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Covariant basis induced by parity for the $(j, 0) \oplus (0, j)$ representation

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In this work, we build a covariant basis for operators acting on the $(j, 0) \oplus (0, j)$ Lorentz group representations. The construction is based on an analysis of the covariant properties of the parity operator, which for these representations transforms as the completely temporal component of a symmetrical tensor of rank $2j$. The covariant properties of parity involve the Jordan algebra of anticommutators of the Lorentz group generators which unlike the Lie algebra is not universal. We make the construction explicit for $j = 1/2$, 1 and $3/2$, reproducing well-known results for the $j = 1/2$ case. We provide an algorithm for the corresponding calculations for arbitrary j. This covariant basis provides an inventory of all the possible interaction terms for gauge and nongauge theories of fields for these representations. In particular, it supplies a single second-rank antisymmetric structure, which in the Poincaré projector formalism implies a single Pauli term arising from gauge interactions and a single (free) parameter g, the gyromagnetic factor. This simple structure predicts that for an elementary particle in this formalism all multipole moments, Q_E^l and Q_M^l, are dictated by the complete algebraic structure of the Lorentz generators and the value of g. We explicitly calculate the multipole moments, for arbitrary j up to $l = 8$. Comparing with results in the literature we find that only the electric charge and magnetic moment of a spin j particle are independent of the Lorentz representation under which it transforms, all higher multipoles being representation dependent. Finally we show that the propagation of the corresponding spin j waves in an electromagnetic background is causal.

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I. INTRODUCTION

The Standard Model of particle physics has three ingredients: a base spacetime, whose symmetries allow us to classify the asymptotically free states; a gauge group, which fixes the number and properties of gauge particles; and a particular spectrum of matter particles, whose number and interactions are largely unfixed [1]. Supersymmetry and extra-dimensional models represent attempts to modify our conventional understanding of the first ingredient, while extensions of the gauge group, including grand unified theories, are extrapolations of the second one which also enlarge the content of spin 1/2 matter fields. The recent discovery of a particle with a mass around 126 GeV at the LHC [2,3] as expected for the Higgs of the Standard Model seems discouraging for both approaches at least in the short run [4], and therefore it may prove valuable to focus on different avenues for going beyond the standard model (SM).

In this concern, it is worthwhile to remark that the SM makes use of only a few representations of the homogeneous Lorentz group. These representations are identified with the three basic types of fields entering this construction: the (0, 0) representation for the Higgs field, the (1/2, 1/2) representation for the gauge fields and the (1/2, 0) and (0, 1/2) representation for the matter fields. Certainly, so far we have no general principle restricting the spin content of these building blocks. Some theories for physics beyond the standard model like supersymmetric models modify this correlation and consider Lorentz representations with high spin content which enter as gauge fields, such as the spinor-vector representation $[(1/2, 0) \oplus (0, 1/2)] \otimes (1/2, 1/2)$.

High spin fields naturally appear also in phenomenology (e.g., the hadronic contribution to the leptonic $g - 2$) and in beyond-the-SM model building (e.g., supergravity and strings). An explanation of the family structure of the SM involving compositeness is also likely to imply the existence of some high spin states. However, the description of high spin ($j > 1$) elementary systems is a longstanding problem in quantum field theory.

Conventional constructions for high spin fields (like Dirac-Fierz-Pauli [5,6] or Rarita-Schwinger [7]) are plagued by well-known problems, which can be traced to ambiguities in the selection and propagation of the desired degrees of freedom. These ambiguities can give rise to inconsistencies like superluminal propagation and other nonlocalities, or the appearance of ghosts [8–14].

Quantum fields which satisfy the cluster decomposition principle [15] are built as induced representations of the semidirect product $\mathfrak{t}_4 \rtimes \mathfrak{so}(1, 3)$ on representations of the Lorentz algebra. This often means that we use fields with redundant or unwanted degrees of freedom. We can generically understand the Velo-Zwanziger problems as a failure to discard these degrees of freedom in the presence of interactions.

We can identify three classes of high spin constructions. There are those in the spirit of Dirac-Fierz-Pauli, where either constrictions or auxiliary fields are used to remove unwanted degrees of freedom in order to exclusively propagate an irreducible representation of the Poincaré group [5–7,16–18]. Then, there are constructions which renounce to fix a single spin and mass, as in Bhabha or Kemmer-Duffin-Petiau [19–22] theories. Finally, we have the Joos-Weinberg formalism where no extra spin degrees of freedom are introduced [23–26] (see [27] for a historical account.)

Although the properties of free fields are only related to the Poincaré quantum numbers, regardless of the Lorentz representations on which the fields are constructed, this is no longer true for interacting fields. Two quantum fields with spin j quanta but defined in different Lorentz representations may share the same asymptotically free properties, but differ in their interaction properties, like their electromagnetic moments [28]. A basic element in the elucidation of the different possibilities to describe interactions of high spin systems is the classification of all possible operators acting in the corresponding Lorentz representation space, which in turn requires the construction of a covariant basis for these operators.

In a recent series of works [29–32], a proposal for the description of arbitrary spin particles was detailed, based on the projection onto subspaces of the Poincaré group for fields with definite quantum numbers transforming in a given representation of the Lorentz group. A key ingredient in the formalism is the construction of the most general space-time antisymmetric operator in the corresponding space. In the general case, the construction of this tensor requires us to classify all operators acting on the chosen representation space in a covariant manner. In this work we solve this problem for particles of arbitrary spin j transforming in the $(j,0)\oplus(0,j)$ representation of the Lorentz group. We find a covariant basis for all operators with internal indexes in these representations. We conclude that there is a single independent antisymmetric operator which gives rise to a unique Pauli-type interaction term for all representations $(j,0)\oplus(0,j)$ and work out the consequences for the multipole electromagnetic moments and the propagation of spin j waves in an electromagnetic background.

Although solving this problem is the main motivation for this work, the scope of the obtained results is beyond this framework since our covariant basis can be helpful in general for the construction of models using the $(j,0)\oplus(0,j)$ representation, either at the elementary level such as models for physics beyond the standard model with elementary high spin matter fields, or at the composite level in effective theories for hadronic interactions or in the description of composite objects in physics beyond the standard model.

The paper is organized as follows: in the next section we find all the representations having only two possible spin values, which are suitable to be used in the Poincaré projector formalism when we impose also to have at most a second order Lagrangian theory. In Sec. III we present the construction of the parity-based covariant basis for $(j,0)\oplus(0,j)$ representations and their connection with the associated Jordan and Lie algebras. In Sec. IV we work out the consequences of this algebraic structure for the electromagnetic multipole moments of an elementary particle described by the Poincaré projector formalism and study the corresponding propagation of spin j waves in an electromagnetic background. Our conclusions are presented in Sec. V.

II. GENERAL POINCARÉ PROJECTORS

The Poincaré algebra is the semidirect product $\mathfrak{t}_4 \rtimes \mathfrak{so}(1,3)$, satisfying the Lie brackets

$$i[M_{\mu\nu},P_\rho]=\eta_{\mu\rho}P_\nu-\eta_{\nu\rho}P_\mu,\quad [P_\mu,P_\nu]=0$$
$$i[M_{\mu\nu},M_{\rho\sigma}]=\eta_{\mu\rho}M_{\nu\sigma}-\eta_{\mu\sigma}M_{\nu\rho}-\eta_{\nu\rho}M_{\mu\sigma}+\eta_{\nu\sigma}M_{\mu\rho}, \tag{1}$$

where indexes run as $\mu=0,\ldots,3$. The $\mathfrak{so}(1,3)$ subalgebra generated by the $M_{\mu\nu}$ is the Lorentz algebra. It is customary to separate the Lorentz generators into *rotations* $\mathbf{J}$ and *boosts* $\mathbf{K}$ operators

$$M^{0i}=K_i,\qquad M^{ij}=\epsilon_{ijk}J_k. \tag{2}$$

The Poincaré algebra has two algebraic invariants with the corresponding Casimir operators given by

$$\mathcal{C}_2=P^\mu P_\mu\qquad \mathcal{C}_4=W^\mu W_\mu. \tag{3}$$

where W_μ stands for the Pauli-Lubanski four-vector

$$W_\mu=\frac{1}{2}\varepsilon_{\mu\sigma\tau\rho}M^{\sigma\tau}P^\rho. \tag{4}$$

The Poincaré one-particle states are then characterized by the quantum numbers coming from

$$\mathcal{C}_2|\Psi\rangle=m^2|\Psi\rangle\qquad \mathcal{C}_4|\Psi\rangle=-m^2j(j+1)|\Psi\rangle, \tag{5}$$

where m denotes the mass and j the spin. Now, the fields which enter our theories are linear combinations of the creation and annihilation operators of these states, defined by the relation $|m^2,j\rangle=a^\dagger(m^2,j)|0\rangle$ and its adjoint. The transformation properties of the creation and annihilation operators under the Poincaré group are fixed by this relation, and in turn they fix (through Poincaré invariance of the scattering matrix and cluster decomposition) the general form for the fields as [15]

$$\psi_l(x)=\int d\Gamma(\kappa e^{ip\cdot x}\omega_l(\Gamma)a^\dagger(\Gamma)+\lambda e^{-ip\cdot x}\omega_l^c(\Gamma)a(\Gamma)), \tag{6}$$

where Γ is the set of labels $[m^2,j,p^\mu,\sigma]$, with σ the quantum number of the little group (i.e., spin projection for massive fields or helicity for massless fields) and κ, λ are constants. The coefficients ω_l and ω_l^c in this expression transform in some representation of the Lorentz group. In order to consider this construction complete, all that remains is to choose coefficients with the appropriate transformation rules and the constants κ, λ with the appropriate values to properly account for discrete symmetries.

Conventionally, we construct our field theories using only a handful of (in general, reducible) Lorentz representations: Dirac spinors, four-vectors, spinor-vectors and higher order tensors for higher spin. We require, however, that these fields carry irreducible representations of the Poincaré group, that is, particles with definite mass and spin. Indeed, it is when this requisite is not properly satisfied that inconsistencies manifest, like superluminal propagation or the appearance of spurious degrees of freedom. The role of equations of motion is then to covariantly ensure that only the desired degrees of freedom are present.

The homogeneous Lorentz algebra $\mathfrak{so}(1,3)$ is locally isomorphic to the direct sum $\mathfrak{su}(2)_A \oplus \mathfrak{su}(2)_B$, spanned by the combinations of rotation $\mathbf{J}$ and boost $\mathbf{K}$ generators

$$\mathbf{A} = \frac{1}{2}(\mathbf{J} - i\mathbf{K}) \qquad \mathbf{B} = \frac{1}{2}(\mathbf{J} + i\mathbf{K}). \tag{7}$$

These two sets commute:

$$[\mathbf{A}, \mathbf{B}] = 0. \tag{8}$$

Therefore, we can label the irreducible Lorentz representations with two $\mathfrak{su}(2)$ numbers (a, b). There are two Casimir operators for the Lorentz group, $M_{\mu\nu}M^{\mu\nu}$ and $M_{\mu\nu}\tilde{M}^{\mu\nu}$, where the dual tensor $\tilde{M}^{\mu\nu}$ is defined as

$$\tilde{M}^{\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}M_{\alpha\beta}. \tag{9}$$

These operators can be recast in terms of the $\mathfrak{su}(2)$ Casimir operators $\mathbf{A}^2$ and $\mathbf{B}^2$ as

$$M_{\mu\nu}M^{\mu\nu} = 4(\mathbf{A}^2 + \mathbf{B}^2) \tag{10}$$

$$M_{\mu\nu}\tilde{M}^{\mu\nu} = -4i(\mathbf{A}^2 - \mathbf{B}^2). \tag{11}$$

The (linear, unitary) parity operator induces the following transformations of the Lorentz generators

$$\Pi\mathbf{J}\Pi^{-1} = \mathbf{J} \tag{12}$$

$$\Pi\mathbf{K}\Pi^{-1} = -\mathbf{K}, \tag{13}$$

so that $\mathbf{A}$ and $\mathbf{B}$ transform into each other:

$$\Pi\mathbf{A}\Pi^{-1} = \mathbf{B} \tag{14}$$

$$\Pi\mathbf{B}\Pi^{-1} = \mathbf{A}. \tag{15}$$

This immediately suggests a broad classification of the Lorentz representations (enlarged by parity) into two groups. First, the (a, a) representations that transform into themselves under the action of parity, for which the second invariant is null, that is, $M_{\mu\nu}\tilde{M}^{\mu\nu} = 0$. These were the representations proposed by Fierz and Pauli to describe arbitrary integer spin $j = 2a$ [6]. Second, we have the reducible $(a, b) \oplus (b, a)$ representations with $a \neq b$, for which (a, b) and (b, a) are exchanged by parity. We call these *chiral* representations, because for them we can define a chirality operator

$$\chi = \frac{i}{4a(a+1) - 4b(b+1)}M_{\mu\nu}\tilde{M}^{\mu\nu}. \tag{16}$$

Since χ is proportional to a Casimir operator of the Lorentz group, we have the commutation rule

$$[\chi, M_{\mu\nu}] = 0. \tag{17}$$

These chiral representations include the Joos-Weinberg representations [24] which correspond to $b = 0$.

The specific form of the projector selecting the degrees of freedom with quantum numbers (m, j) depends on the spin content of the chosen representation. By working in Lorentz representations containing at most two spin sectors, we can build second-order projections in the momenta which can be implemented in Lagrangian form without the addition of constraints or auxiliary degrees of freedom [29].

In general, the representation (a, b) contains states with all spins between $|a - b|$ and $a + b$. Those with at most two spin sectors which are also irreducible representations for parity can be enumerated as follows:

(i) The nonchiral representations $(0, 0)$ and $(\frac{1}{2}, \frac{1}{2})$.
(ii) The single-spin chiral representations $(j, 0) \oplus (0, j)$ with $j \geq \frac{1}{2}$.
(iii) The double-spin chiral representations $(j - \frac{1}{2}, \frac{1}{2}) \oplus (\frac{1}{2}, j - \frac{1}{2})$ with $j > 1$.

For the single-spin $(j, 0) \oplus (0, j)$ representation the Poincaré projector is

$$\mathcal{P}^{\{m,j\}} = \frac{P^2}{m^2}\left[\frac{W^2}{-j(j+1)P^2}\right]. \tag{18}$$

We remark that this projector fixes the appropriate mass and spin quantum numbers but in general other properties of the particles such as parity are not fixed.

On the other hand, for the double-spin representations, we use instead the projector

$$\mathcal{P}^{\{m,j\}} = \left[\frac{W^2}{-2jm^2} - \frac{j(j-1)}{2j}\frac{P^2}{m^2}\right] \tag{19}$$

which removes the unwanted spin $j - 1$ components of the field to ensure that only spin j is propagated [29].

This projection produces second order equations of motion of the form

$$(T_{\mu\nu}P^\mu P^\nu - m^2)\Psi = 0, \tag{20}$$

for both fermions and bosons where the specific form of the tensor $T_{\mu\nu}$ depends on the chosen Lorentz representation. However, as discussed in the previous works [28–30], only the symmetric part of the operator $T_{\mu\nu}$ is fixed by the projector. This requires us to construct the most general space-time antisymmetric operator which is clearly representation dependent. Furthermore, a given Lorentz representation will house, in addition to these antisymmetric operators relevant for the description of gauge interactions, many other operators which can be relevant for nongauge or self-interactions. In this work we aim to classify all of them for the restricted class of single-spin chiral $(j,0)\oplus(0,j)$ representations, in a construction based on the covariant properties of the parity operator.

III. COVARIANT BASIS INDUCED BY PARITY FOR THE $(j,0)\oplus(0,j)$ REPRESENTATION SPACE

The specific representation of operators depends on our choice for the basis; thus we start by fixing our conventions for the basis in $(j,0)\oplus(0,j)$ space. For the "right" representation $(j,0)$ we choose the angular momentum basis $\{|j,m\rangle_R\}$. Similarly, for the "left" representation $(0,j)$ we chose to work with the corresponding $\{|j,m\rangle_L\}$ basis. The Lorentz generators for the $(j,0)$ and $(0,j)$ representations are

$$M_R{}^{0i} = (K_R)_i, \qquad M_R{}^{ij} = \epsilon_{ijk}(J_R)_k, \tag{21}$$

$$M_L{}^{0i} = (K_L)_i, \qquad M_L{}^{ij} = \epsilon_{ijk}(J_L)_k, \tag{22}$$

where $\mathbf{J}_R = \mathbf{J}_L = \boldsymbol{\tau}$ are the conventional $(2j+1)\times(2j+1)$ angular momentum matrices and $\mathbf{K}_R = -\mathbf{K}_L = i\boldsymbol{\tau}$. With this choice the states $\{|j,m\rangle_R, |j,m\rangle_L\}$ form a basis for the direct sum $(j,0)\oplus(0,j)$ representation space which we will denote as *chiral basis* in the following. In this basis the Lorentz generators take the following form:

$$\mathbf{J} = \begin{pmatrix} \boldsymbol{\tau} & 0 \\ 0 & \boldsymbol{\tau} \end{pmatrix}, \qquad \mathbf{K} = \begin{pmatrix} i\boldsymbol{\tau} & 0 \\ 0 & -i\boldsymbol{\tau} \end{pmatrix}. \tag{23}$$

The components of the Lorentz antisymmetric tensor for the $(j,0)\oplus(0,j)$ representation can be written in terms of these matrices as

$$M^{ij} = \epsilon_{ijk}\begin{pmatrix} \tau_k & 0 \\ 0 & \tau_k \end{pmatrix} \qquad M^{0i} = i\begin{pmatrix} \tau_i & 0 \\ 0 & -\tau_i \end{pmatrix}. \tag{24}$$

In the chiral basis, the chirality operator in Eq. (16) takes the diagonal form

$$\chi = \begin{pmatrix} \mathbb{1}_{2j+1} & 0 \\ 0 & -\mathbb{1}_{2j+1} \end{pmatrix}, \tag{25}$$

and the parity operator which swaps the chiral subspaces $(j,0)$ and $(0,j)$ takes the form

$$\Pi = \begin{pmatrix} 0 & \mathbb{1}_{2j+1} \\ \mathbb{1}_{2j+1} & 0 \end{pmatrix}. \tag{26}$$

Thus, parity and chirality anticommute for all $(j,0)\oplus(0,j)$ representations

$$\{\Pi, \chi\} = 0. \tag{27}$$

For these representations the Lorentz generators also satisfy

$$\mathbf{K} = i\chi\mathbf{J}, \tag{28}$$

which can be covariantly written as

$$\tilde{M}^{\mu\nu} = -i\chi M^{\mu\nu}. \tag{29}$$

Now, the parity operator fulfills the relations

$$[\Pi, \mathbf{J}] = 0 \tag{30}$$

$$[\Pi, \mathbf{K}] = 2\Pi\mathbf{K}, \tag{31}$$

or in covariant form

$$[M_{\mu\nu}, \Pi] = i\eta_{0\mu}(2i\Pi M_{0\nu}) - i\eta_{0\nu}(2i\Pi M_{0\mu}). \tag{32}$$

As these commutation rules make clear, parity, while rotating as a scalar, is not a Lorentz scalar under boosts. A straightforward calculation yields the following commutation rules for the object $V_k = 2i\Pi M_{0k}$:

$$\begin{aligned} [M_{ij}, V_k] &= -i\eta_{ik}V_j + i\eta_{jk}V_i \\ [M_{0i}, V_j] &= -2\Pi\{M_{0i}, M_{0j}\}. \end{aligned} \tag{33}$$

The composite object V_k rotates as a vector but it does not behave as a vector under boosts. The transformation properties under boosts involve the anticommutator $\{M_{0i}, M_{0j}\}$ (the Jordan algebra of Lorentz generators), which, in contrast to the Lie algebra, is not universal. Writing Eqs. (33) in covariant notation we get

$$\begin{aligned} [M_{\mu\nu}, V_\rho] = i(&\eta_{\nu\rho}V_\mu - \eta_{\mu\rho}V_\nu) + i\eta_{0\mu}\eta_{0\rho}V_\nu \\ &- i\eta_{0\mu}\eta_{0\nu}V_\rho - i\eta_{0\nu}\eta_{0\rho}V_\mu + i\eta_{0\mu}\eta_{0\nu}V_\rho \\ &- 2i\eta_{0\mu}\Pi\{M_{0\nu}, M_{0\rho}\} + 2i\eta_{0\nu}\Pi\{M_{0\mu}, M_{0\rho}\}). \end{aligned} \tag{34}$$

The appearance of the quantity $\Pi\{M_{0\mu}, M_{0\nu}\}$ on the right-hand side suggests that in general the covariant properties of Π will depend on the transformation properties of the objects

$$t_{\mu_1,\mu_2} = \{M_{0\mu_1}, M_{0\mu_2}\}, \ldots \tag{35}$$

$$t_{\mu_1\ldots\mu_{2j}} = \{t_{\mu_1\ldots\mu_{2j-1}}, M_{0\mu_{2j}}\}. \tag{36}$$

By calculating, for a particular representation, the commutators with the Lorentz generators of the series $(\Pi, \Pi M_{0\mu}, \Pi t_{\mu_1\mu_2}, \ldots)$ we will eventually arrive at a set of objects transforming into themselves. It will be shown below that these operators form a symmetric tensor $S_{\mu_1\ldots\mu_{2j}}$ whose time component $S_{0\ldots0}$ is Π.

In order to make our parity-based construction transparent, we start by studying the simplest case, the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ space which for obvious reasons we denote as Dirac representation in the following. Here and in the following, the inner product in the matrix space is taken to be

$$A\cdot B=\mathrm{Tr}[AB]. \tag{37}$$

A basis for the operators acting on the $(j,0)\oplus(0,j)$ representation space can be obtained from the exterior product of states in the $\{|j,m\rangle_R, |j,m\rangle_L\}$ basis. This set provides a basis for constructing the most general bilinear in the fields with definite Lorentz transformation properties. In particular for $j=1/2$, examining the Lorentz decomposition of the external product of states in this basis we get

$$\left[\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)\right]^2=(0,0)_2\oplus(1,0)\oplus(0,1)\oplus\left(\frac{1}{2},\frac{1}{2}\right)_2.$$

In this equation, the left side stands for the exterior product of the basis states; in the right side we have the Lorentz decomposition of this product, with subscripts denoting multiplicity. It corresponds to a pair of scalars, an antisymmetric tensor, and a pair of four-vectors.

The first operator transforming in the (0, 0) representation is the unit operator $\mathbb{1}$. Given that the chirality operator commutes with the Lorentz generators (17), we can use this operator as the second operator transforming in the (0, 0) representation. Operators transforming in the $(1,0)\oplus(0,1)$ are clearly the Lorentz generators $M_{\mu\nu}$. The remaining operators can be built analyzing the covariant properties of parity.

Recalling the transformation rules in Eq. (34), we need to construct the t_{ij} operators. For the Dirac representation, $\{M_{0i},M_{0j}\}=\frac{1}{2}\delta_{ij}$. Defining the object

$$S_\mu=\eta_{0\mu}\Pi-2i\Pi M_{0\mu}, \tag{38}$$

we can see that it transforms as a four-vector

$$[M_{\rho\sigma},S_\mu]=i\eta_{\mu\rho}S_\sigma-i\eta_{\mu\sigma}S_\rho. \tag{39}$$

Thus we conclude that Π transforms as the zeroth component of the four-vector S^μ.

An important property induced by Eqs. (17) and (27) and the specific form of S^μ in Eq. (38) is

$$\{\chi,S^\mu\}=0, \tag{40}$$

which implies the orthogonality of χ and S_μ. A simple combination of Eqs. (17) and (39) shows that χS_μ transforms also as a four-vector. Finally, a direct calculation yields that these are independent operators.

In summary, for the $(1/2,0)\oplus(0,1/2)$ representation our parity-based construction yields the following covariant basis:

$$\{\mathbb{1},\chi,S_\mu,\chi S_\mu,M_{\mu\nu}\}. \tag{41}$$

It is well known that the conventional sixteen matrices,

$$\{\mathbb{1},\gamma_5,\gamma_\mu,\gamma_5\gamma_\mu,\sigma_{\mu\nu}\}, \tag{42}$$

form a basis for this operator space, with the Lorentz generators for this representation space given by $M_{\mu\nu}=\frac{1}{2}\sigma_{\mu\nu}$. A direct comparison shows that our procedure reproduces the conventional covariant basis with the γ^μ matrices in the Weyl representation, except for an irrelevant 1/2 normalization factor in $M_{\mu\nu}$. It is also clear that the specific form of the operators in Eq. (41) depend on the choice of the basis we use for the states in $(1/2,0)\oplus(0,1/2)$.

In terms of the covariant operator S^μ in Eq. (38), the rest-frame parity projection equation

$$\frac{1}{2}(1\pm\Pi)\psi(0)=\psi(0) \tag{43}$$

transforms, for an arbitrary frame, into the familiar

$$(S_\mu P^\mu\mp m)\psi(p)=0. \tag{44}$$

From this perspective, the Dirac equation is simply the covariant projection over parity-invariant subspaces in the $(1/2,0)\oplus(0,1/2)$ representation space and the Dirac algebra satisfied by the S^μ matrices,

$$\{S^\mu,S^\nu\}=2\eta^{\mu\nu}, \tag{45}$$

is just a manifestation of the covariant properties of the parity operator.

Using the projection over states with well-defined spin and mass produces a condition of the general form in Eq. (20). For the $(1/2,0)\oplus(0,1/2)$ representation, the symmetric part of the $T_{\mu\nu}$ space-time tensor is fixed by the projector to be $\eta_{\mu\nu}$, but the antisymmetric part is not; thus its general form must be constructed in terms of the covariant basis. For this representation the only possibility is a term $gM_{\mu\nu}$, with g arbitrary, which upon gauging produces an interaction

$$gM_{\mu\nu}F^{\mu\nu}. \tag{46}$$

Such a formalism has been studied at one loop for the Abelian and non-Abelian cases [31,32,34].

A. Lorentz structure of the operators acting on $(1,0)\oplus(0,1)$

As a second example, let us proceed in full detail with the $(1,0)\oplus(0,1)$ construction, which then we can generalize to arbitrary j. The conventional angular momentum generators entering Eq. (24) for this representation are

$$\tau_1 = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} \qquad \tau_2 = \frac{i}{2}\begin{pmatrix} 0 & -\sqrt{2} & 0 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$\tau_3 = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{47}$$

A basis for the operators acting on the $(1,0) \oplus (0,1)$ space can be obtained via the external products of the states in the $\{|j,m\rangle_R, |j,m\rangle_L\}$ basis which has the following Lorentz decomposition:

$$[(1,0) \oplus (0,1)]^2 = (0,0)_2 \oplus (1,1)_2 \oplus (1,0) \oplus (0,1) \oplus (2,0) \oplus (0,2). \tag{48}$$

We may identify the two operators transforming in the (0, 0) representation as the unit and chirality operators, and the Lorentz generators with the operators transforming in the $(1,0) \oplus (0,1)$. However, it is not obvious how we can construct the operators transforming in the two (1, 1) and the $(2,0) \oplus (0,2)$ Lorentz representations. In order to construct these operators while aiming to envision the general case, we briefly review the theory of $\mathfrak{so}(1,3)$ Young projectors.

An arbitrary traceless Lorentz tensor of rank r can be decomposed in an orthogonal basis given by all the completely traceless tensors enumerated by all possible Young tableaux of r boxes. These Young tableaux index the representations of the symmetric group S_N. We identify them with the permutation properties of the Lorentz indices of our tensor. For example, a symmetrical tensor $S_{\mu\nu}$ of rank 2 corresponds to a row Young tableau $\begin{array}{|c|c|}\hline \mu & \nu \\ \hline\end{array}$, while an antisymmetrical tensor corresponds to the column Young tableau $\begin{array}{|c|}\hline \mu \\ \hline \nu \\ \hline\end{array}$.

A Young projector associated with some Young tableau is an operator which projects a general tensor into the subspace with the symmetries of the tableau. Since these projectors are built with the metric tensor $\eta_{\mu\nu}$, which is an invariant tensor of the Lorentz algebra, the Young projection is also invariant. Therefore, the subspaces transform separately. This fact is at the root of our decomposition.

We are interested in the chiral representations, which produce either totally symmetrical or self-dual/anti-self-dual tensors. We only need the characterization of the Young diagrams with one and two rows [33]. The dimension of a completely traceless tensor (i.e., a tensor for which every contraction vanishes) in $\mathfrak{so}(1,3)$ is given by the following combinatorial formulas:

$$d_{[n]} = (n+1)^2 \quad \begin{array}{|c|c|c|c|c|}\hline \mu_1 & \mu_2 & \mu_3 & \ldots & \mu_n \\ \hline\end{array} \tag{49}$$

$$d_{[n,m]} = 2n^2 + 4n - 2m^2 + 2 \quad \begin{array}{|c|c|c|c|c|}\hline \mu_1 & \mu_2 & \mu_3 & \ldots & \mu_n \\ \hline \nu_1 & \nu_2 & \ldots & \nu_m \\ \cline{1-4}\end{array}. \tag{50}$$

These diagrams describe either a symmetrical tensor of rank n, or a mixed-symmetry tensor of rank $n+m$. The factor 2 comes about because we are considering both the self-dual and anti-self-dual parts. (For an in-depth discussion of this technical point, see Chapter 9 of [33]).

The Young projector corresponding to a given Young pattern is constructed as the product of the appropriate symmetrizers and antisymmetrizers. For a row or column Young tableau, we have the pure symmetrizers and antisymmetrizers, while for mixed-symmetry tableaux we choose to first antisymmetrize index subsets, and then to symmetrize as appropriate. This choice is not unique, but is convenient for calculations [35,36]. For example, to obtain the Young projector corresponding to the diagram

$$\begin{array}{|c|c|}\hline \mu & \alpha \\ \hline \nu & \beta \\ \hline\end{array} \tag{51}$$

we first antisymmetrize the column pairs, and then symmetrize the row pairs:

$$\mathbb{P}^{\mu\nu\alpha\beta}{}_{\xi\zeta\lambda\kappa} \propto \mathcal{S}^{\mu\alpha}{}_{\rho\sigma}\mathcal{S}^{\nu\beta}{}_{\tau\delta}\mathcal{A}^{\rho\tau}{}_{\xi\zeta}\mathcal{A}^{\sigma\delta}{}_{\lambda\kappa}, \tag{52}$$

with a suitable normalization.

Coming back to the $(1,0) \oplus (0,1)$ representation, we again have two scalars, which are chosen as the unit operator and the chirality operator χ. We expect that the rest of the covariant basis will be decomposable as

$$2\,\begin{array}{|c|c|}\hline & \\ \hline\end{array} \oplus \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array} \oplus \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline\end{array}. \tag{53}$$

This result tells us that the basis we seek is provided by the following operators:

$$\{1, \chi, S_{\mu\nu}, \mathcal{S}_{\mu\nu}, M_{\mu\nu}, C_{\mu\nu\alpha\beta}\}, \tag{54}$$

where $S_{\mu\nu}$ is a symmetric traceless ($S^{\mu}{}_{\mu} = 0$) tensor operator thus having nine independent components which coincide with the degrees of freedom of the (1, 1) Lorentz representation. Similar results hold for the $\mathcal{S}_{\mu\nu}$ tensor operator which also has nine independent components.

As for the $C_{\mu\nu\alpha\beta}$ tensor operator, it has the following symmetries:

$$C_{\mu\nu\alpha\beta} = -C_{\nu\mu\alpha\beta} = -C_{\mu\nu\beta\alpha}, \qquad C_{\mu\nu\alpha\beta} = C_{\alpha\beta\mu\nu}. \tag{55}$$

It is traceless (i.e., the contraction of any pair of indices vanishes), and it satisfies the algebraic Bianchi identity

$$C_{\mu\nu\alpha\beta} + C_{\mu\alpha\beta\nu} + C_{\mu\beta\nu\alpha} = 0. \tag{56}$$

A fourth rank tensor has 256 components but it is easy to convince ourselves that these symmetries restrict this tensor to have only ten independent components, which are precisely the number of degrees of freedom in the $(2,0)\oplus(0,2)$ Lorentz representation. The calculation is simple, and familiar from the Riemann tensor: our tensor is akin to a rank 2 symmetrical tensor in six dimensions, which has 21 independent components, but the traceless condition removes ten, and the Bianchi identity removes one, leaving the aforementioned ten independent components.

Aiming to construct explicitly the tensors in the operator basis in Eq. (54), we calculate the Lorentz commutators of the series Π, ΠM_{0i}, Πt_{ij}, ...

$$\begin{aligned}
[M_{\mu\nu},\Pi] &= -2\eta_{0\mu}\Pi M_{0\nu} - 2\eta_{0\nu}\Pi M_{0\mu}\\
[M_{\mu\nu},\Pi M_{0i}] &= -i\eta_{\mu i}\Pi M_{0\nu} + i\eta_{\nu i}\Pi M_{0\mu} + \eta_{0\mu}\Pi t_{\nu i}\\
&\quad - \eta_{0\nu}\Pi t_{\mu i}\\
[M_{\mu\nu},\Pi t_{ij}] &= -\eta_{0\mu}\Pi t_{\nu ij} + \eta_{0\nu}\Pi t_{\mu ij} - i\eta_{\mu i}\Pi t_{\nu j}\\
&\quad + i\eta_{\nu i}\Pi t_{\mu j} - i\eta_{\mu j}\Pi t_{\nu i} + i\eta_{\nu j}\Pi t_{\mu i}.
\end{aligned} \tag{57}$$

For this representation the following relation holds:

$$t_{\rho\mu\nu} = \eta_{\mu\rho}M_{0\nu} + \eta_{\nu\rho}M_{0\mu} + 2\eta_{\mu\nu}M_{0\rho}. \tag{58}$$

The conclusion is that the operators $(\Pi, \Pi M_{0\mu}, \Pi\{M_{0\mu}, M_{0\nu}\})$ transform into themselves under the Lorentz group, and therefore the symmetric traceless tensor $S^{\mu\nu}$ must be constructed as a linear combination of these operators. It is easy to check that the appropriate combination is

$$S_{\mu\nu} = \Pi\eta_{\mu\nu} - i\Pi(\eta_{0\mu}M_{0\nu} + \eta_{0\nu}M_{0\mu}) - \Pi\{M_{0\mu}, M_{0\nu}\}. \tag{59}$$

These operators are also traceless in the "spinor" space, and consequently orthogonal to the unit operator. Using Eqs. (17) and (27) it is easy to show that

$$\{\chi, S^{\mu\nu}\} = 0, \tag{60}$$

which yields

$$\mathrm{Tr}(\chi S^{\mu\nu}) = 0. \tag{61}$$

Thus χ and $S^{\mu\nu}$ are also orthogonal operators.

A straightforward calculation yields the following commutation relations:

$$\begin{aligned}
i[M^{\mu\nu}, S^{\alpha\beta}] &= \eta^{\mu\alpha}S^{\nu\beta} - \eta^{\nu\alpha}S^{\mu\beta} + \eta^{\mu\beta}S^{\nu\alpha} - \eta^{\nu\beta}S^{\mu\alpha}\\
\{M^{\mu\nu}, S^{\alpha\beta}\} &= \varepsilon^{\mu\nu\sigma\beta}\chi S^{\alpha}{}_{\sigma} + \varepsilon^{\mu\nu\sigma\alpha}\chi S^{\beta}{}_{\sigma}\\
i[S^{\mu\nu}, S^{\alpha\beta}] &= \eta^{\mu\alpha}M^{\nu\beta} + \eta^{\nu\alpha}M^{\mu\beta} + \eta^{\nu\beta}M^{\mu\alpha} + \eta^{\mu\beta}M^{\nu\alpha}.
\end{aligned} \tag{62}$$

Using these relations and Eq. (60) it is possible to show that $\chi S_{\mu\nu}$ is also an independent set of nine orthogonal operators; thus the second traceless symmetric tensor is given by $\mathcal{S}_{\mu\nu} = \chi S_{\mu\nu}$.

Finally, the fourth-order Weyl-like tensor $C_{\mu\nu\alpha\beta}$ can be built from the product $M_{\mu\nu}M_{\alpha\beta}$ by applying the projection operator

$$\begin{array}{|c|c|}\hline \mu & \alpha \\ \hline \nu & \beta \\ \hline \end{array} \propto \mathcal{S}_{\mu\alpha}\mathcal{S}_{\nu\beta}\mathcal{A}_{\mu\nu}\mathcal{A}_{\alpha\beta} \tag{63}$$

which gives

$$T_{\mu\nu\alpha\beta} = 4\{M_{\mu\nu}, M_{\alpha\beta}\} + 2\{M_{\mu\alpha}, M_{\nu\beta}\} - 2\{M_{\mu\beta}, M_{\nu\alpha}\}. \tag{64}$$

Removing the Young-projected trace of this tensor we get

$$\begin{aligned}
C_{\mu\nu\alpha\beta} &= T_{\mu\nu\alpha\beta} + \frac{1}{2}(\eta_{\mu[\alpha}T_{\beta]\nu} - \eta_{\nu[\alpha}T_{\beta]\mu})\\
&\quad - \frac{1}{6}\eta_{\mu[\alpha}\eta_{\beta]\nu}T_{\rho}{}^{\rho}\\
&= T_{\mu\nu\alpha\beta} - 8(\eta_{\mu\alpha}\eta_{\nu\beta} - \eta_{\nu\alpha}\eta_{\mu\beta})
\end{aligned} \tag{65}$$

where

$$T_{\mu\nu} = T_{\mu\beta\nu}{}^{\beta}. \tag{66}$$

Finally, the following tensor

$$\begin{aligned}
C_{\mu\nu\alpha\beta} &= 4\{M_{\mu\nu}, M_{\alpha\beta}\} + 2\{M_{\mu\alpha}, M_{\nu\beta}\} - 2\{M_{\mu\beta}, M_{\nu\alpha}\}\\
&\quad - 8(\eta_{\mu\alpha}\eta_{\nu\beta} - \eta_{\nu\alpha}\eta_{\mu\beta})
\end{aligned} \tag{67}$$

obeys the symmetries in Eqs. (55) and the Bianchi identity in Eq. (56). These relations together with the vanishing of all contractions

$$C_{\mu}{}^{\beta}{}_{\alpha\beta} = 0, \tag{68}$$

leave only ten independent components. A direct calculation shows that this set is orthogonal to the previously constructed operators.

Summarizing our construction for this representation, the parity-based covariant basis for $(1,0)\oplus(0,1)$ is given by the set

$$\{1, \chi, S_{\mu\nu}, \chi S_{\mu\nu}, M_{\mu\nu}, C_{\mu\nu\alpha\beta}\}. \tag{69}$$

Concerning the Poincaré projector formalism, the most general antisymmetric tensor $T_{\mu\nu}$ for this representation can be expanded in terms of the operators in the basis in Eq. (69). Beyond $M_{\mu\nu}$ the only possibilities are the contractions $C_{\mu\nu\alpha\beta}M^{\alpha\beta}$ and $C_{\mu\nu\alpha\beta}\eta^{\alpha\beta}$. The latter contraction vanishes due to the properties of the C tensor and the former is not independent and in turn can be expanded in terms of the covariant basis. Since the only antisymmetric tensor with the appropriate symmetries is the Lorentz generator tensor, this contraction must be proportional to $M_{\mu\nu}$. In summary, the most general antisymmetric tensor for $(1,0)\oplus(0,1)$ representation space is of the form $gM_{\mu\nu}$.

It is clear that high spin brings into the construction commutators and anticommutators of the generators and it is important to realize the algebraic structure of these operators. We start with the Dirac space where the following commutation relations hold:

$$
\begin{aligned}
&[S_\mu, S_\nu] = 4iM_{\nu\mu} \qquad [\chi S_\mu, \chi S_\nu] = 4iM_{\mu\nu}\\
&[\chi S_\mu, S_\nu] = 2\eta_{\mu\nu}\chi \qquad [\chi, S_\mu] = 2\chi S_\mu\\
&[\chi, \chi S_\mu] = 2S_\mu.
\end{aligned}
\tag{70}
$$

For anticommutators, we get the following results:

$$
\begin{aligned}
&\{S_\mu, S_\nu\} = 2\eta_{\mu\nu}\mathbb{1} \qquad \{\chi, S_\mu\} = 0\\
&\{\chi S_\mu, \chi S_\nu\} = -2\eta_{\mu\nu}\mathbb{1} \qquad \{\chi, \chi S_\mu\} = 0\\
&\{\chi S_\mu, S_\nu\} = 4\tilde{M}_{\mu\nu} \qquad \{\chi, \chi\} = 2\mathbb{1}\\
&\{S_\rho, M_{\mu\nu}\} = -\varepsilon_{\rho\mu\nu\alpha}\chi S^\alpha \qquad \{\chi, M_{\mu\nu}\} = 2i\tilde{M}_{\mu\nu}\\
&\{\chi S_\rho, M_{\mu\nu}\} = -\varepsilon_{\rho\mu\nu\alpha}S^\alpha.
\end{aligned}
\tag{71}
$$

These relations are schematically summarized in Tables I and II.

Let us consider now the covariant basis we have constructed for the $(1,0)\oplus(0,1)$ representation. Besides the commutation rules in Eqs. (62), a straightforward calculation yields the following Lie brackets

$$
\begin{aligned}
[\chi S_{\mu\nu}, \chi S_{\mu\nu}] &= i\eta_{\mu\rho}M_{\nu\sigma} + i\eta_{\nu\rho}M_{\mu\sigma} + i\eta_{\mu\sigma}M_{\nu\rho}\\
&\quad + i\eta_{\nu\sigma}M_{\mu\rho}\\
[\chi S_{\mu\nu}, S_{\rho\sigma}] &= \frac{4}{3}\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \frac{1}{2}\eta_{\mu\nu}\eta_{\rho\sigma}\right)\\
&\quad - \frac{i}{6}(\tilde{C}_{\mu\rho\nu\sigma} + \tilde{C}_{\mu\sigma\nu\rho})\\
[\chi, S_{\mu\nu}] &= 2\chi S_{\mu\nu} \qquad [\chi, \chi S_{\mu\nu}] = 2S_{\mu\nu},
\end{aligned}
\tag{72}
$$

TABLE I. Algebraic Lie structure of the Dirac basis.

Lie $[\cdot,\cdot]$	$\mathbb{1}$	χ	S_μ	χS_μ	$M_{\mu\nu}$
$\mathbb{1}$	0	0	0	0	0
χ	0	0	χS_μ	S_μ	0
S_μ	0	χS_μ	$M_{\mu\nu}$	χ	S_μ
χS_μ	0	S_μ	χ	$M_{\mu\nu}$	χS_μ
$M_{\mu\nu}$	0	0	S_μ	χS_μ	$M_{\mu\nu}$

TABLE II. Algebraic Jordan structure of the Dirac basis.

Jordan $\{\cdot,\cdot\}$	$\mathbb{1}$	χ	S_μ	χS_μ	$M_{\mu\nu}$
$\mathbb{1}$	$\mathbb{1}$	χ	S_μ	χS_μ	$M_{\mu\nu}$
χ	χ	$\mathbb{1}$	0	0	$M_{\mu\nu}$
S_μ	S_μ	0	$\mathbb{1}$	$M_{\mu\nu}$	χS_μ
χS_μ	χS_μ	0	$M_{\mu\nu}$	$\mathbb{1}$	S_μ
$M_{\mu\nu}$	$M_{\mu\nu}$	$M_{\mu\nu}$	χS_μ	S_μ	$\mathbb{1}$, χ

and the anticommutators

$$
\begin{aligned}
\{S_{\mu\nu}, S_{\rho\sigma}\} &= \frac{4}{3}\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \frac{1}{2}\eta_{\mu\nu}\eta_{\rho\sigma}\right)\\
&\quad - \frac{1}{6}(C_{\mu\rho\nu\sigma} + C_{\mu\sigma\nu\rho})\\
\{\chi S_{\mu\nu}, \chi S_{\mu\nu}\} &= -\frac{4}{3}\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \frac{1}{2}\eta_{\mu\nu}\eta_{\rho\sigma}\right)\\
&\quad + \frac{1}{6}(C_{\mu\rho\nu\sigma} + C_{\mu\sigma\nu\rho})\\
\{\chi S_{\mu\nu}, S_{\rho\sigma}\} &= \frac{1}{2}(\eta_{\mu\rho}\tilde{M}_{\nu\sigma} + \eta_{\nu\sigma}\tilde{M}_{\mu\rho})\\
&\quad + \frac{1}{2}(\eta_{\mu\sigma}\tilde{M}_{\nu\rho} + \eta_{\nu\rho}\tilde{M}_{\mu\sigma})\\
\{M_{\mu\nu}, M_{\rho\sigma}\} &= \frac{4}{3}(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho})\\
&\quad - \frac{8}{6}i\varepsilon_{\mu\nu\rho\sigma}\chi + \frac{1}{6}C_{\mu\nu\rho\sigma}\\
\{\chi, S_{\mu\nu}\} &= 0 \qquad \{\chi, \chi S_{\mu\nu}\} = 0.
\end{aligned}
\tag{73}
$$

Here $\tilde{C}_{\mu\rho\nu\sigma} = \frac{1}{2}\epsilon_{\mu\rho}{}^{\alpha\beta}C_{\alpha\beta\nu\sigma} = -i\chi C_{\mu\rho\nu\sigma}$. The similarities with the Dirac case can best be seen in Tables III and IV.

B. Lorentz structure of the $(3/2, 0)\oplus(0, 3/2)$

As a final explicit example let us now consider the $j = \frac{3}{2}$ case. The angular momentum matrices are given by

TABLE III. Algebraic Lie structure of the $(1,0)\oplus(0,1)$ basis.

Lie $[\cdot,\cdot]$	$\mathbb{1}$	χ	$S_{\mu\nu}$	$\chi S_{\mu\nu}$	$M_{\mu\nu}$	$C_{\mu\nu\rho\sigma}$
$\mathbb{1}$	0	0	0	0	0	0
χ	0	0	$\chi S_{\mu\nu}$	$S_{\mu\nu}$	0	0
$S_{\mu\nu}$	0	$\chi S_{\mu\nu}$	$M_{\mu\nu}$	χ, $C_{\mu\nu\rho\sigma}$	$S_{\mu\nu}$	$\chi S_{\mu\nu}$
$\chi S_{\mu\nu}$	0	$S_{\mu\nu}$	χ, $C_{\mu\nu\rho\sigma}$	$M_{\mu\nu}$	$\chi S_{\mu\nu}$	$S_{\mu\nu}$
$M_{\mu\nu}$	0	0	$S_{\mu\nu}$	$\chi S_{\mu\nu}$	$M_{\mu\nu}$	$C_{\mu\nu\rho\sigma}$
$C_{\mu\nu\rho\sigma}$	0	0	$\chi S_{\mu\nu}$	$S_{\mu\nu}$	$C_{\mu\nu\rho\sigma}$	$M_{\mu\nu}$

TABLE IV. Algebraic Jordan structure of the $(1,0)\oplus(0,1)$ basis.

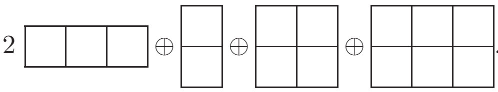
Jordan $\{\cdot,\cdot\}$	$\mathbb{1}$	χ	$S_{\mu\nu}$	$\chi S_{\mu\nu}$	$M_{\mu\nu}$	$C_{\mu\nu\rho\sigma}$
$\mathbb{1}$	$\mathbb{1}$	χ	$S_{\mu\nu}$	$\chi S_{\mu\nu}$	$M_{\mu\nu}$	$C_{\mu\nu\rho\sigma}$
χ	χ	$\mathbb{1}$	0	0	$M_{\mu\nu}$	$C_{\mu\nu\rho\sigma}$
$S_{\mu\nu}$	$S_{\mu\nu}$	0	$\mathbb{1}$, $C_{\mu\nu\rho\sigma}$	$M_{\mu\nu}$	$\chi S_{\mu\nu}$	$S_{\mu\nu}$
$\chi S_{\mu\nu}$	$\chi S_{\mu\nu}$	0	$M_{\mu\nu}$	$\mathbb{1}$, $C_{\mu\nu\rho\sigma}$	$S_{\mu\nu}$	$\chi S_{\mu\nu}$
$M_{\mu\nu}$	$M_{\mu\nu}$	$M_{\mu\nu}$	$\chi S_{\mu\nu}$	$S_{\mu\nu}$	$\mathbb{1}$, χ, $C_{\mu\nu\rho\sigma}$	$M_{\mu\nu}$
$C_{\mu\nu\rho\sigma}$	$C_{\mu\nu\rho\sigma}$	$C_{\mu\nu\rho\sigma}$	$S_{\mu\nu}$	$\chi S_{\mu\nu}$	$M_{\mu\nu}$	$\mathbb{1}$, χ, $C_{\mu\nu\rho\sigma}$

$$\tau_1 = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$
$$\tau_2 = \frac{i}{2}\begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \quad (74)$$
$$\tau_3 = \frac{1}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

In this case the external product of states in the basis decomposes as

$$(0,0)_2 \oplus \left(\frac{3}{2},\frac{3}{2}\right)_2 \oplus (1,0) \oplus (0,1) \oplus (2,0) \oplus (0,2) \oplus (3,0) \oplus (0,3). \quad (75)$$

Here, besides the $\mathbb{1}$ and χ scalars, we have the decomposition

2  (76)

This corresponds to a pair of third-rank totally symmetrical tensors, and the operators

$$\{M_{\mu\nu}, C_{\mu\nu\rho\sigma}, D_{\mu\nu\rho\sigma\alpha\beta}\}. \quad (77)$$

The C tensor is given by Eq. (68) with the appropriate changes in the $M_{\mu\nu}$ operators. In the calculation of the remaining tensors it is useful to define the quantity

$$\{A, B, C\} = ABC + ACB + BAC + BCA + CAB + CBA.$$

The symmetric tensor is constructed along the lines of the $j = 1$ case. We just quote the final result

$$S_{\mu\nu\rho} = \frac{1}{2}\Pi(-\eta_{0\mu}\eta_{0\nu}\eta_{0\rho} + \eta_{\mu\nu}\eta_{0\rho} + \eta_{\mu\rho}\eta_{0\mu} + \eta_{\rho\nu}\eta_{0\mu}) + \frac{i}{9}\Pi[7\eta_{\mu\nu}M_{0\rho} + 7\eta_{\mu\rho}M_{0\nu} + 7\eta_{\nu\rho}M_{0\mu}]$$
$$- i\Pi(\eta_{0\mu}\eta_{0\nu}M_{0\rho} + \eta_{0\mu}\eta_{0\rho}M_{0\nu} + \eta_{0\nu}\eta_{0\rho}M_{0\mu}) + \frac{2i}{9}\Pi\{M_{0\mu}, M_{0\nu}, M_{0\rho}\}. \quad (78)$$

To build the sixth-order tensor $D_{\mu\nu\rho\sigma\alpha\beta}$, we apply the Young projector

$$\begin{array}{|c|c|c|}\hline \mu & \alpha & \rho \\ \hline \nu & \beta & \sigma \\ \hline \end{array}$$

to the product $M_{\mu\nu}M_{\rho\sigma}M_{\alpha\beta}$ to get

$$Y_{\mu\nu\rho\sigma\alpha\beta} = \frac{4}{18}\{M_{\mu\nu}, M_{\rho\sigma}, M_{\alpha\beta}\} + \frac{2}{18}\{M_{\mu\nu}, M_{\rho\alpha}, M_{\sigma\beta}\} - \frac{2}{18}\{M_{\mu\nu}, M_{\rho\beta}, M_{\alpha\sigma}\} + \frac{2}{18}\{M_{\mu\alpha}, M_{\rho\sigma}, M_{\nu\beta}\}$$
$$- \frac{2}{18}\{M_{\mu\beta}, M_{\rho\beta}, M_{\nu\alpha}\} + \frac{2}{18}\{M_{\mu\rho}, M_{\nu\sigma}, M_{\alpha\beta}\} - \frac{2}{18}\{M_{\mu\sigma}, M_{\nu\rho}, M_{\alpha\beta}\} + \frac{1}{18}\{M_{\mu\rho}, M_{\nu\alpha}, M_{\sigma\beta}\}$$
$$- \frac{1}{18}\{M_{\mu\rho}, M_{\nu\beta}, M_{\sigma\alpha}\} + \frac{1}{18}\{M_{\mu\sigma}, M_{\nu\beta}, M_{\rho\alpha}\} - \frac{1}{18}\{M_{\mu\sigma}, M_{\nu\alpha}, M_{\rho\beta}\} + \frac{1}{18}\{M_{\mu\alpha}, M_{\nu\sigma}, M_{\rho\beta}\}$$
$$- \frac{1}{18}\{M_{\mu\alpha}, M_{\nu\rho}, M_{\sigma\beta}\} + \frac{1}{18}\{M_{\mu\beta}, M_{\nu\rho}, M_{\sigma\alpha}\} - \frac{1}{18}\{M_{\mu\beta}, M_{\nu\sigma}, M_{\rho\alpha}\}. \quad (79)$$

As with the C tensor, this tensor is not traceless and we need to remove the Young-projected contractions, which are of the form

$$Y_{\mu}{}^{\rho}{}_{\rho\sigma\alpha\beta} = -\eta_{\mu\sigma}M_{\alpha\beta} + \frac{1}{2}\eta_{\mu\alpha}M_{\beta\sigma} - \frac{1}{2}\eta_{\mu\beta}M_{\alpha\sigma} + \frac{1}{2}\eta_{\sigma\beta}M_{\mu\alpha} - \frac{1}{2}\eta_{\sigma\alpha}M_{\mu\beta} \equiv Y_{\mu\sigma\alpha\beta}. \quad (80)$$

The Young-projected trace is proportional to the following tensor:

$$\Lambda_{\mu\nu\rho\sigma\alpha\beta} = \eta_{\mu\rho}Y_{\nu\sigma\alpha\beta} - \eta_{\nu\rho}Y_{\mu\sigma\alpha\beta} - \eta_{\mu\sigma}Y_{\nu\rho\alpha\beta} + \eta_{\nu\sigma}Y_{\mu\rho\alpha\beta} + \eta_{\mu\alpha}Y_{\nu\beta\rho\sigma} - \eta_{\nu\alpha}Y_{\mu\beta\rho\sigma}$$
$$- \eta_{\mu\beta}Y_{\nu\alpha\rho\sigma} + \eta_{\nu\beta}Y_{\mu\alpha\rho\sigma} + \eta_{\rho\alpha}Y_{\sigma\beta\mu\nu} - \eta_{\sigma\alpha}Y_{\rho\beta\mu\nu} - \eta_{\rho\beta}Y_{\sigma\alpha\mu\nu} + \eta_{\sigma\beta}Y_{\rho\alpha\mu\nu}. \quad (81)$$

The final result for the D tensor is

$$D_{\mu\nu\rho\sigma\alpha\beta} = Y_{\mu\nu\rho\sigma\alpha\beta} + \frac{41}{60}\Lambda_{\mu\nu\rho\sigma\alpha\beta}. \tag{82}$$

This tensor is antisymmetric in the $\mu\nu$, $\rho\sigma$ and $\alpha\beta$ indices; completely symmetric under the exchange of these pairs; and traceless (the contraction of any pair of indices vanishes). It also satisfies the generalized Bianchi identity

$$D_{\mu\nu\rho\sigma\alpha\beta} + D_{\mu\nu\rho\alpha\beta\sigma} + D_{\mu\nu\rho\beta\sigma\alpha} = 0. \tag{83}$$

Notice that the D tensor is like a symmetric third-rank tensor in six dimensions (the six possible values of the antisymmetric pairs of indices), which has 56 components. The traceless condition then removes 36 of those components, and the generalized Bianchi identity removes another six, leaving only 14 independent components which are the degrees of freedom for the $(3,0)\oplus(0,3)$ representation.

Concerning the Poincaré projector formalism for the $(3/2,0)\oplus(0,3/2)$ representation, the antisymmetric part of the space-time tensor $T_{\mu\nu}$ in Eq. (20) must be constructed with the elements of the basis. Here, in principle we can have contractions of C and D tensors among themselves and with products of the metric tensor or of the generators. All these possible products are operators that can be expanded in terms of the basis and the result must be a rank 2 tensor antisymmetric under the exchange $\mu \leftrightarrow \nu$. Since the only rank 2 tensor with this property in the basis is the Lorentz generator tensor, these products must be proportional to $M_{\mu\nu}$; thus electromagnetic properties are in this case also of the form $gM_{\mu\nu}F^{\mu\nu}$ when we use the gauge principle.

C. The general structure of $(j,0)\oplus(0,j)$ fields

In general, for the $(4j+2)$-dimensional representations $(j,0)\oplus(0,j)$ the external product of the states in the basis has the decomposition

$$[(j,0)\oplus(0,j)]^2 = \bigoplus_{i=0}^{2j}[(i,0)\oplus(0,i)]\oplus 2(j,j). \tag{84}$$

We can construct, for every j, a set of operators which form a basis for this square space. In general this set will contain the scalars $\{\mathbb{1},\chi\}$, and a pair of symmetrical tensors transforming as (j,j). The chirality and parity operators are given by Eqs. (25) and (26). Parity turns out to be the time component of the totally symmetric tensor $S^{\mu_1\mu_2\cdots\mu_{2j}}$ transforming as (j,j). In general chirality and parity anticommute which in turn causes that chirality and the symmetric tensor S to also anticommute. The second symmetric tensor is given in general as $\chi S^{\mu_1\mu_2\cdots\mu_{2j}}$. In addition the covariant basis contains the series

$$\bigoplus_{i=1}^{2j}[(i,0)\oplus(0,i)] = \yng(1,1)\oplus\yng(2,2)\oplus\yng(3,3)\oplus\ldots \tag{85}$$

which gives the generators, plus a series of generalizations of the Weyl tensor

$$\{M_{\mu\nu}, C_{\mu\nu\rho\sigma}, D_{\mu\nu\rho\sigma\alpha\beta}, E_{\mu\nu\rho\sigma\alpha\beta\tau\delta},\ldots\}. \tag{86}$$

These tensors are to be constructed by taking the product of $2j$ generators, applying the Young projector and removing all contractions.

Concerning the Poincaré projector formalism for the $(j,0)\oplus(0,j)$, the antisymmetric part of the space-time tensor $T_{\mu\nu}$ in Eq. (20) in general must be constructed with the elements of a basis of this space. When using our covariant basis, due to the properties of the $C, D, E\ldots$ tensors coming from the Young projectors it is clear that the result found for $j = 1/2, 1, 3/2$ is valid for any j. The contractions of these tensors among themselves, with the metric tensor or with the generators yielding a rank 2 antisymmetric tensor, are necessarily proportional to $M_{\mu\nu}$ or they vanish. In consequence, for arbitrary j, the most general space-time tensor in Eq. (20) is given by

$$T_{\mu\nu} = \eta_{\mu\nu} - igM_{\mu\nu}. \tag{87}$$

There are two direct consequences of this result: (i) the multipole electromagnetic moments of a spin j particle in this formalism are dictated by two free parameters, the electric charge e and the gyromagnetic factor g; and (ii) the propagation of spin j waves in an electromagnetic background is causal. In the next section we elaborate on these points.

IV. ELECTROMAGNETIC STRUCTURE AND CAUSAL PROPAGATION FOR THE $(j,0)\oplus(0,j)$ REPRESENTATION

In the Poincaré projector formalism, the Lagrangian for an interacting elementary particle transforming in the $(j,0)\oplus(0,j)$ representation is

$$\mathcal{L} = \overline{D^\mu\psi}T_{\mu\nu}D^\nu\psi - m^2\bar{\psi}\psi, \tag{88}$$

where $\bar{\psi} = \psi^\dagger\Pi$, $D^\mu = \partial^\mu + ieA^\mu$ for a particle of charge e and $T_{\mu\nu}$ stands for a space-time tensor. The Poincaré projector fixes only the symmetric part of this tensor. The antisymmetric part must be constructed in terms of the basis for the operators acting on the $(j,0)\oplus(0,j)$ representation. According to results in the last section, the most general space-time tensor is given by the tensor in Eq. (87). This is the tensor used in [28] to calculate the multipole moments of a particle transforming in these representations for $j = 1/2, 1, 3/2$. It is shown there that the multipole moments are dictated solely by the two parameters appearing in the Lagrangian, the charge e and the gyromagnetic factor g. Our results in the previous sections put these calculations on a firm basis and allow us to generalize them to particles of arbitrary spin j.

The electromagnetic current is simply calculated to be

$$J_\mu(p',\lambda';p,\lambda) = e\bar{u}(p',\lambda')[(p'+p)_\mu + igM_{\mu\nu}(p'-p)^\nu]u(p,\lambda). \quad (89)$$

The multipole moments of this current can be calculated from the charge and current densities using the Breit frame where

$$p' = (\omega/2, \mathbf{q}/2), \qquad p = (\omega/2, -\mathbf{q}/2). \quad (90)$$

In terms of the Breit current defined by

$$J^B_\mu(\mathbf{q}, j, \lambda) = \frac{1}{\omega} J_\mu(\mathbf{p}', \lambda; \mathbf{p}, \lambda), \quad (91)$$

with $\omega = \sqrt{4m^2 + \mathbf{q}^2}$, the electromagnetic moments for a particle of spin j and polarization λ are given by

$$Q^l_E(\mathbf{q}, j, \lambda) = b^{l0}(-i\nabla_\mathbf{q})\varrho_E(\mathbf{q}, j, \lambda)|_{\mathbf{q}=0},$$
$$Q^l_M(\mathbf{q}, s, \lambda) = \frac{1}{l+1} b^{l0}(-i\nabla_\mathbf{q})\varrho_M(\mathbf{q}, j, \lambda)|_{\mathbf{q}=0}, \quad (92)$$

where the electric, $\varrho_E(\mathbf{q}, j, \lambda)$, and the magnetic, $\varrho_M(\mathbf{q}, j, \lambda)$, densities are given by

$$\varrho_E(\mathbf{q}, j, \lambda) = J^0_B(\mathbf{q}, j, \lambda),$$
$$\varrho_M(\mathbf{q}, j, \lambda) = \nabla_\mathbf{q} \cdot [\mathbf{J}^B(\mathbf{q}, j, \lambda) \times \mathbf{q}], \quad (93)$$

and the b^{l0} operators are given by

$$b^{l0}(\mathbf{r}) = l!\sqrt{4\pi/(2l+1)} r^l Y_{l0}(\Omega). \quad (94)$$

Explicitly, for $l = 1, 2, \ldots 8$ these operators read

$$b^{00}(\mathbf{r}) = 1, \quad b^{10}(\mathbf{r}) = z, \quad b^{20}(\mathbf{r}) = 3z^2 - r^2,$$
$$b^{30}(\mathbf{r}) = 3z(5z^2 - 3r^2), \quad b^{40}(\mathbf{r}) = 3(35z^4 - 30z^2r^2 + 3r^4),$$
$$b^{50}(\mathbf{r}) = 15z(63z^4 - 70z^2r^2 + 15r^4),$$
$$b^{60}(\mathbf{r}) = 45(231z^6 - 315z^4r^2 + 105z^2r^4 - 5r^6),$$
$$b^{70}(\mathbf{r}) = 315z(429z^6 - 693z^4r^2 + 315z^2r^4 - 35r^6),$$
$$b^{80}(\mathbf{r}) = 315(6435z^8 - 12012z^6r^2 + 6930z^4r^4 - 1260z^2r^6 + 35r^8). \quad (95)$$

The electromagnetic current can be rewritten as

$$J_\mu(p',\lambda';p,\lambda) = e\bar{u}(0,\lambda')[(p'+p)_\mu B(-p')B(p) + igB(-p')M_{\mu\nu}B(p)(p'-p)^\nu]u(0,\lambda), \quad (96)$$

where $B(p)$ stands for the boost operator.

In the Breit frame we obtain

$$B(-p') = \exp[i\mathbf{K}\cdot\boldsymbol{\varphi}'], \qquad B(p) = \exp[-i\mathbf{K}\cdot\boldsymbol{\varphi}] \quad (97)$$

with

$$\cosh\varphi' = \frac{\omega}{2m} = \cosh\varphi, \qquad \sinh\varphi' = \frac{|\mathbf{q}|}{2m} = \sinh\varphi. \quad (98)$$

In terms of the unitary vector $\mathbf{n} = \mathbf{q}/|\mathbf{q}|$ the corresponding angles are

$$\boldsymbol{\varphi} = -\mathbf{n}\varphi, \qquad \boldsymbol{\varphi}' = \mathbf{n}\varphi, \quad (99)$$

and hence

$$B(-p') = B(p) = \exp[i\mathbf{K}\cdot\mathbf{n}\varphi] \equiv B(\mathbf{q}). \quad (100)$$

The time component of the electromagnetic current then becomes

$$J_0(p',\lambda';p,\lambda) = e\bar{u}(0,\lambda')[(\omega - ig\mathbf{K}\cdot\mathbf{n}|\mathbf{q}|) \times \exp[i2\mathbf{K}\cdot\mathbf{n}\varphi]]u(0,\lambda). \quad (101)$$

For the representations $(j,0)\oplus(0,j)$ the rotations and boosts generators are related as $i\mathbf{K} = -\chi\mathbf{J}$, which when used for the charge density in the Breit frame yield

$$\varrho_E(\mathbf{q}, j, \lambda) = e\bar{u}(0,\lambda')O(\rho, x)u(0,\lambda), \quad (102)$$

where $x = \frac{|\mathbf{q}|}{2m}$, $\rho = \mathbf{J}\cdot\mathbf{n}$ and the operator $O(\rho, x)$ is given by

$$O(\rho, x) = \left(1 + \frac{g\rho\chi x}{\sqrt{1+x^2}}\right)\exp[-2\chi\rho \sin h^{-1}(x)]. \quad (103)$$

The electric multipoles involve the matrix elements of derivatives with respect to q_i of this operator; for this reason, it is convenient to expand $O(\rho, x)$ in powers of x. Expanding and using $\chi^2 = 1$ we get

$$O(\rho, x) = 1 + (g-2)\chi\rho x - 2(g-1)\rho^2x^2 + \frac{\rho}{6}[(12g-8)\rho^2 - (3g-2)]\chi x^3 + \cdots \quad (104)$$

The calculation of the lth multipole requires the lth derivatives of this operator, with only the order x^l term contributing. Using $\{\chi, \Pi\} = 0$, $[\chi, \mathbf{J}] = 0$ and $[\Pi, \mathbf{J}] = 0$, it can be shown that the matrix elements of odd powers of x between states of the same parity vanish. As a consequence, odd electric multipole moments vanish for particles of well-defined parity. Skipping odd terms in the expansion we rewrite the operator in Eq. (103) up to order x^8 as

$$O(\rho, x) = 1 - 2(g-1)\rho^2x^2 - \frac{2(2g-1)}{3}[\rho^4 - \rho^2]x^4 - \frac{4(3g-1)}{45}[\rho^6 - 5\rho^4 + 4\rho^2]x^6 - \frac{2(4g-1)}{315}[\rho^8 - 14\rho^6 + 49\rho^4 - 36\rho^2]x^8 + \ldots \quad (105)$$

The calculation of the electric multipole moments for arbitrary values of j and λ is now straightforward. The first five nonvanishing electric multipole moments, for arbitrary j, are given by

$$Q_E^0(j,\lambda) = e \qquad Q_E^2(j,\lambda) = -\frac{e(g-1)}{m^2}\langle \mathbf{J}^2 - 3J_z^2\rangle$$

$$Q_E^4(j,\lambda) = -\frac{e}{m^4}3(2g-1)\langle 3\mathbf{J}^4 - 30\mathbf{J}^2J_z^2 + 35J_z^4 - 6\mathbf{J}^2 + 25J_z^2\rangle$$

$$Q_E^6(j,\lambda) = -\frac{e}{m^6}(3g-1)45\langle 5\mathbf{J}^6 - 105\mathbf{J}^4J_z^2 + 315\mathbf{J}^2J_z^4 - 231J_z^6 - 40\mathbf{J}^4 + 525\mathbf{J}^2J_z^2 - 735J_z^4 + 60\mathbf{J}^2 - 294J_z^2\rangle$$

$$Q_E^8(j,\lambda) = -\frac{e}{m^8}(4g-1)315\langle 35J^8 - 1260J^6Jz^2 - 700J^6 + 6930J^4Jz^4 + 18270J^4Jz^2 + 3780J^4 - 12012J^2Jz^66 - 64680J^2Jz^4 - 59388J^2Jz^2 - 5040J^2 + 6435Jz^8 + 54054Jz^6 + 93555Jz^4 + 27396Jz^2\rangle \tag{106}$$

where we used the shorthand notation

$$\langle O\rangle \equiv \bar{u}(0,\lambda)Ou(0,\lambda). \tag{107}$$

It is worth remarking that, for a given j, the special combination of $\mathbf{J}^2$ and J_z appearing in $Q_E^l(j,\lambda)$ in Eqs. (106) vanishes for $l > 2j$. This is a consequence of the full algebraic structure of the covariant basis for $(j,0)\oplus(0,j)$ representation, which at this level manifests in the fact that the rotation generators satisfy

$$\prod_{\lambda=-j}^{j}(\mathbf{J}\cdot\mathbf{n} - \lambda) = 0, \tag{108}$$

for an arbitrary unitary vector $\mathbf{n}$. For $\mathbf{n} = \mathbf{k}$ this relation lowers the powers of J_z appearing in $Q_E^l(j,\lambda)$ and causes it to vanish for $l > 2j$. The simplest example is $j = 1/2$ in whose case $J_z^2 = 1/4$. In this case, the combinations of $\mathbf{J}^2$ and J_z appearing in $Q_E^l(j,\lambda)$ for $l > 1$ reduce to the unity operator ($\mathbf{J}^2$ is diagonal) with vanishing coefficient, as can be easily checked. For $j = 1$ we get $J_z^3 = J_z$ in which case the combinations of $\mathbf{J}^2$ and J_z appearing in $Q_E^l(j,\lambda)$ for $l > 2$ reduce to a linear combination of the unity operator and J_z with vanishing coefficients. Similar results are obtained for higher values of j. Therefore, we understand the well-known fact that a spin j particle can have at most $2j$ nonvanishing electric multipole moments as a consequence of the full algebra satisfied by the elements of the basis of operators acting on the $(j,0)\oplus(0,j)$ representation.

As for the magnetic current a similar calculation yields

$$\varrho_M(\mathbf{q},j,\lambda) = ieg[\bar{u}(0,\lambda)[\nabla_{\mathbf{q}}\cdot\mathbf{M}(\mathbf{q},j)]u(0,\lambda)] \tag{109}$$

with

$$\mathbf{M}(\mathbf{q},j) = \frac{1}{\omega}B(\mathbf{q})[(\mathbf{J}\cdot\mathbf{q})\mathbf{q} - |\mathbf{q}|^2\mathbf{J}]B(\mathbf{q}). \tag{110}$$

To evaluate the magnetic multipoles we calculate the derivatives of this operator and then evaluate them at $\mathbf{q} = 0$. Notice that the lth magnetic moment receives contributions only of the term $\mathbf{q}^{l+1}$ in the expansion of $\mathbf{M}(\mathbf{q},j)$. In particular, the calculation of Q_M^1 does not involve the specific structure of the boost operator. For l even, the term $\mathbf{q}^{l+1}$ in the expansion of $\mathbf{M}(\mathbf{q},j)$ contains a factor χ and the even magnetic multipole moments vanish for the same reason as the odd electric moments do. The nonvanishing lowest order magnetic multipoles are given by

$$Q_M^1(j,\lambda) = \frac{eg}{2m}\langle J_z\rangle \qquad Q_M^3(j,\lambda) = \frac{eg}{2m^3}9\langle 3\mathbf{J}^2J_z - 5J_z^3 - J_z\rangle$$

$$Q_M^5(j,\lambda) = \frac{eg}{2m^5}\frac{75}{2}\langle 15\mathbf{J}^4J_z - 70\mathbf{J}^2J_z^3 + 63J_z^5 - 50\mathbf{J}^2J_z + 105J_z^3 + 12J_z\rangle.$$

$$Q_M^7(j,\lambda) = \frac{eg}{2m^7}\frac{2205}{2}\langle 35\mathbf{J}^6J_z - 315\mathbf{J}^4J_z^3 + 693\mathbf{J}^2J_z^5 - 429J_z^7 - 385\mathbf{J}^4J_z + 2205\mathbf{J}^2J_z^3 - 2310J_z^5 + 882\mathbf{J}^2J_z - 2121J_z^3 - 180J_z\rangle. \tag{111}$$

We would like to remark that, beyond the gauge principle, our calculation depends only on the space-time structure of the $(j,0)\oplus(0,j)$ representation: (i) the Poincaré projector, (ii) the explicit form of the Lorentz generators and (iii) the properties of the parity-based covariant basis. This is, therefore, a calculation from first principles, and our main result in this section is that, beyond the electric charge, all multipole moments for an elementary particle in this formalism are dictated by a single Lorentz structure, namely $M_{\mu\nu}$, and consequently all multipole moments are dictated by the value of the corresponding constant, g, the gyromagnetic factor. This is a free parameter in the Poincaré projector formalism but for low values of j it has been fixed to be $g = 2$ [28–32]. On the other hand, there is a variety of consistency arguments for this value to be universal (see [37] for a review and further references) and we consider this value in our numerical computations below.

The existence of relations among multipole moments for elementary particles has been noticed before. The calculation of Q_M^1, Q_E^2 and Q_M^3 for $j = 1/2, 1, 3/2$ in the Poincaré projector formalism was done in Ref. [28]. Our alternative derivation confirms results in this work but the form of Q_M^3 in Eq. (4.52d) of [28] is valid only for $j = 3/2$. The relation between Q_M^3 and the matrix elements of the rotation generators valid for every value of j is given in Eqs. (111) and for $j = 3/2$ agrees with Eq. (4.52d) of [28]. Recently, a systematic calculation of the multipole moments for particles of arbitrary spin j transforming in the Rarita-Schwinger representations was performed in Ref. [38], using a well-motivated ansatz for the electromagnetic current written in terms of the covariant Lorentz structures and multipole form factors $G_{El}(q^2)$ and $G_{Ml}(q^2)$. As remarked there, as defined in this reference, $G_{El}(0)$ and $G_{Ml}(0)$ have an interpretation of electromagnetic multipole moments only for $l = 0, 1, 2$. For $l > 2$ these form factors at zero transferred momentum are related to conventional electromagnetic multipole moments through l-dependent factors. The multipole form factors $G_{El}(0)$ and $G_{Ml}(0)$ calculated in that work are related to our multipole moments in Eqs. (106) and (111) with $\lambda = j$. The appropriate quantities to compare with results obtained in Ref. [38] are [39]

$$\hat{G}_{El} = \frac{m^l}{e}\frac{2^l}{(l!)^2}Q^l_E(j,j), \qquad \hat{G}_{Ml} = \frac{2m^l}{e}\frac{2^l}{(l!)^2}Q^l_M(j,j). \quad (112)$$

We list the values for $\hat{G}_{El}$ and $\hat{G}_{Ml}$ predicted by the Poincaré projector formalism for the lowest values of j and l in Table V. In general, spin j particles transforming in different representations of the Lorentz group have different electromagnetic properties. This has been noticed before in Ref. [28] based on a similar calculation for $j = 1$, $3/2$. The only representation-independent multipole moments are the charge and the magnetic moment. This is not a surprising result because these are the only multipole moments that are independent of the specific structure of the boost operators. The electric charge is completely independent of the Lorentz structure (it is associated to a global symmetry), whereas the magnetic moment is related to the rotation generators, which for a fixed value of j have the same algebraic structure even if they are embedded in different Lorentz representations. Beyond these multipoles, it is clear from Eqs. (101) and (110) that the specific structure of the boost operators becomes important, and since this structure is different for different representations, we should not expect the same multipole moments, a fact reflected in Table V when compared with Table I of Ref. [38]. In general, the quantities in Eq. (112) for the $(j,0)\oplus(0,j)$ representation obtained here are related to those for the Rarita-Schwinger representation obtained in Ref. [38] by

TABLE V. Multipole moments normalized according to Eq. (112).

j	$\hat{G}_{E0}$	$\hat{G}_{M1}$	$\hat{G}_{E2}$	$\hat{G}_{M3}$	$\hat{G}_{E4}$	$\hat{G}_{M5}$	$\hat{G}_{E6}$	$\hat{G}_{M7}$	$\hat{G}_{E8}$
0	1	0	0	0	0	0	0	0	0
1/2	1	1	0	0	0	0	0	0	0
1	1	2	1	0	0	0	0	0	0
3/2	1	3	3	−3	0	0	0	0	0
2	1	4	6	−12	−3	0	0	0	0
5/2	1	5	10	−30	−15	5	0	0	0
3	1	6	15	−60	−45	30	5	0	0
7/2	1	7	21	−105	−105	105	35	−7	0
4	1	8	28	−168	−210	280	140	−56	−7

$$\hat{G}_{El} = \left(1 - \frac{gl}{2}\right)G_{El}(0), \qquad \hat{G}_{Ml} = \frac{gl}{2}G_{Ml}(0). \quad (113)$$

Finally, we would like to remark on another important side result of our construction of the covariant basis for operators acting on the $(j,0)\oplus(0,j)$ representation: the restriction of the antisymmetric part of the space-time tensor $T_{\mu\nu}$ to be given solely by the Lorentz generators' tensor $M_{\mu\nu}$ yields causal propagation of spin j waves in an electromagnetic background. Indeed, this problem has been studied in Ref. [40] for $j = 1$ under the assumption that the most general tensor is given by Eq. (87) for $j = 1$. Once we have proved that this is indeed the case and that this result is valid for arbitrary j, the generalization to the propagation of high spin waves is straightforward. The gauged classical equation of motion for arbitrary j can be rewritten as

$$\left(D^\mu D_\mu + \frac{g}{2}M_{\mu\nu}F^{\mu\nu} + m^2\right)\psi = 0. \quad (114)$$

Notice that Eq. (114) is a set of coupled equations for the $2(2j+1)$ components of the spinor ψ_i. Using the form of the generators in Eq. (24) it is easy to see that all the components have a second time derivative. The characteristic determinant is the determinant of the operator $O(n)$ obtained from the highest derivatives in this equation when derivatives of the field $\partial^\mu\psi$ are replaced by a constant four-vector n^μ. The nature of the propagation of classical waves is determined by the vanishing of the characteristic determinant

$$\det[O(n)] = (n^2)^{2(2j+1)} = 0. \quad (115)$$

The solutions for the timelike components of n^μ are $n^0 = \sqrt{n_x^2 + n_y^2 + n_z^2}$ which are always real and according to the Currant-Hilbert criterion the propagation of spin j waves in the Poincaré projector formalism is causal.

Since all the components ψ_i have a second time derivative either in the free or in the interacting case, as discussed in [40] for the case $j = 1$, this formalism actually describes the dynamics of a degenerate parity doublet. As pointed out by Weinberg [15], when the field is built as in Eq. (6) the properties of the one-particle states described by the field are related to the properties of the coefficients $\omega_l(\Gamma)$, $\omega^c_l(\Gamma)$. Beyond mass and spin, these coefficients furnish a record of the choice made to ensure the desired properties of the field under discrete transformations. In the Poincaré projector formalism, the dynamics is dictated solely by the projection onto Poincaré eigensubspaces and the appropriate degrees of freedom for the single-particle states are fixed through a judicious choice of the coefficients $\omega_l(\Gamma)$, $\omega^c_l(\Gamma)$. The spacetime properties of the parity operator are such that it commutes with the Poincaré projector. This justifies our use of definite-parity states for the calculation of the electromagnetic moments.

V. CONCLUSIONS

In this work we study the transformation properties of the rest-frame parity operator under Lorentz transformations for the $(j, 0) \oplus (0, j)$ representation spaces. We show that while rotating as a scalar, under boosts its transformation properties involve the Jordan algebra of the generators, which is representation dependent. Using these general properties and the Young projectors for $\mathfrak{so}(1, 3)$ we show that the rest-frame parity operator transforms as the completely temporal component of a symmetric traceless tensor of rank $2j$. We provide an algorithm for the calculation of a covariant basis for arbitrary j. For a given j, this basis contains the corresponding identity ($\mathbb{1}$) and chirality (χ) operators, the Lorentz generators ($M_{\mu\nu}$) and two symmetric traceless tensors of rank $2j$. The time component of the first symmetric traceless tensor denoted by $S^{\mu_1\mu_2\cdots\mu_{2j}}$ is precisely the rest-frame parity operator and the second symmetric traceless operator is given by the product $\chi S^{\mu_1\mu_2\cdots\mu_{2j}}$. In addition, for $j > 1$ the basis contains tensor operators with the symmetry properties of the Weyl tensor and its generalizations.

We explicitly construct the basis for $j = 1/2, 1, 3/2$. For $j = 1/2$ we reproduce the conventional Dirac basis and rest-frame parity is the time component of a four-vector operator (in this case the "symmetric" tensor of rank $2j = 1$) that turns out to be the conventional Dirac matrices γ^μ; the chirality operator coincides with the γ^5 Dirac matrix. For $j = 1$ the basis is given by $\{\mathbb{1}, \chi, S_{\mu\nu}, \chi S_{\mu\nu}, M_{\mu\nu}, C_{\mu\nu\alpha\beta}\}$. We explicitly construct the symmetric traceless tensor $S_{\mu\nu}$ and the $C_{\mu\nu\alpha\beta}$ tensor which has the symmetries of the Weyl tensor. These two tensors involve the Jordan algebra of the generators. For $j = 3/2$ the basis contains the operators $\{\mathbb{1}, \chi, S_{\mu\nu\rho}, \chi S_{\mu\nu\rho}, M_{\mu\nu}, C_{\mu\nu\alpha\beta}, D_{\mu\nu\rho\sigma\alpha\beta}\}$. The C tensor has the same form as the $j = 1$ case just replacing the $j = 1$ generators with those of $j = 3/2$. We give explicit expressions for the $S_{\mu\nu\rho}$ and $D_{\mu\nu\rho\sigma\alpha\beta}$ tensors.

The formulation of theories for particles, either elementary or composite, transforming in these representations can be done using our covariant basis which has a clear physical interpretation in terms of parity or chirality properties. In particular, in the Poincaré projector formalism for the $(j, 0) \oplus (0, j)$ representations we find that the antisymmetric part of the involved space-time tensor is given by $M_{\mu\nu}$ for all j. This simple structure yields two direct physical consequences: (i) the multipole moments of elementary spin j particles are not independent, they are dictated by the value of the gyromagnetic factor g, and (ii) the propagation of spin j waves in an electromagnetic background is causal.

We calculate the multipole moments and compare with existing calculations in the literature. We conclude that except for Q_E^0 and Q_M^1 the multipole moments are representation specific. The universality of Q_E^0 is due to a global symmetry while that of Q_M^1 is due to the fact that it involves only the algebraic properties of the generators of rotations which are independent of the chosen Lorentz representation. Higher multipole moments depend on the algebraic structure of the boost generators. The structure is different for different Lorentz representations. This difference in the complete algebraic structure is at the root of the representation dependence of higher multipole moments.

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