Viable $f(T)$ models are practically indistinguishable from ΛCDM

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We investigate the cosmological predictions of several $f(T)$ models, with up to two parameters, at both the background and the perturbation levels. Using current cosmological observations (geometric supernovae type Ia, cosmic microwave background and baryonic acoustic oscillation and dynamical growth data) we impose constraints on the distortion parameter, which quantifies the deviation of these models from the concordance Λ cosmology at the background level. In addition we constrain the growth index γ predicted in the context of these models using the latest perturbation growth data in the context of three parametrizations for γ. The evolution of the best fit effective Newton constant, which incorporates the $f(T)$-gravity effects, is also obtained along with the corresponding 1σ error regions. We show that all the viable parameter sectors of the $f(T)$ gravity models considered practically reduce these models to ΛCDM. Thus, the degrees of freedom that open up to ΛCDM in the context of $f(T)$ gravity models are not utilized by the cosmological data leading to an overall disfavor of these models.

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I. INTRODUCTION

The ΛCDM model is currently the simplest model consistent with practically all cosmological observations. It assumes homogeneity and isotropy on large cosmological scales and the presence of a cosmological constant Λ in the context of general relativity. Despite its simplicity and its overall consistency with observations, ΛCDM has two weak points:

(1) It requires a theoretically unnatural and fine-tuned value for Λ.
(2) It is marginally consistent with some recent large scale cosmological observations (for instance the cosmic microwave background anomalies).

Motivated by these two weak points, a wide range of more complex generalized cosmological models has been investigated. Most of these models reduce to ΛCDM for specific values of their parameters. They can be classified in two broad classes: Modified gravity models constitute the one class (see for instance [1]), with the other being the scalar field dark energy that adheres to general relativity (see for instance [2,3]). Among the variety of modified gravity theories, $f(T)$ gravity has recently gained a lot of attention. It is based on the old formulation of the teleparallel equivalent of general relativity (TEGR) [4–6]. In teleparallel formulations the dynamical fields are the four linearly independent vierbeins, while one uses the curvatureless Weitzenböck connection instead of the torsionless Levi-Civita one. Thus, one can construct the torsion tensor, which includes all the information concerning the gravitational field, and then by suitable contractions one can write down the corresponding Lagrangian density T [5] (assuming invariance under general coordinate transformations, global Lorentz and parity transformations, and requiring up to second-order terms of the torsion tensor). Finally, $f(T)$ gravity arises as a natural extension of TEGR, if one generalizes the Lagrangian to be a function of T [7–9], inspired by the well-known extension of $f(R)$ Einstein-Hilbert action. However, the significant advantage is that although the curvature tensor contains second-order derivatives of the metric and thus $f(R)$ gravity gives rise to fourth-order equations which may lead to pathologies, the torsion tensor includes only products of first derivatives of the vierbeins, giving rise to second-order field equations.

Although TEGR coincides completely with general relativity both at the background and perturbation levels, $f(T)$ gravity exhibits novel structural and phenomenological features. In particular, imposing a cosmological background one can extract various cosmological solutions, consistent with the observable behavior [7–13]. Additionally, imposing spherical geometry one can investigate the spherical, black-hole solutions of $f(T)$ gravity [14]. However, we stress that although TEGR coincides with GR, $f(T)$ gravity does not coincide with $f(R)$ extension, but it rather constitutes a different class of modified gravity.

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One crucial question is what classes of $f(T)$ extensions are allowed by observations. At the theoretical level, the aforementioned cosmological and spherical solutions lead to a variety of such expressions. However, taking into account observational data, either from cosmological [11,12,15,16] as well as from Solar System observations [13], one can show that the deviations from TEGR must be small.

In the present work we are interested in constraining the $f(T)$ forms using the latest cosmological data, both at the background and perturbation levels. In order to do so we need to define the Hubble parameter as a function of redshift. The issue of using iterative techniques in order to treat the Hubble expansion in $f(R)$ gravity has been proposed by Starobinsky in Ref. [17]. Furthermore, in a recent paper some of us [18] used a new iterative approach in order to observationally constrain deviations of $f(R)$ models from ΛCDM and general relativity. In this context, we first showed that all known viable $f(R)$ models may be written as perturbations around ΛCDM with a deviation parameter we called b (for $b = 0$ these models reduce to ΛCDM). Using a novel perturbative iterative technique we were able to construct analytic cosmological expansion solutions of a highly nonlinear and stiff system of ordinary differential equations and impose cosmological observational constraints on the deviation parameter b.

We also showed that the observationally viable $f(R)$ models effectively include the cosmological constant even though they were proposed as being free from a cosmological constant in the original $f(R)$ papers [17,19]. Inspired by our previous similar work on $f(R)$ gravity [18], we extend it to the case of $f(T)$ gravity models and use the standard joint likelihood analysis of the recent supernovae type Ia data (SnIa), the cosmic microwave background (CMB) shift parameters, the baryonic acoustic oscillations (BAO) and the growth rate data provided by the various galaxy surveys. Based on these cosmological observations we identify the viable range of parameters of five previously proposed $f(T)$ models. Additionally, comparing the resulting analytical expressions of the $f(T)$ Hubble parameter with the numerical solutions at low and intermediate redshifts, we verify that our iterative perturbative technique is highly accurate.

The plan of the work is as follows: In Sec. II we briefly discuss the main properties of the $f(T)$ gravity, while in Sec. III we apply the $f(T)$ gravity in a cosmological framework, providing the relevant equations both at the background and perturbation levels. In Sec. IV we present and we analytically elaborate on all the $f(T)$ models of the literature with two parameters (out of which one is independent). In Sec. V we impose observational constraints, utilizing three parametrizations of the growth index. Finally, the main conclusions are summarized in Sec. VI.

II. $f(T)$ GRAVITY

In this section we briefly review the $f(T)$ gravitational paradigm. In this construction the dynamical variables are the vierbein fields $\mathbf{e}_A(x^\mu)$.[1] The vierbeins at each point x^μ of the manifold form an orthonormal basis for the tangent space, that is, $\mathbf{e}_A \cdot \mathbf{e}_B = \eta_{AB}$, with $\eta_{AB} = \mathrm{diag}(1, -1, -1, -1)$, and they can be expressed in terms of the components e_A^μ in a coordinate basis as $\mathbf{e}_A = e_A^\mu \partial_\mu$. Therefore, the metric tensor is obtained from the dual vierbein through

$$g_{\mu\nu}(x) = \eta_{AB} e_\mu^A(x) e_\nu^B(x). \tag{1}$$

While in usual gravitational formalism one uses the torsionless Levi-Civita connection, in the present formulation one uses the curvatureless Weitzenböck connection defined as $\overset{\mathbf{w}}{\Gamma}{}^\lambda_{\nu\mu} \equiv e_A^\lambda \partial_\mu e_\nu^A$ [20], and the corresponding torsion tensor is written as

$$T^\lambda_{\mu\nu} = \overset{\mathbf{w}}{\Gamma}{}^\lambda_{\nu\mu} - \overset{\mathbf{w}}{\Gamma}{}^\lambda_{\mu\nu} = e_A^\lambda(\partial_\mu e_\nu^A - \partial_\nu e_\mu^A). \tag{2}$$

Furthermore, the contorsion tensor, which provides the difference between Weitzenböck and Levi-Civita connections, is given by $K^{\mu\nu}{}_\rho \equiv -\frac{1}{2}(T^{\mu\nu}{}_\rho - T^{\nu\mu}{}_\rho - T_\rho{}^{\mu\nu})$, while for convenience we define $S_\rho{}^{\mu\nu} \equiv \frac{1}{2}(K^{\mu\nu}{}_\rho + \delta^\mu_\rho T^{\alpha\nu}{}_\alpha - \delta^\nu_\rho T^{\alpha\mu}{}_\alpha)$. Finally, imposing coordinate, Lorentz and parity symmetries, and the additional requirement the Lagrangian to be second order in the torsion tensor [5,6], one obtains the teleparallel Lagrangian (called "torsion scalar" too)

$$T \equiv \frac{1}{4} T^{\rho\mu\nu} T_{\rho\mu\nu} + \frac{1}{2} T^{\rho\mu\nu} T_{\nu\mu\rho} - T_{\rho\mu}{}^\rho T^{\nu\mu}{}_\nu. \tag{3}$$

Thus, in the teleparallel gravitational paradigm, all the information concerning the gravitational field is embedded in the torsion tensor $T^\lambda_{\mu\nu}$, which produces the torsion scalar T in a similar way as the curvature Riemann tensor gives rise to the Ricci scalar in standard general relativity.

In the teleparallel equivalent of general relativity the action is just T. However, one can be inspired by the $f(R)$ extensions of general relativity and extend T to a function $T + f(T)$. Therefore, the corresponding action of $f(T)$ gravity reads as

$$I = \frac{1}{16\pi G_N} \int d^4x e[T + f(T)], \tag{4}$$

where $e = \det(e_\mu^A) = \sqrt{-g}$, G_N is the gravitational constant, and we use units where the light speed is equal to 1. Lastly, TEGR and thus general relativity is restored when

[1] Throughout the manuscript, greek indices $\mu, \nu, \ldots$ and capital latin indices $A, B, \ldots$ run over all coordinate and tangent space-time 0, 1, 2, 3, while lowercase latin indices (from the beginning of the alphabet) $a, b, \ldots$ and lowercase latin indices (from the middle of the alphabet) $i, j, \ldots$ run over tangent-space and spatial coordinates 1, 2, 3, respectively.

$f(T) = 0$, while if $f(T) = \text{const}$ we recover general relativity with a cosmological constant.

III. $f(T)$ COSMOLOGY

We now proceed to the cosmological application of $f(T)$ gravity. In order to construct a realistic cosmology we have to incorporate in the action the matter and the radiation sectors, respectively. Therefore, the total action is written as

$$I = \frac{1}{16\pi G_N} \int d^4xe[T + f(T) + L_m + L_r], \tag{5}$$

where the matter and radiation Lagrangians are assumed to correspond to perfect fluids with energy densities ρ_m, ρ_r and pressures P_m, P_r, respectively.

Secondly, in order to examine a universe governed by $f(T)$ gravity, we have to impose the usual homogeneous and isotropic geometry. Therefore, we consider the common choice for the vierbein form, that is,

$$e^A_\mu = \mathrm{diag}(1, a, a, a), \tag{6}$$

which corresponds to a flat Friedmann-Robertson-Walker (FRW) background geometry with metric

$$ds^2 = dt^2 - a^2(t)\delta_{ij}dx^i dx^j, \tag{7}$$

with $a(t)$ the scale factor.

A. Background behavior

Varying the action (5) with respect to the vierbeins we acquire the field equations

$$\begin{aligned} &e^{-1}\partial_\mu(ee^\rho_A S_\rho{}^{\mu\nu})[1 + f_T] + e^\rho_A S_\rho{}^{\mu\nu}\partial_\mu(T)f_{TT} \\ &\quad - [1 + f_T]e^\lambda_A T^\rho{}_{\mu\lambda}S_\rho{}^{\nu\mu} + \frac{1}{4}e^\nu_A[T + f(T)] \\ &= 4\pi G e^\rho_A \overset{\mathbf{em}}{T}{}_\rho{}^\nu, \end{aligned} \tag{8}$$

where $f_T = \partial f/\partial T$, $f_{TT} = \partial^2 f/\partial T^2$, and $\overset{\mathbf{em}}{T}{}_\rho{}^\nu$ stands for the usual energy-momentum tensor.

Inserting the vierbein choice (6) into the field equations (8) we obtain the modified Friedmann equations

$$H^2 = \frac{8\pi G_N}{3}(\rho_m + \rho_r) - \frac{f}{6} + \frac{Tf_T}{3}, \tag{9}$$

$$\dot{H} = -\frac{4\pi G_N(\rho_m + P_m + \rho_r + P_r)}{1 + f_T + 2Tf_{TT}}, \tag{10}$$

where $H \equiv \dot{a}/a$ is the Hubble parameter, with the dot denoting derivatives with respect to the cosmic time t. We mention that in order to bring the Friedmann equations closer to their standard form, we used the relation

$$T = -6H^2, \tag{11}$$

which through (3) arises straightforwardly for a FRW universe.

Observing the form of the first Friedmann equation (9), and comparing to the usual one, we deduce that in the scenario at hand we obtain an effective dark energy sector of (modified) gravitational origin. In particular, one can define the dark energy density and pressure as [9]

$$\rho_{\mathrm{DE}} \equiv \frac{3}{8\pi G_N}\left[-\frac{f}{6} + \frac{Tf_T}{3}\right], \tag{12}$$

$$P_{\mathrm{DE}} \equiv \frac{1}{16\pi G_N}\left[\frac{f - f_T T + 2T^2 f_{TT}}{1 + f_T + 2Tf_{TT}}\right], \tag{13}$$

while its effective equation-of-state parameter reads

$$w = -\frac{f/T - f_T + 2Tf_{TT}}{[1 + f_T + 2Tf_{TT}][f/T - 2f_T]}. \tag{14}$$

In order to quantitatively elaborate the above modified Friedmann equations, and confront them with observations, we follow the usual procedure. Firstly we define

$$E^2(z) \equiv \frac{H^2(z)}{H_0^2} = \frac{T(z)}{T_0}, \tag{15}$$

where $T_0 \equiv -6H_0^2$. Also, we have used the redshift $z = \frac{a_0}{a} - 1$ as the independent variable and denoted by "0" the current value of a quantity (in the following we set $a_0 = 1$). Thus, using also that $\rho_m = \rho_{m0}(1+z)^3$, $\rho_r = \rho_{r0}(1+z)^4$, we can rewrite the first Friedmann equation (9) as

$$E^2(z, \mathbf{r}) = \Omega_{m0}(1+z)^3 + \Omega_{r0}(1+z)^4 + \Omega_{F0}y(z, \mathbf{r}) \tag{16}$$

with

$$\Omega_{F0} = 1 - \Omega_{m0} - \Omega_{r0}, \tag{17}$$

where $\Omega_{i0} = \frac{8\pi G\rho_{i0}}{3H_0^2}$ is the corresponding density parameter at present. Therefore, the effect of the $f(T)$ gravity is quantified by the function $y(z, \mathbf{r})$ (normalized to unity at present time), which depends on Ω_{m0}, Ω_{r0}, as well as on the $f(T)$-form parameters $r_1, r_2, \ldots$, and it is of the form

$$y(z, \mathbf{r}) = \frac{1}{T_0\Omega_{F0}}[f - 2Tf_T]. \tag{18}$$

According to Eq. (11) the additional term (18) in the effective Friedman equation (16) induced by the $f(T)$ term is a function of the Hubble parameter only. Thus, this term is not completely arbitrary and cannot reproduce any arbitrary expansion history. As we will show further below, the interesting point of the current analysis is that the particular range of degrees of freedom representing deviations from ΛCDM in the context of $f(T)$ models is not favored by cosmological observations.

B. Linear matter perturbations

We now briefly discuss the linear matter perturbations of $f(T)$ gravity. We first review the standard treatment of perturbations for general dark energy or modified gravity scenarios. In this analysis, the extra information is quantified by the effective Newton's gravitational constant, which appears in the various observables such as the growth index. Thus, inserting in these expressions the calculated effective Newton's gravitational constant of $f(T)$ gravity, we obtain the corresponding perturbation observables of $f(T)$ cosmology.

In the framework of any dark energy model, including those of modified gravity (''geometrical dark energy''), it is well known that at the subhorizon scales the dark energy component is expected to be smooth, and thus we can consider perturbations only on the matter component of the cosmic fluid [21]. We refer the reader to Refs. [18,22–27] for full details of the calculation, summarizing only the relevant results in this section.

The basic equation which governs the behavior of the matter perturbations in the linear regime is written as

$$\ddot{\delta}_m + 2H\dot{\delta}_m = 4\pi G_{\rm eff}\rho_m\delta_m, \tag{19}$$

where ρ_m is the matter density and $G_{\rm eff}(a) = G_N Q(a)$, with G_N denoting Newton's gravitational constant. That is, the effect of the modified gravity at the linear perturbation level is reflected in an effective Newton's gravitational constant $G_{\rm eff}(a)$, which in general is evolving. Finally, in the above analysis it has been found that $\delta_m(t) \propto D(t)$, where $D(t)$ is the linear growth factor normalized to unity at present time.

In the case of general-relativity-based scalar-field dark energy models, we obviously have $G_{\rm eff}(a) = G_N$ [that is, $Q(a) = 1$] and therefore (19) reduces to the usual time-evolution equation for the mass density contrast [28]. Moreover, in the case of the usual Λ cosmology, one can solve (19) analytically in order to obtain the growth factor [28]

$$D_\Lambda(z) = \frac{5\Omega_{m0}E_\Lambda(z)}{2}\int_z^{+\infty}\frac{(1+u)du}{E_\Lambda^3(u)}, \tag{20}$$

where

$$E_\Lambda(z) = [\Omega_{m0}(1+z)^3 + 1 - \Omega_{m0}]^{1/2} \tag{21}$$

in the matter dominated era.

In general for either dark energy or modified gravity scenarios, a useful tool that simplifies the numerical calculations significantly is the growth rate of clustering [28]

$$F(a) = \frac{d\ln\delta_m}{d\ln a} \simeq \Omega_m^\gamma(a), \tag{22}$$

where γ is the growth index, which is general evolving. The growth index is very important since it can be used to distinguish between general relativity and modified gravity on cosmological scales. Indeed, for a constant dark energy equation of state parameter w, dark energy scenarios in the framework of general relativity the growth index is well approximated by $\gamma \simeq \frac{3(w-1)}{6w-5}$ [23,29–32], which reduces to $\approx 6/11$ for the concordance Λ cosmology ($w = -1$). On the other hand, for the braneworld model of Dvali, Gabadadze and Porrati [33] the growth index becomes $\gamma \approx 11/16$ [31,34–36], for some $f(R)$ gravity models one acquires $\gamma \simeq 0.415 - 0.21z$ for various parameter values [22,37], while for Finsler-Randers cosmology we have $\gamma \approx 9/14$ [38].

Generally, combining Eq. (19) with the first equality of (22) we obtain

$$a\frac{dF(a)}{da} + F(a)^2 + X(a)F(a) = \frac{3}{2}\Omega_m(a)Q(a), \tag{23}$$

with

$$X(a) = \frac{1}{2} - \frac{3}{2}w(a)[1-\Omega_m(a)], \tag{24}$$

where we have used that [2,3,18,39]

$$w(a) = \frac{-1 - \frac{2}{3}a\frac{d\ln E}{da}}{1-\Omega_m(a)}, \tag{25}$$

$$\Omega_m(a) = \frac{\Omega_{m0}a^{-3}}{E^2(a)}, \tag{26}$$

and

$$\frac{d\Omega_m(a)}{da} = \frac{3}{a}w(a)\Omega_m(a)[1-\Omega_m(a)]. \tag{27}$$

Concerning the functional form of the growth index we consider various situations. The simplest one is to use a constant growth index (hereafter Γ_0 model). If we allow γ to be a function of redshift, then Eq. (23) can be expressed in terms of $\gamma = \gamma(z)$ and it is given by

$$-(1+z)\gamma'\ln(\Omega_m) + \Omega_m^\gamma + 3w(1-\Omega_m)\left(\gamma - \frac{1}{2}\right) + \frac{1}{2} = \frac{3}{2}Q\Omega_m^{1-\gamma}, \tag{28}$$

where a prime denotes derivative with respect to redshift. Writing the above equation at the present epoch ($z = 0$) we have

$$-\gamma'(0)\ln(\Omega_{m0}) + \Omega_{m0}^{\gamma(0)} + 3w_0(1-\Omega_{m0})\left[\gamma(0) - \frac{1}{2}\right] + \frac{1}{2} = \frac{3}{2}Q_0\Omega_{m0}^{1-\gamma(0)}, \tag{29}$$

where $Q_0 = Q(z=0)$ and $w_0 = w(z=0)$.

In this work we consider some well known $\gamma(z)$ functional forms (see [40–43]). These parametrizations are

$$\gamma(z) = \begin{cases} \gamma_0, & \Gamma_0 \text{ model,} \\ \gamma_0 + \gamma_1 z, & \Gamma_1 \text{ model,} \\ \gamma_0 + \gamma_1(1-a), & \Gamma_2 \text{ model.} \end{cases} \tag{30}$$

Inserting the Γ_{1-2} formulas into Eq. (29) one can easily write the parameter γ_1 in terms of γ_0:

$$\gamma_1 = \frac{\Omega_{m0}^{\gamma_0} + 3w_0(\gamma_0 - \frac{1}{2})(1 - \Omega_{m0}) - \frac{3}{2}Q_0\Omega_{m0}^{1-\gamma_0} + \frac{1}{2}}{\ln \Omega_{m0}}. \tag{31}$$

Finally, we would like to stress that the Γ_1 parametrization is valid only at relatively low redshifts $0 \leq z \leq 0.5$. Therefore, in the statistical analysis presented below we utilize a constant growth index, namely, $\gamma = \gamma_0 + 0.5\gamma_1$ for $z > 0.5$.

Since we now have the general perturbation formulation, we just need to insert $G_{\rm eff}(a)$, or equivalently $Q(a)$, of $f(T)$ gravity in the above relations. Unlike the $f(R)$ gravity, the effective Newton's parameter in $f(T)$ gravity is not affected by the scale but rather it takes the following form [44]:

$$Q(a) = \frac{G_{\rm eff}(a)}{G_N} = \frac{1}{1 + f_T}, \tag{32}$$

as it arises from the complete perturbation analysis [45]. The above can be understood, as it was shown in Ref. [46], from the fact that the $f(T)$ cosmological scenario can be rewritten as the K-essence model which implies that since we remain at the Jordan frame we do not expect to have a k dependence in the effective Newton's parameter and thus in the growth factor. However, doing a similar exercise for the $f(R)$ gravity [see Eqs. (8)–(10) in Ref. [47]] one can easily find that it corresponds to a scalar-tensor theory, i.e. a nonminimally coupled scalar field which obviously induces a k dependence in the matter density perturbations. Therefore, in the rest of the work we apply the above analysis in the case of $f(T)$, that is, with $Q(a)$ given by (32).

IV. SPECIFIC $f(T)$ MODELS AND THE DEVIATION FROM ΛCDM

In this section we review all the specific $f(T)$ models that have appeared in the literature, with two parameters out of which one is independent. We calculate the function $y(z, \mathbf{r})$ using (18) and their $G_{\rm eff}(a)$ using (32). We quantify the deviation of the function $y(z, \mathbf{r})$ from its ΛCDM value (constant) through a distortion parameter b. The considered models are as follows.

(1) The power-law model of Bengochea and Ferraro (hereafter f_1CDM) [8], with

$$f(T) = \alpha(-T)^b, \tag{33}$$

where α and b are the two model parameters. Substituting this $f(T)$ form into the modified Friedmann equation (9) at present, we obtain

$$\alpha = (6H_0^2)^{1-b}\frac{\Omega_{F0}}{2b-1}, \tag{34}$$

while (18) gives

$$y(z, b) = E^{2b}(z, b). \tag{35}$$

Additionally, the effective Newton's constant from (32) becomes

$$G_{\rm eff}(z) = \frac{G_N}{1 + \frac{b\Omega_{F0}}{(1-2b)E^{2(1-b)}}}. \tag{36}$$

It is evident that for b strictly equal to zero the f_1CDM model reduces to ΛCDM cosmology, namely, $T + f(T) = T - 2\Lambda$ (where $\Lambda = 3\Omega_{F0}H_0^2$, $\Omega_{F0} = \Omega_{\Lambda 0}$), while for $b = 1/2$ it reduces to the Dvali-Gabadadze-Porrati (DGP) ones [33]. Note that in order to obtain an accelerating expansion, it is required that $b < 1$.

(2) The Linder model (hereafter f_2CDM) [9]

$$f(T) = \alpha T_0(1 - e^{-p\sqrt{T/T_0}}), \tag{37}$$

with α and p the two model parameters. In this case from (9) we find that

$$\alpha = \frac{\Omega_{F0}}{1-(1+p)e^{-p}}, \tag{38}$$

and from (18) we acquire

$$y(z, p) = \frac{1-(1+pE)e^{-pE}}{1-(1+p)e^{-p}}, \tag{39}$$

while from (32) we obtain

$$G_{\rm eff}(z) = \frac{G_N}{1 + \frac{\Omega_{F0}pe^{-pE}}{2E[1-(1+p)e^{-p}]}}. \tag{40}$$

Thus, for $p \to +\infty$ the f_2CDM reduces to ΛCDM cosmology, since

$$\lim_{p\to+\infty}[T + f(T)] = T - 2\Lambda. \tag{41}$$

The parameter p of the present f_2CDM model has a different interpretation comparing to b for the f_1CDM model, since the two models are obviously different. However, since in the limiting case they both reduce to ΛCDM paradigm, we can rewrite the present f_2CDM model replacing $p = 1/b$. In this case (39) leads to

$$y(z, b) = \frac{1-(1+\frac{E}{b})e^{-E/b}}{1-(1+\frac{1}{b})e^{-1/b}}, \tag{42}$$

which indeed tends to unity for $b \to 0^+$.

(3) Motivated by exponential $f(R)$ gravity [48], one can construct the following $f(T)$ model (hereafter f_3CDM):

$$f(T) = \alpha T_0(1 - e^{-pT/T_0}), \tag{43}$$

with α and p the two model parameters. In this case we obtain

$$\alpha = \frac{\Omega_{F0}}{1-(1+2p)e^{-p}}, \tag{44}$$

$$y(z, p) = \frac{1-(1+2pE^2)e^{-pE^2}}{1-(1+2p)e^{-p}}, \tag{45}$$

and

$$G_{\rm eff}(z) = \frac{G_N}{1+\frac{\Omega_{F0}pe^{-pE^2}}{1-(1+2p)e^{-p}}}. \tag{46}$$

Similarly to the previous case we can rewrite f_3CDM model using $p = 1/b$, obtaining

$$y(z, b) = \frac{1-(1+\frac{2E^2}{b})e^{-E^2/b}}{1-(1+\frac{2}{b})e^{-1/b}}. \tag{47}$$

Again, we see that for $p \to +\infty$, or equivalently for $b \to 0^+$, the f_3CDM model tends to the ΛCDM cosmology.

(4) The Bamba *et al.* logarithmic model (hereafter f_4CDM) [49]

$$f(T) = \alpha T_0\sqrt{\frac{T}{qT_0}}\ln\left(\frac{qT_0}{T}\right) \tag{48}$$

with α and q the two model parameters. In this case we obtain

$$\alpha = \frac{\Omega_{F0}\sqrt{q}}{2}, \tag{49}$$

$$y(z) = E(z), \tag{50}$$

and

$$G_{\rm eff}(z) = \frac{G_N}{1+\frac{\Omega_{F0}}{2E}[\ln(\frac{\sqrt{q}}{E}) - 1]}. \tag{51}$$

The fact that the distortion function does not depend on the model parameters allows us to write (16) as

$$E(z) = \frac{1}{2}\sqrt{\Omega_{F0}^2 + 4[\Omega_{m0}(1+z)^3 + \Omega_{r0}(1+z)^4]} + \frac{\Omega_{F0}}{2}. \tag{52}$$

Interestingly enough, from the above relation we deduce that at the background level the f_4CDM model coincides with the flat DGP one (with $\Omega_{F0} = \Omega_{\rm DGP}$), which implies that the two nonstandard gravity models are cosmologically equivalent as far as the cosmic expansion is concerned, in spite of the fact that the two models have a completely different geometrical basis. At the perturbative level, however, we do expect to find differences between f_4CDM and DGP, since $G_{\rm eff}(z)$ evolves differently in two models [in flat DGP gravity we have $\frac{G_{\rm eff}(z)}{G_N} = \frac{2+4\Omega_m^2(z)}{3+3\Omega_m^2(z)}$].
Notice that this model does not give ΛCDM cosmology for any value of its parameters. However, in this work we are interested in the viable $f(T)$, in the sense that these $f(T)$ models can describe the matter and dark energy eras as well as they are consistent with the observational data (including Solar System tests), and finally they have stable perturbations. Although these necessary analysis have not yet been performed for all the above $f(T)$ models, a failure of a particular model to pass one of these is enough to exclude it. Therefore, since the present f_4CDM model coincides with DGP at the background level, it inherits its disadvantages concerning the confrontation with observations. Thus, as anticipated from previous studies [50], we verify in the following section that this model is nonviable when tested using the latest cosmological observations.

(5) The hyperbolic-tangent model (hereafter f_5CDM) [51]

$$f(T) = \alpha(-T)^n \tanh\left(\frac{T_0}{T}\right) \tag{53}$$

with α and n the two model parameters. In this case we obtain

$$\alpha = -\frac{\Omega_{F0}(6H_0)^{1-n}}{[2{\rm sech}^2(1) + (1-2n)\tanh(1)]}, \tag{54}$$

$$y(z, n) = E^{2(n-1)}\frac{2{\rm sech}^2(\frac{1}{E^2}) + (1-2n)E^2\tanh(\frac{1}{E^2})}{2{\rm sech}^2(1) + (1-2n)\tanh(1)}, \tag{55}$$

and

$$G_{\rm eff}(z) = \frac{G_N}{1+\frac{\Omega_{F0}E^{2(n-2)}[nE^2\tanh(\frac{1}{E^2}) - {\rm sech}^2(\frac{1}{E^2})]}{2{\rm sech}^2(1)+(1-2n)\tanh(1)}}. \tag{56}$$

The f_5CDM model does not give ΛCDM cosmology for any value of its parameters. However, as we show in the next section, this model is in mild tension with the data as it has a best fit $\chi^2_{\rm min} = (579.583, 580.723, 578.027)$ for the Γ_0, Γ_1 and Γ_2 growth rate parameterizations, respectively, which is significantly larger than that of f_{1-3}CDM and ΛCDM models, respectively (see Table I). Additionally the current $f(T)$ model has one more free parameter. For the reasons developed above we consider it as nonviable (see also akaike information criterion (AIC) test in Table I.)

The above five $f(T)$ forms are the ones that have been used in the literature of $f(T)$ cosmology, possessing up to two parameters, out of which one is independent. Clearly, in principle one could additionally consider their combinations too; however, the appearance of many free parameters would be a significant disadvantage. Therefore, in the present work we focus only on these five elementary *Ansätze*.

As we showed, for the first three the distortion parameter measures the smooth deviation from the ΛCDM model. The other two models do not have ΛCDM cosmology as a limiting case; however, as we show in the next section, they are in tension with observations. Thus, in the rest of this section we focus on the first three models, namely, on f_{1-3}CDM ones.

Having performed the above elaboration of various $f(T)$ models, we can now follow the procedure and iterative techniques of Basilakos, Nesseris, and Perivolaropoulos [18], in which we have shown that all the observationally viable $f(R)$ parameterizations can be expressed as perturbations deviating from ΛCDM cosmology.

For the f_1CDM model there are two different, but complementary, ways we can find analytical approximations for the Hubble parameter. The first method involves doing a Taylor expansion of $E^2(z,b)$ around $b=0$, while in the second we perform the Taylor expansion in the modified Friedman equation directly. Below, we briefly review and test both methods, called M_1 and M_2, respectively.

First, from (16) with (35) we can write explicitly the Hubble parameter for the f_1CDM model as

$$E^2(z,b) = \Omega_{m0}(1+z)^3 + \Omega_{r0}(1+z)^4 + \Omega_{F0}y(z,b), \quad (57)$$

where

$$y(z,b) = E^{2b}(z,b). \quad (58)$$

Obviously, in Eq. (57) if we set b strictly equal to zero, then we get the Hubble parameter for the ΛCDM model

$$E^2(z,0) = \Omega_{m0}(1+z)^3 + \Omega_{r0}(1+z)^4 + \Omega_{F0} \equiv E^2_\Lambda(z). \quad (59)$$

Now, performing a Taylor expansion, up to second order, on $E^2(z,b)$ around $b=0$ and with the help of (57) we arrive at

$$\begin{aligned} E^2(z,b) &= E^2(z,0) + \frac{dE^2(z,b)}{db}\bigg|_{b=0} b + \frac{d^2E^2(z,b)}{db^2}\bigg|_{b=0}\frac{b^2}{2} + \cdots \\ &= E^2_\Lambda(z) + \Omega_{F0}\frac{dy(z,b)}{db}\bigg|_{b=0} b + \Omega_{F0}\frac{d^2y(z,b)}{db^2}\bigg|_{b=0}\frac{b^2}{2} + \cdots. \end{aligned} \quad (60)$$

TABLE I. Statistical results of the overall likelihood analysis: The first column indicates the $f(T)$ model, the second column the $\gamma(z)$ parametrizations appearing in Sec. III A, the third and fourth columns provide the Ω_{m0} and b best values, and the fifth and sixth columns show the γ_0 and γ_1 best fit values. In all cases we have used $\sigma_8 = 0.8$. The last three columns present the goodness-of-fit statistics ($\chi^2_{\rm min}$, AIC and $|\Delta\text{AIC}| = |\text{AIC}_\Lambda - \text{AIC}_{f(T)}|$). All the error estimates come from the inverse of the Fisher matrix, called the covariance matrix, and are by definition symmetric.

Expansion model	Parametrization model	Ω_{m0}	b	γ_0	γ_1	$\chi^2_{\rm min}$	AIC	$\lvert\Delta\text{AIC}\rvert$
ΛCDM	Γ_0	0.272 ± 0.003		0.597 ± 0.046	0	574.227	578.227	0
	Γ_1	0.272 ± 0.003		0.567 ± 0.066	0.116 ± 0.191	573.861	579.861	1.634
	Γ_2	0.272 ± 0.003		0.561 ± 0.068	0.183 ± 0.269	573.767	579.767	1.540
f_1CDM [8]:	Γ_0	0.274 ± 0.008	-0.017 ± 0.083	0.602 ± 0.052	0	574.203	580.203	1.976
	Γ_1	0.275 ± 0.008	-0.029 ± 0.088	0.558 ± 0.067	0.187 ± 0.205	573.817	581.817	3.590
	Γ_2	0.275 ± 0.008	-0.030 ± 0.089	0.564 ± 0.069	0.213 ± 0.287	573.640	581.640	3.413
f_2CDM [9]:	Γ_0	0.272 ± 0.004	0.121 ± 0.184	0.596 ± 0.047	0	574.250	580.250	2.023
	Γ_1	0.272 ± 0.003	0.086 ± 0.301	0.566 ± 0.066	0.116 ± 0.191	573.863	581.863	3.636
	Γ_2	0.272 ± 0.003	0.078 ± 0.375	0.561 ± 0.068	0.183 ± 0.269	573.768	581.768	3.541
f_3CDM [48]:	Γ_0	0.273 ± 0.003	0.097 ± 0.155	0.597 ± 0.046	0	574.223	580.223	1.996
	Γ_1	0.273 ± 0.003	0.010 ± 0.324	0.570 ± 0.067	0.099 ± 0.192	573.852	581.852	3.625
	Γ_2	0.273 ± 0.003	0.024 ± 0.183	0.562 ± 0.068	0.185 ± 0.269	573.749	581.749	3.522
f_4CDM [49]:	Γ_0	0.202 ± 0.002		0.417 ± 0.031	0	704.481	708.481	130.254
	Γ_1	0.202 ± 0.002		0.468 ± 0.053	-0.171 ± 0.136	702.865	708.865	130.638
	Γ_2	0.202 ± 0.002		0.467 ± 0.052	-0.224 ± 0.134	703.047	709.047	130.820
f_5CDM [51]:	Γ_0	0.283 ± 0.006	0.226 ± 0.066	0.567 ± 0.049	0	579.583	585.583	7.356
	Γ_1	0.277 ± 0.006	0.298 ± 0.049	0.550 ± 0.065	0.099 ± 0.191	580.723	588.723	10.496
	Γ_2	0.287 ± 0.007	0.193 ± 0.074	0.570 ± 0.070	0.263 ± 0.298	578.027	586.027	7.800

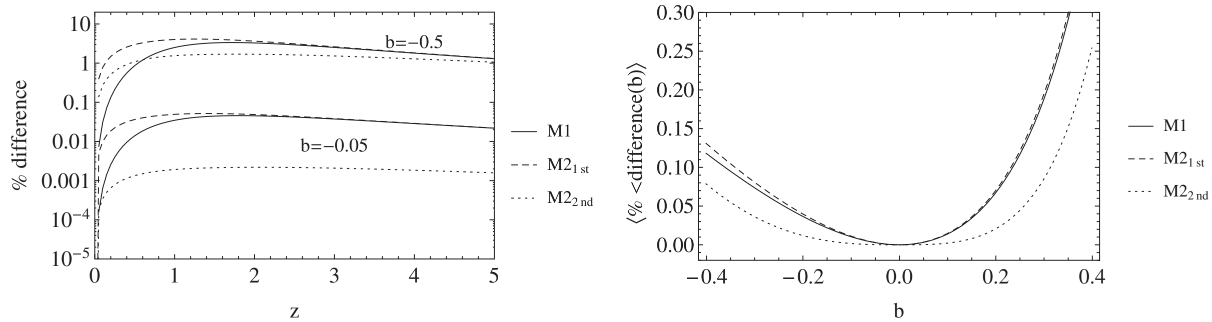

FIG. 1. Left: The percent difference (z, b) between the numerical solution of Eqs. (16) and (35) and the analytical approximations of Eqs. (64) and (65) as a function of z, for various values of the parameter b for both methods M_1 (at first and second order) and M_2. Right: The average percent difference $\langle \text{difference}(b)\rangle$ between the numerical solution of Eqs. (16) and (35) and the analytical approximations of Eqs. (64) and (65) as a function of the parameter b. In this case, the average over the redshift is taken in the range $z \in [0, 100]$.

The terms involving the derivatives of $y(z, b)$ can readily be calculated from Eq. (58) as

$$\frac{dy(z,b)}{db} = 2E(z,b)^{2b}\left\{\frac{b}{E(z,b)}\frac{dE(z,b)}{db} + \ln[E(z,b)]\right\}, \quad (61)$$

and evaluating the above equation for $b = 0$ we have

$$\frac{dy(z,b)}{db}\bigg|_{b=0} = 2\ln[E(z,0)] = \ln[E_\Lambda^2(z)]. \quad (62)$$

Similarly for the second derivative term we have

$$\frac{d^2y(z,b)}{db^2}\bigg|_{b=0} = \frac{2\Omega_{F0}\ln[E_\Lambda^2(z)]}{E_\Lambda^2(z)} + \ln[E_\Lambda^2(z)]^2. \quad (63)$$

Thus, the Taylor expansion up to second order for the first method M_1 becomes

$$E^2(z,b) = E_\Lambda^2(z) + \Omega_{F0}\ln[E_\Lambda^2(z)]b + \Omega_{F0}\left\{\frac{2\Omega_{F0}\ln[E_\Lambda^2(z)]}{E_\Lambda^2(z)} + \ln[E_\Lambda^2(z)]^2\right\}\frac{b^2}{2} + \cdots. \quad (64)$$

The second method M_2 involves performing a Taylor expansion in the modified Friedman equation (57) directly. For the details in this case we refer the interested reader to the Appendix and just present the result here:

$$E^2(z,b) = -b\Omega_F \mathcal{W}_k\left(-\frac{e^{-\frac{E_\Lambda(z)^2}{b\Omega_F}}}{b\Omega_F}\right), \quad (65)$$

where $\mathcal{W}_k(\omega)$ is the Lambert function defined via $\omega \equiv \mathcal{W}_k(\omega)e^{\mathcal{W}_k(\omega)}$ for all complex numbers ω. The Lambert function has branch-cut discontinuities, so the different branches are indicated by the integer k. Our solution has $k = 0$ (the principal branch) for $b \leq 0$ and $k = -1$ for $b > 0$.[2]

[2]The Lambert function $\mathcal{W}_k(\omega)$ is defined in MATHEMATICA as ProductLog[k, ω] and can be evaluated to arbitrary precision for integer values of k and real or complex values of ω.

In order to examine the accuracy of the approximations of (64) and (65), we calculate the average percent deviation from the exact numerical solution of (57), defined as

$$\langle \text{difference}(b)\rangle = \left\langle 100 \cdot \left(1 - \frac{E^2_{\text{approx}}(z,b)}{E^2_{\text{numeric}}(z,b)}\right)\right\rangle, \quad (66)$$

where the average is taken over redshifts in the range $z \in [0, 100]$. In Fig. 1 we show the corresponding results. In particular, on the left plot we show the percent difference between the numerical solution of Eqs. (16) and (35) and the analytical approximations of Eqs. (64) and (65) as a function of z, for various values of the parameter b for both methods M_1, at first (dashed line) and second order (dotted line) and M_2 (solid black line). As it can be seen, at redshifts $z \lesssim 2$ method M_2 is significantly better than the first-order M_1, but overall, obviously the second-order method M_2 is much better than the other two.

On the right plot we show the average percent difference $\langle \text{difference}(b)\rangle$ between the numerical solution of Eqs. (16) and (35) and the analytical approximations of Eqs. (64) and (65) as a function of the parameter b. In this case, the average over the redshift is taken in the range $z \in [0, 100]$. Clearly, on average the second-order method is significantly better than the other two methods, the first-order M_1 and the M_2. Thus, we conclude that the second-order series expansion of Eq. (64) around ΛCDM for the f_1CDM model is a very good approximation, especially for realistic values of the parameter b.

Unfortunately, for the f_2CDM and f_3CDM models it is not possible to analytically obtain similar expressions, due to the presence of terms like $\sim e^{-1/b}$, which do not admit a Taylor expansion around $b \sim 0$. However, as mentioned earlier, they both have the ΛCDM model as a limit for $b \to 0^+$.

V. OBSERVATIONAL CONSTRAINTS

In this section we perform a complete and detailed observational analysis of the above five $f(T)$ models. In particular, we implement a joint statistical analysis with the appropriate Akaike information criterion [52], involving the latest expansion data (SnIa [53], BAO [54,55] and the 9-year WMAP CMB shift parameter [56]) and the growth data (as collected by [18]). The likelihood analysis, the Akaike information criterion, the expansion data, the growth data and the corresponding covariances can be found in Table I and Sec. IV of our previous work [18]. Moreover, we mention that since in order to deal with the growth data we need to know the value of σ_8, which is the rms mass fluctuation on $R_8 = 8h^{-1}$ Mpc scales at redshift $z = 0$, we treat σ_8 either as $\sigma_8 = 0.8$ or as a free parameter. This analysis is significantly improved, comparing to previous observational constraining of $f(T)$ gravity [11,15,49,51].

Let us now provide a presentation of our statistical results. In Table I we give the resulting best fit parameters for the various $f(T)$ models under study (we impose here $\sigma_8 = 0.8$), in which we also show the corresponding quantities for ΛCDM for comparison.

It is clear that utilizing the combination of the most recent growth data set with the expansion cosmological data, we can put tight constraints on (Ω_m, γ). In all cases the best fit value $\Omega_m = 0.272 \pm 0.003$ is in a very good agreement with the one found by WMAP9 + SPT + ACT, that is, $\Omega_m = 0.272$ [56].

In particular, we find the following.

(a) Γ_0 *parametrization.*—Regarding the ΛCDM cosmological model our best fit value growth is $\gamma = 0.597 \pm 0.046$ that is in a good agreement with previous studies [18,57–61]. Concerning the $f(T)$ models we obtain $(\gamma, b) = (0.602 \pm 0.052, -0.017 \pm 0.083)$, $(\gamma, b) = (0.596 \pm 0.047, 0.121 \pm 0.184)$ and $(\gamma, b) = (0.597 \pm 0.046, 0.097 \pm 0.155)$ for the f_1CDM, f_2CDM and f_3CDM models, respectively, with a reduced $\chi^2_{\rm min}$ of $\sim$574.2. In Fig. 2 we show the 1σ, 2σ and 3σ confidence contours in the (Ω_m, b) plane, while in Fig. 3 we present the corresponding contours in the (Ω_m, γ) plane.

(b) Γ_1 *parametrization.*—In the case of the concordance Λ cosmology we find $\gamma_0 = 0.567 \pm 0.066$ and $\gamma_1 = 0.116 \pm 0.191$ with $\chi^2_{\rm min} \simeq 573.861$ which are in agreement with previous studies [18,32,59,62,63]. For the f_1CDM, f_2CDM and f_3CDM models the corresponding likelihood functions peak at $(b, \gamma_0, \gamma_1) = (-0.029 \pm 0.088, 0.558 \pm 0.067, 0.187 \pm 0.205)$ with $\chi^2_{\rm min} \simeq 573.817$, $(b, \gamma_0, \gamma_1) = (0.086 \pm 0.301, 0.566 \pm 0.066, 0.116 \pm 0.191)$ with $\chi^2_{\rm min} \simeq 573.863$ and $(b, \gamma_0, \gamma_1) = (0.010 \pm 0.324, 0.570 \pm 0.067, 0.099 \pm 0.192)$ with $\chi^2_{\rm min} \simeq 573.852$, respectively. In Fig. 4 we present the corresponding 1σ, 2σ and 3σ contours in the (γ_0, γ_1) plane.

(c) Γ_2 *parametrization.*—In the case of ΛCDM model we have $\gamma_0 = 0.561 \pm 0.068$, $\gamma_1 = 0.183 \pm 0.269$ ($\chi^2_{\rm min} \simeq 573.767$), while for the f_1CDM we obtain $b = -0.030 \pm 0.089$, $\gamma_0 = 0.564 \pm 0.069$, $\gamma_1 = 0.213 \pm 0.287$ ($\chi^2_{\rm min} \simeq 573.640$), for the f_2CDM gravity model we find $b = 0.150 \pm 0.096$, $\gamma_0 = 0.560 \pm 0.068$, $\gamma_1 = 0.181 \pm 0.271$ ($\chi^2_{\rm min} \simeq 573.921$) and finally for the f_3CDM model we have we find $b = 0.024 \pm 0.183$, $\gamma_0 = 0.562 \pm 0.068$, $\gamma_1 = 0.185 \pm 0.269$ ($\chi^2_{\rm min} \simeq 573.749$). In Fig. 5 we present the corresponding 1σ, 2σ and 3σ contours in the (γ_0, γ_1) plane.

We stress here that in all three previous $f(T)$ models, namely, f_{1-3}CDM ones, the parameter b which quantifies the deviation from ΛCDM cosmology is constrained in a very narrow window around 0. Thus, although these three models are consistent with observations, their viable forms are practically indistinguishable from ΛCDM and therefore their new degrees of freedom are disfavored by data.

Finally, in Fig. 6 we show the likelihood contours for f_4CDM model, which as discussed in Sec. IV coincides

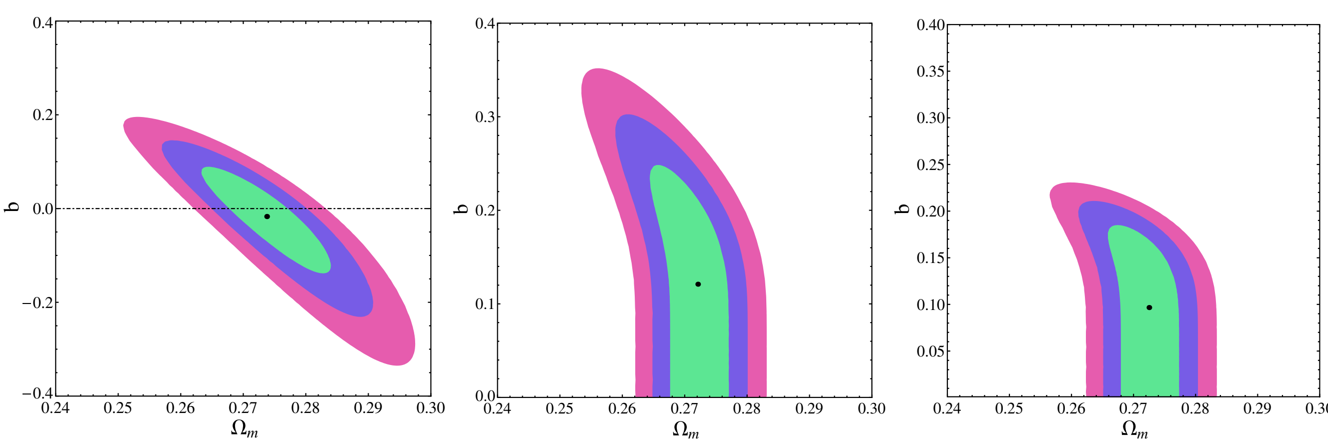

FIG. 2 (color online). Likelihood contours for $\delta\chi^2 \equiv \chi^2 - \chi^2_{\rm min}$ equal to 2.30, 6.18 and 11.83, corresponding to 1σ, 2σ and 3σ confidence levels, in the (Ω_m, b) plane for the Γ_0 growth rate parametrization and the f_1CDM (left), f_2CDM (middle) and f_3CDM (right) models. In all cases the black point corresponds to the best fit. In this plot and in the ones that follow we have set the parameters that are not shown to their best fit values for the corresponding model (see Table I).

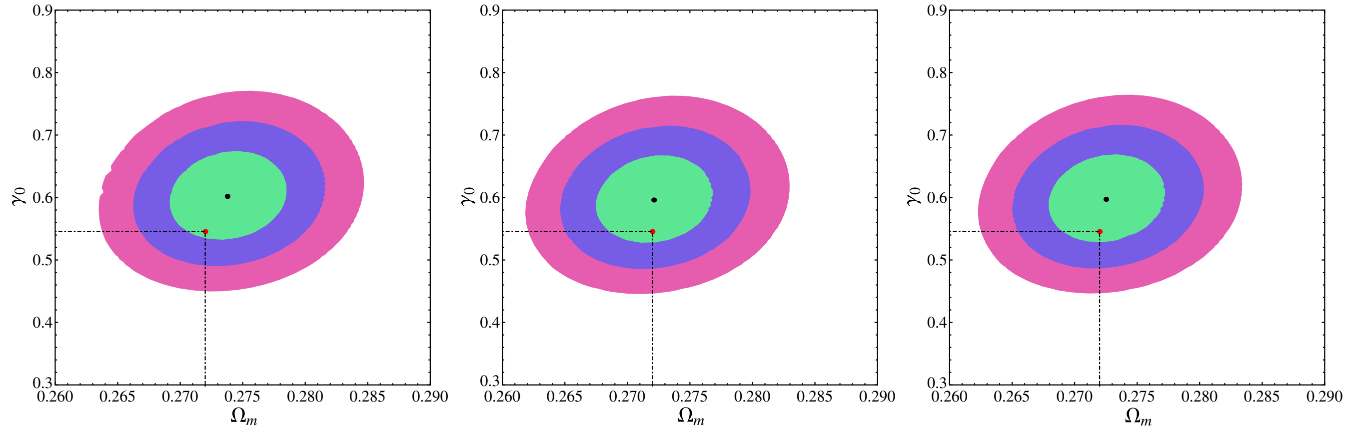

FIG. 3 (color online). Likelihood contours for $\delta\chi^2 \equiv \chi^2 - \chi^2_{\rm min}$ equal to 2.30, 6.18 and 11.83, corresponding to 1σ, 2σ and 3σ confidence levels, in the (Ω_m, γ) plane for the Γ_0 growth rate parametrization and the f_1CDM (left), f_2CDM (middle) and f_3CDM (right) models. In all cases the red point corresponds to $(\Omega_m, \gamma) = (0.272, 6/11)$.

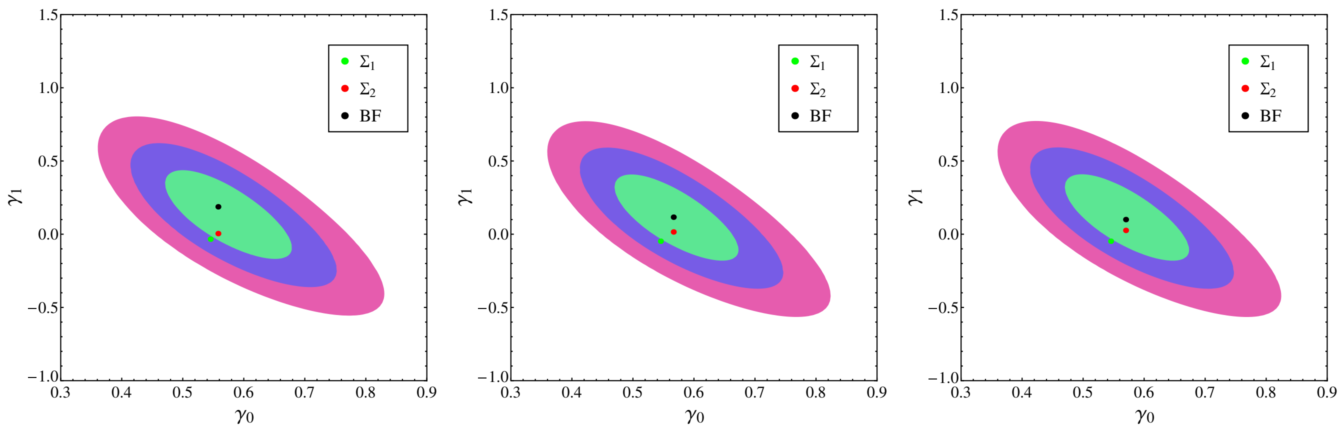

FIG. 4 (color online). Likelihood contours for $\delta\chi^2 \equiv \chi^2 - \chi^2_{\rm min}$ equal to 2.30, 6.18 and 11.83, corresponding to 1σ, 2σ and 3σ confidence levels, in the (γ_0, γ_1) plane for the Γ_1 growth rate parametrization and for the f_1CDM (left), f_2CDM (middle) and f_3CDM (right) models. We also include the theoretical ΛCDM (γ_0, γ_1) values given by $\Sigma_1 = (6/11, \gamma_1(6/11, \Omega_{m0,bf}))$ and $\Sigma_2 = (\gamma_{0,bf}, \gamma_1(\gamma_{0,bf}, \Omega_{m0,bf}))$.

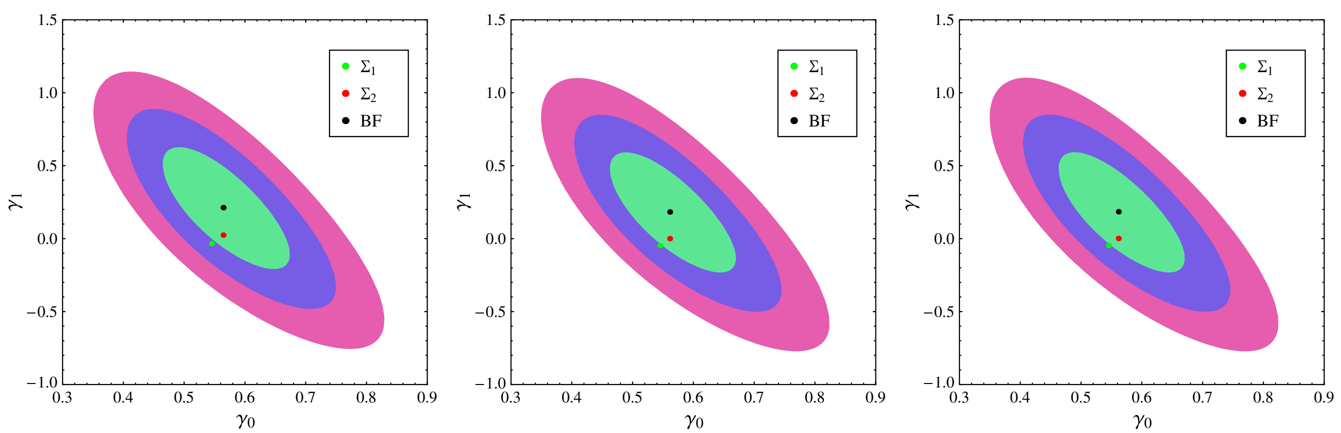

FIG. 5 (color online). Likelihood contours for $\delta\chi^2 \equiv \chi^2 - \chi^2_{\rm min}$ equal to 2.30, 6.18 and 11.83, corresponding to 1σ, 2σ and 3σ confidence levels, in the (γ_0, γ_1) plane for the Γ_2 growth rate parametrization and for the f_1CDM (left), f_2CDM (middle) and f_3CDM (right) models. We also include the theoretical ΛCDM (γ_0, γ_1) values given by $\Sigma_1 = (6/11, \gamma_1(6/11, \Omega_{m0,bf}))$ and $\Sigma_2 = (\gamma_{0,bf}, \gamma_1(\gamma_{0,bf}, \Omega_{m0,bf}))$.

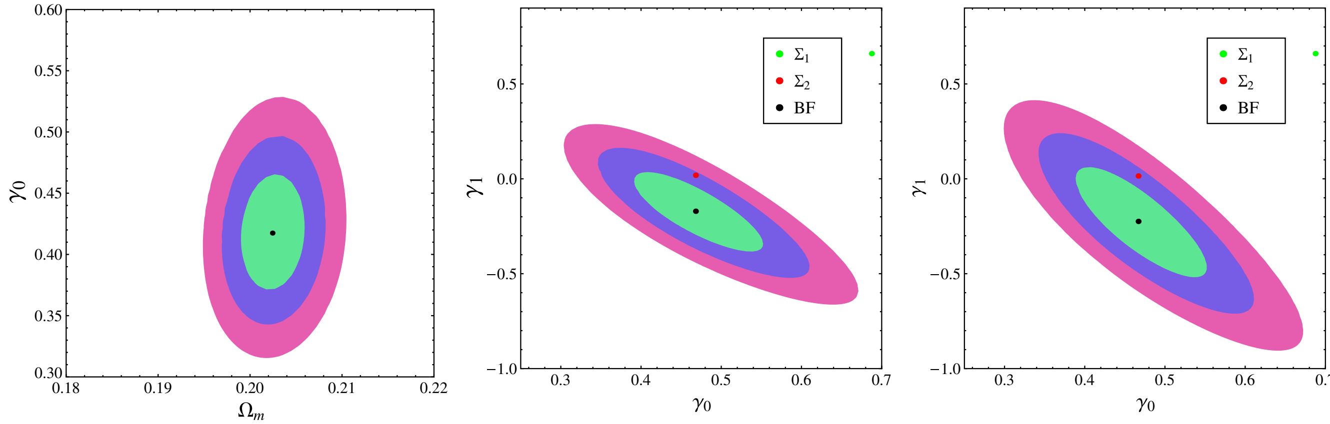

FIG. 6 (color online). Likelihood contours for $\delta\chi^2 \equiv \chi^2 - \chi^2_{\min}$ equal to 2.30, 6.18 and 11.83, corresponding to 1σ, 2σ and 3σ confidence levels, for the f_4CDM model in the (Ω_m, γ_0) plane (left) and the (γ_0, γ_1) plane (middle) and (right). We also include the theoretical ΛCDM (γ_0, γ_1) values given by $\Sigma_1 = (11/16, \gamma_1(11/16, \Omega_{m0,bf}))$ (with the value $\gamma_0 = 11/16$ corresponding to the DGP) and $\Sigma_2 = (\gamma_{0,bf}, \gamma_1(\gamma_{0,bf}, \Omega_{m0,bf}))$. As was mentioned in the text, the difference between the DGP (green point) and f_4CDM (black point) is due to the different $G_{\rm eff}(z)$, which affects the evolution of the matter density perturbations.

with DGP at the background level, and thus it shares its observational disadvantages and therefore we consider it as nonviable. In the same lines, as we can see from Table I, for f_5CDM model we obtain the best fits $\chi^2_{\min} = (579.583, 580.723, 578.027)$ for the Γ_0, Γ_1 and Γ_2 growth-rate parameterizations, respectively, while it additionally has one more free parameter than ΛCDM. Thus, this model is in tension with the data.

For completeness, in Figs. 7–9 we present a comparison of the observed and theoretical evolution of the growth rate $f\sigma_8(z) = F(z)\sigma_8(z)$, the evolution of the growth index $\gamma(z) - \frac{6}{11}$ and the evolution of the $G_{\rm eff}(z)$, respectively.

Finally, in order to enhance the validity of the above results, we repeat the whole analysis by using σ_8 as a free parameter. As expected, we find that the corresponding results are in good agreement, within 1σ, with those of $\sigma_8 = 0.8$ (see Table I). In particular, we find the following.

In the case of the ΛCDM,

(i) for the Γ_0 model: $\chi^2 = 573.254$, $\Omega_m = 0.272 \pm 0.003$, $\gamma_0 = 0.523 \pm 0.0858$, $\sigma_8 = 0.761 \pm 0.038$;

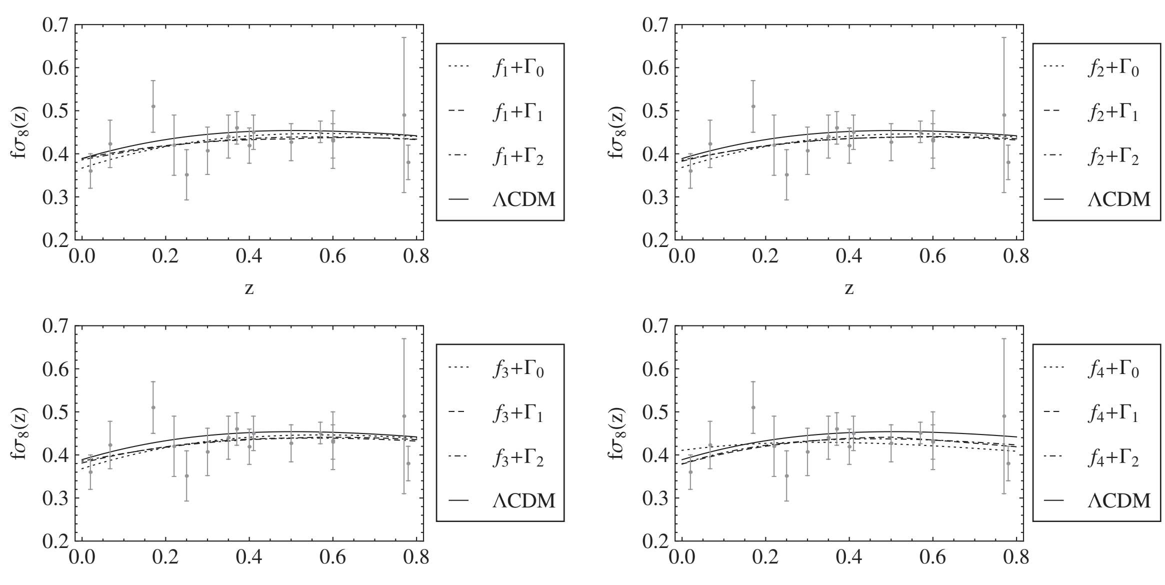

FIG. 7. Comparison of the observed and theoretical evolution of the growth rate $f\sigma_8(z) = F(z)\sigma_8(z)$ for the f_{1-4}CDM models [f_1CDM (top left), f_2CDM (top right), f_3CDM (bottom left), f_4CDM (bottom right)] and the various growth rate parameterizations. The dotted, dashed and dot-dashed lines correspond to the best fit Γ_0, Γ_1 and Γ_2 parametrizations while the solid black line corresponds to the exact solution of Eq. (19) for $f\sigma_8(z)$ for the ΛCDM model for $\Omega_m = 0.273$ [56].

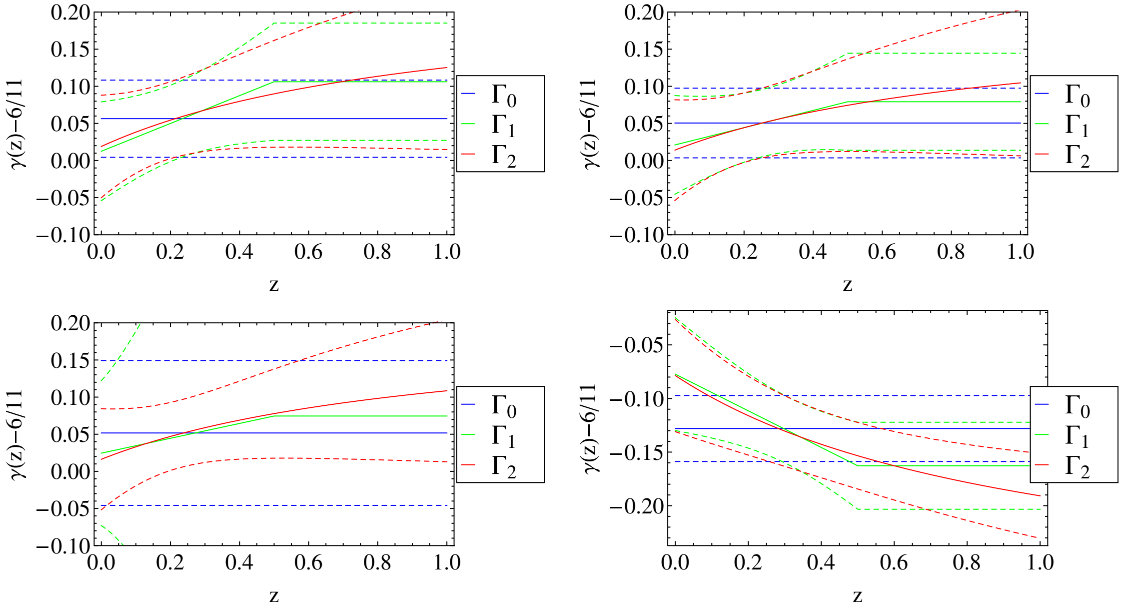

FIG. 8 (color online). The evolution of the growth index $\gamma(z) - \frac{6}{11}$ for the $f_{1\text{–}4}$CDM models [f_1CDM (top left), f_2CDM (top right), f_3CDM (bottom left), f_4CDM (bottom right)] and the various growth rate parameterizations. The lines correspond to Γ_0 (blue), Γ_1 (green), and Γ_2 (red).

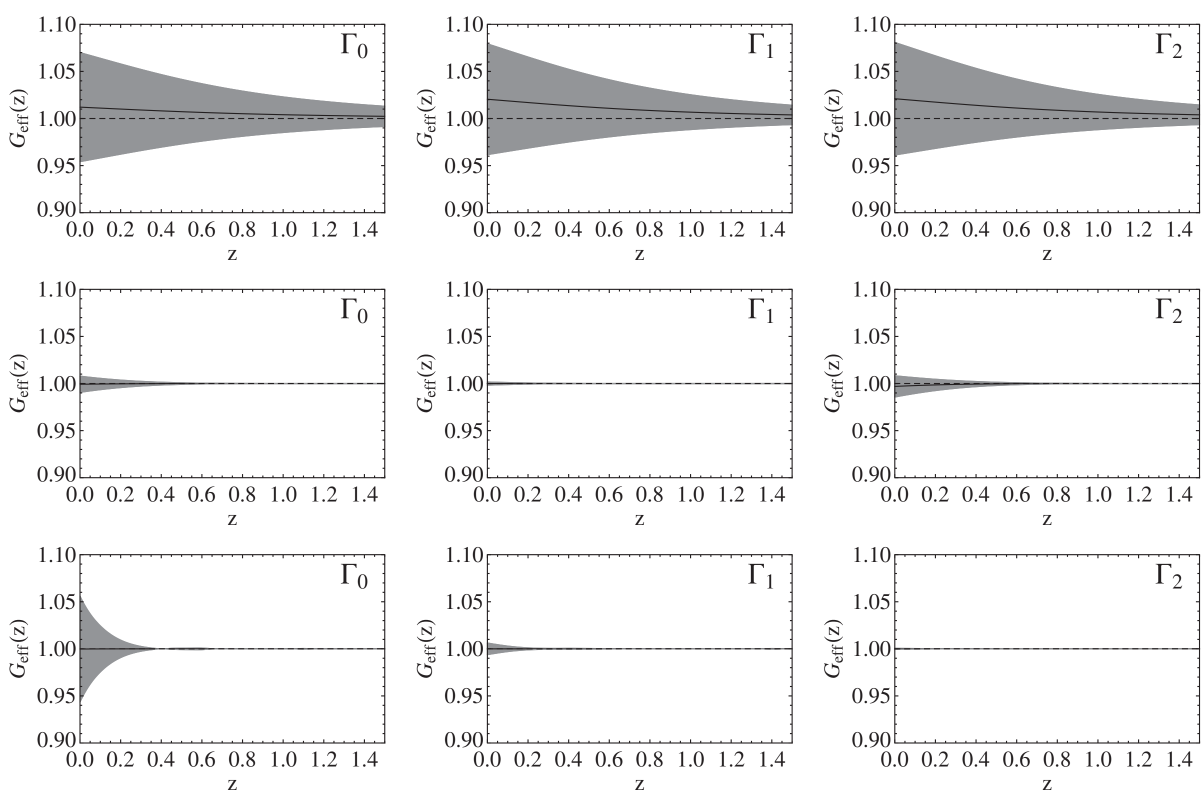

FIG. 9. The evolution of the $G_{\mathrm{eff}}(z)$ for the $f_{1\text{–}3}$CDM models and the various growth rate parameterizations considered in the text, f_1CDM (top), f_2CDM (middle), f_3CDM (bottom), for all three growth rate parametrizations Γ_0 (left), Γ_1 (middle), and Γ_2 (right). The remarkable agreement between $G_{\mathrm{eff}}(z)$ and unity for the f_2CDM and f_3CDM models is easily explained by the fact that these models exhibit little deviation from ΛCDM, as is easily seen in Table I.

(ii) for the Γ_1 model: $\chi^2 = 572.618$, $\Omega_m = 0.272 \pm 0.003$, $\gamma_0 = 0.485 \pm 0.098$, $\gamma_1 = -0.398 \pm 0.502$, $\sigma_8 = 0.694 \pm 0.087$;

(iii) for the Γ_2 model: $\chi^2 = 572.652$, $\Omega_m = 0.272 \pm 0.003$, $\gamma_0 = 0.483 \pm 0.097$, $\gamma_1 = -0.633 \pm 0.815$, $\sigma_8 = 0.685 \pm 0.097$.

In the case of the f_1CDM,

(i) for the Γ_0 model: $\chi^2 = 573.618$, $\Omega_m = 0.274 \pm 0.008$, $b = -0.019 \pm 0.087$, $\gamma_0 = 0.586 \pm 0.090$, $\sigma_8 = 0.783 \pm 0.041$;

(ii) for the Γ_1 model: $\chi^2 = 576.124$, $\Omega_m = 0.281 \pm 0.009$, $b = -0.099 \pm 0.109$, $\gamma_0 = 0.582 \pm 0.092$, $\gamma_1 = 0.680 \pm 0.443$, $\sigma_8 = 0.752 \pm 0.070$;

(iii) for the Γ_2 model: $\chi^2 = 573.756$, $\Omega_m = 0.281 \pm 0.008$, $b = -0.098 \pm 0.104$, $\gamma_0 = 0.569 \pm 0.103$, $\gamma_1 = 0.077 \pm 0.872$, $\sigma_8 = 0.774 \pm 0.114$.

In the case of the f_2CDM,

(i) for the Γ_0 model: $\chi^2 = 573.264$, $\Omega_m = 0.272 \pm 0.003$, $b = 0.101 \pm 0.186$, $\gamma_0 = 0.523 \pm 0.086$, $\sigma_8 = 0.762 \pm 0.038$;

(ii) for the Γ_1 model: $\chi^2 = 572.618$, $\Omega_m = 0.272 \pm 0.003$, $b = 0.052 \pm 2.833$, $\gamma_0 = 0.485 \pm 0.098$, $\gamma_1 = -0.398 \pm 0.502$, $\sigma_8 = 0.694 \pm 0.087$;

(iii) for the Γ_2 model: $\chi^2 = 572.817$, $\Omega_m = 0.272 \pm 0.003$, $b = 0.040 \pm 10.476$, $\gamma_0 = 0.500 \pm 0.113$, $\gamma_1 = -0.599 \pm 1.022$, $\sigma_8 = 0.699 \pm 0.127$.

In the case of the f_3CDM,

(i) for the Γ_0 model: $\chi^2 = 573.224$, $\Omega_m = 0.273 \pm 0.003$, $b = 0.050 \pm 2.561$, $\gamma_0 = 0.523 \pm 0.086$, $\sigma_8 = 0.761 \pm 0.038$;

(ii) for the Γ_1 model: $\chi^2 = 572.599$, $\Omega_m = 0.273 \pm 0.003$, $b = 0.051 \pm 2.264$, $\gamma_0 = 0.485 \pm 0.098$, $\gamma_1 = -0.398 \pm 0.502$, $\sigma_8 = 0.694 \pm 0.087$;

(iii) for the Γ_2 model: $\chi^2 = 572.636$, $\Omega_m = 0.273 \pm 0.003$, $b = 0.039 \pm 4.180$, $\gamma_0 = 0.486 \pm 0.098$, $\gamma_1 = -0.598 \pm 0.817$, $\sigma_8 = 0.688 \pm 0.098$.

In the case of the f_4CDM,

(i) for the Γ_0 model: $\chi^2 = 703.539$, $\Omega_m = 0.202 \pm 0.002$, $\gamma_0 = 0.490 \pm 0.083$, $\sigma_8 = 0.856 \pm 0.061$;

(ii) for the Γ_1 model: $\chi^2 = 702.419$, $\Omega_m = 0.202 \pm 0.002$, $\gamma_0 = 0.399 \pm 0.113$, $\gamma_1 = -0.418 \pm 0.401$, $\sigma_8 = 0.703 \pm 0.134$;

(iii) for the Γ_2 model: $\chi^2 = 702.501$, $\Omega_m = 0.202 \pm 0.002$, $\gamma_0 = 0.379 \pm 0.123$, $\gamma_1 = -0.733 \pm 0.713$, $\sigma_8 = 0.667 \pm 0.154$.

In the case of the f_5CDM,

(i) for the Γ_0 model: $\chi^2 = 577.279$, $\Omega_m = 0.285 \pm 0.006$, $b = 0.217 \pm 0.067$, $\gamma_0 = 0.550 \pm 0.086$, $\sigma_8 = 0.765 \pm 0.038$;

(ii) for the Γ_1 model: $\chi^2 = 577.176$, $\Omega_m = 0.287 \pm 0.007$, $b = 0.189 \pm 0.076$, $\gamma_0 = 0.524 \pm 0.092$, $\gamma_1 = 0.057 \pm 0.470$, $\sigma_8 = 0.758 \pm 0.083$;

(iii) for the Γ_2 model: $\chi^2 = 575.983$, $\Omega_m = 0.287 \pm 0.007$, $b = 0.189 \pm 0.076$, $\gamma_0 = 0.489 \pm 0.090$, $\gamma_1 = -0.717 \pm 0.743$, $\sigma_8 = 0.674 \pm 0.078$.

Lastly, we would like to emphasize that in all cases explored here the value of $\mathrm{AIC}_\Lambda(\sim 578.3)$ is smaller than the corresponding one for the various $f(T)$ models, which implies that the usual ΛCDM cosmology ($\gamma_\Lambda = 0.597$) seems to provide a better fit than the $f_{1\text{–}3}$CDM gravity models the expansion and the growth data. On the other hand, the $|\Delta\mathrm{AIC}| = |\mathrm{AIC}_\Lambda - \mathrm{AIC}_{f_{1-3}(T)}|$ values point that the growth data can be consistent with the $f_{1\text{–}3}$CDM gravity models. We stress here that the f_4CDM and f_5CDM models seem to be disfavored by the current data.

VI. DISCUSSION AND CONCLUSIONS

We have investigated a wide range of different $f(T)$ models, with up to two parameters, both at the background and at the perturbation level. The functional forms of $f(T)$ considered in this work cover practically all the functional forms considered in the literature so far. Despite the fact that the $f(T)$ gravity can be derived from the principle of least action the corresponding $f(T)$ functional forms are phenomenological and even though they do not correspond to a firm theoretical model they cover a wide range of independent functional forms. Thus they represent a wide range of degrees of freedom describing deviations from ΛCDM in the context of $f(T)$ models.

Following our previous work Basilakos, Nesseris, and Perivolaropoulos [18] corresponding to $f(R)$ gravity, we calculated the function $y(z, b)$ which quantifies the deviation from ΛCDM cosmology at the background level. We also obtained the growth index and the effective Newton constant, which incorporate the $f(T)$ gravity effects at the perturbation level. Furthermore, we utilized the recent expansion and growth data, implementing the Akaike information criterion and three different parametrizations for the growth index, in order to constraint the parameters of these $f(T)$ models.

Our results show that all viable $f(T)$ gravity models hardly deviate from the ΛCDM paradigm. In particular, among the five examined models, the power-law one [8] (f_1CDM), the exponential-square-root one [9] (f_2CDM) and the exponential one (f_3CDM) possess ΛCDM cosmology as a limiting case. It is only this limit that is favored by cosmological observations. In fact, the detailed observational confrontation showed that these three models at best fit behave as small perturbations around the concordance ΛCDM cosmology, with the parameter b, which quantifies the deviation from ΛCDM, constrained in a very narrow window around 0. The other two $f(T)$ models, namely, the logarithmic one [49] (f_4CDM) and the hyperbolic-tangent one [51] (f_5CDM), do not possess ΛCDM as a limiting case. We showed that both are in tension with the data. In fact, we have demonstrated that (f_4CDM) coincides with the DGP model at the background level, whose inconsistency between distance measures and horizon scale growth is well known [50] and also demonstrated by our results.

The derived requirement of fine-tuning of the $f(T)$ constructions at the ΛCDM, based on cosmological constraints, would probably be further amplified if we had considered in addition their consistency with Solar System tests, which constitute another powerful source of constraints against any deviation from general relativity. At this point we would like to make a comment concerning the Lorentz invariance of $f(T)$ theories. As was shown in [64], for general $f(T)$ modifications the field equations are not invariant under local Lorentz transformations, unless $f(T)$ is a constant or a linear-in-T function, in which case we reobtain general relativity (that is, ΛCDM) and local Lorentz invariance is restored. This feature imposes strict constraints on the viable $f(T)$ forms, since the observational bounds on gravitational Lorentz violation are very narrow [65]. As we have already mentioned above, confrontation with Solar System data implies that the nontrivial $f(T)$ modification must be significantly small [13]. In the present analysis we were interested in performing a pure confrontation of $f(T)$ theories with cosmological data, without imposing any other theoretical constraints. Thus, from another point of view we verified again that in all viable $f(T)$ scenarios the nontrivial $f(T)$ modifications are so small that these constructions are practically indistinguishable from ΛCDM. Clearly, taking into account the above Lorentz violation discussion strengthens our result that all viable $f(T)$ almost coincide with ΛCDM.

It is therefore safe to conclude that although at early times the additional degrees of freedom provided by $f(T)$ constructions may play an important role and improve the inflationary behavior, at late times these extra degrees of freedom do not appear to be consistent with the degrees of freedom favored by nature.

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APPENDIX: DERIVATION OF EQ. (65)

We can rewrite Eq. (57) as

$$\begin{aligned} E^2(z) &= \Omega_{m0}(1+z)^3 + \Omega_{r0}(1+z)^4 + \Omega_{F0}E^{2b}(z) \\ &= \Omega_{m0}(1+z)^3 + \Omega_{r0}(1+z)^4 + \Omega_{F0} - \Omega_{F0} \\ &\quad + \Omega_{F0}E^{2b}(z) \\ &= E^2_\Lambda(z) + \Omega_{F0}[E^{2b}(z) - 1], \end{aligned} \tag{A1}$$

where $E^2_\Lambda(z)$ is given by Eq. (59) and in the second line we added and subtracted Ω_{F0}.

Now, in this case we assume that the Hubble parameter $\frac{H^2}{H_0^2} \equiv E^2(z)$ depends on b only implicitly via the Friedmann equation (59). In other words, we consider b and $E^2(z)$ to be independent, and thus any derivatives with respect to b are zero. Hence, performing a Taylor expansion of (A1) up to second order around $b = 0$ we acquire

$$\begin{aligned} E^2(z) &= E^2_\Lambda(z) + \ln[E^2(z)]\Omega_{F0}b \\ &\quad + \frac{1}{2}\ln[E^2(z)]^2\Omega_{F0}b^2 + \cdots. \end{aligned} \tag{A2}$$

If we keep only the first-order term and solve for $E^2(z)$, we obtain

$$E^2(z, b) = -b\Omega_{F0}\mathcal{W}_k\left(-\frac{e^{-\frac{E_\Lambda(z)^2}{b\Omega_{F0}}}}{b\Omega_{F0}}\right), \tag{A3}$$

where $\mathcal{W}_k(\omega)$ is the Lambert function defined via $\omega \equiv \mathcal{W}_k(\omega)e^{\mathcal{W}_k(\omega)}$ for all complex numbers ω. The Lambert function has branch-cut discontinuities, so the different branches are indicated by the integer k. Our solution has $k = 0$ (the principal branch) for $b \leq 0$ and $k = -1$ for $b > 0$.

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