Take the inflaton to be homogeneous and its value everywhere in space, ϕ, to be the "coordinate."
(iii) Take the inflaton momentum to also be homogeneous, and its value, $\dot{\phi}$, is proportional to the conjugate momentum.

From this setup, one sees that we are performing a type of semi- or "quasiclassical" analysis. For the emergent universe scenario in particular, this type of analysis will shed light on the question of stability beyond purely classical considerations by being the next step in sophistication. The emergent universe's behavior in the asymptotic past should be that of a static universe, which then evolves ever so slowly for an infinite amount of time. It makes sense then to treat the background classically and independently from the scalar field. However, the delicate tuning of the scalar field's kinetic energy leads us to consider any

[1]The mass dimension of other quantities is $[\phi] = 1$, $[\sigma_0] = 1$, $[T] = 2$.

[2]The dominance of the curvature term in the past, for sufficiently long inflation, allows such a solution with $K = +1$ [6].

[3]For example, it seems likely that nonlinear coupling between the modes would leak power from inhomogeneous modes to the homogeneous mode, making the universe effectively unstable to all perturbations [7].

small deviations, especially over the infinite amount of time. Thus, we treat the scalar field in a quantum mechanical manner. And, as we shall see in the following sections, our setup is enough to see a serious instability or inconsistency in the emergent universe's evolution.

The Lagrangian[4] is

$$L = 2\pi a^3(t)\left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right). \tag{7}$$

After performing the usual Legendre transformation, with conjugate momentum $p \equiv 2\pi^2 a^3(t)\dot{\phi}$, the Hamiltonian is

$$H = \frac{1}{2}\frac{p^2}{2\pi^2 a^3(t)} + 2\pi^2 a^3(t)V(\phi). \tag{8}$$

We now define our "wave function" as $\psi(\phi, t)$, which satisfies the Schrödinger equation from the above Hamiltonian

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{1}{2\pi^2 a^3}\frac{\partial^2\psi}{\partial\phi^2} + 2\pi^2 a^3 V(\phi)\psi. \tag{9}$$

Taking the potential to have at most quadratic terms and after the following ("conformal time"-like) time coordinate change,

$$T = \int \frac{1}{a^3(t')}\,\mathrm{d}t', \tag{10}$$

we rewrite the Schrödinger equation as

$$i\frac{\partial\psi}{\partial T} = -\frac{1}{4\pi^2}\frac{\partial^2\psi}{\partial\phi^2} + u(\phi, T)\psi, \tag{11}$$

with the potential $u(\phi, T)$ given by

$$u(\phi, T) = 2\pi^2 a^6(T)(c_2\phi^2 + c_1\phi + c_0), \tag{12}$$

with the only T dependence coming from the prefactor $2\pi^2 a^6(T)$ and the ϕ dependence explicit.

We now assume a Gaussian wave packet form for ψ, which we parametrize as

$$\psi(\phi, T) = A(T)e^{-B(T)[\phi - f(T)]^2}, \tag{13}$$

with A, B, and f as arbitrary functions of T to be solved for. We plug ψ into the Schrödinger equation, and by matching coefficients of each power of ϕ, we have a set of differential equations for A, B, and f (for more details, see Ref. [5]).

It will also be useful to think of the wave function in momentum space (the conjugate momentum, as defined above), which is given by the usual Fourier transform,

$$\tilde{\psi}(p, T) = \tilde{A}(T)e^{-[p + 2iB(T)f(T)]^2/4B(T)}, \tag{14}$$

with $\tilde{A}(T) \equiv A(T)\exp[-B(T)f(T)^2]/\sqrt{2}B(T)$.

[4]Note that this is not a Lagrangian *density*; we have already integrated over space, hence the volume factor of $2\pi a^3(t)$.

IV. INSTABILITY OF THE EMERGENT UNIVERSE SCENARIO

We can now apply the above framework to the emergent universe. We will model the potential as being composed of sections that are completely flat and sections with a constant slope. This could model a potential that is perfectly flat as $\phi \to -\infty$, connected to a sloping portion at $\phi > 0$, or one constructed out of segments that have a slope approaching zero as $\phi \to -\infty$. In either case, we will consider the field at some time $T = -T_0$ to be in a Gaussian wave packet centered at ϕ_0 with initial spread σ_0 and moving with velocity v_0 (which may be zero). We then evolve the wave packet to later times.

Formulated this way, stability concerns arise almost immediately. For a given value of $V(\phi)$, only one precise value of $\dot{\phi}$ yields stability, yet ϕ and $p \propto \dot{\phi}$ are subject to an uncertainty relation.[5] In momentum space, the wave packet has nonzero width $\Delta \propto 1/\sigma_0$, so at a given time, unless $\Delta \to 0$, there is an infinitesimal probability of a measurement yielding the value of $\dot{\phi}$ which gives stability. With probability approaching unity, the field velocity would have a value for which the universe would evolve away from the emergent dynamics, into either empty de Sitter space or a big crunch. Yet in the $\Delta \to 0$ limit, the value of ϕ is completely uncertain, and one cannot describe the situation as a single classical universe.

Similarly, if we *assume* that the universe can always be treated in a quasiclassical way (as implicitly assumed by Ref. [3]), it should have compact support in both field and field-velocity space. We can then ask the following: if the wave function at time $-T_0 \to -\infty$ is a wave packet of finite width in both ϕ and $\dot{\phi}$, is there *any* probability that it will evolve into a quasiclassical configuration at time $T = 0$ at which inflation starts? This can be computed within our model for either a flat or a constant-slope potential, as detailed in the next section and in the Appendix.

A. In a flat or linear potential

As a simplest calculation, we study the evolution of a wave packet in a completely flat potential ($c_2 = c_1 = 0$), which mimics the asymptotic behavior of the emergent universe. In this case we will need to give the initial wave packet some "kick" to have an initial nonzero velocity.

The initial state (at time $T = -T_0$) is a Gaussian wave packet centered at ϕ_0 with initial spread σ_0 and moving with velocity v_0:

$$\psi(\phi, -T_0) = \left(\frac{1}{2\pi\sigma_0^2}\right)^{1/4} e^{\frac{-(\phi-\phi_0)^2}{4\sigma_0^2} + iv_0\phi_0}. \tag{15}$$

[5]The wave packets are initially minimum uncertainty wave packets, saturating the uncertainty relationship between ϕ and p, $\sigma_\phi\sigma_p \geq 1/2$. As the packet evolves in the following potentials, however, σ_ϕ grows while σ_p is constant.

At a time T, the wave packet evolves to

$$\psi(\phi, T) = \frac{1}{(2\pi\sigma_0^2)^{1/4}} \exp\bigg\{-v_0^2\sigma_0^2 + iv_0\phi_0 - \frac{\pi^2}{\sigma(T)}(\phi - \phi_0 - 2iv_0\sigma_0^2)^2 - \frac{i}{2}\int_{-T_0}^{T} \frac{1}{\sigma(T')}[1 + 2\sigma(T')c(T')]\mathrm{d}T'\bigg\}, \quad (16)$$

with

$$\sigma(T) \equiv 4\pi^2\sigma_0^2 + i(T_0 + T), \quad (17)$$

and $c(T) \equiv 2\pi^2 a^6(T)c_0$. The center of the wave packet, which is $\langle\phi\rangle$, moves with constant velocity v as[6]

$$\langle\phi\rangle = \phi_0 + \frac{(T_0 + T)v_0}{2\pi^2}, \quad (18)$$

and the quantum mechanically uncertainty in ϕ is

$$\sigma_\phi^2 = |\sigma|^2/16\pi^4\sigma_0^2, \quad (19)$$

while the conjugate momentum has

$$\langle p\rangle = v_0, \qquad \sigma_p = \frac{1}{4\sigma_0}. \quad (20)$$

We now want to calculate the probability that a well-formed initial wave packet at $T = -T_0$, centered at $\phi = \phi_0$, will evolve into another "nice" wave packet some time later, $T = 0$, in this potential. For the final state to compare with, we will use the initial wave packet with its center shifted to be lined up with the wave packet that evolved after time T_0 (to time $T = 0$). In other words, the initial state is $\psi(\phi, -T_0)$, which evolves into $\psi(\phi, 0)$, which we compare to $\psi(\phi, -T_0)|_{\phi_0 = T_0 v_0/2\pi^2}$. The probability we are calculating is

$$P = |\langle\phi, -T_0|_{\phi_0 = T_0 v_0/2\pi^2}|e^{-iHT_0}|\phi, -T_0\rangle|^2 = |\langle\phi, -T_0|_{\phi_0 = Tv_0/2\pi^2}|\phi, 0\rangle|^2. \quad (21)$$

Before specifying the scale factor, the probability is

$$P = \left(\frac{T_0^2}{\chi^2} + 1\right)^{-1/2}, \quad (22)$$

with $\chi \equiv 8\pi^2\sigma_0^2$. For a static universe, the scale factor is a constant, which we set to a_0, and $T = t/a_0^3$. For a long evolution, we take $T_0 \gg \chi$, and the probability, to leading order in t, is

$$P \approx \frac{a_0^3\chi}{t_0} \ll 1. \quad (23)$$

Therefore, for a long evolution ($-t_0 \to -\infty$) in a constant potential, the probability of a wave packet evolving into another well-defined wave packet after a time t_0 is much less than 1, falling like $1/t_0$.

Since the wave packet spreads in ϕ space, the probability could be greater if we allow the shifted final state wave packet's width to vary from σ_0. Call the width σ_1. The probability (before specifying a scale factor) is then

$$P = \frac{8\pi^2\sigma_0\sigma_1}{\sqrt{T_0^2 + 16\pi^4(\sigma_0^2 + \sigma_1^2)^2}}. \quad (24)$$

This can be maximized to go like $\sigma_0/\sqrt{T_0}$, but only if $\sigma_1 \propto \sqrt{T_0}$, which does not correspond to a well-defined classical configuration for large T_0; this is a more precise version of the argument at the beginning of this section.

We also consider a linear potential with slope $-b$. Here we will very briefly summarize the results, while the details of this calculation can be found in the Appendix. The field now starts at rest and accelerates down the potential. The final state we take the overlap with will also have a velocity. If this velocity exactly matches the velocity of the wave packet which was evolved in the potential, then the probability is the same as above, Eq. (23). If the velocities do not match, there is an additional exponential suppression which depends on the slope $-b$. Any constraint on the final velocity of the wave packet will then also constrain the length of the linear potential (i.e., amount of time the field evolves in the linear potential).

V. DISCUSSION AND CONCLUSIONS

The emergent universe paradigm represents an intriguing effort to construct a cosmology without a past classical singularity. In this paper we have analyzed a version of this model in which a scalar field evolves in a potential that is flat or has a constant slope to approximate the asymptotic behavior of the emergent universe potential. Assuming a Gaussian wave packet form for the wave function of the homogenous mode of the inflaton, we have derived the evolution of the wave packet in these two types of potentials with a fixed background geometry. We then answered the following question: what is the probability of a well-defined initial wave packet evolving into well-defined state after a time t_0? In both cases the probability is proportional to $1/t_0$ for large t_0. The emergent universe is built on an infinite past, and thus this probability goes to zero.

It thus appears inconsistent to have both a well-defined semiclassical approximation to the field and also have infinite past nonsingular time. If the field has a well-defined value at any given time (which might be posed as a boundary condition), then evolving back in time, the wave functional was spread over a range of values. The field velocity will also always have a spread of different values, most of which do not balance the negative pressure term. If we were to then include gravity, at yet earlier times,

[6]This is an example of an expression we rederived that is also in Ref. [5], where we differ by a factor of 1/2 in front of T (the other notable instances are the probability densities in Ref. [5]). We believe this is a typo in these papers, but these factors make no difference for our analysis.

the universe would presumably be a superposition dominated by expanding (from a singularity) or contracting states.

We stress that we have analyzed only one version of the emergent universe, with a simplified model. Nonetheless, we believe that the effect that this analysis points to may be rather generic. For example, consider alternative theories of gravity. The emergent universe has been studied extensively in theories such as Hořava–Lifshitz, $f(R)$, loop quantum gravity,[7] and others (see, for instance, Refs. [8–11], respectively). There have also been several studies of the stability of the Einstein static universe in alternative theories (see Ref. [12], for example). However, in our framework we have, in a sense, decoupled gravity—it enters only when assessing the affect of the spreading wave functional. Even in alternative theories in which the Einstein static universe is more stable than in standard general relativity, we anticipate that once the wave functional has spread enough, the geometry must follow, and the spacetime becomes classically ill-defined as well as containing portions corresponding to singularities. Therefore, this seems like a generic (and perhaps expected, given our construction of the scenario) problem with such an eternal and precisely tuned inflationary scheme.

To avoid this behavior, the field velocity would have to be stabilized by some mechanism at the correct value, while still allowing for the field value to evolve appropriately. The potential would have to remain constant for a static universe, and thus some sort of (classical) driving and damping terms seem to be necessary. It would also still be difficult to arrange the appropriate initial conditions. It is not immediately obvious how one can successfully achieve this. Another alternative is perhaps a tunneling scenario (for instance, Ref. [13]). However, then the universe is necessarily not eternal.

Models in which the field dynamics and material content are very different would require separate analysis but may lead to a similar basic conclusion. For example, Graham *et al.* [14] construct static and oscillating universes with a specific nonperfect-fluid energy component that are stable against small perturbations. However, Mithani and Vilenkin [15] have shown that this model is unstable to decay via tunneling.

Although we have analyzed only one version of the emergent universe, we would argue that our analysis is pointing to a more general problem: it is very difficult to devise a system—especially a quantum one—that does nothing "forever," then evolves. A truly stationary or periodic quantum state, which would last forever, would never evolve, whereas one with any instability will not endure for an indefinite time. Moreover, the tendency of quantum effects to destabilize even classically stable configurations suggests that, even if an emergent model were possible, it would have to be posed at the quantum (and quantum-gravitational) level, largely undermining the motivation to provide an early state in which quantum gravitational effects are not crucial.

[7] In a quantum theory of gravity, the quantum aspect of our analysis may be modified. In this case, our framework would take place in a classical GR limit of quantum gravity (for instance, the size of the initial universe is large) or cosmological setting.

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APPENDIX: LINEAR POTENTIAL CALCULATION

Here we analyze a potential with a small slope, so the field rolls down the potential without a need for any initial push as in Sec. IVA. The potential is given by

$$u(\phi, T) = 2\pi^2 a^6(T)(-b\phi + c), \tag{A1}$$

and the initial condition is similar to the flat potential case,

$$\psi(\phi, -T_0) = \left(\frac{1}{2\pi\sigma_0^2}\right)^{1/4} e^{\frac{-(\phi-\phi_0)^2}{4\sigma_0^2}}. \tag{A2}$$

The wave function that solves the Schrödinger equation in this potential with this initial condition is

$$\begin{aligned}\psi(\phi, T) = \frac{1}{(2\pi\sigma_0^2)^{1/4}} \exp\Bigg\{ & \frac{-\pi^2}{\sigma(T)}\Bigg(\phi - \phi_0 \\ & - ib\int_{-T_0}^{T} \sigma(T')a^6(T')\mathrm{d}T'\Bigg)^2 \\ & - \frac{i}{2}\int_{-T_0}^{T} \frac{1}{\sigma(T')}\Bigg[1 + 4\pi^2\Bigg(c - b\phi_0 \\ & - ib^2\int_{-T_0}^{T'} \sigma(T'')a^6(T'')\mathrm{d}T''\Bigg)\sigma(T')a^6(T')\Bigg]\mathrm{d}T'\Bigg\},\end{aligned} \tag{A3}$$

with the same $\sigma(T)$ as in Eq. (17).

We compute the same probability as in Sec. IVA, except that in this case the wave packet's center moves as

$$\langle\phi\rangle = \phi_0 + \frac{b(T_0 + T)^2}{2}, \tag{A4}$$

and the momentum changes as

$$\langle p \rangle = 2\pi^2 a_0^6 b(T_0 + T), \tag{A5}$$

where we again assume a constant scale factor (set to a_0). The uncertainties, σ_ϕ and σ_p, are the same as for the constant potential.

Compared to the flat potential, here the field velocity increases with time, as

$$\langle \dot{\phi} \rangle = b(t_0 + t). \tag{A6}$$

If one wants the field velocity to remain below some critical value (e.g., a slow roll condition), then this constrains both the potential and the amount of time the field evolves in the potential.

To compute the probability as we did previously, the final state shifted wave packet needs an additional phase to account for a change in the momentum,

$$\exp[2\pi^2 ixa_0^6 b(T_0 + T)\phi], \tag{A7}$$

where x is an arbitrary positive real number scaling the momentum of the wave packet. With $\chi \equiv 8\pi^2\sigma_0^2$ as in Sec. IV A, we find the probability at $T = 0$ to be

$$P = \frac{\chi}{\sqrt{T_0^2 + \chi^2}} \exp\left[-\pi^2\chi a_0^{12} b^2 (x-1)^2 \frac{T_0^2(T_0^2 + \frac{1}{2}\chi^2)}{T_0^2 + \chi^2}\right]. \tag{A8}$$

For large t_0 (again $T_0 = t_0/a_0^3 \gg \chi$), the probability is

$$P \approx \frac{a_0^3\chi}{t_0} \exp[-\pi^2\chi a_0^6 b^2 (x-1)^2 t_0^2], \tag{A9}$$

which falls exponentially fast, unless $x = 1$ (maximizing the probability with respect to x). In this case the wave packets have the same final momentum, and the probability reduces to the result of the flat potential,[8]

$$P \approx \frac{a_0^3\chi}{t_0}. \tag{A10}$$

Therefore, at best the linear potential can have the same probability, proportional to $1/t_0$, as the flat potential.

[8] As a check, setting $b = 0$, so the potential is a constant, also reproduces the result of the previous section.

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