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Fermions with Lorentz-violating operators of arbitrary dimension

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The theoretical description of fermions in the presence of Lorentz and *CPT* violation is developed. We classify all Lorentz- and *CPT*-violating and invariant terms in the quadratic Lagrange density for a Dirac fermion, including operators of arbitrary mass dimension. The exact dispersion relation is obtained in closed and compact form, and projection operators for the spinors are derived. The Pauli Hamiltonians for particles and antiparticles are extracted, and observable combinations of operators are identified. We characterize and enumerate the coefficients for Lorentz violation for any operator mass dimension via a decomposition using spin-weighted spherical harmonics. The restriction of the general theory to various special cases is presented, including isotropic models, the nonrelativistic and ultrarelativistic limits, and the minimal Standard-Model Extension. Expressions are derived in several limits for the fermion dispersion relation, the associated fermion group velocity, and the fermion spin-precession frequency. We connect the analysis to some other formalisms and use the results to extract constraints from astrophysical observations on isotropic ultrarelativistic spherical coefficients for Lorentz violation.

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I. INTRODUCTION

The invariance of the laws of nature under Lorentz transformations is well established, being based on an extensive series of investigations originating in classic tests such as the Michelson–Morley, Kennedy–Thorndike, Ives–Stilwell, and Hughes–Drever experiments [1–4]. Interest in precision tests of relativity has experienced a renewal in recent years, following the realization that tiny departures from Lorentz invariance could arise in a fundamental theory such as strings [5]. During this period, experiments using techniques from many subfields have achieved striking sensitivities to a variety of effects from Lorentz violation [6].

The general framework characterizing violations of Lorentz invariance is the Standard-Model Extension (SME) [7,8], which is a realistic effective quantum field theory incorporating General Relativity and the Standard Model. Terms in the SME violating *CPT* symmetry also violate Lorentz invariance [9], so the SME also characterizes *CPT* violation. Each Lorentz-violating term in the SME action is a coordinate-independent scalar density involving a Lorentz-violating operator contracted with a controlling coefficient. The mass dimension d of the operator fixes the dimensionality of the corresponding coefficient. In the popular scenario with General Relativity and the Standard Model emerging as the low-energy limit of an underlying theory of quantum gravity at the Planck scale $M_P \sim 10^{19}$ GeV, terms with larger d can plausibly be viewed as higher-order corrections in a series approximating the underlying physics. Other scenarios can also be envisaged.

The focus of the present work is Lorentz violation in fermions. The realistic nature of the SME means that it can readily be applied to analyze observational and experimental data, but existing studies of Lorentz violation with fermions are primarily concerned with the minimal SME, obtained by restricting attention to operators of renormalizable dimensions $d \leq 4$. To date, the minimal SME has been adopted as the theoretical framework in searches for Lorentz violation in the fermion sector involving electrons [10], protons and neutrons [11], muons [12], neutrinos [13], quarks [14], and gravitational couplings of various species [8,15]. Discussions in the literature of the nonminimal SME fermion sector are more limited. The general structure and properties of the nonminimal neutrino sector have been investigated [16], and results are known for some special nonminimal SME-based models [17–21], including ones with nonminimal fermion interactions [22]. However, a complete description of the nonminimal SME fermion sector remains an open issue.

In the present work, we seek to address this gap in the literature by extending the existing treatment of nonminimal Lorentz violation to include quadratic fermion operators of arbitrary mass dimension d, thereby opening the path for additional searches for Lorentz violation. To achieve a reasonable scope, we restrict the analysis to flat spacetime with a Dirac-type action invariant under spacetime translations and phase rotations, so that energy, momentum, and charge are conserved. This scope suffices for applications to many experimental situations involving fermions and can be applied to studies of matter following methods used in the minimal sector [23]. It also serves as a basis for further theoretical investigations of foundational aspects of Lorentz violation, including mathematical topics such as the underling pseudo–Riemann-Finsler geometry [24] and physical issues such as causality and stability [25,26], where operators of large d can dominate the associated physics. Other applications are expected to include phenomenological topics such as radiative corrections in Lorentz-violating models, where loop processes involving terms of a given d can naturally induce effective operators of different dimensions. Our results are also potentially relevant for some proposed theories naturally generating effective field theories dominated by SME operators of dimension $d > 4$, such as supersymmetric Lorentz-violating models [27] or noncommutative quantum electrodynamics [28], in which the corresponding SME operators have $d \geq 6$ [29].

The primary goal of this work is to develop the quadratic nonminimal SME fermion sector to the point where practical applications become feasible. This requires extracting key information from the general SME action, including basic features of fermion behavior in the presence of Lorentz violation. Typical applications are expected to involve measurements of aspects of fermion propagation, such as times of flight or spin-precession rates, and studies of fermion energy levels in systems such as atoms. The former require characterizing the anisotropy, dispersion, and birefringence in fermion propagation, which can conveniently be addressed via the dispersion relation, while the latter can be addressed by studying induced level shifts using the perturbative Hamiltonian for Lorentz violation. Here, we obtain the exact dispersion relation and the perturbative Hamiltonian, and we develop a methodology to study the corresponding effects using a decomposition in spherical harmonics. This permits a classification of all observables in terms of four sets of coefficients for Lorentz violation having straightforward rotation properties, which is expected to simplify future experimental analyses.

This paper is organized as follows. The general quadratic action for a Dirac field is studied in Sec. II. The basic framework is reviewed in Sec. II A, while the role of field redefinitions in determining physical observables is determined in Sec. II B. The exact vacuum dispersion relation is obtained in a closed and compact form in Sec. II C, and some of its physical properties are described in Sec. II D. Covariant projection operators for the spinor solutions to the modified Dirac equation are presented in Sec. II E. We then turn to the construction of the Hamiltonians for particles and antiparticles, deriving expressions for both in Sec. III A and converting them to explicitly covariant forms in Sec. III B. Taking advantage of the approximate rotational symmetry relevant for many applications, we perform in Sec. IV a decomposition of the Hamiltonian in spin-weighted spherical harmonics. This calculation yields a complete set of observable coefficients for Lorentz violation, cataloged according to properties of the corresponding operators. We develop the isotropic limit for the perturbative Hamiltonian and present the general isotropic Lagrange density for operator dimensions $d = 3, 4, 5, 6$ in both Cartesian and spherical coefficients. In Sec. V, we turn to a description of various special cases of the framework, including the nonrelativistic and ultrarelativistic limits and the minimal SME. Section VI contains applications of the results to dispersion, group velocity, and birefringence, along with a discussion of connections between the nonminimal fermion sector of the SME and other field theoretic and kinematical results in the literature. We also provide a compilation of existing astrophysical limits on isotropic Lorentz violation translated into constraints on spherical SME coefficients. Section VII summarizes the results obtained in this work. Throughout this paper, we adopt conventions matching those of the prior studies of nonminimal Lorentz violation in Refs. [16,30].

II. SINGLE DIRAC FERMION

In this section, we consider the effective action for a single Dirac fermion, allowing for operators of arbitrary dimension. Attention is restricted to terms that are quadratic in the fermion field, which gives rise to a linear theory. Features of the corresponding modified Dirac equation are also considered, including observability and field redefinitions. We derive an explicit expression for the exact dispersion relation, and we determine an approximation valid to leading order in Lorentz violation. The result reveals features including anisotropy, dispersion, and birefringence. Leading-order expressions for the eigenspinors are also presented.

A. Basics

Given the conventional Dirac Lagrange density, the effective theory describing the fermion behavior in the presence of general Lorentz violation can be obtained by adding terms formed from tensor operators contracted with coefficients for Lorentz violation [7]. The coefficients play the role of background fields generating the Lorentz violation, and the resulting theory is coordinate independent. For a single Dirac fermion ψ of mass m_ψ, this construction and the requirement of a linear theory imply that the action S extends the usual Dirac action for ψ by a quadratic functional of ψ and its derivatives,

$$\begin{aligned} S &= \int \mathcal{L} d^4x, \\ \mathcal{L} &= \frac{1}{2}\bar{\psi}(\gamma^\mu i\partial_\mu - m_\psi + \hat{\mathcal{Q}})\psi + \text{H.c.}, \end{aligned} \tag{1}$$

where $\hat{\mathcal{Q}}$ is a 4×4 spinor-matrix operator involving derivatives $i\partial_\mu$. Without loss of generality, $\hat{\mathcal{Q}}$ can be taken to obey the Hermiticity condition $\hat{\mathcal{Q}} = \gamma_0 \hat{\mathcal{Q}}^\dagger \gamma_0$. Since $\hat{\mathcal{Q}}$ is general, it includes both all Lorentz-invariant and all Lorentz-violating effects. The latter may be Planck suppressed and in any case are generically tiny, so we treat $\hat{\mathcal{Q}}$ as a perturbative contribution when needed to insure that deviations from the conventional Dirac situation are small. In particular, this implies that any extra modes associated with the higher-order derivatives in $\hat{\mathcal{Q}}$ can be neglected for practical purposes.

The operator $\hat{\mathcal{Q}}$ can describe effects of Lorentz violation arising either spontaneously or explicitly. Spontaneous Lorentz violation occurs when tensor fields dynamically acquire vacuum expectation values [5], which play the role of background tensors in the operator $\hat{\mathcal{Q}}$. In contrast, explicit Lorentz violation involves background tensors in $\hat{\mathcal{Q}}$ that are externally prescribed. In both cases, the operator $\hat{\mathcal{Q}}$ can in principle depend on spacetime position. However, to maintain invariance of the action (1) under spacetime translations and hence preserve energy-momentum conservation, we require here that the operator $\hat{\mathcal{Q}}$ is spacetime constant. This insures a focus on pure Lorentz violation and minimizes complications in analyses at both the theoretical and experimental levels. In the context of spontaneous Lorentz violation, imposing spacetime translation symmetry is equivalent to disregarding solitonic background fields along with any massive and Nambu–Goldstone modes [31]. In certain situations these modes play the role of the photon in Einstein–Maxwell theory [8,32], the graviton [33], or other force mediators [34,35], so in these cases some care may be needed in interpreting results for spacetime-constant $\hat{\mathcal{Q}}$.

Note that spacetime constancy of $\hat{\mathcal{Q}}$ can be either an exact feature of the model or an approximation to dominant or averaged effects in a more complete theory. The complete theory may even be fully Lorentz invariant. Existing or hypothetical forces typically give rise to effects with dominant contributions appearing as backgrounds in a given experimental situation, which can serve as effective Lorentz violation in a phenomenological description. For instance, in a local laboratory, the gravitational force produces a direction dependence that plays the role of explicit Lorentz violation in the corresponding effective theory. Hypothetical ultraweak forces can in principle be constrained or even detected in this way. For example, sharp constraints on torsion have been obtained by studying the effective Lorentz violation associated with a torsion background [36]. In general, viable models for Lorentz-invariant interactions generating effective operators of the form $\hat{\mathcal{Q}}$ must be consistent with known constraints on Lorentz violation [6].

In this subsection, we perform a decomposition of $\hat{\mathcal{Q}}$ that ultimately permits the enumeration and characterization of the coefficients for Lorentz violation appearing in the Lagrange density (1). Expanding $\hat{\mathcal{Q}}$ in the basis of 16 Dirac matrices explicitly reveals the spin content,

$$\begin{aligned}\hat{\mathcal{Q}} &= \sum_I \hat{\mathcal{Q}}^I \gamma_I \\ &= \hat{\mathcal{S}} + i\hat{\mathcal{P}}\gamma_5 + \hat{\mathcal{V}}^\mu \gamma_\mu + \hat{\mathcal{A}}^\mu \gamma_5 \gamma_\mu + \frac{1}{2}\hat{\mathcal{T}}^{\mu\nu}\sigma_{\mu\nu},\end{aligned} \tag{2}$$

where the 16 operators $\hat{\mathcal{Q}}^I = \{\hat{\mathcal{S}}, \hat{\mathcal{P}}, \hat{\mathcal{V}}^\mu, \hat{\mathcal{A}}^\mu, \hat{\mathcal{T}}^{\mu\nu}\}$ are Dirac-scalar functions of the derivatives $i\partial_\mu$ with mass dimension 1. In momentum space, each operator $\hat{\mathcal{Q}}^I$ can be viewed as a series of terms,

$$\hat{\mathcal{Q}}^I = \sum_{d=3}^{\infty} \mathcal{Q}^{(d)I\alpha_1\alpha_2\ldots\alpha_{d-3}} p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_{d-3}}, \tag{3}$$

with $p_\mu = i\partial_\mu$. All the coefficients $\mathcal{Q}^{(d)I\alpha_1\alpha_2\ldots\alpha_{d-3}}$ are spacetime independent and have dimension $4 - d$. Also, they can all be assumed real by Hermiticity. Note that any of these coefficients proportional to combinations of products of the Lorentz-invariant tensors $\eta^{\mu\nu}$ and $\epsilon^{\kappa\lambda\mu\nu}$ correspond to Lorentz-invariant operators in the theory (1).

Often, it is convenient to work with an alternative decomposition of $\hat{\mathcal{Q}}$ that parallels the formalism widely used for the single-fermion limit of the minimal SME [7]. This parallel suggests writing

$$\gamma^\nu p_\nu - m_\psi + \hat{\mathcal{Q}} = \hat{\Gamma}^\nu p_\nu - \hat{M}, \tag{4}$$

where $\hat{\Gamma}^\nu p_\nu$ and $\hat{M}$ consist of operators of even and odd mass dimension, respectively. Decomposing these operators in terms of the basis of 16 Dirac matrices yields

$$\begin{aligned}\hat{\Gamma}^\nu &= \gamma^\nu + \hat{c}^{\mu\nu}\gamma_\mu + \hat{d}^{\mu\nu}\gamma_5\gamma_\mu + \hat{e}^\nu + i\hat{f}^\nu\gamma_5 + \frac{1}{2}\hat{g}^{\kappa\lambda\nu}\sigma_{\kappa\lambda}, \\ \hat{M} &= m_\psi + \hat{m} + i\hat{m}_5\gamma_5 + \hat{a}^\mu\gamma_\mu + \hat{b}^\mu\gamma_5\gamma_\mu + \frac{1}{2}\hat{H}^{\mu\nu}\sigma_{\mu\nu}.\end{aligned} \tag{5}$$

In these expressions, the operators $\hat{c}^{\mu\nu}$, $\hat{d}^{\mu\nu}$ are *CPT* even and dimensionless; $\hat{e}^\mu$, $\hat{f}^\mu$, $\hat{g}^{\mu\rho\nu}$ are *CPT* odd and dimensionless; $\hat{m}$, $\hat{m}_5$, $\hat{H}^{\mu\nu}$ are *CPT* even and of dimension 1; and $\hat{a}^\mu$, $\hat{b}^\mu$ are *CPT* odd and of dimension 1. If desired, a chiral mass term $im_5\gamma_5$ can be added to $\hat{M}$, but in many situations this can be absorbed into m_ψ via a chiral rotation without loss of generality, and so we omit it from Eq. (5). The operators $\hat{m}$ and $\hat{m}_5$ consist solely of higher-derivative terms of nonrenormalizable dimension, but all the others appearing in Eq. (5) have equivalents in the minimal SME.

In Eq. (4), the operator $\hat{\Gamma}^\nu$ is contracted with p_ν. This implies that the operators $\hat{c}^{\mu\nu}$, $\hat{d}^{\mu\nu}$, $\hat{e}^\mu$, $\hat{f}^\mu$, $\hat{g}^{\mu\rho\nu}$ are also contracted with p_ν, and it motivates the introduction of contracted operators via

$$\begin{aligned}&\hat{c}^\mu = \hat{c}^{\mu\nu}p_\nu, \qquad \hat{d}^\mu = \hat{d}^{\mu\nu}p_\nu, \qquad \hat{e} = \hat{e}^\nu p_\nu, \\ &\hat{f} = \hat{f}^\nu p_\nu, \qquad \hat{g}^{\kappa\lambda} = \hat{g}^{\kappa\lambda\nu}p_\nu.\end{aligned} \tag{6}$$

The notation for each operator has been chosen so that its *CPT* handedness corresponds to that of its analog in the minimal SME. In terms of these operators, we find

TABLE I. Operators and coefficients for a Dirac fermion.

Operator	Type	d	CPT	Cartesian coefficients	Number
$\hat{m}$	Scalar	odd, ≥ 5	Even	$m^{(d)\alpha_1\ldots\alpha_{d-3}}$	$d(d-1)(d-2)/6$
$\hat{m}_5$	Pseudoscalar	odd, ≥ 5	Even	$m_5^{(d)\alpha_1\ldots\alpha_{d-3}}$	$d(d-1)(d-2)/6$
$\hat{a}^\mu$	Vector	odd, ≥ 3	Odd	$a^{(d)\mu\alpha_1\ldots\alpha_{d-3}}$	$2d(d-1)(d-2)/3$
$\hat{b}^\mu$	Pseudovector	odd, ≥ 3	Odd	$b^{(d)\mu\alpha_1\ldots\alpha_{d-3}}$	$2d(d-1)(d-2)/3$
$\hat{c}^\mu$	Vector	even, ≥ 4	Even	$c^{(d)\mu\alpha_1\ldots\alpha_{d-3}}$	$2d(d-1)(d-2)/3$
$\hat{d}^\mu$	Pseudovector	even, ≥ 4	Even	$d^{(d)\mu\alpha_1\ldots\alpha_{d-3}}$	$2d(d-1)(d-2)/3$
$\hat{e}$	Scalar	even, ≥ 4	Odd	$e^{(d)\alpha_1\ldots\alpha_{d-3}}$	$d(d-1)(d-2)/6$
$\hat{f}$	Pseudoscalar	even, ≥ 4	Odd	$f^{(d)\alpha_1\ldots\alpha_{d-3}}$	$d(d-1)(d-2)/6$
$\hat{g}^{\mu\nu}$	Tensor	even, ≥ 4	Odd	$g^{(d)\mu\nu\alpha_1\ldots\alpha_{d-3}}$	$d(d-1)(d-2)$
$\hat{H}^{\mu\nu}$	Tensor	odd, ≥ 3	Even	$H^{(d)\mu\nu\alpha_1\ldots\alpha_{d-3}}$	$d(d-1)(d-2)$

$$\hat{\mathcal{S}} = \hat{e} - \hat{m}, \qquad \hat{\mathcal{P}} = \hat{f} - \hat{m}_5, \qquad \hat{\mathcal{V}}^\mu = \hat{c}^\mu - \hat{a}^\mu,$$
$$\hat{\mathcal{A}}^\mu = \hat{d}^\mu - \hat{b}^\mu, \qquad \hat{\mathcal{T}}^{\mu\nu} = \hat{g}^{\mu\nu} - \hat{H}^{\mu\nu}. \tag{7}$$

These expressions provide the explicit link between the decompositions (2) and (4).

Each of the 10 component operators $\hat{e}$, $\hat{m}$, $\hat{f}$, $\hat{m}_5$, $\hat{c}^\mu$, $\hat{a}^\mu$, $\hat{d}^\mu$, $\hat{b}^\mu$, $\hat{g}^{\mu\nu}$, $\hat{H}^{\mu\nu}$ can be expanded in Cartesian momentum components following the form of Eq. (3), yielding 10 infinite series of real coefficients. For example, the operator $\hat{c}^\mu$ can be written as

$$\hat{c}^\mu = \sum_{d\text{ even}} c^{(d)\mu\alpha_1\ldots\alpha_{d-3}} p_{\alpha_1}\cdots p_{\alpha_{d-3}}. \tag{8}$$

Each term in this sum involves a coefficient $c^{(d)\mu\alpha_1\ldots\alpha_{d-3}}$, for which the index μ controls the spin nature of the operator and the $d-3$ symmetric indices $\alpha_1\ldots\alpha_{d-3}$ control the momentum dependence. In the analogous expansions for the photon sector [30], the spin and momentum dependence are intertwined by gauge symmetry, which complicates the counting of components. Here, however, the number of independent coefficients in $c^{(d)\mu\alpha_1\ldots\alpha_{d-3}}$ for each d can be obtained directly as $2d(d-1)(d-2)/3$. The coefficients for the nine other operators can be treated similarly. Table I lists the 10 operators, their corresponding coefficients, and some of their properties.

B. Field redefinitions

In the context of the minimal SME, the freedom to choose coordinates and to redefine fields while leaving the physics unchanged makes some coefficients for Lorentz violation physically unobservable [7,8,35,37–40]. This feature extends to the nonminimal sector. The effects of a coordinate choice, which amounts to selecting the sector in which the effective background spacetime metric has the usual diagonal Minkowski form, are analogous to those in the minimal SME and imply 10 combinations of coefficients are always unobservable. Also, the freedom to redefine the fermion ψ by a position-dependent phase,

$$\psi = \exp(ix^\mu v_\mu)\psi' \tag{9}$$

for a suitable v_μ, can be used as in the minimal SME to remove four constant coefficients coupling like a gauge potential. However, the freedom to make field redefinitions involving the spinor space, which eliminates and recombines certain coefficients in the Lagrange density, is more involved when the nonminimal sector is incorporated.

Here, we consider field redefinitions of the form

$$\psi = (1 + \hat{Z})\psi', \tag{10}$$

where $\hat{Z}$ is an arbitrary p-dependent operator. For this redefinition to leave the physics unaffected, the dominant modes in the Lagrange density must remain dominant in the redefined theory, and so the perturbative assumption for the operator $\hat{\mathcal{Q}}$ in the Dirac action (1) must be maintained. This implies that $\hat{Z}$ itself must be perturbative.

Under the redefinition (10), the operator in the Dirac action (1) acquires a new form,

$$\psi^\dagger\gamma_0(p\cdot\gamma - m_\psi + \hat{\mathcal{Q}})\psi \approx \psi'^\dagger\gamma_0(p\cdot\gamma - m_\psi + \hat{\mathcal{Q}}')\psi', \tag{11}$$

where

$$\hat{\mathcal{Q}}' = \hat{\mathcal{Q}} + (p\cdot\gamma - m_\psi)\hat{Z} + \gamma_0\hat{Z}^\dagger\gamma_0(p\cdot\gamma - m_\psi). \tag{12}$$

To explore the implications of this structure, it is useful to split $\hat{Z}$ into a Hermitian piece $\hat{X}$ and an anti-Hermitian piece $\hat{Y}$, defined according to

$$\hat{Z} = \hat{X} + i\hat{Y}, \qquad \hat{X} = \frac{1}{2}(\hat{Z} + \gamma_0\hat{Z}^\dagger\gamma_0),$$
$$\hat{Y} = \frac{1}{2i}(\hat{Z} - \gamma_0\hat{Z}^\dagger\gamma_0), \tag{13}$$

where both $\hat{X}$ and $\hat{Y}$ obey the same Hermiticity condition as the $\hat{\mathcal{Q}}$ operator, $\hat{X} = \gamma_0\hat{X}^\dagger\gamma_0$, $\hat{Y} = \gamma_0\hat{Y}^\dagger\gamma_0$. The operator $\hat{\mathcal{Q}}'$ is then given by

$$\hat{\mathcal{Q}}' = \hat{\mathcal{Q}} - 2m_\psi\hat{X} + p_\mu\{\gamma^\mu, \hat{X}\} + ip_\mu[\gamma^\mu, \hat{Y}]. \tag{14}$$

This shows that a suitable choice of $\hat{X}$ or $\hat{Y}$ can combine with $\hat{\mathcal{Q}}$ to reduce the observable content of $\hat{\mathcal{Q}}'$.

To determine explicitly which pieces of $\hat{\mathcal{Q}}$ are affected, we can decompose both $\hat{X}$ and $\hat{Y}$ in the basis of 16 Dirac matrices,

$$\hat{X} = \hat{X}_S + i\hat{X}_P\gamma_5 + \hat{X}_V^\mu\gamma_\mu + \hat{X}_A^\mu\gamma_5\gamma_\mu + \frac{1}{2}\hat{X}_T^{\mu\nu}\sigma_{\mu\nu},$$
$$\hat{Y} = \hat{Y}_S + i\hat{Y}_P\gamma_5 + \hat{Y}_V^\mu\gamma_\mu + \hat{Y}_A^\mu\gamma_5\gamma_\mu + \frac{1}{2}\hat{Y}_T^{\mu\nu}\sigma_{\mu\nu}. \tag{15}$$

Each component of $\hat{X}$ and $\hat{Y}$ in these expansions can be considered independently. The component $\hat{Y}_S$ evidently has no effect on $\hat{\mathcal{Q}}$, but the other nine generate field redefinitions mixing various Lorentz-violating operators and acting to remove some of them at leading order. In what follows, we apply each field redefinition in turn, determining the changes $\delta\hat{\mathcal{Q}} = \hat{\mathcal{Q}}' - \hat{\mathcal{Q}}$ and identifying the resulting effects.

First, consider transformations involving the components of $\hat{X}$. A nonzero $\hat{X}_S$ gives

$$\delta\hat{\mathcal{S}} = -2m_\psi\hat{X}_S, \qquad \delta\hat{\mathcal{V}}^\mu = 2\hat{X}_S p^\mu, \tag{16}$$

showing that for $m_\psi \neq 0$ the scalar operator $\hat{\mathcal{S}}$ can be removed by absorbing it into the vector $\hat{\mathcal{V}}^\mu$. Using $\hat{X}_P$ instead produces

$$\delta\hat{\mathcal{P}} = -2m_\psi\hat{X}_P, \tag{17}$$

which reveals that the pseudoscalar operator $\hat{\mathcal{P}}$ can also be removed. A nonzero $\hat{X}_V^\mu$ gives

$$\delta\hat{\mathcal{S}} = 2p_\mu\hat{X}_V^\mu, \qquad \delta\hat{\mathcal{V}}^\mu = -2m_\psi\hat{X}_V^\mu, \tag{18}$$

which reconfirms that the scalar and vector operators mix under field redefinitions. Using $\hat{X}_A^\mu$ yields

$$\delta\hat{\mathcal{A}}^\mu = -2m_\psi\hat{X}_A^\mu, \qquad \delta\hat{\mathcal{T}}^{\mu\nu} = -2\epsilon^{\mu\nu\rho\sigma}p_\rho\hat{X}_{A\sigma}, \tag{19}$$

so the pseudovector operators $\hat{\mathcal{A}}^\mu$ can be absorbed into the tensor ones. Finally, a nonzero $\hat{X}_T^{\mu\nu}$ gives

$$\delta\hat{\mathcal{A}}^\mu = -\epsilon^{\mu\nu\rho\sigma}p_\nu\hat{X}_{T\rho\sigma}, \qquad \delta\hat{\mathcal{T}}^{\mu\nu} = -2m_\psi\hat{X}_T^{\mu\nu}, \tag{20}$$

again showing that the pseudovector and tensor operators mix.

Next, we turn to transformations involving the components of $\hat{Y}$. Taking nonzero $\hat{Y}_S$ has no effect, as mentioned above. Using $\hat{Y}_P$ gives

$$\delta\hat{\mathcal{A}}^\mu = 2p^\mu\hat{Y}_P, \tag{21}$$

which permits the removal of the component of $\hat{\mathcal{A}}^\mu$ proportional to p^μ. A nonzero $\hat{Y}_V^\mu$ gives

$$\delta\hat{\mathcal{T}}^{\mu\nu} = 2p^{[\mu}\hat{Y}_V^{\nu]}. \tag{22}$$

In the minimal sector, this can be used to remove the trace component of $g^{(4)\mu\nu\rho}$. More generally, the coefficients appearing in the expansion of the dual $\tilde{\hat{\mathcal{T}}}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\hat{\mathcal{T}}_{\rho\sigma}$ can be split into pieces that transform under two different representations of the Lorentz group, with one set antisymmetric in the first three indices and the other antisymmetric in the first two indices with vanishing antisymmetrization on any three indices. The above field redefinition with $\hat{Y}_V^\mu$ can be used to remove the first piece. Taking instead a nonzero $\hat{Y}_A^\mu \neq 0$ gives

$$\delta\hat{\mathcal{P}} = -2p_\mu Y_A^\mu, \tag{23}$$

which reconfirms that the pseudoscalar operator $\hat{\mathcal{P}}$ can be removed. Finally, using $\hat{Y}_T^{\mu\nu}$ leads to

$$\delta\hat{\mathcal{V}}^\mu = 2\hat{Y}_T^{\mu\nu}p_\nu. \tag{24}$$

In this case, the coefficients appearing in the expansion of $\hat{\mathcal{V}}^\mu$ can be split into a piece that is totally symmetric and one with mixed symmetry that is antisymmetric in the first two indices. The field redefinition with $\hat{Y}_T^{\mu\nu}$ allows the removal of the piece with mixed symmetry.

We thus see that the physical observables in the quadratic fermion theory (1) are restricted to pieces of $\hat{\mathcal{V}}^\mu$ and $\hat{\mathcal{T}}^{\mu\nu}$. The relationships (7) show these observables correspond to parts of $\hat{a}^\mu$, $\hat{c}^\mu$, $\hat{g}^{\mu\nu}$, and $\hat{H}^{\mu\nu}$. This feature parallels results for the neutrino sector, where the propagation of neutrinos is controlled by four effective coefficients of these types despite the multiple flavors, the mixing, and the handedness of the fermions [16]. It also reduces correctly to known results in the minimal SME [7,8].

Using the field redefinitions, we can define a canonical set of effective operators representing physical observables in the quadratic theory (1),

$$\hat{\mathcal{S}}_{\rm eff} = 0, \qquad \hat{\mathcal{P}}_{\rm eff} = 0, \qquad \hat{\mathcal{A}}^\mu_{\rm eff} = 0,$$
$$\hat{\mathcal{V}}^\mu_{\rm eff} = \left(\hat{\mathcal{V}}^\mu + \frac{1}{m_\psi}p^\mu\hat{\mathcal{S}}\right)_{[0]},$$
$$\tilde{\hat{\mathcal{T}}}^{\mu\nu}_{\rm eff} = \left(\tilde{\hat{\mathcal{T}}}^{\mu\nu} + \frac{1}{m_\psi}p^{[\mu}\hat{\mathcal{A}}^{\nu]}\right)_{[2]}, \tag{25}$$

where the subscript $[n]$ indicates that the coefficients appearing in the operator expansion are restricted to an irreducible representation antisymmetric in the first n indices. The relationships (7) imply the corresponding definitions

$$\hat{a}^\mu_{\rm eff} = \left(\hat{a}^\mu - \frac{1}{m_\psi}p^\mu\hat{e}\right)_{[0]} = \sum_d a^{(d)\mu\alpha_1\ldots\alpha_{d-3}}_{\rm eff}p_{\alpha_1}\cdots p_{\alpha_{d-3}},$$
$$\hat{c}^\mu_{\rm eff} = \left(\hat{c}^\mu - \frac{1}{m_\psi}p^\mu\hat{m}\right)_{[0]} = \sum_d c^{(d)\mu\alpha_1\ldots\alpha_{d-3}}_{\rm eff}p_{\alpha_1}\cdots p_{\alpha_{d-3}},$$
$$\tilde{\hat{g}}^{\mu\nu}_{\rm eff} = \left(\tilde{\hat{g}}^{\mu\nu} - \frac{1}{m_\psi}p^{[\mu}\hat{b}^{\nu]}\right)_{[2]} = \sum_d \tilde{g}^{(d)\mu\nu\alpha_1\ldots\alpha_{d-3}}_{\rm eff}p_{\alpha_1}\cdots p_{\alpha_{d-3}},$$
$$\tilde{\hat{H}}^{\mu\nu}_{\rm eff} = \left(\tilde{\hat{H}}^{\mu\nu} - \frac{1}{m_\psi}p^{[\mu}\hat{d}^{\nu]}\right)_{[2]} = \sum_d \tilde{H}^{(d)\mu\nu\alpha_1\ldots\alpha_{d-3}}_{\rm eff}p_{\alpha_1}\cdots p_{\alpha_{d-3}}. \tag{26}$$

Note that the analysis of field redefinitions naturally leads to expansions of the duals $\tilde{\hat{g}}^{\mu\nu}_{\rm eff}$, $\tilde{\hat{H}}^{\mu\nu}_{\rm eff}$ rather than the tensor operators $\hat{g}^{\mu\nu}_{\rm eff}$, $\hat{H}^{\mu\nu}_{\rm eff}$ directly.

In terms of the fundamental coefficients, the effective coefficients are

$$a^{(d)\mu\alpha_1\ldots\alpha_{d-3}}_{\rm eff} = \Big(a^{(d)\mu\alpha_1\ldots\alpha_{d-3}} - \frac{1}{m_\psi}\eta^{\mu\alpha_1} e^{(d-1)\alpha_2\ldots\alpha_{d-3}}\Big)_{[0]},$$
$$c^{(d)\mu\alpha_1\ldots\alpha_{d-3}}_{\rm eff} = \Big(c^{(d)\mu\alpha_1\ldots\alpha_{d-3}} - \frac{1}{m_\psi}\eta^{\mu\alpha_1} m^{(d-1)\alpha_2\ldots\alpha_{d-3}}\Big)_{[0]},$$
$$\tilde{g}^{(d)\mu\nu\alpha_1\ldots\alpha_{d-3}}_{\rm eff} = \Big(\tilde{g}^{(d)\mu\nu\alpha_1\ldots\alpha_{d-3}} - \frac{1}{m_\psi}\eta^{\alpha_1[\mu} b^{(d-1)\nu]\alpha_2\ldots\alpha_{d-3}}\Big)_{[2]},$$
$$\tilde{H}^{(d)\mu\nu\alpha_1\ldots\alpha_{d-3}}_{\rm eff} = \Big(\tilde{H}^{(d)\mu\nu\alpha_1\ldots\alpha_{d-3}} - \frac{1}{m_\psi}\eta^{\alpha_1[\mu} d^{(d-1)\nu]\alpha_2\ldots\alpha_{d-3}}\Big)_{[2]}. \qquad (27)$$

In these equations, the dual coefficients are defined by

$$\tilde{g}^{(d)\mu\nu\alpha_1\ldots\alpha_{d-3}} = \frac{1}{2}\epsilon^{\mu\nu}{}_{\rho\sigma} g^{(d)\rho\sigma\alpha_1\ldots\alpha_{d-3}},$$
$$\tilde{H}^{(d)\mu\nu\alpha_1\ldots\alpha_{d-3}} = \frac{1}{2}\epsilon^{\mu\nu}{}_{\rho\sigma} H^{(d)\rho\sigma\alpha_1\ldots\alpha_{d-3}}. \qquad (28)$$

Also, the subscript [0] indicates symmetrization on all indices, while [2] indicates symmetrization on $\nu\alpha_1\ldots\alpha_{d-3}$ followed by antisymmetrization on $\mu\nu$.

The above results demonstrate, for example, that leading-order signals from $\hat{b}^\mu$ can be absorbed into those from $\tilde{\hat{g}}^{\mu\nu}$, while signals from $\hat{d}^\mu$ merge with those of $\tilde{\hat{H}}^{\mu\nu}$. As an illustration, the $d = 4$ terms in $\hat{d}^\mu$ can be absorbed into the $d = 5$ terms in $\tilde{\hat{H}}^{\mu\nu}$, giving rise to effective coefficients $\tilde{H}^{(5)\mu\alpha_1\alpha_2\alpha_3}_{\rm eff}$. This example also reveals the potentially surprising result that an operator naively having renormalizable dimension and hence lying in the minimal SME may in fact most naturally be regarded as belonging to the nonminimal sector and having nonrenormalizable dimension. Note also that the Cartesian coefficients $m_5^{(d)\alpha_1\ldots\alpha_{d-3}}$ and $f^{(d)\alpha_1\ldots\alpha_{d-3}}$ have no observable role. This is consistent with known results for the minimal case [8,39,40]. Moreover, additional field redefinitions or coordinate choices can further reduce the number of observable effects. For example, the phase redefinition (9) shows that the effective coefficient $a^{(3)\mu}_{\rm eff}$ is unobservable. All the effective operators, their Cartesian effective coefficients, and some of their properties are compiled in Table II.

A few of the effective coefficients correspond to Lorentz-invariant operators in the theory (1). They must be formed from combinations of products of the Lorentz-invariant tensors $\eta^{\mu\nu}$ and $\epsilon^{\mu\nu\rho\sigma}$ multiplied by constant scalars. The coefficients $a^{(d)\mu\alpha_1\ldots\alpha_{d-3}}_{\rm eff}$ and $\tilde{g}^{(d)\mu\nu\alpha_1\ldots\alpha_{d-3}}_{\rm eff}$ both have an odd number of indices, so they all produce Lorentz-violating effects. Inspection reveals that the symmetries of the coefficients $\tilde{H}^{(d)\mu\nu\alpha_1\ldots\alpha_{d-3}}_{\rm eff}$ preclude constructing them in terms of invariant tensors as well. The only option for generating Lorentz-invariant operators is therefore to use the coefficients $c^{(d)\mu\alpha_1\ldots\alpha_{d-3}}_{\rm eff}$ constructed as completely symmetrized products of the metric,

$$c^{(d)\mu\alpha_1\ldots\alpha_{d-3}}_{\rm eff,LI} = \frac{1}{(d-2)!} c^{(d)}_{\rm LI}\, \eta^{(\mu\alpha_1}\eta^{\alpha_2\alpha_3}\ldots\eta^{\alpha_{d-4}\alpha_{d-3})}. \qquad (29)$$

This reveals that there is exactly one Lorentz-invariant effective operator at each even dimension $d = 4, 6, \ldots$. No Lorentz-invariant effective operators exist for odd d.

The addition of interactions or the presence of a non-Minkowski background typically changes the set of physical observables by affecting the implementation of field redefinitions. For generality in what follows, we therefore present calculations and results with all coefficients explicitly included. However, expressions relevant for physical measurements can be expected to yield only observable quantities. For example, the effective coefficients appearing in the Hamiltonian derived in Sec. III below are compatible with this structure of observables.

C. Exact vacuum dispersion relation

The action (1) leads to the modified Dirac equation

$$(p\cdot\gamma - m_\psi + \hat{\mathcal{Q}})\psi = 0, \qquad (30)$$

where $\hat{\mathcal{Q}}$ can be viewed as the expression (2). Formally, the exact dispersion relation for plane-wave solutions in the vacuum is found by requiring that the determinant of the modified Dirac operator vanishes,

$$\det(p\cdot\gamma - m_\psi + \hat{\mathcal{Q}}) = 0. \qquad (31)$$

This condition determines the propagation of spinor wave packets in the presence of Lorentz-violating operators of arbitrary dimension.

TABLE II. Effective operators and effective coefficients for a Dirac fermion.

Operator	Type	d	CPT	Cartesian coefficients	Number
$\hat{a}^\mu_{\rm eff}$	Vector	Odd, ≥ 3	Odd	$a^{(d)\mu\alpha_1\ldots\alpha_{d-3}}_{\rm eff}$	$(d+1)d(d-1)/6$
$\hat{c}^\mu_{\rm eff}$	Vector	Even, ≥ 4	Even	$c^{(d)\mu\alpha_1\ldots\alpha_{d-3}}_{\rm eff}$	$(d+1)d(d-1)/6$
$\tilde{\hat{g}}^{\mu\nu}_{\rm eff}$	Tensor	Even, ≥ 4	Odd	$\tilde{g}^{(d)\mu\nu\alpha_1\ldots\alpha_{d-3}}_{\rm eff}$	$(d+1)d(d-2)/2$
$\tilde{\hat{H}}^{\mu\nu}_{\rm eff}$	Tensor	Odd, ≥ 3	Even	$\tilde{H}^{(d)\mu\nu\alpha_1\ldots\alpha_{d-3}}_{\rm eff}$	$(d+1)d(d-2)/2$

An explicit form for the dispersion relation (31) can be obtained by direct calculation. One method proceeds by adopting a chiral representation of the Dirac matrices and breaking the modified Dirac operator into 2×2 blocks A, B, C, D,

$$p \cdot \gamma - m_\psi + \hat{\mathcal{Q}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{32}$$

It is convenient to introduce the notation $\sigma^\mu = (\sigma^0, \sigma^j)$, where σ^0 is the 2×2 identity matrix and σ^j are the usual three Pauli matrices. The adjoint matrices are $\bar{\sigma} = (\sigma^0, -\sigma^j)$, and they satisfy the basic identity

$$\bar{\sigma}_\mu \sigma_\nu = \eta_{\mu\nu} + \frac{i}{2}\epsilon_{\mu\nu\kappa\lambda}\bar{\sigma}^k \sigma^\lambda. \tag{33}$$

The block decomposition can then be written

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \hat{\mathcal{S}}_- + \frac{i}{2}\hat{\mathcal{T}}_-^{\mu\nu}\sigma_\mu\bar{\sigma}_\nu & \hat{\mathcal{V}}_-^\mu \sigma_\mu \\ \hat{\mathcal{V}}_+^\mu \bar{\sigma}_\mu & \hat{\mathcal{S}}_+ + \frac{i}{2}\hat{\mathcal{T}}_+^{\mu\nu}\bar{\sigma}_\mu\sigma_\nu \end{pmatrix}, \tag{34}$$

where

$$\hat{\mathcal{S}}_\pm = -m_\psi + \hat{\mathcal{S}} \pm i\hat{\mathcal{P}}, \qquad \hat{\mathcal{V}}_\pm^\mu = p^\mu + \hat{\mathcal{V}}^\mu \pm \hat{\mathcal{A}}^\mu,$$
$$\hat{\mathcal{T}}_\pm^{\mu\nu} = \frac{1}{2}(\hat{\mathcal{T}}^{\mu\nu} \pm i\tilde{\hat{\mathcal{T}}}^{\mu\nu}). \tag{35}$$

Here, $\tilde{\hat{\mathcal{T}}}^{\mu\nu}$ is the dual of $\hat{\mathcal{T}}^{\mu\nu}$. Note that $\hat{\mathcal{T}}_\pm^{\mu\nu} = \pm i\tilde{\hat{\mathcal{T}}}_\pm^{\mu\nu}$ are the two chiral components of the tensor operator $\hat{\mathcal{T}}^{\mu\nu}$.

The determinant (31) can be obtained from the block form (32) using the identity

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD) + \det(BC) - \mathrm{tr}(\bar{B}A\bar{C}D), \tag{36}$$

where $\bar{B} = \mathrm{adj}(B)$ and $\bar{C} = \mathrm{adj}(C)$ are matrix adjoints. This quantity can be directly evaluated using the basic result (33) and the subsidiary identities

$$\begin{aligned} &\mathrm{tr}(\bar{\sigma}_\mu\sigma_\nu\bar{\sigma}_k\sigma_\lambda) = 2\eta_{\mu\nu\kappa\lambda}, \\ &\mathrm{tr}(\bar{\sigma}_\mu\sigma_\nu\bar{\sigma}_k\sigma_\lambda\bar{\sigma}_\rho\sigma_\sigma) = 2\eta_{\mu\nu\kappa}{}^{\tau}\eta_{\tau\lambda\rho\sigma}, \\ &\eta_{\mu\nu\kappa\lambda}T_-^{\nu\kappa} = 4T_{-\mu\lambda}, \qquad \eta_{\mu\nu\kappa\lambda}T_+^{\kappa\lambda} = -4T_{+\mu\nu}, \end{aligned} \tag{37}$$

where $\eta_{\mu\nu\kappa\lambda}$ is defined by

$$\eta_{\mu\nu\kappa\lambda} = \eta_{\mu\nu}\eta_{\kappa\lambda} - \eta_{\mu\kappa}\eta_{\nu\lambda} + \eta_{\mu\lambda}\eta_{\nu\kappa} - i\epsilon_{\mu\nu\kappa\lambda}. \tag{38}$$

The calculation outlined above yields an explicit form for the exact dispersion relation (31) of the modified Dirac operator. We find

$$\begin{aligned} &(\hat{\mathcal{S}}_-^2 - \hat{\mathcal{T}}_-^2)(\hat{\mathcal{S}}_+^2 - \hat{\mathcal{T}}_+^2) + \hat{\mathcal{V}}_-^2\hat{\mathcal{V}}_+^2 \\ &\quad - 2\hat{\mathcal{V}}_- \cdot (\hat{\mathcal{S}}_- + 2i\hat{\mathcal{T}}_-) \cdot (\hat{\mathcal{S}}_+ - 2i\hat{\mathcal{T}}_+) \cdot \hat{\mathcal{V}}_+ = 0, \end{aligned} \tag{39}$$

where $\hat{\mathcal{T}}_\pm^2 = \hat{\mathcal{T}}_\pm^{\mu\nu}\hat{\mathcal{T}}_{\pm\mu\nu}$. This compact expression holds for a Dirac field experiencing Lorentz violation involving operators of arbitrary mass dimension. It reduces correctly to the well-known result for the renormalizable theory [25] and its nonrelativistic limit [23]. In terms of the effective operators (25), we obtain

$$\begin{aligned} 0 = &(p + \hat{\mathcal{V}}_{\mathrm{eff}})^4 + (m_\psi^2 - \hat{\mathcal{T}}_{\mathrm{eff}-}^2)(m_\psi^2 - \hat{\mathcal{T}}_{\mathrm{eff}+}^2) \\ &- 2(p + \hat{\mathcal{V}}_{\mathrm{eff}}) \cdot (m_\psi - 2i\hat{\mathcal{T}}_{\mathrm{eff}-}) \\ &\cdot (m_\psi + 2i\hat{\mathcal{T}}_{\mathrm{eff}+}) \cdot (p + \hat{\mathcal{V}}_{\mathrm{eff}}), \end{aligned} \tag{40}$$

where $\hat{\mathcal{T}}_{\mathrm{eff}\pm}^{\mu\nu} = \frac{1}{2}(\hat{\mathcal{T}}_{\mathrm{eff}}^{\mu\nu} \pm i\tilde{\hat{\mathcal{T}}}_{\mathrm{eff}}^{\mu\nu})$.

The superficially quartic nature of the dispersion relation (39) reflects the usual presence of the four independent Dirac spinors, representing two spin projections for each of the particle and antiparticle modes. However, viewed as a function of p_μ, the dispersion relation (39) represents an algebraic variety $\mathcal{R}(p_\mu)$ of arbitrarily high order rather than the usual Dirac quartic. When the coefficients for Lorentz violation are small, four roots of $\mathcal{R}$ appear as small corrections to the four roots of the usual Dirac equation, while the remaining roots represent high-frequency modes that are physically uninteresting. This behavior is analogous to that found for the exact covariant dispersion relation for photons in the presence of Lorentz-violating operators of arbitrary dimension, which is given as Eq. (30) of Ref. [30]. We remark in passing that the explicit dispersion relation (39) can be expected to have an interpretation in terms of the geodesic motion of a classical particle in a Finsler spacetime, paralleling the existing treatment of the renormalizable case [24,40].

D. Properties

For many practical purposes, and to gain insight about the physical content of the exact result (39), it is useful to consider the approximate dispersion relation valid at leading order in Lorentz violation. Since Lorentz violation is expected to be small, $\hat{\mathcal{Q}}$ can be taken as a perturbation on the conventional Dirac operator $p \cdot \gamma - m_\psi$. The dispersion relation can therefore be expanded in the small operators $\hat{\mathcal{S}}$, $\hat{\mathcal{P}}$, $\hat{\mathcal{V}}^\mu$, $\hat{\mathcal{A}}^\mu$, $\hat{\mathcal{T}}^{\mu\nu}$.

At leading order, this expansion yields the approximate dispersion relation

$$p^2 - m_\psi^2 \approx 2(-m_\psi\hat{\mathcal{S}} - p \cdot \hat{\mathcal{V}} \pm \Upsilon), \tag{41}$$

where

$$\begin{aligned} \Upsilon^2 &= (p \cdot \hat{\mathcal{A}})^2 - m_\psi^2\hat{\mathcal{A}}^2 - 2m_\psi p \cdot \tilde{\hat{\mathcal{T}}} \cdot \hat{\mathcal{A}} + p \cdot \tilde{\hat{\mathcal{T}}} \cdot \tilde{\hat{\mathcal{T}}} \cdot p \\ &= p \cdot \tilde{\hat{\mathcal{T}}}_{\mathrm{eff}} \cdot \tilde{\hat{\mathcal{T}}}_{\mathrm{eff}} \cdot p. \end{aligned} \tag{42}$$

Solving for the energy E gives

$$E \approx E_0 - \frac{m_\psi \hat{\mathcal{S}} + p \cdot \hat{\mathcal{V}}}{E_0} \pm \frac{\Upsilon}{E_0} = E_0 - \frac{p \cdot \hat{\mathcal{V}}_{\rm eff}}{E_0} \pm \frac{\Upsilon}{E_0}, \qquad (43)$$

where ${E_0}^2 = m_\psi^2 + \boldsymbol{p}^2$. The results in this section are valid for E_0 of either sign, but in subsequent sections we take $E_0 > 0$. Note that the terms $\hat{\mathcal{V}}$, $\hat{\mathcal{S}}$, and Υ depend on the 4-momentum, which at the relevant order in Lorentz violation can be taken as $p^\mu \approx (E_0, \boldsymbol{p})$ on the right-hand side of this equation.

The two sign choices for E_0 correspond to particle and antiparticle modes, so in the presence of nonzero Lorentz violation the dispersion relation (43) can have four nondegenerate solutions for each $\boldsymbol{p}$. The usual spin degeneracy of a free Dirac fermion is broken when Υ is nonzero, which requires pseudovector or tensor operators for Lorentz violation. In contrast, the scalar and vector operators for Lorentz violation can shift the energy but preserve the spin degeneracy. The degeneracy between particles and antiparticles is broken when any of these operators have nonzero CPT-odd components. Note, however, that the pseudoscalar operator plays no role in the leading-order dispersion relation.

The dispersion relation (43) describes various kinds of deviations from the conventional Lorentz-covariant behavior of a massive fermion. Many are analogous to effects appearing in the nonminimal photon sector of the SME [30,41]. Among them are anisotropy, dispersion, and birefringence.

Anisotropy is a consequence of violation of rotation invariance, which implies the properties of the fermion depend on the momentum orientation $\hat{\boldsymbol{p}}$. For example, the group velocity $\boldsymbol{v}_g = \partial E/\partial \boldsymbol{p}$ becomes a direction-dependent quantity. We emphasize that in practice anisotropy is always present in models with physical Lorentz violation, even ones formulated as being rotation invariant in a particular frame, because boosts induce rotations. For example, any laboratory frame is instantaneously boosted by the Earth's rotation and revolution about the Sun, and these boosts necessarily introduce anisotropy.

When a dispersion relation is nonlinear, component waves in a packet travel at different phase velocities $\boldsymbol{v}_p = \boldsymbol{p}/E$. This dispersion is a familiar feature for a conventional massive fermion, and most Lorentz-violating operators are dispersive. Indeed, the only nondispersive terms in the Lagrange density (1) are those with a single derivative. However, certain dispersive terms are unobservable at leading order in Lorentz violation. For example, the dispersive operators contained in the pseudoscalar $\hat{\mathcal{P}}$ play no role in the dispersion relation (43). Also, the restriction $E \approx E_0$ on the right-hand side of this equation implies that some dispersive operators in $\hat{\mathcal{V}}$, $\hat{\mathcal{S}}$, and Υ produce effects in vacuum propagation that are unobservable at leading order. Changing the boundary conditions or introducing a medium leaves unaffected the basic dispersive nature of an operator, which is associated with its derivative structure. However, the corresponding change in the physics can trigger dispersion controlled by coefficients for Lorentz violation that at leading order are unobservable in the vacuum case. This is analogous to the situation for photon propagation [30].

In the presence of Lorentz violation, the fermion spin projections can mix during propagation because spin may no longer be conserved. Following the terminology for the analogous mixing of photon spins in Lorentz-violating electrodynamics, we refer to this spin mixing as fermion birefringence. It occurs whenever a particular solution to the dispersion relation is associated with only one low-energy mode instead of the usual two degenerate spin modes. The dispersion relation (43) holds for plane-wave solutions obeying the usual boundary conditions for vacuum propagation, and its form implies that fermion birefringence occurs whenever the combination Υ of coefficients given in Eq. (42) is nonzero. Note, however, that other situations such as a fermion trapped in a spherical container can involve different boundary conditions and hence can lead to spin-mixing effects controlled by different combinations of coefficients. Also, the presence of a medium such as matter or a background electromagnetic field can be expected to modify the combination of coefficients controlling fermion birefringence, which again parallels the situation for Lorentz-violating effects in photon propagation [30].

E. Spinors

A basic feature of the conventional Dirac equation is that the four linearly independent eigenspinors can be written using covariant projection operators as

$$u_\pm(p, n) = P_\pm \Lambda_+ \psi, \qquad v_\pm(p, n) = P_\mp \Lambda_- \psi, \qquad (44)$$

where the projection operators $\Lambda_\pm$ select positive- and negative-energy states, while

$$P_\pm = \frac{1}{2}(1 \pm \gamma_5 n \cdot \gamma) \qquad (45)$$

project the spin along a polarization vector n^μ satisfying $n^2 = -1$ and $n \cdot p = 0$ but otherwise having arbitrary orientation. The freedom in the choice of the unit spacelike transverse vector n^μ reflects the spin degeneracy of the eigenspinors. However, in the presence of perturbative Lorentz violation, the breaking of spin degeneracy for $\Upsilon \neq 0$ implies that each solution to the dispersion relation becomes an eigenmode having a definite spin polarization. The polarization projection operators $P_\pm$ must therefore involve a vector n^μ with a definite orientation. Next, we obtain an approximation representation of n^μ.

The birefringent term involving $\pm\Upsilon$ can be isolated from the modified Dirac equation at leading order in Lorentz violation by acting on the left with the operator $(p + \hat{\mathcal{V}}) \cdot \gamma + (m_\psi - \hat{\mathcal{S}})$. Using both the modified Dirac equation and the result (41) in the form $[(p + \hat{\mathcal{V}})^2 - (m_\psi - \hat{\mathcal{S}})^2]^2 \approx 4\Upsilon^2$ permits the elimination of terms at second order in Lorentz-violating operators. This generates the equation

$$[\pm\Upsilon - \gamma_5(p\cdot\hat{\mathcal{A}} - m_\psi\hat{\mathcal{A}}\cdot\gamma - p\cdot\tilde{\hat{\mathcal{T}}}\cdot\gamma)]\psi \approx 0, \tag{46}$$

which for $\Upsilon \neq 0$ motivates the definition

$$P_\pm \approx \frac{1}{2}\left(1 \pm \frac{\gamma_5(p\cdot\hat{\mathcal{A}} - m_\psi\hat{\mathcal{A}}\cdot\gamma - p\cdot\tilde{\hat{\mathcal{T}}}\cdot\gamma)}{\Upsilon}\right). \tag{47}$$

A short calculation reveals that

$$[\gamma_5(p\cdot\hat{\mathcal{A}} - m_\psi\hat{\mathcal{A}}\cdot\gamma - p\cdot\tilde{\hat{\mathcal{T}}}\cdot\gamma)]^2 = \Upsilon^2, \tag{48}$$

ensuring that $P_\pm$ are indeed orthogonal projection operators.

To express the projectors (47) in the form (45), we use $m_\psi\psi \approx p\cdot\gamma\psi$ and thereby identify the spacelike vector n^μ as

$$n^\mu \approx \frac{p\cdot\hat{\mathcal{A}}p^\mu - m_\psi^2\hat{\mathcal{A}}^\mu + m_\psi\tilde{\hat{\mathcal{T}}}^{\mu\nu}p_\nu}{m_\psi\Upsilon} \equiv \frac{1}{\Upsilon}N^\mu, \tag{49}$$

which satisfies $n^2 = -1$ and $n\cdot p \approx 0$, as required. The spacelike vector

$$N^\mu = \left(\tilde{\hat{\mathcal{T}}}^{\mu\nu} + \frac{1}{m_\psi}p^{[\mu}\hat{\mathcal{A}}^{\nu]}\right)p_\nu = \tilde{\hat{\mathcal{T}}}^{\mu\nu}_{\text{eff}}p_\nu\psi \tag{50}$$

obeying $N^2 = -\Upsilon^2$ and $N\cdot p \approx 0$ is introduced for notational convenience in what follows. The expression for n^μ, which is at zeroth order in Lorentz violation, fixes the dominant polarization required for a solution to approximate the exact eigenspinor. Note that if $\Upsilon = 0$ the derivation breaks down, but the spin degeneracy is then restored, and so n^μ can be approximated as a unit spacelike transverse vector, as usual. Note also that the subscripts on the projections $P_\pm$ correspond to the signs in the modified dispersion relation (43), so polarizing a fermion along n^μ increases the energy while the opposite polarization decreases it.

III. HAMILTONIAN

The construction of the exact Hamiltonian associated with the full theory (1) is complicated by the higher-order time derivatives that appear. For most practical applications, however, it suffices to obtain an effective Hamiltonian that describes correctly the behavior at leading order in Lorentz violation. We present here a perturbative derivation of the Hamiltonian via a generalization of the standard approach, and we extract the relativistic combinations of coefficients that it contains.

A. Construction

The goal of the standard approach to constructing the Hamiltonian is to find a unitary transformation $U = U(\boldsymbol{p})$ converting the modified Dirac equation (30) to the form

$$U\gamma_0(p\cdot\gamma - m_\psi + \hat{\mathcal{Q}})U^\dagger U\psi = (E - H)U\psi = 0, \tag{51}$$

where $E \equiv p_0$ and the 4×4 relativistic Hamiltonian H is block diagonal or "even" with vanishing 2×2 off-diagonal "odd" blocks. This decouples the positive and negative energy states, and the diagonal blocks give the 2×2 relativistic Hamiltonians describing particles and antiparticles. We adopt the chiral representation, in which the matrices γ^μ are block off diagonal, so we seek U such that Eq. (51) involves only an even number of γ^μ matrices. Since the Lorentz violation is perturbative, it is useful to write

$$H = H_0 + \delta H, \tag{52}$$

where $H_0 = \gamma_0(\boldsymbol{p}\cdot\boldsymbol{\gamma} + m_\psi)$ is the usual 4×4 Dirac Hamiltonian for the Lorentz-invariant case and δH contains the Lorentz-violating modifications.

Consider first the usual Lorentz-invariant case with $\hat{\mathcal{Q}} = 0$ and $\delta H = 0$. An appropriate transformation $U = VW = WV$ is the product of the two commuting transformations

$$V = \frac{1 + \gamma_0\gamma_5}{\sqrt{2}}, \qquad W(\boldsymbol{p}) = \frac{E_0 + m_\psi + \boldsymbol{p}\cdot\boldsymbol{\gamma}}{\sqrt{2E_0(E_0 + m_\psi)}}, \tag{53}$$

with $E_0 = \sqrt{\boldsymbol{p}^2 + m_\psi^2} > 0$. Direct calculation shows that this transformation gives the expected block-diagonal Hamiltonian

$$H_0 = -\gamma_5 E_0 = \begin{pmatrix} E_0 & 0 \\ 0 & -E_0 \end{pmatrix}. \tag{54}$$

The upper 2×2 block describes positive-energy particles with Hamiltonian $h_0 = E_0$, while after the usual reinterpretation, the lower negative-energy block gives the Hamiltonian $\bar{h}_0 = E_0$ for positive-energy antiparticles.

In the Lorentz-violating case, if $\hat{\mathcal{Q}}$ is nonzero but contains only the operators $\hat{\mathcal{S}}$ and $\hat{\mathcal{V}}^\mu$, then the same procedure can be used to perform the block diagonalization. It suffices to replace p_μ with $p_\mu + \hat{\mathcal{V}}_\mu$ and m_ψ with $m_\psi - \hat{\mathcal{S}}$ in the transformation (53). In contrast, the general case involving also nonzero $\hat{\mathcal{P}}$, $\hat{\mathcal{A}}^\mu$, and $\hat{\mathcal{T}}^{\mu\nu}$ is challenging. However, a perturbative treatment can be adopted to implement the block diagonalization at leading order in Lorentz violation.

To zeroth order, the transformation U is given by the product VW. So, we start by applying this,

$$\begin{aligned} &VW\gamma_0(p\cdot\gamma - m_\psi + \hat{\mathcal{Q}})W^\dagger V^\dagger \\ &\quad = E + \gamma_5 E_0 + VW\gamma_0\hat{\mathcal{Q}}W^\dagger V^\dagger. \end{aligned} \tag{55}$$

The Lorentz-invariant terms are block diagonal, but the last term contains both even and odd parts. The even part of any matrix M can be extracted by applying the even matrix γ_5 to give $M_{\text{even}} = (M + \gamma_5 M \gamma_5)/2$. To remove the odd part at first order in Lorentz violation, we can therefore modify U by an additional small transformation,

$$U = \Big(1 + \frac{1}{4E_0}[\gamma_5, VW\gamma_0 \hat{\mathcal{Q}} W^\dagger V^\dagger]\Big) VW. \tag{56}$$

This gives

$$\begin{aligned} &U\gamma_0(p\cdot\gamma - m_\psi + \hat{\mathcal{Q}})U^\dagger \\ &\quad = E + \gamma_5 E_0 + (VW\gamma_0\hat{\mathcal{Q}}W^\dagger V^\dagger)_{\text{even}} \end{aligned} \tag{57}$$

at first order in Lorentz violation. We can now identify the leading-order block-diagonal Lorentz-violating Hamiltonian as

$$\delta H = -(VW\gamma_0\hat{\mathcal{Q}}W^\dagger V^\dagger)_{\text{even}}. \tag{58}$$

Substituting for $\hat{\mathcal{Q}}$ using Eq. (2) and performing some explicit calculations, we obtain the result

$$\begin{aligned} \delta H = \frac{1}{E_0}[&m_\psi \hat{\mathcal{S}}\gamma_5 - E_0\hat{\mathcal{V}}^0 - \hat{\mathcal{V}}^j p^j \gamma_5 + \hat{\mathcal{A}}^0 p^j \gamma^j \gamma_0 \\ &+ m_\psi \hat{\mathcal{A}}^j \gamma^j \gamma_0 \gamma_5 + \hat{\mathcal{A}}^j p^j p^k \gamma^k \gamma_0 \gamma_5/(E_0 + m_\psi) \\ &+ i p^j \hat{\mathcal{T}}^{0k} \gamma^j \gamma^k + i \hat{\mathcal{T}}^{0j} p^j - E_0 \hat{\tilde{\mathcal{T}}}^{0j} \gamma^j \gamma_0 \\ &+ \hat{\tilde{\mathcal{T}}}^{0j} p^j p^k \gamma^k \gamma_0/(E_0 + m_\psi)]. \end{aligned} \tag{59}$$

The upper 2×2 block of this operator represents the leading-order perturbation δh to the positive-energy Hamiltonian for particles,

$$\delta h = \frac{\Delta + \mathbf{\Sigma}\cdot\boldsymbol{\sigma}}{E_0}, \tag{60}$$

where

$$\Delta = -m_\psi \hat{\mathcal{S}} - E_0 \hat{\mathcal{V}}^0 + p^j \hat{\mathcal{V}}^j = -p_\mu \hat{\mathcal{V}}^\mu_{\text{eff}} \tag{61}$$

and the spin dependence is controlled by

$$\begin{aligned} \Sigma^j &= \hat{\mathcal{A}}^0 p^j - m_\psi \hat{\mathcal{A}}^j - \hat{\mathcal{A}}^k p^k p^j/(E_0 + m_\psi) \\ &\quad - E_0 \hat{\tilde{\mathcal{T}}}^{0j} - \hat{\tilde{\mathcal{T}}}^{jk} p^k + \hat{\tilde{\mathcal{T}}}^{0k} p^k p^j/(E_0 + m_\psi) \\ &= -E_0 \hat{\tilde{\mathcal{T}}}^{0j}_{\text{eff}} - \hat{\tilde{\mathcal{T}}}^{jk}_{\text{eff}} p^k + \hat{\tilde{\mathcal{T}}}^{0k}_{\text{eff}} p^k p^j/(E_0 + m_\psi). \end{aligned} \tag{62}$$

These results reduce to those established in Ref. [23] for operators of minimal dimension $d = 3$ and $d = 4$, as expected. We emphasize that the 2×2 Hamiltonian

$$h = h_0 + \delta h \tag{63}$$

is fully relativistic.

After reinterpretation, the lower 2×2 block of Eq. (59) gives the change $\delta\bar{h}$ to the positive-energy Hamiltonian for antiparticles,

$$\delta\bar{h} = \frac{\bar{\Delta} + \mathbf{\Sigma}\cdot\bar{\boldsymbol{\sigma}}}{E_0}, \tag{64}$$

where

$$\bar{\Delta} = -m_\psi \hat{\mathcal{S}} + E_0 \hat{\mathcal{V}}^0 + p^j \hat{\mathcal{V}}^j \tag{65}$$

and

$$\begin{aligned} \bar{\Sigma}^j &= \hat{\mathcal{A}}^0 p^j + m_\psi \hat{\mathcal{A}}^j + \hat{\mathcal{A}}^k p^k p^j/(E_0 + m_\psi) \\ &\quad - E_0 \hat{\tilde{\mathcal{T}}}^{0j} + \hat{\tilde{\mathcal{T}}}^{jk} p^k + \hat{\tilde{\mathcal{T}}}^{0k} p^k p^j/(E_0 + m_\psi). \end{aligned} \tag{66}$$

Note that in the Lorentz-violating terms, we can take $p_0 \approx E_0$ for particles and $p_0 \approx -E_0$ for antiparticles because corrections to these approximations contribute only at second order. Since the physical antiparticle 3-momentum is $-\boldsymbol{p}$, the corresponding physical 4-momentum can be taken to be $-p_\mu$. This implies that the antiparticle Hamiltonian

$$\bar{h} = \bar{h}_0 + \delta\bar{h} \tag{67}$$

can be obtained from h by changing the sign of all coefficients for CPT-odd operators, as expected.

The Lorentz-violating portion of the transformation (56) can be expressed in an alternative form by commuting VW through to the right. This gives

$$U = VW\Big(1 - \frac{1}{4E_0^2}[H_0, \gamma_0\hat{\mathcal{Q}}]\Big). \tag{68}$$

We then find

$$\begin{aligned} &U\gamma_0(p\cdot\gamma - m_\psi + \hat{\mathcal{Q}})U^\dagger \\ &\quad = VW(E - H_0 + \Lambda_+\gamma_0\hat{\mathcal{Q}}\Lambda_+ + \Lambda_-\gamma_0\hat{\mathcal{Q}}\Lambda_-)W^\dagger V^\dagger, \end{aligned} \tag{69}$$

where $\Lambda_\pm = (1 \pm H_0/E_0)/2$ are the usual projection operators for energy. This equation reveals that the net effect of the Lorentz-violating part of U is to remove the portions of $\hat{\mathcal{Q}}$ mixing the usual particle and antiparticle states.

B. Coefficients

The explicit nature of the terms (61) and (62) in the perturbation Hamiltonian δh obscures the relativistic combinations of coefficients from which they are formed. A more elegant form for δh that displays these combinations can be obtained using the relativistic polarization vector N^μ defined in Eq. (49).

The spin vector Σ^j is related to N^μ by

$$\begin{aligned} \Sigma^j &= N^j - \frac{N^0 p^j}{E_0 + m_\psi} = N^j - \frac{N^k p^k p^j}{E_0(E_0 + m_\psi)} \\ &= N^j_\perp + \frac{m_\psi}{E_0} N^j_\parallel, \end{aligned} \tag{70}$$

where $\boldsymbol{N}_\perp$ and $\boldsymbol{N}_\parallel$ are the components of $\boldsymbol{N}$ perpendicular and parallel to $\boldsymbol{p}$, respectively. Note that both $\boldsymbol{N}_\perp$ and

$m_\psi \boldsymbol{N}_\parallel$ remain finite even in the massless limit. The magnitude of the spin vector is $|\boldsymbol{\Sigma}| = \Upsilon$, so the spin-dependent energy shifts are Υ/E_0 for spin along $\boldsymbol{\Sigma}$ and $-\Upsilon/E_0$ for spin opposite $\boldsymbol{\Sigma}$, as expected.

To gain further insight, consider a massive particle in its rest frame with $N^0 = 0$ and $\boldsymbol{N}^2 = \Upsilon^2$, and introduce the rest-frame polarization unit vector $\boldsymbol{N}'$. Boosting to an arbitrary frame then gives

$$N^0 = \frac{|\boldsymbol{p}|}{m_\psi}|\boldsymbol{N}'_\parallel|, \qquad \boldsymbol{N} = \boldsymbol{N}'_\perp + \frac{E_0}{m_\psi}\boldsymbol{N}'_\parallel, \tag{71}$$

where $\boldsymbol{N}'_\parallel$ and $\boldsymbol{N}'_\perp$ are the projections of $\boldsymbol{N}'$ parallel and perpendicular to $\boldsymbol{p}$, respectively. Comparing to Eq. (70) reveals that $\boldsymbol{\Sigma}$ is the rest-frame $\boldsymbol{N}$ vector,

$$\boldsymbol{\Sigma} = \boldsymbol{N}'. \tag{72}$$

Note that $\boldsymbol{N}'$ depends on p_μ because the required boost varies with p_μ.

The above considerations permit us to write $\boldsymbol{\Sigma}\cdot\boldsymbol{\sigma}$ as

$$\boldsymbol{\Sigma}\cdot\boldsymbol{\sigma} = -N^\mu \tau_\mu, \tag{73}$$

where

$$\tau^0 = \frac{\boldsymbol{p}\cdot\boldsymbol{\sigma}}{m_\psi}, \qquad \tau^j = \sigma^j + \frac{p^j \tau^0}{E_0 + m_\psi}. \tag{74}$$

The perturbation Hamiltonian (60) therefore takes the form

$$\delta h = \frac{\Delta - N^\mu \tau_\mu}{E_0}, \tag{75}$$

showing that the spin-dependent Lorentz violation is fixed by the relativistic polarization vector N^μ given in Eq. (49).

We can now expand Δ and N^μ in powers of momentum p^μ to extract the effective coefficients for Lorentz violation that appear in the Hamiltonian. In practice, it is convenient to split Δ and N^μ into CPT-odd and CPT-even pieces for this purpose.

Expanding Δ yields

$$\Delta = \sum_d \Delta^{(d)\alpha_1\ldots\alpha_{d-2}} p_{\alpha_1}\ldots p_{\alpha_{d-2}}, \tag{76}$$

where even and odd d are associated with CPT-even and CPT-odd Lorentz violations, respectively, and the coefficients $\Delta^{(d)\alpha_1\ldots\alpha_{d-2}}$ have mass dimension $4-d$. Separating the CPT-even and CPT-odd parts and substituting the definitions (26) into the expression (76) gives

$$\begin{aligned}\Delta_{\rm odd} &\equiv \hat{a}^\mu_{\rm eff} p_\mu = \sum_d a^{(d)\alpha_1\ldots\alpha_{d-2}}_{\rm eff} p_{\alpha_1}\ldots p_{\alpha_{d-2}},\\ \Delta_{\rm even} &\equiv -\hat{c}^\mu_{\rm eff} p_\mu = -\sum_d c^{(d)\alpha_1\ldots\alpha_{d-2}}_{\rm eff} p_{\alpha_1}\ldots p_{\alpha_{d-2}},\end{aligned} \tag{77}$$

where the effective coefficients are given in terms of fundamental coefficients by Eq. (27).

Similarly, expanding N^μ yields

$$N^\mu = \sum_d N^{(d)\mu\alpha_1\ldots\alpha_{d-2}} p_{\alpha_1}\ldots p_{\alpha_{d-2}}, \tag{78}$$

where now even and odd d are associated with CPT-odd and CPT-even Lorentz violations, respectively, with the coefficients $N^{(d)\mu\alpha_1\ldots\alpha_{d-2}}$ having mass dimension $4-d$. Note that constant N^μ is forbidden by the restriction $p\cdot N = 0$. Separating the CPT-even and CPT-odd parts and combining the definitions (26) and the result (50) gives

$$\begin{aligned}N^\mu_{\rm odd} &\equiv \tilde{\hat{g}}^{\mu\nu}_{\rm eff} p_\nu = \sum_d \tilde{g}^{(d)\mu\alpha_1\ldots\alpha_{d-2}}_{\rm eff} p_{\alpha_1}\ldots p_{\alpha_{d-2}},\\ N^\mu_{\rm even} &\equiv -\tilde{\hat{H}}^{\mu\nu}_{\rm eff} p_\nu = -\sum_d \tilde{H}^{(d)\mu\alpha_1\ldots\alpha_{d-2}}_{\rm eff} p_{\alpha_1}\ldots p_{\alpha_{d-2}},\end{aligned} \tag{79}$$

where again the effective coefficients are given in terms of fundamental coefficients by Eq. (27).

The above analysis reveals that the perturbative Hamiltonian δh in Eq. (60) can conveniently be split into four pieces according to

$$\begin{aligned}\delta h &= h_a + h_c + h_g + h_H,\\ &= \frac{1}{E_0}(\hat{a}^\nu_{\rm eff} - \hat{c}^\nu_{\rm eff} - \tilde{\hat{g}}^{\mu\nu}_{\rm eff}\tau_\mu + \tilde{\hat{H}}^{\mu\nu}_{\rm eff}\tau_\mu)p_\nu,\end{aligned} \tag{80}$$

where the explicit expansions for $\hat{a}^\mu_{\rm eff}$, $\hat{c}^\mu_{\rm eff}$, $\tilde{\hat{g}}^{\mu\nu}_{\rm eff}$, and $\tilde{\hat{H}}^{\mu\nu}_{\rm eff}$ are given by Eqs. (77) and (79). Each of the four component Hamiltonians is uniquely specified by spin and CPT properties: the spin-independent terms h_a and h_c are CPT odd and CPT even, respectively, as are the spin-dependent terms h_g and h_H. Note that the structure of the results obtained above is compatible with the discussion in Sec. II B concerning field redefinitions and physical observables.

IV. SPHERICAL DECOMPOSITION

The complexity of the two-component perturbative Hamiltonian (80) and the appearance of coefficients with numerous indices make a general analysis of physical implications unwieldy for arbitrary d. Some of the difficulties can be alleviated by performing a spherical-harmonic decomposition of the Hamiltonian. For example, a typical experimental application involves a transformation from a noninertial laboratory frame to the canonical Sun-centered inertial frame [6,37,42], which is generically dominated by rotations and is therefore simpler in spherical basis. For each d, the spherical-harmonic decomposition yields a set of coefficients equivalent to those introduced in Sec. II A but having comparatively simple rotation properties. This permits a systematic classification of the coefficients affecting the dynamics and is also advantageous because rotation violations are a key signature of Lorentz violation.

A. Basics

Since the Hamiltonian (80) is expressed in momentum space, the relevant spherical coordinates also lie in this space. We can introduce spherical polar angles θ, ϕ via the unit 3-momentum vector $\hat{\boldsymbol{p}} = \boldsymbol{p}/|\boldsymbol{p}|$ written in the form $\hat{\boldsymbol{p}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Rotation scalars can then be expanded in terms of the usual spherical harmonics ${}_0Y_{jm}(\hat{\boldsymbol{p}}) \equiv Y_{jm}(\theta, \phi)$. However, the expansion of rotation tensors requires some form of generalized spherical harmonics. We adopt here the spin-weighted spherical harmonics ${}_sY_{jm}(\hat{\boldsymbol{p}}) \equiv {}_sY_{jm}(\theta, \phi)$, which permit the spherical decomposition of tensors in the helicity basis. The spin weight s of an irreducible tensor is defined as the negative of its helicity and is limited by $|s| \le j$. A summary of properties of the spin-weighted spherical harmonics is given in Appendix A of Ref. [30].

In the perturbative Hamiltonian (80), h_a and h_c transform as scalars under rotations, while h_g and h_H are spin dependent through the quantity $\boldsymbol{\Sigma}\cdot\boldsymbol{\sigma} = -N^\mu\tau_\mu$ and so have nontrivial rotation properties. To perform the expansion in spin-weighted spherical harmonics, we therefore require the decomposition of $\boldsymbol{\Sigma}\cdot\boldsymbol{\sigma}$ in the helicity basis. The helicity basis vectors are defined as $\hat{\boldsymbol{\epsilon}}_r = \hat{\boldsymbol{\epsilon}}^r = \hat{\boldsymbol{p}}$ and $\hat{\boldsymbol{\epsilon}}_\pm = \hat{\boldsymbol{\epsilon}}^\mp = (\hat{\boldsymbol{\theta}} \pm i\hat{\boldsymbol{\phi}})/\sqrt{2}$, where $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ are the usual unit vectors associated with the polar angle θ and azimuthal angle ϕ. The helicity decomposition is

$$\boldsymbol{\Sigma}\cdot\boldsymbol{\sigma} = \Sigma_w\sigma^w = \Sigma^w\sigma_w = \Sigma^-\sigma_- + \Sigma^r\sigma_r + \Sigma^+\sigma_+, \tag{81}$$

where the repeated index w is summed over $w = +, r, -$, and $\sigma_w = \hat{\boldsymbol{\epsilon}}_w\cdot\boldsymbol{\sigma}$, $\sigma^w = \hat{\boldsymbol{\epsilon}}^w\cdot\boldsymbol{\sigma}$. The component $\Sigma_r = \hat{\boldsymbol{\epsilon}}_r\cdot\boldsymbol{\Sigma}$ is a rotational scalar with spin weight zero and can therefore be expanded in the usual spherical harmonics ${}_0Y_{jm}(\hat{\boldsymbol{p}})$. The components $\Sigma_\pm = \hat{\boldsymbol{\epsilon}}_\pm\cdot\boldsymbol{\Sigma}$ have spin weight $s = \pm1$ and can be expanded in the harmonics ${}_{\pm1}Y_{jm}(\hat{\boldsymbol{p}})$, while the components $\Sigma^\pm = \hat{\boldsymbol{\epsilon}}^\pm\cdot\boldsymbol{\Sigma} = \Sigma_\mp$ have helicity ±1. The Pauli matrices in the helicity basis are

$$\sigma_r = \sigma^r = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix},$$
$$\sigma_\pm = \sigma^\mp = \frac{1}{\sqrt{2}}\begin{pmatrix} -\sin\theta & (\cos\theta\pm1)e^{-i\phi} \\ (\cos\theta\mp1)e^{i\phi} & \sin\theta \end{pmatrix}. \tag{82}$$

Up to constants, σ_r is the helicity operator, and $\sigma_\pm$ are helicity ladder operators. To see this, consider the special frame in which $\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{x}}$, $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{y}}$, $\hat{\boldsymbol{p}} = \hat{\boldsymbol{z}}$ and so

$$\sigma_r = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_+ = \sqrt{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \sqrt{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{83}$$

This shows that σ_+ raises the helicity and σ_- lowers it. Acting with $\boldsymbol{\Sigma}\cdot\boldsymbol{\sigma}$ on a spinor ϕ with components $(\phi_\uparrow, \phi_\downarrow)$ having helicities $(+1/2, -1/2)$, respectively, gives

$$\boldsymbol{\Sigma}\cdot\boldsymbol{\sigma}\begin{pmatrix} \phi_\uparrow \\ \phi_\downarrow \end{pmatrix} = \begin{pmatrix} \Sigma^r\phi_\uparrow + \sqrt{2}\Sigma^+\phi_\downarrow \\ -\Sigma^r\phi_\downarrow + \sqrt{2}\Sigma^-\phi_\uparrow \end{pmatrix}. \tag{84}$$

The first component maintains its positive helicity because $\Sigma^r\phi_\uparrow$ is the product of objects with helicity 0 and $+1/2$, while $\Sigma^+\phi_\downarrow$ is the product of objects with helicity $+1$ and $-1/2$. Similarly, the second component remains an object of helicity $-1/2$. We can conclude that $\Sigma^r = \Sigma_r$ generates helicity-dependent effects without changing the helicity, $\Sigma^+ = \Sigma_-$ is associated with a raising of helicity, and $\Sigma^- = \Sigma_+$ is associated with a lowering of helicity.

B. Decomposition

We can now proceed with the spherical decomposition of the perturbative Hamiltonian (80). The components h_a and h_c are rotational scalars and so can be expanded in the usual spherical polar coordinates as

$$h_a = \sum_{dnjm} E_0{}^{d-3-n}|\boldsymbol{p}|^n\,{}_0Y_{jm}(\hat{\boldsymbol{p}})a^{(d)}_{njm},$$
$$h_c = -\sum_{dnjm} E_0{}^{d-3-n}|\boldsymbol{p}|^n\,{}_0Y_{jm}(\hat{\boldsymbol{p}})c^{(d)}_{njm}. \tag{85}$$

In contrast, the spin-dependent component Hamiltonians h_g and h_H transform nontrivially under rotations and must therefore first be separated into spin-weighted components,

$$h_g = (h_g)_w\sigma^w, \qquad h_H = (h_H)_w\sigma^w. \tag{86}$$

At the end of this subsection, we show that these components have expansions

$$(h_g)_r = -m_\psi\sum_{dnjm} E_0{}^{d-4-n}|\boldsymbol{p}|^n\,{}_0Y_{jm}(\hat{\boldsymbol{p}})(n+1)g^{(d)(0B)}_{njm},$$
$$(h_g)_\pm = \sum_{dnjm} E_0{}^{d-3-n}|\boldsymbol{p}|^n\,{}_{\pm1}Y_{jm}(\hat{\boldsymbol{p}}) \times\left[\pm\sqrt{\frac{j(j+1)}{2}}g^{(d)(0B)}_{njm} \pm g^{(d)(1B)}_{njm} + ig^{(d)(1E)}_{njm}\right],$$
$$(h_H)_r = m_\psi\sum_{dnjm} E_0{}^{d-4-n}|\boldsymbol{p}|^n\,{}_0Y_{jm}(\hat{\boldsymbol{p}})(n+1)H^{(d)(0B)}_{njm},$$
$$(h_H)_\pm = -\sum_{dnjm} E_0{}^{d-3-n}|\boldsymbol{p}|^n\,{}_{\pm1}Y_{jm}(\hat{\boldsymbol{p}}) \times\left[\pm\sqrt{\frac{j(j+1)}{2}}H^{(d)(0B)}_{njm} \pm H^{(d)(1B)}_{njm} + iH^{(d)(1E)}_{njm}\right]. \tag{87}$$

The full perturbative Hamiltonian (80) is therefore given by the expansion

$$\delta h = h_a + h_c + (h_g)_+ \sigma^+ + (h_g)_r \sigma^r + (h_g)_- \sigma^- + (h_H)_+ \sigma^+ + (h_H)_r \sigma^r + (h_H)_- \sigma^-, \quad (88)$$

where the component Hamiltonians are given by Eqs. (85) and (87).

The properties and index ranges of the eight sets of spherical coefficients appearing in these expansions are summarized in Table III. All coefficients have mass dimension $4-d$. The first column of the table lists the coefficients. The second column specifies the *CPT* handedness of the corresponding operators. The third column gives the behavior of the operators under parity, where operators with E-type parity acquire a sign $(-1)^j$ and those with B-type parity acquire a sign $(-1)^{j+1}$. The next three columns list the allowed ranges of d, n, and j, while the final column provides the number of independent coefficients appearing for each d.

The coefficients given in the expansions (85) and (87) and listed in Table III comprise the set of observable quantities at leading order in Lorentz violation. Each set of coefficients $\mathcal{K}_{jm}$ obeys the complex conjugation relation

$$\mathcal{K}^*_{jm} = (-1)^m \mathcal{K}_{j(-m)}, \quad (89)$$

which stems from the reality of the underlying tensors in momentum space and ultimately from the Hermiticity of the Hamiltonian (80) and the theory (1). The coefficients have comparatively simple properties under rotations, which can be implemented using the standard Wigner rotation matrices in parallel with the treatments for the photon and neutrino sectors given in Sec. V of Ref. [30] and Sec. VI of Ref. [16]. As an example, the relation between coefficients $\mathcal{K}^{\rm lab}_{jm}$ in a standard laboratory frame with x axis pointing south and y axis pointing east to coefficients $\mathcal{K}_{jm}$ in the canonical Sun-centered frame [6,37,42] is

$$\mathcal{K}^{\rm lab}_{jm} = \sum_{m'} e^{im'\omega_\oplus T_\oplus} d^{(j)}_{mm'}(-\chi) \mathcal{K}_{jm'}, \quad (90)$$

where $\omega_\oplus$ is the sidereal rotation frequency of the Earth, $T_\oplus$ is the sidereal time, the quantities $d^{(j)}_{mm'}$ are the "little" Wigner matrices given in Eq. (136) of Ref. [30], and χ is the colatitude of the laboratory in the northern hemisphere. This expression only involves a linear combination mixing the azimuthal components labeled by m'.

The subset of isotropic coefficients can be identified by imposing $j = m = 0$ in the spherical-harmonic expansion of the Hamiltonian (80). In this limit, the helicity-flipping pieces of the Hamiltonian vanish. This is because helicity ± 1 is incompatible with $j = 0$ or, equivalently, because the spin weight is limited by $|s| \le j$. As a result, the perturbative isotropic Hamiltonian takes the form

$$\delta \mathring{h} = \mathring{h}_a + \mathring{h}_c + (\mathring{h}_g)_r \sigma^r + (\mathring{h}_H)_r \sigma^r, \quad (91)$$

where

$$\begin{aligned} \mathring{h}_a &= \sum_{dn} E_0{}^{d-3-n} |\boldsymbol{p}|^n \mathring{a}^{(d)}_n, \\ \mathring{h}_c &= -\sum_{dn} E_0{}^{d-3-n} |\boldsymbol{p}|^n \mathring{c}^{(d)}_n, \\ (\mathring{h}_g)_r &= -m_\psi \sum_{dn} E_0{}^{d-4-n} |\boldsymbol{p}|^n \mathring{g}^{(d)}_n, \\ (\mathring{h}_H)_r &= m_\psi \sum_{dn} E_0{}^{d-4-n} |\boldsymbol{p}|^n \mathring{H}^{(d)}_n. \end{aligned} \quad (92)$$

In these expressions, the isotropic coefficients are related to the spherical coefficients through

$$\mathring{a}^{(d)}_n = \frac{1}{\sqrt{4\pi}} a^{(d)}_{n00}, \qquad \mathring{c}^{(d)}_n = \frac{1}{\sqrt{4\pi}} c^{(d)}_{n00},$$
$$\mathring{g}^{(d)}_n = \frac{1}{\sqrt{4\pi}} (n+1) g^{(d)(0B)}_{n00}, \qquad \mathring{H}^{(d)}_n = \frac{1}{\sqrt{4\pi}} (n+1) H^{(d)(0B)}_{n00}. \quad (93)$$

These relations give equivalent representations for the spherical coefficients with $j = 0$ listed in Table III. All

TABLE III. Spherical coefficients for Lorentz violation.

Coefficient	*CPT*	Parity type	d	n	j	Number
$a^{(d)}_{njm}$	Odd	E	Odd, ≥ 3	$0, 1, \ldots, d-2$	$n, n-2, n-4, \ldots \ge 0$	$\frac{1}{6}(d+1)d(d-1)$
$c^{(d)}_{njm}$	Even	E	Even, ≥ 4	$0, 1, \ldots, d-2$	$n, n-2, n-4, \ldots \ge 0$	$\frac{1}{6}(d+1)d(d-1)$
$g^{(d)(0B)}_{njm}$	Odd	B	Even, ≥ 4	$0, 1, \ldots, d-3$	$n+1, n-1, n-3, \ldots \ge 0$	$\frac{1}{6}(d+1)d(d-1)-1$
$g^{(d)(1B)}_{njm}$	Odd	B	Even, ≥ 4	$2, 3, \ldots, d-2$	$n-1, n-3, n-5, \ldots \ge 1$	$\frac{1}{6}(d-2)(d^2-d-3)$
$g^{(d)(1E)}_{njm}$	Odd	E	Even, ≥ 4	$1, 2, \ldots, d-2$	$n, n-2, n-4, \ldots \ge 1$	$\frac{1}{6}(d+2)d(d-2)$
$H^{(d)(0B)}_{njm}$	Even	B	Odd, ≥ 3	$0, 1, \ldots, d-3$	$n+1, n-1, n-3, \ldots \ge 0$	$\frac{1}{6}(d+1)d(d-1)-1$
$H^{(d)(1B)}_{njm}$	Even	B	Odd, ≥ 5	$2, 3, \ldots, d-2$	$n-1, n-3, n-5, \ldots \ge 1$	$\frac{1}{6}(d+1)(d-1)(d-3)$
$H^{(d)(1E)}_{njm}$	Even	E	Odd, ≥ 3	$1, 2, \ldots, d-2$	$n, n-2, n-4, \ldots \ge 1$	$\frac{1}{6}(d-1)(d^2+d-3)$
$\mathring{a}^{(d)}_n$	Odd	Even	Odd, ≥ 3	$0, 2, 4, \ldots, d-3$	0	$\frac{1}{2}(d-1)$
$\mathring{c}^{(d)}_n$	Even	Even	Even, ≥ 4	$0, 2, 4, \ldots, d-2$	0	$\frac{1}{2}d$
$\mathring{g}^{(d)}_n$	Odd	Odd	Even, ≥ 4	$1, 3, 5, \ldots, d-3$	0	$\frac{1}{2}(d-2)$
$\mathring{H}^{(d)}_n$	Even	Odd	Odd, ≥ 5	$1, 3, 5, \ldots, d-4$	0	$\frac{1}{2}(d-3)$

the isotropic coefficients have mass dimension $4-d$. Their index ranges and counting are summarized in Table III. Note that the result (29) implies exactly one linear combination of $\mathring{c}_n^{(d)}$ at each even d controls a Lorentz-invariant operator.

To illustrate the connection between the Cartesian and isotropic coefficients in the context of the Lagrange density (1), we can consider the explicit form of the effective isotropic theory for the first few dimensions $d = 3, 4, 5, 6$. At $d = 3$ only one term exists,

$$\hat{\mathcal{Q}}_{\rm eff}^{(3)} = -a_{\rm eff}^{(3)0}\gamma_0 \equiv -\mathring{a}_0^{(3)}\gamma_0, \tag{94}$$

representing an isotropic CPT-violating operator. Note, however, that the phase redefinition (9) can be used to show this term has no observable effects, as discussed in Sec. II B. At $d = 4$ there are three independent isotropic terms, given by

$$\hat{\mathcal{Q}}_{\rm eff}^{(4)} = c_{\rm eff}^{(4)00}p^0\gamma^0 + \frac{1}{3}c_{\rm eff}^{(4)jj}p^k\gamma^k + \frac{1}{3}i\tilde{g}_{\rm eff}^{(4)0jj}p^k\gamma_5\sigma^{0k}$$
$$\equiv \mathring{c}_0^{(4)}p^0\gamma^0 + \mathring{c}_2^{(4)}p^k\gamma^k - i\mathring{g}_1^{(4)}p^k\gamma_5\sigma^{0k}. \tag{95}$$

Since d is even, one combination of the coefficients $\mathring{c}_n^{(4)}$ must be associated with a Lorentz-invariant operator, and it is

$$c_{\rm eff}^{(4)00} - c_{\rm eff}^{(4)jj} = \mathring{c}_0^{(4)} - 3\mathring{c}_2^{(4)}. \tag{96}$$

At $d = 5$, the theory also contains three independent isotropic terms,

$$\hat{\mathcal{Q}}_{\rm eff}^{(5)} = -a_{\rm eff}^{(5)000}p^0p^0\gamma^0 - \frac{1}{3}a_{\rm eff}^{(5)0jj}(p^kp^k\gamma^0 + 2p^0p^k\gamma^k)$$
$$-\frac{2}{3}i\tilde{H}_{\rm eff}^{(5)0j0j}p^0p^k\gamma_5\sigma^{0k}$$
$$\equiv -\mathring{a}_0^{(5)}p^0p^0\gamma^0 - \frac{1}{3}\mathring{a}_2^{(5)}(p^kp^k\gamma^0 + 2p^0p^k\gamma^k)$$
$$+ i\mathring{H}_1^{(5)}p^0p^k\gamma_5\sigma^{0k}, \tag{97}$$

all of which are Lorentz violating. This is the lowest dimension d at which the effective coefficients $\mathring{H}_n^{(d)}$ appear. Finally, at $d = 6$ there are five isotropic terms,

$$\hat{\mathcal{Q}}_{\rm eff}^{(6)} = c_{\rm eff}^{(6)0000}p^0p^0p^0\gamma^0 + c_{\rm eff}^{(6)00jj}(p^0p^kp^k\gamma^0 + p^0p^0p^k\gamma^k) + \frac{1}{5}c_{\rm eff}^{(6)jjkk}p^lp^lp^n\gamma^n + i\tilde{g}_{\rm eff}^{(6)0j00j}p^0p^0p^k\gamma_5\sigma^{0k}$$
$$+\frac{1}{5}i\tilde{g}_{\rm eff}^{(6)0jjkk}p^lp^lp^n\gamma_5\sigma^{0n}$$
$$\equiv \mathring{c}_0^{(6)}p^0p^0p^0\gamma^0 + \frac{1}{2}\mathring{c}_2^{(6)}(p^0p^kp^k\gamma^0 + p^0p^0p^k\gamma^k) + \mathring{c}_4^{(6)}p^lp^lp^n\gamma^n - i\mathring{g}_1^{(6)}p^0p^0p^k\gamma_5\sigma^{0k} - i\mathring{g}_3^{(6)}p^lp^lp^n\gamma_5\sigma^{0n}. \tag{98}$$

At this dimension another Lorentz-invariant trace appears, associated with the coefficient combination

$$c_{\rm eff}^{(6)0000} - 2c_{\rm eff}^{(6)00jj} + c_{\rm eff}^{(6)jjkk} = \mathring{c}_0^{(6)} - \mathring{c}_2^{(6)} + 5\mathring{c}_4^{(6)}. \tag{99}$$

More generally, both even dimensions $d = 2k$ and odd dimensions $d = 2k+1$ have $2k-1$ independent isotropic terms. For even dimensions one combination is Lorentz invariant, and the number of independent CPT-even and CPT-odd Lorentz-violating operators is the same. For odd dimensions all terms are Lorentz violating, and the CPT-odd Lorentz-violating operators number one more than the CPT-even ones.

The remainder of this section derives the results (87). The reader uninterested in the derivation can proceed directly to the discussion of dispersion and birefringence in Sec. VI A.

The coefficients $\tilde{g}_{\rm eff}^{(d)\mu\nu\alpha_1\ldots\alpha_{d-3}}$ are antisymmetric in the first two indices and symmetric in the remaining indices, and their appearance in the operator $\tilde{g}_{\rm eff}^{\mu\nu}p_\nu$ in conjunction with p_ν implies that they can be taken to vanish under antisymmetrization of any three indices. The operator $\tilde{g}_{\rm eff}^{\mu\nu}$ therefore obeys the Maxwell-like equation $\partial^\lambda\tilde{g}_{\rm eff}^{\mu\nu} + \partial^\mu\tilde{g}_{\rm eff}^{\nu\lambda} + \partial^\nu\tilde{g}_{\rm eff}^{\lambda\mu} = 0$, which in turn constrains the coefficients in the spherical expansion.

To understand this constraint, it is useful to define a pseudovector $\mathcal{E}^j = \tilde{g}_{\rm eff}^{j0}$ and a vector $\mathcal{B}^j = -\epsilon^{jkl}\tilde{g}_{\rm eff}^{kl}/2$, in terms of which the constraint equation resembles the homogeneous Maxwell equations,

$$\boldsymbol{\nabla}\times\boldsymbol{\mathcal{E}} + \partial_0\boldsymbol{\mathcal{B}} = 0, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{B}} = 0. \tag{100}$$

Prior to imposing these constraint equations, the spherical-harmonic expansion of the operators $\mathcal{E}^j$ and $\mathcal{B}^j$ can be written as

$$\mathcal{E}_r = \sum_{dnjm} E_0^{d-3-n}|\boldsymbol{p}|^n Y_{jm}(\hat{\boldsymbol{p}})(\mathcal{E}^{(d)})_{njm}^{(0B)},$$
$$\mathcal{E}_\pm = \sum_{dnjm} E_0^{d-3-n}|\boldsymbol{p}|^n {}_{\pm1}Y_{jm}(\hat{\boldsymbol{p}})(\pm(\mathcal{E}^{(d)})_{njm}^{(1B)} + i(\mathcal{E}^{(d)})_{njm}^{(1E)}),$$
$$\mathcal{B}_r = \sum_{dnjm} E_0^{d-3-n}|\boldsymbol{p}|^n Y_{jm}(\hat{\boldsymbol{p}})(\mathcal{B}^{(d)})_{njm}^{(0E)},$$
$$\mathcal{B}_\pm = \sum_{dnjm} E_0^{d-3-n}|\boldsymbol{p}|^n {}_{\pm1}Y_{jm}(\hat{\boldsymbol{p}})(\pm(\mathcal{B}^{(d)})_{njm}^{(1E)} + i(\mathcal{B}^{(d)})_{njm}^{(1B)}). \tag{101}$$

Equation (100) imply interrelations between the six sets of coefficients in these expansions.

The first equation of Eq. (100) yields two constraints on E-type coefficients and one on B-type coefficients,

$$
\begin{aligned}
(\mathcal{B}^{(d)})^{(0E)}_{njm} &= -\frac{\sqrt{2j(j+1)}}{n+2}(\mathcal{B}^{(d)})^{(1E)}_{njm},\\
(\mathcal{E}^{(d)})^{(1E)}_{njm} &= \frac{d-2-n}{n+1}(\mathcal{B}^{(d)})^{(1E)}_{(n-1)jm},\\
(\mathcal{E}^{(d)})^{(1B)}_{njm} &= -\frac{\sqrt{2j(j+1)}}{2(n+1)}(\mathcal{E}^{(d)})^{(0B)}_{njm} - \frac{d-2-n}{n+1}(\mathcal{B}^{(d)})^{(1B)}_{(n-1)jm}.
\end{aligned} \tag{102}
$$

The second of Eqs. (100) ensures vanishing divergence of $\mathcal{B}$ but provides no additional constraints. Careful consideration of the index ranges of all the coefficients reveals that we can choose $(\mathcal{E}^{(d)})^{(0B)}_{njm}$, $(\mathcal{B}^{(d)})^{(1B)}_{njm}$, and $(\mathcal{B}^{(d)})^{(1E)}_{njm}$ to be a set of independent coefficients. Consequently, the expansions (101) become

$$
\begin{aligned}
\mathcal{E}_r &= \sum_{dnjm} E_0{}^{d-3-n}|\boldsymbol{p}|^n Y_{jm}(\hat{\boldsymbol{p}})(\mathcal{E}^{(d)})^{(0B)}_{njm},\\
\mathcal{E}_\pm &= \sum_{dnjm} E_0{}^{d-3-n}|\boldsymbol{p}|^n {}_{\pm1}Y_{jm}(\hat{\boldsymbol{p}})\\
&\quad\times\bigg[\mp\frac{\sqrt{2j(j+1)}}{2(n+1)}(\mathcal{E}^{(d)})^{(0B)}_{njm}\\
&\qquad+\frac{d-2-n}{n+1}(\mp(\mathcal{B}^{(d)})^{(1B)}_{(n-1)jm}+i(\mathcal{B}^{(d)})^{(1E)}_{(n-1)jm})\bigg],\\
\mathcal{B}_r &= \sum_{dnjm} E_0{}^{d-3-n}|\boldsymbol{p}|^n Y_{jm}(\hat{\boldsymbol{p}})(-)\frac{\sqrt{2j(j+1)}}{n+2}(\mathcal{B}^{(d)})^{(1E)}_{njm},\\
\mathcal{B}_\pm &= \sum_{dnjm} E_0{}^{d-3-n}|\boldsymbol{p}|^n {}_{\pm1}Y_{jm}(\hat{\boldsymbol{p}})\Big(\pm(\mathcal{B}^{(d)})^{(1E)}_{njm}+i(\mathcal{B}^{(d)})^{(1B)}_{njm}\Big).
\end{aligned} \tag{103}
$$

The helicity-basis components of the Hamiltonian h_g can now be written as

$$
\begin{aligned}
(h_g)_r &= \frac{m_\psi}{E_0{}^2}\hat{p}^j\tilde{\tilde{g}}^{j\nu}_{\text{eff}}p_\nu = \frac{m_\psi}{E_0}\mathcal{E}_r,\\
(h_g)_\pm &= \frac{1}{E_0}\hat{\epsilon}^j_\pm\tilde{\tilde{g}}^{j\nu}_{\text{eff}}p_\nu = \frac{1}{E_0}(E_0\mathcal{E}_\pm \pm i|\boldsymbol{p}|\mathcal{B}_\pm).
\end{aligned} \tag{104}
$$

We can now choose the convenient match

$$
\begin{aligned}
(\mathcal{E}^{(d)})^{(0B)}_{njm} &= -(n+1)g^{(d)(0B)}_{njm},\\
(\mathcal{B}^{(d)})^{(1B)}_{(n-1)jm} &= -\frac{n+1}{d-1}g^{(d)(1B)}_{njm},\\
(\mathcal{B}^{(d)})^{(1E)}_{(n-1)jm} &= \frac{n+1}{d-1}g^{(d)(1E)}_{njm},
\end{aligned} \tag{105}
$$

which gives the first two equations in Eq. (87). The calculation for the CPT-even operators $\tilde{\tilde{H}}^{\mu\nu}_{\text{eff}}$ is similar to that for $\tilde{\tilde{g}}^{\mu\nu}_{\text{eff}}$, up to an overall sign.

V. LIMITING CASES

For many applications it is appropriate to consider limiting cases of the Lorentz-violating Hamiltonian (80). In this section, we consider in turn the nonrelativistic limit, the ultrarelativistic case, and the restriction to the minimal SME.

A. Nonrelativistic

The nonrelativistic limit of the Lorentz-violating Hamiltonian (80) can be obtained directly from the spherical-harmonic expansions obtained in Sec. IV by expanding the energy E_0 in the usual power series in $|\boldsymbol{p}|$,

$$
E_0 \approx m_\psi + \frac{|\boldsymbol{p}|^2}{2m_\psi} - \frac{|\boldsymbol{p}|^4}{8m_\psi^3} + \cdots. \tag{106}
$$

In many common physics applications, this series can be truncated as desired, but here it entangles contributions from different dimensions d into any given power n of the momentum $|\boldsymbol{p}|^n$. Some care is therefore required in constructing the nonrelativistic limit.

Consider first h_a. Substituting the nonrelativistic series (106) for E_0 produces

$$
\begin{aligned}
h_a &= \sum_{njm}|\boldsymbol{p}|^n {}_0Y_{jm}(\hat{\boldsymbol{p}})\\
&\quad\times\bigg(\sum_d m_\psi^{d-3-n}\sum_{k\le n/2}\binom{(d-3-n+2k)/2}{k}a^{(d)}_{(n-2k)jm}\bigg),
\end{aligned} \tag{107}
$$

where $\binom{j}{k}$ denotes a binomial coefficient. The summation over k represents the linear combination of coefficients at dimension d appearing in the nonrelativistic limit. The sum over d gives the combination of coefficients of different dimensions contributing to the momentum dependence $|\boldsymbol{p}|^n$. The expression (107) can be viewed as an expansion in the momentum magnitude $|\boldsymbol{p}|^n$ and direction $\hat{\boldsymbol{p}}$ involving nonrelativistic coefficients consisting of the terms in parentheses. Each such nonrelativistic coefficient is a superposition of the original spherical coefficients with fixed values of j and m but summed over d and k. We denote these nonrelativistic coefficients by a^{NR}_{njm}, thereby obtaining the nonrelativistic form of h_a. The same reduction can be applied to obtain the nonrelativistic form of all terms in the Hamiltonian (80).

The result of this procedure is the perturbative nonrelativistic Hamiltonian

$$
\begin{aligned}
\delta h^{\text{NR}} &= h^{\text{NR}}_a + h^{\text{NR}}_c + (h^{\text{NR}}_g)_+\sigma^+ + (h^{\text{NR}}_g)_r\sigma^r\\
&\quad+(h^{\text{NR}}_g)_-\sigma^- + (h^{\text{NR}}_H)_+\sigma^+ + (h^{\text{NR}}_H)_r\sigma^r\\
&\quad+(h^{\text{NR}}_H)_-\sigma^-,
\end{aligned} \tag{108}
$$

where the spin-independent terms take the form

TABLE IV. Nonrelativistic coefficients for Lorentz violation.

Coefficient	n	j	Number
$a^{\rm NR}_{njm}$	≥ 0	$n, n-2, n-4, \ldots \geq 0$	$\frac{1}{2}(n+1)(n+2)$
$c^{\rm NR}_{njm}$	≥ 0	$n, n-2, n-4, \ldots \geq 0$	$\frac{1}{2}(n+1)(n+2)$
$g^{\rm NR(0B)}_{njm}$	≥ 0	$n+1, n-1, n-3, \ldots \geq 0$	$\frac{1}{2}(n+2)(n+3)$
$g^{\rm NR(1B)}_{njm}$	≥ 0	$n+1, n-1, n-3, \ldots \geq 1$	$\frac{1}{2}(n+1+\iota_n)(n+4-\iota_n)$
$g^{\rm NR(1E)}_{njm}$	≥ 1	$n, n-2, n-4, \ldots \geq 1$	$\frac{1}{2}(n+1-\iota_n)(n+2+\iota_n)$
$H^{\rm NR(0B)}_{njm}$	≥ 0	$n+1, n-1, n-3, \ldots \geq 0$	$\frac{1}{2}(n+2)(n+3)$
$H^{\rm NR(1B)}_{njm}$	≥ 0	$n+1, n-1, n-3, \ldots \geq 1$	$\frac{1}{2}(n+1+\iota_n)(n+4-\iota_n)$
$H^{\rm NR(1E)}_{njm}$	≥ 1	$n, n-2, n-4, \ldots \geq 1$	$\frac{1}{2}(n+1-\iota_n)(n+2+\iota_n)$
$\mathring{a}^{\rm NR}_n$	Even, ≥ 0	0	1
$\mathring{c}^{\rm NR}_n$	Even, ≥ 0	0	1
$\mathring{g}^{\rm NR}_n$	Odd, ≥ 0	0	1
$\mathring{H}^{\rm NR}_n$	Odd, ≥ 0	0	1

$$h_a^{\rm NR} = \sum_{njm} |\boldsymbol{p}|^n {}_0Y_{jm}(\hat{\boldsymbol{p}}) a^{\rm NR}_{njm}, \qquad h_c^{\rm NR} = -\sum_{njm} |\boldsymbol{p}|^n {}_0Y_{jm}(\hat{\boldsymbol{p}}) c^{\rm NR}_{njm}, \tag{109}$$

and the spin-dependent terms are

$$\begin{aligned} (h_g^{\rm NR})_r &= -\sum_{njm} |\boldsymbol{p}|^n {}_0Y_{jm}(\hat{\boldsymbol{p}}) g^{\rm NR(0B)}_{njm}, \qquad (h_g^{\rm NR})_\pm = \sum_{njm} |\boldsymbol{p}|^n {}_{\pm1}Y_{jm}(\hat{\boldsymbol{p}})(\pm g^{\rm NR(1B)}_{njm} + i g^{\rm NR(1E)}_{njm}), \\ (h_H^{\rm NR})_r &= \sum_{njm} |\boldsymbol{p}|^n {}_0Y_{jm}(\hat{\boldsymbol{p}}) H^{\rm NR(0B)}_{njm}, \qquad (h_H^{\rm NR})_\pm = -\sum_{njm} |\boldsymbol{p}|^n {}_{\pm1}Y_{jm}(\hat{\boldsymbol{p}})(\pm H^{\rm NR(1B)}_{njm} + i H^{\rm NR(1E)}_{njm}). \end{aligned} \tag{110}$$

The nonrelativistic coefficients are related to the spherical coefficients via

$$a^{\rm NR}_{njm} = \sum_d m_\psi^{d-3-n} \sum_{k\leq n/2} \binom{(d-3-n+2k)/2}{k} a^{(d)}_{(n-2k)jm}, \tag{111}$$

together with an identical equation relating the coefficients $c^{\rm NR}_{njm}$ to $c^{(d)}_{njm}$, and via

$$\begin{aligned} g^{\rm NR(0B)}_{njm} &= \sum_d m_\psi^{d-3-n} \sum_{k\leq n/2} (n-2k+1) \binom{(d-4-n+2k)/2}{k} g^{(d)(0B)}_{(n-2k)jm}, \\ g^{\rm NR(1B)}_{njm} &= \sum_d m_\psi^{d-3-n} \sum_{k\leq n/2} \binom{(d-3-n+2k)/2}{k} \left[g^{(d)(1B)}_{(n-2k)jm} + \sqrt{\frac{j(j+1)}{2}} g^{(d)(0B)}_{(n-2k)jm} \right], \\ g^{\rm NR(1E)}_{njm} &= \sum_d m_\psi^{d-3-n} \sum_{k\leq n/2} \binom{(d-3-n+2k)/2}{k} g^{(d)(1E)}_{(n-2k)jm}, \end{aligned} \tag{112}$$

together with three identical equations relating the coefficients $H^{\rm NR(0B)}_{njm}$ to $H^{(d)(0B)}_{njm}$, $H^{\rm NR(1B)}_{njm}$ to $H^{(d)(1B)}_{njm}$ and $H^{(d)(0B)}_{njm}$, and $H^{\rm NR(1E)}_{njm}$ to $H^{(d)(1E)}_{njm}$. In all these expressions, the entanglement of coefficients with different dimensions d arising from the series (106) is manifest. For example, the coefficients $a^{(d)}_{111}$ contribute to all $j = m = 1$ terms at order $|\boldsymbol{p}|^n$ for $n = 1, 3, 5, \ldots$. However, for all coefficients the minimum d required to produce anisotropies with a particular j is $j+2$. This means that probing effects with large j and n offers sensitivity to Lorentz violation involving large d that is independent of results from lower values of n.

Some properties of the nonrelativistic coefficients are summarized in Table IV. The first column lists the coefficients, while the second column shows the allowed values of n. The third column lists the allowed range of j, while the last column specifies the number of independent coefficients for each n value. In this column, $\iota_n = 1$ for even n, and $\iota_n = 0$ for odd n. All nonrelativistic coefficients have mass dimension $1-n$.

In the isotropic limit, the component nonrelativistic Hamiltonians reduce to

$$\overset{\circ}{h}{}_a^{\rm NR} = \sum_n |\boldsymbol{p}|^n \overset{\circ}{a}{}_n^{\rm NR}, \qquad \overset{\circ}{h}{}_c^{\rm NR} = -\sum_n |\boldsymbol{p}|^n \overset{\circ}{c}{}_n^{\rm NR},$$
$$(\overset{\circ}{h}{}_g^{\rm NR})_r = -\sum_n |\boldsymbol{p}|^n \overset{\circ}{g}{}_n^{\rm NR}, \qquad (\overset{\circ}{h}{}_H^{\rm NR})_r = \sum_n |\boldsymbol{p}|^n \overset{\circ}{H}{}_n^{\rm NR}. \tag{113}$$

The nonrelativistic isotropic coefficients in these expressions are related to the spherical isotropic coefficients by

$$\overset{\circ}{a}{}_n^{\rm NR} = \frac{1}{\sqrt{4\pi}} a_{n00}^{\rm NR}, \qquad \overset{\circ}{c}{}_n^{\rm NR} = \frac{1}{\sqrt{4\pi}} c_{n00}^{\rm NR},$$
$$\overset{\circ}{g}{}_n^{\rm NR} = \frac{1}{\sqrt{4\pi}} g_{n00}^{{\rm NR}(0B)}, \qquad \overset{\circ}{H}{}_n^{\rm NR} = \frac{1}{\sqrt{4\pi}} H_{n00}^{{\rm NR}(0B)}. \tag{114}$$

The nonrelativistic isotropic coefficients have mass dimension $1-n$, and their index and counting properties are provided in Table IV. None correspond to Lorentz-invariant operators.

B. Ultrarelativistic

A detailed discussion of the ultrarelativistic limit in the context of neutrinos is given in Ref. [16]. Here, we consider the single-fermion ultrarelativistic limit of the Hamiltonian (80). Expanding E_0 gives

$$E_0 \approx |\boldsymbol{p}| + \frac{m_\psi^2}{2|\boldsymbol{p}|} - \frac{m_\psi^4}{8|\boldsymbol{p}|^3} + \cdots. \tag{115}$$

However, substitution of this full series into the spherical decomposition of the perturbative Hamiltonian generates an expansion in powers of $|\boldsymbol{p}|^d$ instead of $|\boldsymbol{p}|^n$, and the result retains the coefficient complexity of the exact expressions (85) and (87). For example, substitution of the full series (115) into the term h_a produces

$$h_a = \sum_{djm} |\boldsymbol{p}|^{d-3} {}_0Y_{jm}(\hat{\boldsymbol{p}}) \sum_{nk} \binom{(d-3-n+2k)/2}{k} m_\psi^{2k} a_{njm}^{(d+2k)}, \tag{116}$$

showing that coefficients with arbitrary n contribute at each d. This situation differs from the nonrelativistic limit, where substitution of the full analogous series (106) leads to a simplification of the coefficient structure. We therefore limit attention here to the dominant term in the series (115), which yields the perturbation Hamiltonian in the ultrarelativistic limit to order m_ψ.

To see the effect of taking this ultrarelativistic limit, consider first h_a. For $E_0 \to |\boldsymbol{p}|$ the above expression reduces to

$$h_a \approx \sum_{djm} |\boldsymbol{p}|^{d-3} {}_0Y_{jm}(\hat{\boldsymbol{p}}) \bigg(\sum_n a_{njm}^{(d)} \bigg), \tag{117}$$

which takes the form of an expansion in $|\boldsymbol{p}|$ and $\hat{\boldsymbol{p}}$ with ultrarelativistic coefficients consisting of the term in parentheses. We denote these coefficients by $a_{jm}^{{\rm UR}(d)}$. They are superpositions of spherical coefficients with different values of n. Repeating this limiting procedure produces the ultrarelativistic limit of all terms in the Hamiltonian (80).

The resulting perturbative ultrarelativistic Hamiltonian has the form

$$\delta h^{\rm UR} = h_a^{\rm UR} + h_c^{\rm UR} + (h_g^{\rm UR})_+ \sigma^+ + (h_g^{\rm UR})_r \sigma^r + (h_g^{\rm UR})_- \sigma^- + (h_H^{\rm UR})_+ \sigma^+ + (h_H^{\rm UR})_r \sigma^r + (h_H^{\rm UR})_- \sigma^-, \tag{118}$$

where the spin-independent terms are

$$h_a^{\rm UR} = \sum_{djm} |\boldsymbol{p}|^{d-3} {}_0Y_{jm}(\hat{\boldsymbol{p}}) a_{jm}^{{\rm UR}(d)}, \tag{119}$$

$$h_c^{\rm UR} = -\sum_{djm} |\boldsymbol{p}|^{d-3} {}_0Y_{jm}(\hat{\boldsymbol{p}}) c_{jm}^{{\rm UR}(d)} \tag{120}$$

and the spin-dependent terms are

$$\begin{aligned}
(h_g^{\rm UR})_r &= -m_\psi \sum_{djm} |\boldsymbol{p}|^{d-4} {}_0Y_{jm}(\hat{\boldsymbol{p}}) \bigg[g_{jm}^{{\rm UR}(d)(0B)} + \sqrt{\frac{2j}{j+1}} g_{jm}^{{\rm UR}(d)(1B)} \bigg], \\
(h_g^{\rm UR})_\pm &= \sum_{djm} |\boldsymbol{p}|^{d-3} {}_{\pm1}Y_{jm}(\hat{\boldsymbol{p}}) [\pm g_{jm}^{{\rm UR}(d)(1B)} + i g_{jm}^{{\rm UR}(d)(1E)}], \\
(h_H^{\rm UR})_r &= m_\psi \sum_{djm} |\boldsymbol{p}|^{d-4} {}_0Y_{jm}(\hat{\boldsymbol{p}}) \bigg[H_{jm}^{{\rm UR}(d)(0B)} + \sqrt{\frac{2j}{j+1}} H_{jm}^{{\rm UR}(d)(1B)} \bigg], \\
(h_H^{\rm UR})_\pm &= -\sum_{djm} |\boldsymbol{p}|^{d-3} {}_{\pm1}Y_{jm}(\hat{\boldsymbol{p}}) [\pm H_{jm}^{{\rm UR}(d)(1B)} + i H_{jm}^{{\rm UR}(d)(1E)}].
\end{aligned} \tag{121}$$

Most of the ultrarelativistic coefficients $\mathcal{K}_{jm}^{{\rm UR}(d)}$ are related to the spherical coefficients $\mathcal{K}_{njm}^{(d)}$ by expressions of the form

$$\mathcal{K}_{jm}^{{\rm UR}(d)} = \sum_n \mathcal{K}_{njm}^{(d)}. \tag{122}$$

However, for the B-type coefficients it is convenient to define

$$g_{jm}^{\mathrm{UR}(d)(0B)} = \sum_n \left[(n+1-j) g_{njm}^{(d)(0B)} - \sqrt{\frac{2j}{j+1}} g_{njm}^{(d)(1B)} \right], \qquad g_{jm}^{\mathrm{UR}(d)(1B)} = \sum_n \left[g_{njm}^{(d)(1B)} + \sqrt{\frac{j(j+1)}{2}} g_{njm}^{(d)(0B)} \right],$$
$$H_{jm}^{\mathrm{UR}(d)(0B)} = \sum_n \left[(n+1-j) H_{njm}^{(d)(0B)} - \sqrt{\frac{2j}{j+1}} H_{njm}^{(d)(1B)} \right], \qquad H_{jm}^{\mathrm{UR}(d)(1B)} = \sum_n \left[H_{njm}^{(d)(1B)} + \sqrt{\frac{j(j+1)}{2}} H_{njm}^{(d)(0B)} \right]. \tag{123}$$

Each ultrarelativistic coefficient has mass dimension $4-d$.

Information about the index ranges and counting of the ultrarelativistic coefficients is collected in Table V. The first column lists the coefficients, the next two provide the allowed ranges of d and j, and the final column gives the number of independent coefficients at each d.

In the isotropic limit, the ultrarelativistic Hamiltonian components become

$$\mathring{h}_a^{\mathrm{UR}} = \sum_d |\boldsymbol{p}|^{d-3} \mathring{a}^{\mathrm{UR}(d)}, \qquad \mathring{h}_c^{\mathrm{UR}} = -\sum_d |\boldsymbol{p}|^{d-3} \mathring{c}^{\mathrm{UR}(d)},$$
$$(\mathring{h}_g^{\mathrm{UR}})_r = -m_\psi \sum_d |\boldsymbol{p}|^{d-4} \mathring{g}^{\mathrm{UR}(d)},$$
$$(\mathring{h}_H^{\mathrm{UR}})_r = m_\psi \sum_d |\boldsymbol{p}|^{d-4} \mathring{H}^{\mathrm{UR}(d)}. \tag{124}$$

The connection between the ultrarelativistic isotropic coefficients and the spherical isotropic coefficients is

$$\mathring{a}^{\mathrm{UR}(d)} = \frac{1}{\sqrt{4\pi}} a_{00}^{\mathrm{UR}(d)}, \qquad \mathring{c}^{\mathrm{UR}(d)} = \frac{1}{\sqrt{4\pi}} c_{00}^{\mathrm{UR}(d)},$$
$$\mathring{g}^{\mathrm{UR}(d)} = \frac{1}{\sqrt{4\pi}} g_{00}^{\mathrm{UR}(d)(0B)}, \qquad \mathring{H}^{\mathrm{UR}(d)} = \frac{1}{\sqrt{4\pi}} H_{00}^{\mathrm{UR}(d)(0B)}. \tag{125}$$

The ultrarelativistic isotropic coefficients have mass dimension $4-d$, and there is no more than 1 coefficient of any given type at each d. None of them correspond to Lorentz-invariant operators. The allowed dimensions d are given in Table V.

We remark in passing that the above results differ in detail from those obtained in the analysis of the nonminimal neutrino sector in Ref. [16]. The differences arise because the neutrino treatment involves Dirac- and Majorana-type couplings of multiple flavors of left-handed fermions, while the present discussion involves a single Dirac fermion without helicity restriction.

C. Minimal SME

The minimal SME in flat spacetime [7] consists of operators of renormalizable dimension $d = 3, 4$. In the present context, this involves the Cartesian coefficients $a^{(3)\mu}$, $b^{(3)\mu}$, $c^{(4)\mu\nu}$, $d^{(4)\mu\nu}$, $e^{(4)\mu}$, $f^{(4)\mu}$, $g^{(4)\lambda\mu\nu}$, and $H^{(3)\mu\nu}$. Of these, the coefficient $f^{(4)\mu}$ plays no observable role and can be disregarded [39,40], as described in Sec. II B. Restricting attention to this coefficient set, the Cartesian expansion introduced in Sec. II A can be matched to the spherical-harmonic one presented in Sec. IV. This produces a set of relations connecting the minimal Cartesian and the spherical coefficients for Lorentz violation.

The Cartesian expansion of the perturbative Hamiltonian (75) is given by Eqs. (76) and (78). For the spin-independent piece involving Δ, we can use the expressions (27) for the effective coefficients in Eq. (77) to project onto the minimal SME terms with $d = 3$ and $d = 4$, giving

$$\Delta^{(3)} = a^{(3)\mu} p_\mu, \qquad \Delta^{(4)} = -c^{(4)\mu\nu} p_\mu p_\nu,$$
$$\Delta^{(5)} = -\frac{p^2}{m_\psi} e^{(4)\mu} p_\mu. \tag{126}$$

For the spin-dependent terms in N^μ, combining Eq. (27) with Eq. (79) and projecting onto the minimal SME coefficients yields

$$N^{(3)\mu} = -\tilde{H}^{(3)\mu\nu} p_\nu,$$
$$N^{(4)\mu} = \tilde{g}_{\mathrm{eff}}^{(4)\mu\nu\lambda} p_\nu p_\lambda = \Big(\tilde{g}^{(4)\mu\nu\lambda} - \frac{1}{m_\psi} \eta^{\lambda[\mu} b^{(3)\nu]} \Big) p_\nu p_\lambda,$$
$$N^{(5)\mu} = \frac{1}{m_\psi} (p^\mu d^{(4)\nu\lambda} p_\nu p_\lambda - p^2 d^{(4)\mu\nu} p_\nu). \tag{127}$$

Note that in these expressions we are using the dual coefficients $\tilde{g}^{(4)\lambda\mu\nu}$ and $\tilde{H}^{(3)\mu\nu}$ introduced in the expansions (26). Since in the minimal case $\tilde{g}^{(4)\mu\nu\lambda}$ and $b^{(3)\mu}$ always appear in the same linear combination $\tilde{g}_{\mathrm{eff}}^{(4)\mu\nu\lambda}$, we use the

TABLE V. Ultrarelativistic coefficients for Lorentz violation.

Coefficient	d	j	Number
$a_{jm}^{\mathrm{UR}(d)}$	Odd, ≥ 3	$0 \leq j \leq d-2$	$(d-1)^2$
$c_{jm}^{\mathrm{UR}(d)}$	Even, ≥ 4	$0 \leq j \leq d-2$	$(d-1)^2$
$g_{jm}^{\mathrm{UR}(d)(0B)}$	Even, ≥ 4	$0 \leq j \leq d-3$	$(d-2)^2$
$g_{jm}^{\mathrm{UR}(d)(1B)}$	Even, ≥ 4	$1 \leq j \leq d-2$	$(d-2)d$
$g_{jm}^{\mathrm{UR}(d)(1E)}$	Even, ≥ 4	$1 \leq j \leq d-2$	$(d-2)d$
$H_{jm}^{\mathrm{UR}(d)(0B)}$	Odd, ≥ 5	$0 \leq j \leq d-3$	$(d-2)^2$
$H_{jm}^{\mathrm{UR}(d)(1B)}$	Odd, ≥ 3	$1 \leq j \leq d-2$	$(d-2)d$
$H_{jm}^{\mathrm{UR}(d)(1E)}$	Odd, ≥ 3	$1 \leq j \leq d-2$	$(d-2)d$
$\mathring{a}^{\mathrm{UR}(d)}$	Odd, ≥ 3	0	1
$\mathring{c}^{\mathrm{UR}(d)}$	Even, ≥ 4	0	1
$\mathring{g}^{\mathrm{UR}(d)}$	Even, ≥ 4	0	1
$\mathring{H}^{\mathrm{UR}(d)}$	Even, ≥ 5	0	1

latter in making the matches that follow. Note also that the Lorentz-invariant trace $d^{(4)\mu\nu}\eta_{\mu\nu}$ is absent from Eq. (127).

We can now match these results to the spherical expansion of the perturbative Hamiltonian (75) as written in the form (88). This gives relations between the spherical coefficients and the Cartesian ones for the minimal SME. To express compactly some results, it is convenient to define an azimuthal spin vector,

$$\boldsymbol{x}_\pm = \hat{\boldsymbol{x}} \mp i\hat{\boldsymbol{y}}. \tag{128}$$

Considering first the match for spin-independent effects, we find the four coefficients $a_{\text{eff}}^{(3)\mu}$ are related to the four spherical coefficients $a_{njm}^{(3)}$ by

$$\begin{aligned} a_{000}^{(3)} &= \sqrt{4\pi}a_{\text{eff}}^{(3)t}, & a_{11(-1)}^{(3)} &= -\sqrt{\frac{2\pi}{3}}x_-^j a_{\text{eff}}^{(3)j}, \\ a_{110}^{(3)} &= -\sqrt{\frac{4\pi}{3}}a_{\text{eff}}^{(3)z}, & a_{111}^{(3)} &= \sqrt{\frac{2\pi}{3}}x_+^j a_{\text{eff}}^{(3)j}. \end{aligned} \tag{129}$$

The coefficients $c^{(4)\mu\nu}$ are related to the spherical coefficients $c_{njm}^{(4)}$ by

$$\begin{aligned} c_{000}^{(4)} &= \sqrt{4\pi}c^{(4)tt}, & c_{11(-1)}^{(4)} &= -\sqrt{\frac{8\pi}{3}}x_-^j c^{(4)tj}, \\ c_{110}^{(4)} &= -\sqrt{\frac{16\pi}{3}}c^{(4)tz}, & c_{111}^{(4)} &= \sqrt{\frac{8\pi}{3}}x_+^j c^{(4)tj}, \\ c_{200}^{(4)} &= \sqrt{\frac{4\pi}{9}}c^{(4)jj}, & c_{22(-2)}^{(4)} &= \sqrt{\frac{2\pi}{15}}x_-^j x_-^k c^{(4)jk}, \\ c_{22(-1)}^{(4)} &= \sqrt{\frac{8\pi}{15}}x_-^j c^{(4)jz}, & c_{220}^{(4)} &= \sqrt{\frac{4\pi}{5}}\left(c^{(4)zz} - \frac{1}{3}c^{(4)jj}\right), \\ c_{221}^{(4)} &= -\sqrt{\frac{8\pi}{15}}x_+^j c^{(4)jz}, & c_{222}^{(4)} &= \sqrt{\frac{2\pi}{15}}x_+^j x_+^k c^{(4)jk}. \end{aligned} \tag{130}$$

Note that this set of 10 coefficients contains the trace combination $\eta_{\mu\nu}c^{(4)\mu\nu}$ associated with a Lorentz-invariant operator. This trace can be removed by adding the constraint $c_{000}^{(4)} = 3c_{200}^{(4)}$ involving the two isotropic components of $c_{njm}^{(4)}$. Finally, as is apparent from Eq. (126), the minimal Cartesian coefficients $e^{(4)\mu}$ act effectively as $d = 5$ coefficients $a^{(5)\mu}$ according to

$$\begin{aligned} a_{200}^{(5)} &= -a_{000}^{(5)} = \frac{1}{m_\psi}\sqrt{4\pi}e^{(4)t}, \\ a_{31(-1)}^{(5)} &= -a_{11(-1)}^{(5)} = -\frac{1}{m_\psi}\sqrt{\frac{2\pi}{3}}x_-^j e^{(4)j}, \\ a_{310}^{(5)} &= -a_{110}^{(5)} = -\frac{1}{m_\psi}\sqrt{\frac{4\pi}{3}}e^{(4)z}, \\ a_{311}^{(5)} &= -a_{111}^{(5)} = \frac{1}{m_\psi}\sqrt{\frac{2\pi}{3}}x_+^j e^{(4)j}. \end{aligned} \tag{131}$$

Turning next to the match for spin-dependent effects, we begin with the terms involving $\tilde{g}_{\text{eff}}^{(4)\mu\nu\rho}$. Disregarding the unobservable totally antisymmetric part leaves 20 Cartesian coefficients, which can be connected to the twelve B-type spherical coefficients $g_{njm}^{(4)(0B)}$, $g_{njm}^{(4)(1B)}$ and the eight E-type spherical coefficients $g_{njm}^{(4)(1E)}$. The nine B-type spherical coefficients $g_{njm}^{(4)(0B)}$ are related to Cartesian ones by

$$\begin{aligned} g_{01(-1)}^{(4)(0B)} &= \sqrt{\frac{2\pi}{3}}x_-^j \tilde{g}_{\text{eff}}^{(4)tjt}, \qquad g_{010}^{(4)(0B)} = \sqrt{\frac{4\pi}{3}}\tilde{g}_{\text{eff}}^{(4)tzt}, \\ g_{011}^{(4)(0B)} &= -\sqrt{\frac{2\pi}{3}}x_+^j \tilde{g}_{\text{eff}}^{(4)tjt}, \qquad g_{100}^{(4)(0B)} = -\sqrt{\frac{\pi}{9}}\tilde{g}_{\text{eff}}^{(4)tjj}, \\ g_{12(-2)}^{(4)(0B)} &= -\sqrt{\frac{\pi}{30}}x_-^j x_-^k \tilde{g}_{\text{eff}}^{(4)tjk}, \\ g_{12(-1)}^{(4)(0B)} &= -\sqrt{\frac{\pi}{30}}x_-^j (\tilde{g}_{\text{eff}}^{(4)tjz} + \tilde{g}_{\text{eff}}^{(4)tzj}), \\ g_{120}^{(4)(0B)} &= -\sqrt{\frac{\pi}{5}}\left(\tilde{g}_{\text{eff}}^{(4)tzz} - \frac{1}{3}\tilde{g}_{\text{eff}}^{(4)tjj}\right), \\ g_{121}^{(4)(0B)} &= \sqrt{\frac{\pi}{30}}x_+^j (\tilde{g}_{\text{eff}}^{(4)tjz} + \tilde{g}_{\text{eff}}^{(4)tzj}), \\ g_{122}^{(4)(0B)} &= -\sqrt{\frac{\pi}{30}}x_+^j x_+^k \tilde{g}_{\text{eff}}^{(4)tjk}. \end{aligned} \tag{132}$$

The three B-type spherical coefficients $g_{njm}^{(4)(1B)}$ are given by the equations

$$\begin{aligned} g_{21(-1)}^{(4)(1B)} &= -\sqrt{\frac{\pi}{6}}x_-^j \tilde{g}_{\text{eff}}^{(4)jkk}, \qquad g_{210}^{(4)(1B)} = -\sqrt{\frac{\pi}{3}}\tilde{g}_{\text{eff}}^{(4)zjj}, \\ g_{211}^{(4)(1B)} &= \sqrt{\frac{\pi}{6}}x_+^j \tilde{g}_{\text{eff}}^{(4)jkk}, \end{aligned} \tag{133}$$

while the expressions for the eight E-type spherical coefficients are

$$g^{(4)(1E)}_{11(-1)} = -i\sqrt{\frac{3\pi}{2}} x^j_- \tilde{g}^{(4)jzt}_{\rm eff}, \quad g^{(4)(1E)}_{110} = -i\sqrt{\frac{3\pi}{4}} x^j_+ x^k_- \tilde{g}^{(4)jkt}_{\rm eff},$$
$$g^{(4)(1E)}_{111} = -i\sqrt{\frac{3\pi}{2}} x^j_+ \tilde{g}^{(4)jzt}_{\rm eff}, \quad g^{(4)(1E)}_{22(-2)} = -i\sqrt{\frac{\pi}{10}} x^j_- x^k_- \tilde{g}^{(4)zjk}_{\rm eff},$$
$$g^{(4)(1E)}_{22(-1)} = i\sqrt{\frac{2\pi}{5}} x^j_- \left(\tilde{g}^{(4)jzz}_{\rm eff} - \frac{1}{2}\tilde{g}^{(4)jkk}_{\rm eff}\right),$$
$$g^{(4)(1E)}_{220} = i\sqrt{\frac{3\pi}{20}} x^j_+ x^k_- \tilde{g}^{(4)jkz}_{\rm eff},$$
$$g^{(4)(1E)}_{221} = i\sqrt{\frac{2\pi}{5}} x^j_+ \left(\tilde{g}^{(4)jzz}_{\rm eff} - \frac{1}{2}\tilde{g}^{(4)jkk}_{\rm eff}\right),$$
$$g^{(4)(1E)}_{222} = i\sqrt{\frac{\pi}{10}} x^j_+ x^k_+ \tilde{g}^{(4)zjk}_{\rm eff}. \tag{134}$$

Using the pseudotensor nature of the coefficients $\tilde{g}^{(4)\mu\nu\rho}_{\rm eff}$, one can verify that the E-type and B-type coefficients are associated with operators having parity $(-1)^j$ and $(-1)^{j+1}$, respectively.

The six components of the antisymmetric Cartesian coefficients $\tilde{H}^{(3)\mu\nu}$ can be used to obtain the three B-type spherical coefficients $H^{(3)(0B)}_{njm}$ and the three E-type spherical coefficients $H^{(3)(1E)}_{njm}$. The three B-type ones are given by

$$H^{(3)(0B)}_{01(-1)} = \sqrt{\frac{2\pi}{3}} x^j_- \tilde{H}^{(3)tj},$$
$$H^{(3)(0B)}_{010} = \sqrt{\frac{4\pi}{3}} \tilde{H}^{(3)tz}, \tag{135}$$
$$H^{(3)(0B)}_{011} = -\sqrt{\frac{2\pi}{3}} x^j_+ \tilde{H}^{(3)tj},$$

while the three E-type ones are found to be

$$H^{(3)(1E)}_{11(-1)} = -i\sqrt{\frac{2\pi}{3}} x^j_- \tilde{H}^{(3)jz},$$
$$H^{(3)(1E)}_{110} = -i\sqrt{\frac{\pi}{3}} x^j_+ x^k_- \tilde{H}^{(3)jk}, \tag{136}$$
$$H^{(3)(1E)}_{111} = -i\sqrt{\frac{2\pi}{3}} x^j_+ \tilde{H}^{(3)jz}.$$

The remaining 15 Cartesian coefficients $d^{(4)\mu\nu}$ specify 15 independent spherical coefficients $H^{(5)(0B)}_{njm}$ and $H^{(5)(1E)}_{njm}$ corresponding to operators with mass dimension $d = 5$, as can be seen from Eq. (127). The six B-type spherical coefficients $H^{(5)(0B)}_{njm}$ with even n are given by

$$H^{(5)(0B)}_{01(-1)} = -\frac{1}{m_\psi}\sqrt{\frac{2\pi}{3}} x^j_- d^{(4)jt}, \quad H^{(5)(0B)}_{010} = -\frac{1}{m_\psi}\sqrt{\frac{4\pi}{3}} d^{(4)zt},$$
$$H^{(5)(0B)}_{011} = \frac{1}{m_\psi}\sqrt{\frac{2\pi}{3}} x^j_+ d^{(4)jt}, \quad H^{(5)(0B)}_{21(-1)} = -\frac{1}{3m_\psi}\sqrt{\frac{2\pi}{3}} x^j_- d^{(4)tj},$$
$$H^{(5)(0B)}_{210} = -\frac{1}{3m_\psi}\sqrt{\frac{4\pi}{3}} d^{(4)tz}, \quad H^{(5)(0B)}_{211} = \frac{1}{3m_\psi}\sqrt{\frac{2\pi}{3}} x^j_+ d^{(4)tj}. \tag{137}$$

The six B-type spherical coefficients $H^{(5)(0B)}_{njm}$ with odd n are determined in terms of the six symmetric parts of $d^{(4)jk}$ to be

$$H^{(5)(0B)}_{100} = \frac{\sqrt{\pi}}{m_\psi}\left(d^{(4)tt} + \frac{1}{3} d^{(4)jj}\right),$$
$$H^{(5)(0B)}_{12(-2)} = \frac{1}{m_\psi}\sqrt{\frac{\pi}{30}} x^j_- x^k_- d^{(4)jk},$$
$$H^{(5)(0B)}_{12(-1)} = \frac{1}{m_\psi}\sqrt{\frac{\pi}{30}} x^j_- (d^{(4)jz} + d^{(4)zj}),$$
$$H^{(5)(0B)}_{120} = \frac{1}{m_\psi}\sqrt{\frac{\pi}{5}}\left(d^{(4)zz} - \frac{1}{3} d^{(4)jj}\right), \tag{138}$$
$$H^{(5)(0B)}_{121} = -\frac{1}{m_\psi}\sqrt{\frac{\pi}{30}} x^j_+ (d^{(4)jz} + d^{(4)zj}),$$
$$H^{(5)(0B)}_{122} = \frac{1}{m_\psi}\sqrt{\frac{\pi}{30}} x^j_+ x^k_+ d^{(4)jk}.$$

The three E-type spherical coefficients $H^{(5)(1E)}_{njm}$ are specified in terms of the antisymmetric part of $d^{(4)jk}$ by

$$H^{(5)(1E)}_{11(-1)} = -\frac{i}{m_\psi}\sqrt{\frac{\pi}{6}} x^j_- (d^{(4)jz} - d^{(4)zj}),$$
$$H^{(5)(1E)}_{110} = -\frac{i}{m_\psi}\sqrt{\frac{\pi}{12}} x^j_+ x^k_- (d^{(4)jk} - d^{(4)kj}), \tag{139}$$
$$H^{(5)(1E)}_{111} = -\frac{i}{m_\psi}\sqrt{\frac{\pi}{6}} x^j_+ (d^{(4)jz} - d^{(4)zj}).$$

Finally, we remark that the remaining nonzero spherical coefficients can be constructed as combinations of the 15 above independent ones according to

$$H^{(5)(1B)}_{21m} = -H^{(5)(0B)}_{01m} - H^{(5)(0B)}_{21m}, \quad H^{(5)(1B)}_{32m} = -\sqrt{3} H^{(5)(0B)}_{12m},$$
$$H^{(5)(1E)}_{31m} = -H^{(5)(1E)}_{11m}. \tag{140}$$

Note that some spherical coefficients remain zero, as expected from the analysis in Sec. II B showing that the absorption of the pseudovector operators $\hat{\mathcal{A}}^\mu$ involves only parts of the tensor operators $\hat{\mathcal{T}}^{\mu\nu}$.

VI. APPLICATIONS

Given the spherical decomposition and the various limiting cases, several immediate applications become feasible. In this section, we begin by revisiting the topics of dispersion and birefringence discussed in Sec. II D, presenting quantitative expressions for the dispersion relation, the group velocity, and the fermion spin precession in various limits. We next take advantage of the generality of the SME framework to make connections to other special models in the literature, which yields some interesting insights. With these results in hand, we can then translate existing astrophysical limits on isotropic Lorentz violation in the fermion sector into constraints on isotropic spherical SME coefficients, thereby revealing the relationships between the various approaches and the scope of the coverage of coefficient space.

A. Dispersion and birefringence

Using the spherical decomposition to extend the discussion in Sec. II D, we can generate expressions for the dispersion relation, including several useful limiting cases. We can also determine the group velocity of a fermion wave packet and the spin precession of the fermion induced by birefringence.

For simplicity, we begin by neglecting spin-dependent effects. Using the result (41) expressed in the spherical basis and with the birefringent contributions set to zero, the dispersion relation can be written as

$$p^2 - m_\psi^2 = 2E_0 \delta E, \tag{141}$$

where

$$\delta E = \sum_{dnjm} {E_0}^{d-3-n} |\boldsymbol{p}|^n {}_0Y_{jm}(a_{njm}^{(d)} - c_{njm}^{(d)}). \tag{142}$$

The modified group velocity $\boldsymbol{v}_g = \partial E/\partial \boldsymbol{p}$ obeys

$$|\boldsymbol{v}_g| = \frac{|\boldsymbol{p}|}{E_0} + \sum_{dnjm} ((d-3)p^2 + nm_\psi^2) \times {E_0}^{d-5-n} |\boldsymbol{p}|^{n-1} {}_0Y_{jm}(\hat{\boldsymbol{p}})(a_{njm}^{(d)} - c_{njm}^{(d)}). \tag{143}$$

Note that either increases or decreases in the velocity are possible, depending on the signs of the coefficients and on the direction of travel. Note also that the corresponding expressions for antifermions involve opposite signs for the coefficients $a_{njm}^{(d)}$.

Including spin dependence is straightforward in the isotropic limit. The isotropic dispersion relation also takes the form (141). Denoting E as $\mathring{E}$ for this case, we have

$$\begin{aligned}\delta\mathring{E} &= \sum_{dn} {E_0}^{d-3-n} |\boldsymbol{p}|^n (\mathring{a}_n^{(d)} - \mathring{c}_n^{(d)}) \\ &\quad \pm m_\psi \sum_{dn} {E_0}^{d-4-n} |\boldsymbol{p}|^n (-\mathring{g}_n^{(d)} + \mathring{H}_n^{(d)}) \\ &= \sum_{dn} {E_0}^{d-3-n} |\boldsymbol{p}|^n (\mathring{a}_n^{(d)} \mp m_\psi \mathring{g}_n^{(d+1)} - \mathring{c}_n^{(d)} \pm m_\psi \mathring{H}_n^{(d+1)}).\end{aligned} \tag{144}$$

In these expressions, the upper and lower signs correspond to positive and negative helicities, respectively. The modified group velocity in this case is

$$|\mathring{\boldsymbol{v}}_g| = \frac{|\boldsymbol{p}|}{E_0} + \sum_{dn} ((d-3)p^2 + nm_\psi^2) {E_0}^{d-5-n} \times |\boldsymbol{p}|^{n-1} (\mathring{a}_n^{(d)} \mp m_\psi \mathring{g}_n^{(d+1)} g_n - \mathring{c}_n^{(d)} \pm m_\psi \mathring{H}_n^{(d+1)}). \tag{145}$$

The results for antiparticles take the same form but with opposite signs for the coefficients $\mathring{a}_n^{(d)}$ and $\mathring{g}_n^{(d+1)}$. We thus see that the two helicities for each fermion species and the two for the corresponding antifermions all experience generically distinct dispersion relations and group velocities.

Many applications involve fermions at high energies, where the ultrarelativistic limit may be appropriate. In this limit, the dispersion relation takes the form

$$p^2 - m_\psi^2 = 2|\boldsymbol{p}|\delta E^{\rm UR}, \tag{146}$$

where

$$\delta E^{\rm UR} = \sum_d |\boldsymbol{p}|^{d-3} (\mathring{a}^{{\rm UR}(d)} \mp m_\psi \mathring{g}^{{\rm UR}(d+1)} - \mathring{c}^{{\rm UR}(d)} \pm m_\psi \mathring{H}^{{\rm UR}(d+1)}) \tag{147}$$

in terms of the ultrarelativistic coefficients defined in Table V. The modified group velocity is

$$|\boldsymbol{v}_g^{\rm UR}| = 1 + \sum_d (d-3)|\boldsymbol{p}|^{d-4} (\mathring{a}^{{\rm UR}(d)} \mp m_\psi \mathring{g}^{{\rm UR}(d+1)} - \mathring{c}^{{\rm UR}(d)} \pm m_\psi \mathring{H}^{{\rm UR}(d+1)}). \tag{148}$$

The above expansions have some intriguing consequences. One popular approach in the literature focuses on isotropic modifications to p^2 or $\delta\mathring{E}$ involving powers only of $|\boldsymbol{p}|$, restricted to dimensions $d \le 5$ or $d \le 6$. A potentially surprising feature in this context is that the expansions of p^2 and of $\delta\mathring{E}$ produce two completely different limits of the general theory. As can be seen from Eqs. (141) and (144), expanding p^2 in this way requires imposing the condition $n = d-2$, while expanding $\delta\mathring{E}$ requires $n = d-3$ instead, so the two expansions involve distinct coefficients. Explicitly, we find

$$p^2 - m^2 = \mp 2m_\psi \mathring{g}_1^{(4)}|\boldsymbol{p}| - 2\mathring{c}_2^{(4)}|\boldsymbol{p}|^2 \mp 2m_\psi \mathring{g}_3^{(6)}|\boldsymbol{p}|^3 - 2\mathring{c}_4^{(6)}|\boldsymbol{p}|^4 + \cdots \tag{149}$$

and

$$\delta\mathring{E} = \mathring{a}_0^{(3)} \pm m_\psi \mathring{H}_1^{(5)}|\boldsymbol{p}| + \mathring{a}_2^{(5)}|\boldsymbol{p}|^2 \pm m_\psi \mathring{H}_3^{(7)}|\boldsymbol{p}|^3 + \cdots, \tag{150}$$

showing that the two approaches have orthogonal content. Note that both expansions contain terms with odd and even powers of $|\boldsymbol{p}|$, but the first involves only operators of even dimension d while the second involves only operators of odd d. The attribution of operator dimensionality in this way is an automatic and natural consequence of the freedom to use field redefinitions to absorb some effects into others, discussed in Sec. II B.

The isotropic expansions of p^2 and $\delta\mathring{E}$ can be arranged to match if we stipulate *a priori* that only ultrarelativistic physics is relevant. In the ultrarelavistic limit, we obtain

$$\begin{aligned} p^2 - m^2 &= 2(\mathring{a}^{\mathrm{UR}(3)} \mp m_\psi \mathring{g}^{\mathrm{UR}(4)})|\boldsymbol{p}| \\ &\quad + 2(-\mathring{c}^{\mathrm{UR}(4)} \pm m_\psi \mathring{H}^{\mathrm{UR}(5)})|\boldsymbol{p}|^2 \\ &\quad + 2(\mathring{a}^{\mathrm{UR}(5)} \mp m_\psi \mathring{g}^{\mathrm{UR}(6)})|\boldsymbol{p}|^3 \\ &\quad + 2(-\mathring{c}^{\mathrm{UR}(6)} \pm m_\psi \mathring{H}^{\mathrm{UR}(7)})|\boldsymbol{p}|^4 + \cdots \\ &= 2|\boldsymbol{p}|\delta E^{\mathrm{UR}}. \end{aligned} \tag{151}$$

This expression reveals that the natural attribution of operator dimensionalities in the ultrarelavistic expansions involves a mixing of operators of different d at each power of $|\boldsymbol{p}|$.

The components h_g and h_H of the perturbative Hamiltonian give rise to birefringence, which can be viewed as a Larmor-like precession of the spin $\boldsymbol{S}$ as the particle travels. Writing the expressions (86) in the form $h_g = \boldsymbol{h}_g \cdot \boldsymbol{\sigma}$ and $h_H = \boldsymbol{h}_H \cdot \boldsymbol{\sigma}$, the rate of change of the spin expectation value of a particle state localized in momentum space is given via the commutator of the Hamiltonian h with the spin $\boldsymbol{S}$ as

$$\frac{d\langle \boldsymbol{S}\rangle}{dt} = \langle i[h, \boldsymbol{S}]\rangle \approx 2(\boldsymbol{h}_g + \boldsymbol{h}_H) \times \langle \boldsymbol{S}\rangle. \tag{152}$$

The precession frequency is then $\boldsymbol{\omega} = 2(\boldsymbol{h}_g + \boldsymbol{h}_H)$. This generalizes the result obtained for muon precession and used to extract constraints on muon Lorentz violation from storage-ring data [12]. In the helicity basis, we can write the result (152) in component form as

$$\frac{d\langle S^u\rangle}{dt} = \epsilon^{uvw} 2(h_g + h_H)_v \langle S_w\rangle, \tag{153}$$

where u, v, w range over components labeled by $(+, r, -)$ and the nonzero components of the antisymmetric tensor ϵ^{uvw} are specified by $\epsilon^{+r-} = -i$.

In the isotropic limit, the nonzero spin-dependent terms are given by $(\mathring{h}_g)_r$ and $(\mathring{h}_H)_r$. The helicity states then become stationary states, and the expression for the spin precession takes the simple form

$$\frac{d\langle S^\pm\rangle}{dt} = \mp 2i(\mathring{h}_g + \mathring{h}_H)_r \langle S^\pm\rangle, \tag{154}$$

where

$$(\mathring{h}_g + \mathring{h}_H)_r = -m_\psi \sum_{dn} E_0{}^{d-4-n} |\boldsymbol{p}|^n (\mathring{g}_n^{(d)} - \mathring{H}_n^{(d)}). \tag{155}$$

The expectation value $\langle S^r\rangle$ of the helicity remains constant in this limit.

B. Connections to other formalisms

A few special models containing quadratic fermion operators with $d > 4$ can be found in the existing literature. The generality of the SME-based analysis presented above implies that any special model based on standard field theory can be described using a selected subset of the Cartesian coefficients in Table I or, equivalently, of the spherical coefficients in Table III. The SME framework also incorporates several kinematical frameworks in a field-theoretic context. In this subsection, we summarize some of these links, treating first field-theoretic models and then kinematical formalisms.

1. Field-theoretic models

Consider first special models defined via a Lagrange density for a Dirac fermion of mass m_ψ that contains quadratic fermion operators with $d > 4$. Examples in the literature include models with a few specific Lorentz-violating operators of dimensions $d = 5$ and $d = 6$. Here, we identify the correspondence between these models and the SME coefficients for Lorentz violation.

One special model with quadratic Dirac operators is given by Myers and Pospelov [17]. This model involves two $d = 5$ operators for Lorentz violation constructed using a timelike vector n^μ, which fixes a preferred frame, and two corresponding parameters η_1/M_P, η_2/M_P. Matching the operators to the SME framework reveals that the nonzero Cartesian coefficients for Lorentz violation are given by

$$a^{(5)\mu\alpha\beta} = \frac{\eta_1}{M_P} n^\mu n^\alpha n^\beta, \qquad b^{(5)\mu\alpha\beta} = -\frac{\eta_2}{M_P} n^\mu n^\alpha n^\beta. \tag{156}$$

In the preferred frame with $n^\mu = (1, 0, 0, 0)$, the model is isotropic and can therefore be matched to isotropic spherical coefficients in the SME. We find the correspondence

$$\mathring{a}_0^{(5)} = \frac{\eta_1}{M_P}, \qquad \mathring{g}_1^{(6)} = -\frac{\eta_2}{m_\psi M_P}. \tag{157}$$

The model therefore involves two of the eight possible observable isotropic degrees of freedom with $d = 5$ and

$d = 6$ listed in Table III and displayed explicitly in Eqs. (97) and (98): one of the three for $d = 5$, and one of the five for $d = 6$. Note that the parameter η_2/M_P is most naturally viewed as an observable isotropic $d = 6$ coefficient due to the freedom to make field redefinitions absorbing all $\hat{b}^\mu$ coefficients, as discussed in Sec. II B.

An extension of this model is given by Mattingly [18], who uses the notation $u^\alpha \equiv n^\alpha$, $E_P \equiv M_P$. In addition to the two operators (156), this extension includes two others with $d = 5$ parametrized by $\alpha_L^{(5)}/M_P$, $\alpha_R^{(5)}/M_P$ and four more with $d = 6$ controlled by the real parameters $\alpha_L^{(6)}/M_P$, $\alpha_R^{(6)}/M_P$, $\tilde{\alpha}_L^{(6)}/M_P$, $\tilde{\alpha}_R^{(6)}/M_P$. Matching the $d = 5$ terms to the Cartesian coefficients in the SME yields nonzero contributions

$$m^{(5)\alpha\beta} = -\frac{(\alpha_L^{(5)} + \alpha_R^{(5)})}{2M_P} n^\alpha n^\beta, \qquad m_5^{(5)\alpha\beta} = -i\frac{(\alpha_L^{(5)} - \alpha_R^{(5)})}{2M_P} n^\alpha n^\beta. \tag{158}$$

In the SME framework, Hermiticity requires $\alpha_R^{(5)} = \alpha_L^{(5)*}$. This condition appears to have been overlooked in the literature. If $\alpha_L^{(5)}$ and $\alpha_R^{(5)}$ are both real, then only $m^{(5)\alpha\beta}$ is nonzero; if both parameters are imaginary, then only $m_5^{(5)\alpha\beta}$ is nonzero; while even when both parameters are complex only two degrees of freedom appear. For the $d = 6$ terms, the corresponding nonzero Cartesian coefficients in the SME are given by

$$c^{(6)\mu\alpha\beta\gamma} = \frac{(\alpha_L^{(6)} + \alpha_R^{(6)})}{2M_P^2} n^\mu n^\alpha n^\beta n^\gamma + \frac{(\tilde{\alpha}_L^{(6)} + \tilde{\alpha}_R^{(6)})}{2M_P^2} n^\mu n^\alpha \eta^{\beta\gamma},$$
$$d^{(6)\mu\alpha\beta\gamma} = \frac{(\alpha_L^{(6)} - \alpha_R^{(6)})}{2M_P^2} n^\mu n^\alpha n^\beta n^\gamma + \frac{(\tilde{\alpha}_L^{(6)} - \tilde{\alpha}_R^{(6)})}{2M_P^2} n^\mu n^\alpha \eta^{\beta\gamma}. \tag{159}$$

This model is also isotropic in the preferred frame with $n^\mu = (1, 0, 0, 0)$. Matching all the additional terms to the isotropic spherical coefficients in the SME in this frame gives

$$\mathring{c}_0^{(6)} = \frac{\alpha_L^{(5)} + \alpha_R^{(5)}}{2m_\psi M_P} + \frac{\alpha_L^{(6)} + \alpha_R^{(6)} + \tilde{\alpha}_L^{(6)} + \tilde{\alpha}_R^{(6)}}{2M_P^2},$$
$$\mathring{c}_2^{(6)} = -\frac{\alpha_L^{(5)} + \alpha_R^{(5)}}{2m_\psi M_P} - \frac{\tilde{\alpha}_L^{(6)} + \tilde{\alpha}_R^{(6)}}{2M_P^2},$$
$$\mathring{H}_1^{(7)} = \frac{\alpha_L^{(6)} - \alpha_R^{(6)} + \tilde{\alpha}_L^{(6)} - \tilde{\alpha}_R^{(6)}}{2m_\psi M_P^2},$$
$$\mathring{H}_3^{(7)} = -\frac{\tilde{\alpha}_L^{(6)} - \tilde{\alpha}_R^{(6)}}{2m_\psi M_P^2}. \tag{160}$$

This match reveals that the couplings $\alpha_L^{(5)}$, $\alpha_R^{(5)}$ are most naturally viewed as a single real observable isotropic coupling at $d = 6$, involving observable effects that are inseparable from those governed by the coefficient sum $\tilde{\alpha}_L^{(6)} + \tilde{\alpha}_R^{(6)}$. The combination associated with $m_5^{(5)\alpha\beta}$ in Eq. (158) has no observable effects, as shown in Sec. II B. Also, the four degrees of freedom in the real parameters $\alpha_L^{(6)}$, $\alpha_R^{(6)}$, $\tilde{\alpha}_L^{(6)}$, $\tilde{\alpha}_R^{(6)}$ are most naturally interpreted as two of the five observable isotropic coefficients with $d = 6$ and two of the five with $d = 7$.

Another special model involving quadratic Dirac operators is considered by Rubtsov, Satunin, and Sibiryakov [19]. This isotropic model contains a $d = 4$ term and a $d = 6$ term, with parameters $\varkappa$ and g. It corresponds to the SME framework in the limit

$$c^{(4)jk} = -\varkappa\delta^{jk}, \qquad c^{(6)jklm} = -g\delta^{jk}\delta^{lm}. \tag{161}$$

In the preferred frame, the match to the isotropic spherical coefficients in the SME is

$$\mathring{c}_2^{(4)} = -\varkappa, \qquad \mathring{c}_4^{(6)} = -g/M^2, \tag{162}$$

showing that the model involves another of the five possible isotropic operators for $d = 6$ displayed in Table III and Eq. (98).

A more general model focusing on $d = 5$ operators is given by Bolokhov and Pospelov [20], who limit attention to operators that cannot be reduced to ones with $d < 5$ using the equations of motion. The model involves quadratic fermion operators expressed in terms of parameters $h_1^{\alpha\beta}$, $h_2^{\alpha\beta}$, $C_1^{\mu\alpha\beta}$, $C_2^{\mu\alpha\beta}$, $E_1^{\mu\nu\alpha\beta}$, and $E_4^{(d)\alpha\mu\nu\beta}$. These parameters form a subset of the $d = 5$ Cartesian coefficients listed in Table I. Explicitly, we find the relations

$$m^{(5)\alpha\beta} = 2h_1^{\alpha\beta}, \qquad m_5^{(5)\alpha\beta} = -2ih_2^{\alpha\beta},$$
$$a^{(5)\mu\alpha\beta} = 6C_1^{\mu\alpha\beta}, \qquad b^{(5)\mu\alpha\beta} = -6C_2^{\mu\alpha\beta},$$
$$H^{(5)\mu\nu\alpha\beta} = 12E_1^{\mu\nu\alpha\beta} + 16E_4^{(d)\alpha\mu\nu\beta}. \tag{163}$$

Counting the degrees of freedom in each relation is instructive. The parameters $h_1^{\alpha\beta}$ and $h_2^{\alpha\beta}$ are symmetric, and so each have 10 independent components, matching the SME counting for $m^{(5)\alpha\beta}$ and $m_5^{(5)\alpha\beta}$. The parameters $C_1^{\mu\alpha\beta}$ and $C_2^{\mu\alpha\beta}$ are totally symmetric, giving 20 + 20 independent components, whereas the SME coefficients $a^{(5)\mu\alpha\beta}$ and $b^{(5)\mu\alpha\beta}$ contain a total of 40 + 40 degrees of freedom. We remark in passing that the 20 parameters $C_1^{\mu\nu\rho}$ correspond to the 20 on-shell effective coefficients $a_{\rm eff}^{(5)\alpha\beta\gamma}$ in Eq. (77). The parameter $E_1^{\mu\nu\alpha\beta}$ is antisymmetrized in $\mu\nu$ and then symmetrized in $\mu\nu\beta$, giving a total of 45 independent components, while $E_4^{\alpha\mu\nu\beta}$ is antisymmetrized in $\alpha\mu\nu$ and then symmetrized in $\alpha\beta$, generating 15 independent components. The total is 45 + 15 = 60, matching the counting for the SME coefficients $H^{(5)\mu\nu\alpha\beta}$. Note, however, that the 45 components of $E_1^{\mu\nu\alpha\beta}$ cannot be matched to the 45 on-shell SME effective coefficients $H_{\rm eff}^{(5)\mu\alpha\beta\gamma}$ in Eq. (79).

2. Kinematical formalisms

In the context of the photon sector, Sec. IV F of Ref. [30] discusses the relationship between the SME and several kinematical formalisms purporting to describe aspects of Lorentz violation based on modifications of the transformation laws. In this subsection, we revisit these discussions briefly in light of the insights provided by the nonminimal fermion sector.

One kinematical approach involves models called deformed special relativities (DSR). These are defined as smooth nonlinear momentum-space representations of the usual Lorentz transformations, which is known to imply that they have no observable consequences beyond conventional special relativity [43]. All corresponding coefficients for Lorentz violation in the photon sector are explicitly constructed in Sec. IV F 3 of Ref. [30] and are indeed found to be unobservable. Since DSR models are sector-independent by definition, a parallel analysis holds for the nonminimal fermion sector discussed in the present work, and so further consideration of DSR models in this context provides no new insights.

Another kinematical approach, the Robertson–Mansouri–Sexl (RMS) formalism [44], does describe certain physical deviations from special relativity. The RMS formalism can be viewed as a special limit of the SME requiring flat spacetime, the existence of a universal preferred frame U in which light is conventional, and only isotropic Lorentz violation affecting clocks and rods in U. The three RMS parameters a, b, d are experiment dependent unless identical clocks and rods in the same physical states are used as the reference standards, so caution is required in comparing results from different experiments. For any given experiment, the RMS parameters can in principle be expressed in terms of SME coefficients by incorporating the underlying physics of the clocks and rods. The mapping from the SME to the RMS formalism is described in Sec. IV F 2 of Ref. [30].

The development of the nonminimal fermion sector in the present work offers the opportunity to investigate further the relationship between the RMS formalism and the SME by considering effects from the fermion content of clocks and rods. In general, the behavior of physical clocks and rods is complicated and determined by properties of their component particles and the forces involved. A detailed SME description is therefore necessary for a careful treatment of Lorentz violation in this context. However, a phenomenological treatment in the SME vein using simple model clocks and rods can illustrate some of the basic features to be expected from Lorentz violation and their role in the RMS formalism.

Consider first a clock in conventional special relativity that ticks at a frequency ω_0 in a comoving inertial frame. In a different boosted frame, the frequency $\omega \equiv p^0$ of the clock and the wave vector $\boldsymbol{p}$ of its oscillations obey a dispersion-type relation $p_\mu p^\mu \equiv \omega^2 - \boldsymbol{p}^2 = \omega_0^2$, where the invariant ω_0 plays the role of a particle mass. In the presence of Lorentz violation, this dispersion relation becomes modified. Ignoring possible spin effects for simplicity, the modified relation can be written

$$p^2 = \omega_0^2 + 2\hat{a}_c - 2\hat{c}_c, \tag{164}$$

where $\hat{a}_c$ and $\hat{c}_c$ are p^μ-dependent effective Lorentz-violating corrections associated with CPT-odd and CPT-even operators, respectively. If the clock is a single fermion of mass m_ψ, Eq. (164) can be viewed as a special limit of the modified dispersion relation (41) derived in Sec. II D [45]. To match to the RMS formalism, we must further restrict the clock dispersion relation by assuming the existence of a preferred universal frame U in which the physics describing the clock is isotropic. The dispersion relation (164) then takes the form

$$p^2 = \omega_0^2 + 2\sum_{dn} \omega^{d-2-n} |\boldsymbol{p}|^n ((\mathring{a}_c)^{(d)}_n - (\mathring{c}_c)^{(d)}_n), \tag{165}$$

involving only the isotropic effective coefficients for Lorentz violation $(\mathring{a}_c)^{(d)}_n$ and $(\mathring{c}_c)^{(d)}_n$. The allowed values of d and n and the coefficient counting are the same as those given in the last four rows of Table III.

Suppose a clock obeying (165) moves at a constant speed v in the x direction relative to U. In the comoving inertial frame, the clock wave 4-vector can be written as $k^\mu = (\omega_c, 0, 0, 0)$, where ω_c denotes the ticking frequency. In the frame U, the wave 4-vector takes the form $p^\mu = (\gamma\omega_c, \gamma v \omega_c, 0, 0)$. Combining this expression with the dispersion relation (165), the velocity-dependent ratio of the clock ticking frequencies $\omega_c(v)$ and $\omega_c(0) \equiv \omega_c$ is

$$\frac{\omega_c(v)}{\omega_c(0)} = 1 + \sum_{dn} \omega_0^{d-4} (v^n \gamma^{d-2} - \delta_{n0}) ((\mathring{a}_c)^{(d)}_n - (\mathring{c}_c)^{(d)}_n). \tag{166}$$

In the SME, the frequency $\omega_c(v)$ is frame dependent. In the RMS formalism, however, this clock serves as the time standard, with all other times measured relative to it. The ratio (166) reduces to 1 for vanishing v and is an even function of v because n is even, in agreement with RMS postulates.

Next, consider a rod in conventional special relativity with length and orientation specified by a rest-frame wave vector $\boldsymbol{k}_0$ and a corresponding wave 4-vector $k^\mu = (0; \boldsymbol{k}_0)$. A simple choice of rod is the Compton wavelength of a single particle, which could be of a species different from any involved in the time standard. Other rod choices are possible, such as one formed from particles with frequencies locked to an internal clock. In a boosted frame, the wave 4-vector p^μ of the rod obeys the dispersion-type relation $p^2 = -|\boldsymbol{k}_0|^2$. In the presence of Lorentz violation, and assuming as before the existence of a universal preferred frame U as required by the RMS formalism, the modified dispersion relation for the rod can be written as

$$p^2 = -|\boldsymbol{k}_0|^2 + 2\sum_{dn} \omega^{d-2-n}|\boldsymbol{p}|^n((\mathring{a}_r)^{(d)}_n - (\mathring{c}_r)^{(d)}_n), \quad (167)$$

where the allowed values of d and n and the coefficient counting parallel those for the clock.

If a rod with wave vector $\boldsymbol{k}_r$ in a comoving frame moves at speed v in the x direction relative to the frame U, then its wave 4-vector in U is $p^\mu = (\gamma v k_r^x, \gamma k_r^x, k_r^y, k_r^z)$. Using the dispersion relation (167) reveals that the velocity-dependent ratio of the wave-vector magnitudes $|\boldsymbol{k}_r(\boldsymbol{v})|$ and $|\boldsymbol{k}_r(0)| \equiv |\boldsymbol{k}_r|$ is

$$\frac{|\boldsymbol{k}_r(\boldsymbol{v})|}{|\boldsymbol{k}_r(0)|} = 1 - \sum_{dn} |\boldsymbol{k}_0|^{d-4}\Big(\gamma^{d-2} v_\parallel^{d-2-n}(1-v_\perp^2)^{n/2} - \delta_{n,d-2}\Big) \times ((\mathring{a}_r)^{(d)}_n - (\mathring{c}_r)^{(d)}_n), \quad (168)$$

where $v_\parallel$ and $v_\perp$ are the components of the boost velocity parallel and perpendicular to the rod, respectively. This expression characterizes the variations in rod length in different Lorentz frames, explicitly showing the orientation and velocity dependence arising in the SME context. In contrast, the rod serves as the length standard in the RMS formalism, with all other lengths measured relative to it.

The result (168) illuminates some aspects of the RMS formalism. The coefficients $(\mathring{c}_r)^{(d)}_n$ associated with CPT-even effects have indices d and n taking only even values, and hence they introduce dilations involving only even powers of v. This is in agreement with the RMS postulates. However, the coefficients $(\mathring{a}_r)^{(d)}_n$ controlling CPT-odd effects produce shifts that are odd in v, so boosts in opposite directions give different effects. This possibility lies outside the RMS formalism despite its origin in comparatively simple isotropic Lorentz violations in U.

The expression (168) has another significant implication: the rod length measured in RMS coordinates is the same when the rod is oriented along any of the three coordinate axes, but it typically differs for other orientations. This feature appears to have been overlooked in the literature. It emerges here in the context of a simple SME-based model, but the dependence of the ratio (168) on parallel and perpendicular velocities suggests it is a generic aspect of Lorentz violation. In particular, the RMS transformation assumes one rod is aligned along the boost axis and the other two are perpendicular. The expression (168) therefore implies that nonstandard choices of rod orientation lie outside the RMS formalism because they cannot be linked to the frame U via a transformation of the RMS form. This is problematic for laboratory experiments attempting to report bounds in the RMS language because the results of any measurement are meaningful only when the chosen length standards are correctly aligned with a particular boost and moreover only when this alignment is maintained throughout the measurement process. This requirement is challenging and perhaps impossible to satisfy in practice due to the rotation and orbital revolution of the Earth and to the motion of the solar system relative to the frame U.

For the simple model with clocks of type (165) and rods of type (167) with RMS-compatible orientations, we can construct explicitly the RMS transformation T from U to the boosted frame and identify the RMS parameters a, b, d and hence the factors α, β, and δ multiplying their v^2 components. Assuming Einstein synchronization, T takes the form

$$T = \begin{pmatrix} a\gamma^2 & -av\gamma^2 & 0 & 0 \\ -bv & b & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & d \end{pmatrix}, \quad (169)$$

where a, b, and d are functions of v that reduce to $a = 1/\gamma$, $b = \gamma$, and $d = 1$ in the Lorentz-invariant limit. The RMS transformation can be viewed as the product $T = C\Lambda$ of a standard Lorentz transformation Λ from U to the comoving Lorentz frame with coordinate dilations C scaling space and time relative to the chosen clocks and rods [30]. In terms of RMS functions, the scaling matrix C is diagonal with entries $(a\gamma, b/\gamma, d, d)$. The ratio (166) then implies that the time-dilation function a is given by

$$\begin{aligned} a &= \frac{1}{\gamma} + \frac{1}{\gamma}\sum_{dn} \omega_0^{d-4}(v^n\gamma^{d-2} - \delta_{n0})((\mathring{a}_c)^{(d)}_n - (\mathring{c}_c)^{(d)}_n) \\ &= 1 + v^2\Big[-\frac{1}{2} + \sum_d \omega_0^{d-4}\Big(\frac{d-2}{2}(\mathring{a}_c)^{(d)}_0 \\ &\quad - \frac{d-2}{2}(\mathring{c}_c)^{(d)}_0 + (\mathring{a}_c)^{(d)}_2 - (\mathring{c}_c)^{(d)}_2\Big)\Big] + O(v^4). \end{aligned} \quad (170)$$

The coefficient of v^2 is the expression for the RMS parameter α in terms of SME coefficients for Lorentz violation in this simple model.

To find the RMS functions b and d for spatial dilations, consider first a rod oriented along the boost direction x and two rods in the orthogonal y and z directions. For the rod lying along the x axis, the ratio (168) gives

$$\begin{aligned} b &= \gamma - \gamma\sum_{dn} |\boldsymbol{k}_0|^{d-4}(\gamma^{d-2}v^{d-2-n} - \delta_{n,d-2})((\mathring{a}_r)^{(d)}_n - (\mathring{c}_r)^{(d)}_n) \\ &= 1 - v\sum_d |\boldsymbol{k}_0|^{d-4}(\mathring{a}_r)^{(d)}_{d-3} \\ &\quad + v^2\Big[\frac{1}{2} + \sum_d |\boldsymbol{k}_0|^{d-4}\Big((\mathring{c}_r)^{(d)}_{d-4} + \frac{d-2}{2}(\mathring{c}_r)^{(d)}_{d-2}\Big)\Big] + O(v^3). \end{aligned} \quad (171)$$

The term linear in the boost v stems from CPT violation and lies outside the RMS formalism, as discussed above. The coefficient of the term quadratic in v^2 is the RMS parameter β.

Next, consider a rod lying along the y or z axis. In the simple model with the ratio (168), no dilation along these directions is produced, and so the RMS parameter d is found to be $d = 1$, implying $\delta = 0$. However, the result (168) accounts only for modifications arising from the coupling of the intrinsic wavelengths of the rod components to the Lorentz-violating vacuum. A more realistic phenomenological description of a rod must also allow for couplings of the rod bulk properties such as its macroscopic momentum or spin. For example, if the rod has mass M, then its bulk 4-momentum in the boosted frame takes the form $P^\mu = M\gamma(1, \boldsymbol{v})$. Suppose the effective dispersion relation for the rod can be written as

$$p^2 = -|\boldsymbol{k}_0|^2 - 2C_r \boldsymbol{P}^2 \qquad (172)$$

instead of the result (167). The modification vanishes when the rod is at rest in U but otherwise leads to an isotropic rod distortion given by

$$\frac{b}{\gamma} = d = \frac{|\boldsymbol{k}_r(v)|}{|\boldsymbol{k}_r(0)|} = 1 + |\boldsymbol{k}_0|^{-2} C_r M^2 \gamma^2 v^2 = 1 + (|\boldsymbol{k}_0|^{-2} C_r M^2) v^2 + O(v^4). \qquad (173)$$

In this case, the RMS parameter d is nonzero, and the coefficient multiplying v^2 is the parameter $\delta = \beta - \frac{1}{2}$. Note that in more realistic models, the parameters β and δ are independent and both nonzero. For example, the simple phenomenological model obtained by adding the two modifications (167) and (173) generates independent nonzero parameters β and δ.

C. Astrophysical bounds

A number of papers in the literature obtain bounds on various kinds of isotropic Lorentz violation from astrophysical observations. A few of these are based on field-theoretic models, but the bulk use an approach based on isotropic dispersion relations. The results obtained in Secs. VI A and VI B 1 make feasible a translation of these various bounds into constraints on isotropic spherical coefficients in the SME. This translation also clarifies the relationships between the different bounds and reveals the coverage of the available coefficient space achieved to date.

Since all the astrophysical bounds are obtained at high energies, it is appropriate to work in the ultrarelativistic limit of the SME, with dispersion relation given by Eq. (151). The existing bounds only involve operator dimensions $d \leq 6$. The possibility of helicity dependence is disregarded by many authors, so it is also appropriate to set to zero the coefficients $\mathring{g}^{\mathrm{UR}(6)}$ and $\mathring{H}^{\mathrm{UR}(5)}$ in these cases.

TABLE VI. Astrophysical limits on isotropic SME coefficients. Units are GeV^{4-d}.

Dimension	Sector	Lower bound	Coefficient	Upper bound	Source
$d=4$	Electron		$\mathring{c}_e^{\mathrm{UR}(4)}$	$<1.5\times10^{-15}$	[46]
		$-5\times10^{-13}<$	$\mathring{c}_e^{\mathrm{UR}(4)}$		[47]
		$-1.3\times10^{-15}<$	$\mathring{c}_e^{\mathrm{UR}(4)}$	$<2\times10^{-16}$	[48]
		$-1.2\times10^{-16}<$	$\mathring{c}_e^{\mathrm{UR}(4)}$		[49]
		$-6\times10^{-20}<$	$\mathring{c}_e^{\mathrm{UR}(4)}$		[50]
	Proton	$-5\times10^{-23}<$	$\mathring{c}_p^{\mathrm{UR}(4)}$		[46]
			$\mathring{c}_p^{\mathrm{UR}(4)}$	$<5\times10^{-24}$	[47]
		$-2\times10^{-22}<$	$\mathring{c}_p^{\mathrm{UR}(4)}$		[49]
			$\mathring{c}_p^{\mathrm{UR}(4)}$	$<4.5\times10^{-23}$	[51]
		$-9.8\times10^{-22}<$	$\mathring{c}_p^{\mathrm{UR}(4)} - \mathring{c}_e^{\mathrm{UR}(4)}$	$<9.8\times10^{-22}$	[52]
	Quark	$-1\times10^{-23}<$	$\mathring{c}_q^{\mathrm{UR}(4)}$	$<1.8\times10^{-21}$	[53]
		$-1\times10^{-23}<$	$\mathring{c}_q^{\mathrm{UR}(4)} - 2\mathring{c}_e^{\mathrm{UR}(4)}$	$<2\times10^{-20}$	[53]
$d=5$	Electron		$\mathring{a}_e^{\mathrm{UR}(5)}$	$<6.5\times10^{-27}$	[54]
		$-3.5\times10^{-27}<$	$\mathring{a}_e^{\mathrm{UR}(5)}$		[55]
		$-1\times10^{-34}<$	$\mathring{a}_e^{\mathrm{UR}(5)} - m_e \mathring{g}_e^{\mathrm{UR}(6)}$	$<1\times10^{-34}$	[53]
		$-4\times10^{-25}<$	$\mathring{a}_e^{\mathrm{UR}(5)} \pm m_e \mathring{g}_e^{\mathrm{UR}(6)}$	$<4\times10^{-25}$	[56]
		$-1\times10^{-20}<$	$\mathring{a}_e^{\mathrm{UR}(5)}$	$<2.8\times10^{-17}$	[21]
	Muon	$-1\times10^{-34}<$	$\mathring{a}_\mu^{\mathrm{UR}(5)} - m_\mu \mathring{g}_\mu^{\mathrm{UR}(6)}$	$<1\times10^{-34}$	[53]
	Tau	$-2\times10^{-33}<$	$\mathring{a}_\tau^{\mathrm{UR}(5)} - m_\tau \mathring{g}_\tau^{\mathrm{UR}(6)}$	$<2\times10^{-33}$	[53]
$d=6$	Electron	$-8.5\times10^{-20}<$	$\mathring{c}_e^{\mathrm{UR}(6)}$	$<2.5\times10^{-23}$	[53]
		$-5.4\times10^{-14}<$	$\mathring{g}_e^{\mathrm{UR}(6)}$	$<5.4\times10^{-14}$	[21]
	Muon	$-8.5\times10^{-20}<$	$\mathring{c}_\mu^{\mathrm{UR}(6)}$	$<2.5\times10^{-23}$	[53]
	Proton	$-3.4\times10^{-45}<$	$\mathring{c}_p^{\mathrm{UR}(6)}$	$<3.4\times10^{-42}$	[57]
	Quark	$-6.3\times10^{-23}<$	$\mathring{c}_q^{\mathrm{UR}(6)}$	$<1.7\times10^{-22}$	[53]

The reported bounds involve a variety of particle species, including electrons, muons, taus, protons, and quarks. For the latter, all the partonic quarks are assumed to have the same dispersion relation. Since the present focus is on fermions, Lorentz violation in bosons such as pions or photons is neglected for simplicity when making conversions.

Table VI compiles some resulting constraints on isotropic Lorentz violation in the SME. The first two columns of this table list the operator dimension and the sector of the SME involved. The table includes constraints on minimal SME operators with $d = 4$ as well as on ones with nonminimal dimensions. The next three columns of the table contain the constraints on ultrarelativistic isotropic spherical coefficients obtained from existing bounds. The coefficients for different particle species are distinguished with a subscript denoting the species in question. The final column provides the source from which the constraint is extracted.

The table reveals that the existing bounds span different coefficients. However, of the seven types of possible isotropic ultrarelativistic spherical coefficients with $d \leq 6$, namely, $\mathring{a}^{\mathrm{UR}(3)}$, $\mathring{c}^{\mathrm{UR}(4)}$, $\mathring{g}^{\mathrm{UR}(4)}$, $\mathring{a}^{\mathrm{UR}(5)}$, $\mathring{H}^{\mathrm{UR}(5)}$, $\mathring{c}^{\mathrm{UR}(6)}$, and $\mathring{g}^{\mathrm{UR}(6)}$, constraints exist on at most four of them for any one species. We see that even within the very restrictive assumption of isotropic ultrarelativistic Lorentz violation, much of the coefficient space is unconstrained to date. Also notably lacking are limits for neutral fermions, including neutrons and other baryons. We remark in passing that numerous constraints exist on nonisotropic minimal fermion operators [6], including some extracted from studies of mesons and some at impressive sensitivities. Nonetheless, the experimental coverage of SME coefficients in the fermion sector is at present limited to a tiny fraction of the available possibilities.

VII. SUMMARY

In this work, the general quadratic theory of a single Dirac fermion in the presence of Lorentz violation has been developed. Our discussion began with the construction and basic properties of the theory (1), including two useful decompositions of the general spinor-matrix operator $\hat{\mathcal{Q}}$ for Lorentz violation. The first reveals the different spin content via the operators $\hat{\mathcal{S}}$, $\hat{\mathcal{P}}$, $\hat{\mathcal{V}}^\mu$, $\hat{\mathcal{A}}^\mu$, $\hat{\mathcal{T}}^{\mu\nu}$, while the second displays CPT and other properties via the notation $\hat{m}$, $\hat{m}_5$, $\hat{a}^\mu$, $\hat{b}^\mu$, $\hat{c}^{\mu\nu}$, $\hat{d}^{\mu\nu}$, $\hat{e}^\mu$, $\hat{f}^\mu$, $\hat{g}^{\mu\rho\nu}$, $\hat{H}^{\mu\nu}$ paralleling the conventions in the minimal SME. Table I compiles some features of the corresponding coefficients for Lorentz violation. In Sec. II B, we show that the physical observables in the pure quadratic theory (1) are restricted to pieces of $\hat{\mathcal{V}}^\mu$ and $\hat{\mathcal{T}}^{\mu\nu}$, generalizing known results for the minimal SME and for the nonminimal neutrino sector.

We next constructed the exact dispersion relation for a fermion wave packet, obtaining the closed and compact form (39). For some practical applications, an approximate expression for the energy valid at leading order in Lorentz violation is useful, and this is provided in Eq. (43). The form of this equation reveals that fermions experience anisotropy, dispersion, and birefringence when in the presence of Lorentz violation. The covariant projection operator yielding the spinor polarization is derived, and the corresponding relativistic polarization vector is given in Eq. (49).

With these key results in hand, we next turned to the construction of the particle and antiparticle Hamiltonians associated with the theory (1). The 2×2 Hamiltonian for particles is given as Eq. (60) in Sec. III A, while that for antiparticles is in Eq. (64). Using the relativistic polarization vector, we can reduce the structure of these expressions to the conceptually simple form (80), which separates the particle Hamiltonian into four pieces according to spin and CPT properties.

Despite its conceptual simplicity, the explicit form of the Hamiltonian (80) involves coefficients with numerous indices and is unwieldy for many practical applications. In Sec. IV, we have taken advantage of the approximate rotation symmetry present in many experimental situations to decompose the Hamiltonian in spherical harmonics. The result (88) involves eight sets of spherical coefficients that characterize all types of Lorentz violation for a single Dirac fermion. Table III summarizes the basic properties of these coefficients. Their comparatively simple properties under rotation, exemplified in Eq. (90), make them well suited to explicit analyses. The isotropic limit of the perturbative Hamiltonian, which can be useful in some treatments, is obtained in Eq. (91), and the corresponding isotropic Lagrange density for operator dimensions $d = 3$, 4, 5, 6 is given in Eq. (94) through (98).

For many practical purposes, limiting cases of the general formalism are useful. Section V extracts the nonrelativistic and ultrarelativistic cases and their isotropic limits. The nonrelativistic Hamiltonian is given in Eq. (108), and the corresponding coefficients are summarized in Table IV. The ultrarelativistic Hamiltonian is presented in Eq. (118), and Table V lists properties of its coefficients. This section also explicitly connects the spherical decomposition for operators of renormalizable dimension with standard expressions for the minimal SME.

The final technical discussions in this paper concern immediate applications of our results. In Sec. VI A, the issue of dispersion and birefringence is revisited in the spherical language. The dispersion relation, group velocity, and the spin-precession rate (152) are derived in compact forms in various limiting cases. We then address in Sec. VI B the relationships between the present general framework and some special field-theoretic models and kinematical approaches in the literature. The combination of the above results permits translation of a wide variety of existing astrophysical bounds on isotropic Lorentz violation into constraints on isotropic spherical SME coefficients, which are compiled in Table VI.

Overall, the results in this paper offer a comprehensive theoretical framework for investigations of Lorentz and *CPT* violation involving quadratic fermion operators. The physical effects identified here provide a basis for future experimental searches. Numerous types of Lorentz and *CPT* violation are unconstrained to date, and the prospects for exploration and the potential for discovery remain bright.

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