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Dimension four wins the same game as the standard model group

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In a previous article Don Bennett and I looked for, found, and proposed a game in which the standard model gauge group $S(U(2) \times U(3))$ gets singled out as the "winner." This "game" means that the by Nature chosen gauge group should be just the one that has the maximal value for a quantity, which is a modification of the ratio of the quadratic Casimir for the adjoint representation and that for a "smallest" faithful representation. Here I propose to extend this "game" to construct a corresponding game between different potential dimensions for space-time. The idea is to formulate how the same competition as the one between the potential gauge groups would run out, if restricted to the potential Lorentz or Poincare groups achievable for different dimensions of space-time d. The remarkable point is that it is the experimental space-time dimension 4 which wins. So the same function defined over Lie groups seems to single out *both* the gauge group *and* the dimension of space-time in nature. This seems a rather strange coincidence, unless there really is some similar physical reason behind causing our game-variable (or goal variable) to be selected to be maximal. It has crudely to do with that the groups preferred are easily represented on very "small" but yet faithful representations.

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I. INTRODUCTION

The main idea of the present series of articles is to seek some game that at the same time can select out the gauge group observed in nature and the dimension of space time. Let us suppose we should have the standard model group $S(U(2) \times U(3))$ in nature, and also the gauge group (whatever that means) of the (gravitational) general relativity by saying that nature has chosen the "winner" in this game. That is to say we look for a group-characteristic quantity ("goal quantity") which happens to be the largest possible for both the gauge group of the standard model and a group associate with the Lorentz transformations (or somehow with the gauge transformations in general relativity), e.g., the Lorentz group. We could then claim that such a goal quantity specifies both the gauge group for the standard model and the Lorentz group, thereby meaning the dimension of space-time. If the quantity is reasonably simple, this could be an explanation for both the gauge group and the dimension of space-time. We could then answer: Why do we have in nature just the standard model group $S(U(2) \times U(3))$ and why just 4 space-time dimensions? In the previous article [1] we sought in this way to invent a game or rather a "goal quantity," which were at first the ratio C_A/C_F of the quadratic Casimirs for the group in question for the adjoint representation to the quadratic Casimir of some "small" but still faithful representation in such a way that this ratio would take its largest value for the by nature chosen (gauge) group.

Both ourselves and others earlier have made other attempts to find arguments pointing out both the gauge group [2] and the dimension [3–10]. We shall shortly review earlier works in the appendix.

In Sec. II we shall review the previous work [1].

Actually N. Brene and I had already earlier proposed another game that essentially pointed also to the standard model gauge *group* being the winner [2], but it is the more recent proposal with the quadratic Casimirs or rather their ratio C_A/C_F which we seek to generalize to determine the dimension of space-time in this article. This concept of the gauge group for general relativity may be a bit imprecise, and so I want at first to simplify it a little bit by making a few *ad hoc* decisions to extract a group–essentially the gauge group–of general relativity, even if the definition of this concept is not completely clear yet.

A first candidate, which is for me rather attractive for the purpose, is simply the Lorentz group, meaning the group of Lorentz boosts and rotations.

You could consider the attitude of the present article and the foregoing one in the series [1] as attempts to extract the information as discussed in the article [11] contained in the group structure of the standard model gauge group and the dimension of space-time. Of course the hope could be that one would in this way learn about the true theory, that might be behind the standard model by finding some regularity (as we may say we do in the present series of papers).

The reader should consider these different proposals for a quantity to maximize (= use as goal quantity) as rather closely related versions of a quantity suggested by perhaps a bit of a vague idea being improved successively. From our point of view the translational generators in the Poincare group are a bit difficult and so we seek to treat them only in an approximate way. One should have in mind, that the basic idea is: The group selected by nature

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is the one that is counted in a ‘‘normalization determined from the Lie algebra of the group’’ and can be said to have a faithful representation (F) the matrices of which move as little as possible when the group element being represented moves around in the group.

Let me at least clarify this statement:

As usual we mean by representations *linear* representations. Thus we really consider homomorphisms of the group into a subset of matrices (with matrix multiplication as the group composition law). The requirement of the representation being faithful then means that this group of matrices shall actually be an isomorphic image of the original group. Now on a system of matrices we have a natural metric, namely the metric in which the distance between two matrices **A** and **B** is given by the square root of the trace of the numerical square of the difference

$$\text{dist} = \sqrt{\text{tr}((\mathbf{A}-\mathbf{B})(\mathbf{A}-\mathbf{B})^{+})}. \quad (1)$$

To make a comparison of one group and some representation of it with another group and its representation with respect to how fast the representation matrices move for a given motion of the group elements, we need a normalization giving us a well-defined metric on the groups, with respect to that which we can ask for the rate of variation of the representations. In my short statement I suggested that this ‘‘normalization should be determined from the Lie algebra of the group.’’ This means, more precisely, that one shall consider the *adjoint* representation, which is in fact completely given by the Lie algebra, and then use the same distance concept as we just proposed for the matrix representation $\sqrt{\text{tr}((\mathbf{A}-\mathbf{B})(\mathbf{A}-\mathbf{B})^{+})}$. In this way the quantity to minimize would be the ratio of the motion-distance in the representation, e.g., F, and in the Lie algebra representation, i.e., the adjoint representation. But that ratio is just for infinitesimal motions $\sqrt{C_F/C_A}$. So instead of talking about what to minimize, if we inverted it and claimed we should maximize, we would get $\sqrt{C_A/C_F}$ to be maximized. Of course the square root does not matter, and we thus obtain in this way a means to look at the ratio C_A/C_F as a measure for the motion of an element in the group compared to the same element motion on the representation.

It is not unreasonable to think that a group represented so that the representation matrix moves less with the group element is more likely to become a good symmetry for a theory than one with more variation of the representation matrix. If one imagines that the potential groups become good symmetries by accident, then at least it would be less of an accident required the less the degrees of freedom moves around under the to the group corresponding symmetry (approximately). It is the suggestion of the speculation inspired by our result that it is a matter of it being easier to get some groups as better symmetries than others; and those with the biggest C_A/C_F should be the easiest to become symmetries by accident. That is indeed the speculation behind the present article as well as the previous one [1] that symmetries may appear by accident (then perhaps be strengthened to be exact by some means [7,12]).

But let us stress that you can also look at the present work and the previous one in the following phenomenological philosophy:

We wonder, why Nature has chosen just $4\,(=3+1)$ dimensions and why Nature—at the present experimentally accessible scale at least—has chosen just the standard model group $S(U(2)\times U(3))$? Then we speculate that there might be some quantity characterizing groups, which measures how well they ‘‘are suited ’’ to be the groups for Nature. And then we begin to seek that quantity as being some function defined on the class of abstract groups—i.e., giving a number for each abstract (Lie?) group—of course by proposing for ourselves at least various versions or ideas for what such a relatively simple function defined on the abstract Lie groups could be. Then the present works—this paper and the previous one [1]—represent the present status of the search: We found that with small variations the types of such functions representing the spirit of the little motion of the ‘‘best’’ faithful representation, i.e., essentially the largest C_A/C_F, turned out truly to bring Nature’s choices to be the winners.

In this sense we may then claim that we have found by phenomenology that at least the ‘‘direction’’ of a quantity like C_A/C_F or light modifications of it is a very good quantity to make up a ‘‘theory’’ for why we have got the groups we got.

In the following Sec. II we review the main results of the C_A/C_F quantity, which in the previous article we studied for the various Lie groups in order to discuss that the standard model group could be made to be favored. In Sec. III we then extract and concentrate on those groups that can be Lorentz groups. The main content of both these sections are actually the tables listing the results of the quantities proposed to be maximized for the relevant groups. In Sec. IV we resume and conclude that actually we may be on the track to have found a common reason or explanation for the gauge group of the standard model and for why we have $3+1$ dimensions.

In the appendix we have put a review of previous attempt to argue for why we have just 4 dimensions.

II. OUR PREVIOUS NUMBERS

In the previous work by D. Bennett and myself [1] we essentially collected the ratios (related to the Dynkin index [13]) C_A/C_F, where we, for the representation of the group in question G, selected that representation F, which would give the largest value for this ratio C_A/C_F. (In the table we give in a few cases two proposals for F, but really it is what one would loosely call the smallest faithful representation). We shall keep in mind that this ratio is only well defined for the simple Lie groups; and it is thus only for the simple groups we could make a clean table as the one just below. For semisimple Lie groups it is strictly speaking needed to specify a replacement quantity, that can be the needed generalization to semisimple Lie groups. One shall naturally construct a logarithmic average [see formula (40) or (16) for what we mean by a "logarithmic average"] weighted with the dimensions of the various simple group factors contained in the semisimple Lie group written as a product of its simple invariant subgroups. Extending the generalization even to the inclusion of $U(1)$ factors in the Lie group gets even a bit more arbitrary, but we did choose the rule of counting the $U(1)$ factors as if they had C_A/C_F equal to unity [which was the case at first but then we argued for a correction factor derived from the way the elements in the center of the group are identified (by division out a subgroup), and the combined result of these rules became equivalent to the introduction of formally taking $C_A/C_F \to e_A^2/e_F^2$ as described below in item IV in Sec. II A). The problem with the $U(1)$'s, the Abelian groups, is that the adjoint quadratic Casimir C_A is just zero and does not provide a good normalization. Although we have to declare formally C_A/C_F to be unity at first for $U(1)$, we take the opportunity to—and we think it is very natural—to include a correction depending not only on the Lie algebra but also on the group structure in a way roughly describing that a $U(1)$ representation with a small "charge" is "smaller" than one with a larger charge in a very similar way to the way in which a small quadratic Casimir signals a "small" representation.

Here we give our (essentially Dynkin index) ratios for the simple Lie groups:

Our ratio of adjoint to "simplest" (or smallest) quadratic Casimirs C_A/C_F

$$\frac{C_A}{C_F}\bigg|_{A_n} = \frac{2(n+1)^2}{n(n+2)} = \frac{2(n+1)^2}{(n+1)^2-1} = \frac{2}{1-\frac{1}{(n+1)^2}}, \tag{2}$$

$$\frac{C_A}{C_{F\,\text{vector}}}\bigg|_{B_n} = \frac{2n-1}{n} = 2-\frac{1}{n}, \tag{3}$$

$$\frac{C_A}{C_{F\,\text{spinor}}}\bigg|_{B_n} = \frac{2n-1}{\frac{2n^2+n}{8}} = \frac{16n-8}{n(2n+1)}, \tag{4}$$

$$\frac{C_A}{C_F}\bigg|_{C_n} = \frac{n+1}{n/2+1/4} = \frac{4(n+1)}{2n+1}, \tag{5}$$

$$\frac{C_A}{C_{F\,\text{vector}}}\bigg|_{D_n} = \frac{2(n-1)}{n-1/2} = \frac{4(n-1)}{2n-1}, \tag{6}$$

$$\frac{C_A}{C_{F\,\text{spinor}}}\bigg|_{D_n} = \frac{2(n-1)}{\frac{2n^2-n}{8}} = \frac{16(n-1)}{n(2n-1)}, \tag{7}$$

$$\frac{C_A}{C_F}\bigg|_{G_2} = \frac{4}{2} = 2, \tag{8}$$

$$\frac{C_A}{C_F}\bigg|_{F_4} = \frac{9}{6} = \frac{3}{2}, \tag{9}$$

$$\frac{C_A}{C_F}\bigg|_{E_6} = \frac{12}{\frac{26}{3}} = \frac{18}{13}, \tag{10}$$

$$\frac{C_A}{C_F}\bigg|_{E_7} = \frac{18}{\frac{57}{4}} = \frac{72}{57} = \frac{24}{19}, \tag{11}$$

$$\frac{C_A}{C_F}\bigg|_{E_8} = \frac{30}{30} = 1. \tag{12}$$

For the calculation of Eqs. (2)–(12) see [14,15].

In Eqs. (2)–(12) we have of course used the conventional notation for the classification of Lie algebras, wherein the index n on the capital letter denotes the rank of the Lie algebra, and:

(i) A_n is $SU(n+1)$.
(ii) B_n is the odd dimension orthogonal group Lie algebra for $SO(2n+1)$ or for its covering group Spin(2n + 1).
(iii) C_n are the symplectic Lie algebras.
(iv) D_n is the even dimension orthogonal Lie algebra for $SO(2n)$ or its covering group Spin($2n$),
(v) while F_4, G_2, and E_n for $n = 6, 7, 8$ are the exceptional Lie algebras.

The words *spinor* or *vector* following in the index the letter F which itself denotes the "small" representation, i.e., most promising for giving a small quadratic Casimir C_F—means that we have used for F respectively the spinor and vector representation.

It may be reassuring to check that our goal quantity for the simple groups C_A/C_F becomes the same for the cases of isomorphic Lie algebras such as $B_2 \cong C_2$ and $C_3 \cong A_3$.

A. Development of the gauge group determination proposal

It may be best to describe the proposal for the quantity to be maximized for the gauge group by describing how a phenomenological discussion adjusting small problems can be guided toward the final rule. Let me review the work [1] as a successive discussion of larger and larger classes of groups toward finding a goal quantity that would make the standard model group win the game of making it maximal.

However, before that I want to be a bit concrete and present the typical type of group that we consider a possibility as a gauge group. Indeed we imagine that it can be written as a cross product of Lie groups with at the end some subgroup of the center being divided out.

In [1], we were overall satisfied by considering a Lie group of the form of a cross product of some $U(1)$ groups and some simple Lie groups that were finally modified by dividing out some discrete subgroup of the center. That is to say we have in mind groups of the form

$$G = (U(1) \times U(1) \times \cdots \times U(1) \times SU(2)(\text{say}) \times \cdots \ldots \times G_{\max})/D, \quad (13)$$

where the $\times$-product runs over a number of occurrences of all the possible simple Lie groups as classified by their Dynkin diagrams up to the last one for the group G in question here denoted $G_{\max}$. We imagine using the covering groups for the Lie algebras in question and take only the compact groups. Finally then the group [16] rather than Lie algebra structure is achieved by dividing some discrete subgroup D out of the center of the group achieved by the cross product without modification. This division out of a subgroup from the center only has significance for the group but not for the Lie algebra. The gauge fields in a gauge field theory *a priori* only depend on the Lie algebra, but we, as a very important point in our works, assign a physical significance to even the Lie group by making use of the fact that the group structure restricts the representations that are allowed. Thus one can in a phenomenological way read off the group structure (and thereby what were divided out) by studying the representations occurring as representations for the matter fields. It is, e.g., the empirical charge quantization rule

$$y/2 + I_W + \text{“triality”}/3 = 0(\text{mod } 1), \quad (14)$$

where I_W is the weak isospin and y the hypercharge, and it tells that the electric charge $Q = T_{3W} + y/2$ is integer for particles with zero triality, written as "triality," and becomes $1/3$ modulo unity for triality 1(mod3), while -1/3(mod 1) for "triality" = 2 (mod 3).

The reader should have in mind that the typical covering groups as, e.g., $SU(N) = A_{N-1}$ has often a nontrivial center with a finite number of elements. For example, $SU(N)$ has a center isomorphic to the group Z_N of the integer numbers counted modulo N. The whole center in the cross product thus becomes the cross product of the typically discrete centers for the simple Lie groups crossed further with the $U(1)$'s which each of them are all center (since they are Abelian). For example the center of the cross product $U(1) \times SU(2) \times SU(3)$ that shall be used to produce the standard model group (and which has the standard model Lie algebra) is $U(1) \times Z_2 \times Z_3$. We divide the covering group $\mathbf{R} \times SU(2) \times SU(3)$ by the discrete subgroup D of the center of this group generated by the element [2π, 1(mod2), 1(mod3)]. Hereby one obtains a factor group which only has as representations a subset of the representations of the Lie algebra or the covering group, *but still has the representations realized physically in the Standard Model*.

The following successive proposals are then made for larger and larger subsets of the groups of the type considered, beginning in I with the simple Lie groups, then in II the semisimple, etc.:

(I) The ground idea for the goal quantity is the ratio C_A/C_F in which the symbols C_A and C_F are the quadratic Casimirs for the group in question for, respectively, the adjoint representation A and another representation F, which then in the search for a maximal ratio C_A/C_F will lead to choosing F with minimal C_F. To avoid F being the trivial representation, we shall require F to be faithful.
This simple starting proposal C_A/C_F for defining a "goal quantity" to seek the maximum is really only working for simple non-Abelian Lie groups. (For other Lie groups it will need some improvements to be a good and well-defined quantity)
In fact the reader shall have in mind that
   (1) The ratio C_A/C_F does not suffer from the normalization problem of the generators representing the Lie algebra, because we take the ratio so that scaling the convention for the Lie algebra basis, if changed, will change the numerator C_A and the denominator C_F by the same factor.
   (2) However, at first even this ratio C_A/C_F is only well defined for a simple Lie group. In the case of even a still semisimple Lie group there is, namely, an ambiguity in the normalization of the basis vectors for the Lie algebra of one of the simple components relative to another simple component. So just dividing two Casimirs is not sufficient to make a normalizaton convention independent ratio, as it were in the simple Lie algebra case.
   (3) If we do not specify the normalizations of the basis vectors and thereby their representations, then we get a notation dependent quantity for the (quadratic) Casimir operators and thus the quadratic Casimirs.
   (4) If we have $U(1)$ as factors in the $\times$-product, then we have the obvious trouble with our first proposal C_A/C_F that the adjoint representation of $U(1)$ is trivial, or rather the Abelian Lie group $U(1)$ has no (meaningful) adjoint representation and thus C_A becomes meaningless for $U(1)$.

(II) Next let us improve the first proposal C_A/C_F for the goal quantity by generalizing it in a good way to the semi-simple Lie groups.
The problem which we first have to solve in extending in a meaningful way the proposed quantity is to ignore at first the $U(1)$ groups and restrict ourselves to semisimple groups to find some way of defining a quantity like one for the simple Lie group or the algebra of well-defined quantity C_A/C_F.
Since for a simple group the ratio C_A/C_F is well defined, the obvious idea to make an analogous expression for a semisimple one, which is just a $\times$-product of several simple Lie groups $S_1 \times S_2 \times \cdots S_n$, is taking some sort of average over the

separate simple groups S_i of the quantities $C_A/C_F|_{S_i}$ for the various simple groups. The proposal that we thought was reasonable was to average logarithmically [see (40) or (16) to get an idea what ''logarithmic averaging'' means] and weighting with the dimension of the Lie groups. That means we proposed to use as the average that should replace the C_A/C_F for the simple group in the semisimple case

$$\text{“}C_A/C_F \text{ replacement for } S_1 \times S_2 \times \cdots \times S_n\text{”} \tag{15}$$

$$= (C_A/C_F|_{S_1})^{\frac{\dim(S_1)}{\sum_i \dim(S_i)}} * (C_A/C_F|_{S_2})^{\frac{\dim(S_2)}{\sum_i \dim(S_i)}} * \cdots * (C_A/C_F|_{S_n})^{\frac{\dim(S_n)}{\sum_i \dim(S_i)}}. \tag{16}$$

It is of course *a priori* an *ad hoc* choice to weight just with the dimensions $\dim(S_i)$ of the simple groups S_i, but is an extremely natural choice. However, it is the type of choice we could revise if we should look for some little adjustment of our proposal to make it more successful.

You may consider the quadratic Casimir as representing a metric tensor describing distances in the representation space for coordinates originating from the Lie algebra or Lie group. If one would think of the volume [of dimension $\dim(G)$ of course] for a faithful representation, F, relative to the adjoint representation volume, it would be $(C_F/C_A)^{\dim(G)/2}$. So you could see our quantity “C_A/C_F replacement for $S_1 \times S_2 \times \cdots \times S_n$” as being the $\dim(S_1 \times S_2 \times \cdots \times S_n)/2$th root of the volume ratio of the adjoint representation A, i.e., Vol(A) relative to that of the faithful representation F, i.e., Vol(F),

$$\text{“}C_A/C_F \text{ replacement for } S_1 \times S_2 \times \cdots \times S_n\text{”} = \left(\frac{\mathrm{Vol}(A)}{\mathrm{Vol}(F)}\right)^{\frac{1}{\dim(G)}}. \tag{17}$$

This simple and nice interpretation supports aesthetically the use of the dimensionality of the various simple Lie groups being used to weight the logarithmic average. We can, instead of talking at first about the quadratic Casimirs, say that we talk about the volume ratio of the adjoint representation and the representation F from the start. The ratio of the volumes of two representations in the natural metric defined above in (1) is a very simple and beautiful quantity. We then take the root of (half) the dimension of the group to make it depend on the structure of the various simple subgroups rather than on the total number of them or their dimension in a too strong way. By using this root choice (17) we obtain the good feature that you cannot obtain the large quantity just by taking a group with a high dimension by taking, for instance, a cross product of a lot of groups. For single simple groups we also achieve that our root quantity becomes just the C_A/C_F, from which we started. And you can even see from the series of Eqs. (2)–(12) above that in the limit of the rank going to infinity, the various series of infinitely many simple Lie groups have our quantity go nicely to 2.[1] In this way we get a very balanced quantity, favoring at first neither large nor small dimensions for the group dramatically.

Hereby we think we have proposed a very nice and beautiful quantity for the semisimple groups.

(III) Next we have the problem with groups having $U(1)$ factors:

For the $U(1)$ our C_A/C_F hardly makes any sense, and so we have to invent a replacement essentially arbitrarily. In order to do that let us keep in mind that whenever our quantity C_A/C_F makes sense, it is for trivial reasons always bigger than unity. We could namely always as a special possibility for the faithful representation F use the adjoint representation itself, in which case the ratio $C_A/C_F = C_A/C_A = 1$. So since we shall choose the representation F so as to maximize the ratio C_A/C_F, it must always be larger than or equal to this possibility value 1.

When we invent a value for the replacement of the C_A/C_F for the Abelian group $U(1)$, we must at least choose the value larger than or equal to 1 in order not to violate the trivial lower bound.

Since all representations of an Abelian group are one-dimensional, there is with respect to dimension only one representation and thus only one choice for F. Therefore the first proposal is to replace C_A/C_F by the for trivial reasons minimal value 1. However, truly an Abelian group $U(1)$ has a series of different representations given by a ''charge'' e.

In other words we propose to choose:

$$\text{“}C_A/C_F \text{ replacement for } U(1)\text{”} = 1. \tag{18}$$

It is then the obvious generalization to the groups being cross products of $U(1)$'s with a semisimple group that we shall average this 1 logarithmically

[1] According to our rule, one shall choose the representation F to be the faithful representation making C_A/C_F maximal for the given group. Whether to choose the spinor or vector possibility for F according to this rule will shift for the Spin(N) $\approx SO(N)$ groups meaning the B_n and D_n at Spin(8) $\approx SO(8)$ for which the spinor and the vector representations are isomorphic. For the asymptotic case of large ranks n shall one as F use the vector possibility, and that gives the limit 2.

with the C_A/C_F's for the simple groups weighting as above with the dimension of the Lie groups.

This means that we have now come to the proposal:

$$\text{“}C_A/C_F \text{ replacement for } U(1)\times\cdots \times U(1)\times S_1\times\cdots\times S_n\text{”} \tag{19}$$

$$= 1^{\frac{1}{\#U(1)'s+\sum_i \dim(S_i)}} * \cdots * 1^{\frac{1}{\#U(1)'s+\sum_i \dim(S_i)}} \tag{20}$$

$$* (C_A/C_F|_{S_1})^{\frac{\dim(S_1)}{\#U(1)'s+\sum_i \dim(S_i)}} * \cdots * (C_A/C_F|_{S_n})^{\frac{\dim(S_n)}{\#U(1)'s+\sum_i S_i}} \tag{21}$$

$$= (C_A/C_F|_{S_1})^{\frac{\dim(S_1)}{\#U(1)'s+\sum_i \dim(S_i)}} * \cdots * (C_A/C_F|_{S_n})^{\frac{\dim(S_n)}{\#U(1)'s+\sum_i S_i}}. \tag{22}$$

Here $\#U(1)'s$ means the number of $U(1)$ factors in the $\times$-product forming the group G under study/evaluation with respect to the goal quantity. Of course the full dimension of the Lie group G is just

$$\dim(G) = \#U(1)'s + \sum_i \dim(S_i). \tag{23}$$

At this stage it will not with respect to maximizing the goal quantity "C_A/C_F replacement for $U(1)\times\cdots\times U(1)\times S_1\times\cdots\times S_n$" pay to have any $U(1)$'s at all. So we have at this stage in developing the goal quantity no chance of making a group that, like the standard model has an invariant Abelian subgroup $U(1)$, has any chance of winning the game of maximization.

(IV) Improvement for Abelian and the group structure: In order to make it at all pay for the maximization to have at least some factor $U(1)$ like the standard model happens to have, we must be open for the possibility of letting a $U(1)$ invariant subgroup contribute more than just the absolute minimum 1 as a replacement for its meaningless C_A/C_F. But the irreducible representations of the $U(1)$ are all just one-dimensional representations with the single element in the unitary matrix being just a phase factor $\exp(ie\delta)$, where δ is the phase describing the element in $U(1)$ and e is a "charge" for the representation in question. It is well known that the various representations of $U(1)$ are characterized by such "charges" e. The quadratic Casimirs are given as the square of the "charges" $C_R \to e_R^2$, where e_R is the "charge" for the representation R. Now let us keep in mind that for our purpose of studying gauge groups, we mainly have in mind the charges for various particles, and that when we give a physical meaning to the gauge group rather than just to the gauge Lie algebra we do that on the basis of phenomenology suggested restrictions on the representations/particles occurring in the model. We can then ask the questions: What is the lowest nonzero charge e_A say on a particle in accordance with the restrictions from the group structure, when this particle has no non-Abelian transformations (i.e., when it transforms trivially, i.e., not at all under the other factors in the group)? We can also ask what is the absolutely (numerically) smallest "charge" e_F say on any particle allowed under the group rule (whatever its couplings to the non-Abelian Lie groups might be)? The indexes suggested here were chosen to form a "replacement" for the C_A/C_F for a $U(1)$ being instead of the one first proposed, now improved to e_A^2/e_F^2. We can only obtain this ratio e_A^2/e_F^2 to be different from unity by having a gauge group obtained by dividing out a discrete subgroup of the center of the starting pure cross product. Actually this choice is very reasonable in as far as the e_F charge is the smallest possible nontrivial charge quite analogous to ours in the non-Abelian case of the representation F being the smallest faithful one. So the only *ad hoc* choice is to replace the adjoint representation for the non-Abelian case by the smallest representation of the $U(1)$ that does *not* mix up with non-Abelian groups. We think this is pretty much the simplest reasonable replacement.

Choosing this procedure we get to the final proposal:

$$\text{“}C_A/C_F \text{ replacement for } U(1)^{(1)}\times\cdots\times U(1)^{(m)} \times S_1\times\cdots\times S_n\text{”} = \tag{24}$$

$$= \left(\frac{e_A^{(1)2}}{e_F^{(1)2}} * \cdots \frac{e_A^{(m)2}}{e_F^{(m)2}} * \frac{C_A}{C_F}\bigg|_{S_1}^{\dim(S_1)} * \cdots * \frac{C_A}{C_F}\bigg|_{S_n}^{\dim(S_n)}\right)^{\frac{1}{\#U(1)'s+\sum_i \dim(S_i)}}. \tag{25}$$

Here of course $m = \#U(1)'s$ and the index in the round brackets on the charges enumerates the various $U(1)$ factors in the cross product. It should be kept in mind that the nontrivial (i.e., not just 1) ratios $\frac{e_A^{(i)2}}{e_F^{(i)2}}$ only come into play when a discrete subgroup of the center of the cross product has been divided out. Remember that I in reviewing the work [2] told that this division out of a discrete subgroup of the center were in the "skewness" estimation [2] an important ingredient in reducing the symmetry and thus got favored by asking for "skewness." So letting this outdivision be favored in the game thus favors the standard model group, which has relatively much such outdivision. This is how we introduced in this last step a favoring of the "division out," although we started out with a

C_A/C_F ratio which was purely dependent on the Lie algebra.

B. Standard model group wins

Using Eqs. (2)–(12) inserting it into Eq. (25) we may now contemplate which group should win the "game" of obtaining the largest value for the goal quantity (25).

First we see from Eq. (12) that the groups favored even in the system that were made balanced in such a way that the ratio C_A/C_F goes to a constant—actually 2—for very large ranks $r \to \infty$ are the small rank ones: $SU(2) = A_1$ is the winner among the simple group with its $C_A/C_F = \frac{2}{1-\frac{1}{(n+1)^2}}$ for A_n which for $n = 1$ gives 8/3.

The next among the simplest Lie groups is $SO(5) = B_2 = C_2$ with $C_A/C_F = 12/5$, which is obtained by using the spinor representation for B_2 giving $\frac{16n-8}{n(2n+1)}|_{n=2} = \frac{12}{5}$, while the vector representation gives the less competitive $C_A/C_F|_{B_2\text{ vector}} = 3/5$, and we can check that the isomorphic C_2 also gives as it should 12/5. First at the third place we find the Lie group $SU(3) = A_2$ with its $C_A/C_F|_{A_2} = 9/4$ the group we would hope to win over the $SO(5)$, because it is $SU(3)$ and not $SO(5)$ which occurs in the standard model.

But now in the competition between the $SO(5)$ and the $SU(3)$ comes in some help for $SU(3)$ in our final proposal:

(i) *Dimension of $SU(3)$ is lower than that of $SO(5)$.* Comparing the semisimple groups formed by crossing $SU(2)$ with respectively $SO(5)$ and $SU(3)$ we obtain the C_A/C_F ratios when weighted according to (16) to be

$$\text{“}\frac{C_A}{C_F}\text{ replacement”}|_{SU(3)\times SO(5)} = \left(\frac{8}{3}\right)^{3/13} * \left(\frac{12}{5}\right)^{10/13} = 2.459068704 \quad (26)$$

$$\text{“}\frac{C_A}{C_F}\text{ replacement”}|_{SU(3)\times SU(3)} = \left(\frac{8}{3}\right)^{3/11} * \left(\frac{9}{4}\right)^{8/11} = 2.356709384 \quad (27)$$

The $(\frac{12}{5}/\frac{9}{4} - 1) * 100\% = 6.6666667\%$ higher value for $SO(5)$ over $SU(3)$ is by the logarithmic dimensional weighting reduced to $(\frac{2.459068704}{2.356709384} - 1) * 100\% = 4.343315332\%$.

(ii) *Involving a $U(1)$ and the division out of a central subgroup.*
According to the details of the definition of our "goal quantity" when involving $U(1)$ cross product factors, we have the possibility of obtaining e_A^2/e_F^2 to the power of the inverse of the dimension from (25). As by our definitional choice above, the difference between the charge e_A and e_F is that e_A should be represented with only the $U(1)$ charge but trivial under the non-Abelian groups, while e_F can be chosen for any faithful representation, the ratio e_A/e_F can only be bigger than unity by involving a rule for allowed representations of the group (rather than just the Lie algebra). That is to say we need to involve the center of one of the covering groups of the non-Abelian Lie groups. We can then obtain, e.g., for $SU(3)$—which has a center isomorphic to the group of integers modulo 3, i.e., Z_3—a division out of a Z_3 and get a ratio of 3 for e_A over e_F if we wish. The reader should check that using our formula (25) for the goal quantity and imagining various groups obtained by various division-outs of the center, we can only divide out say Z_2 once with only one $U(1)$ cross product factor if we want to get the effect of this division out for the ratios e_A/e_F obtainable, whereas we can manage to get both Z_2 and Z_3 to give rise to effective factors of the type $(e_A/e_F)^2$, even with only one $U(1)$. With just one $U(1)$ factor we can thus gain a factor $(e_A^2/e_F^2)^{1/\dim(G)}$ which gives $(1/3^2)^{1/\dim(G)}$ for $SU(3)$ and $(1/2^2)^{1/\dim(G)}$ for $SU(2)$ in the goal quantity (where $\dim(G)$ is the dimension of the full group). However, once we have already gotten such a gain from $SU(2)$ we cannot gain one more from $SO(5)$ unless we incorporate yet another $U(1)$. In this way $SU(3)$ gets favored not only by having a Z_3 isomorphic center compared to the only Z_2 isomorphic center of the $SO(5)$ covering group Spin(5), meaning a $3^2 = 9$ factor compared to the $2^2 = 4$ only for $SO(5)$, but the $SO(5)$ cannot get its Z_2 in play without one more $U(1)$. So in reality now $SU(3)$ gets in front by a factor 9 (before one takes the $\dim(G)$'th root.).

Let us now compare the two groups obtained from the semisimple ones in (27) by cross multiplying them with a $U(1)$ and successively dividing appropriately a discrete group out of the center:
We calculate the following goal quantities

$$\text{“}C_A/C_F\text{ replacement for }(SU(2)\times SU(3)\times U(1))/Z_6\text{”} = \left(6^2 * \left(\frac{8}{3}\right)^3 * \left(\frac{9}{4}\right)^8\right)^{\frac{1}{12}} \quad (28)$$

$$= 2.957824511 \quad (29)$$

$$\text{“}C_A/C_F\text{ replacement for }(SU(2)\times \mathrm{Spin}(5)\times U(1))/Z_2\text{“} = \left(2^2 * \left(\frac{8}{3}\right)^3 * \left(\frac{12}{5}\right)^{10}\right)^{\frac{1}{14}} \quad (30)$$

$$= 2.54602555 \quad (31)$$

Here Spin(5) just stands for the covering group of $SO(5)$, the numbers for the three involved simple Lie groups are the C_A/C_F ratios, respectively 8/3, 9/4, and 12/5 for $SU(2)$, $SU(3)$, and Spin(5) [or just think $SO(5)$]. The numbers 36 and 4 come from the "charge" ratio and are essentially the squares of the number of elements in the divided out subgroup in the cases here. The final root taking is of course because of the averaging with the dimension of the full group—respectively 12 and 14—to finally be divided out of the logarithm.

It might be nice to have in mind what the significance of, e.g., the factor 3 in the "charge ratio" e_A/e_F due to the $SU(3)$ contributes, namely, a factor 9 before the 12th root is taken. Indeed $9^{\frac{1}{12}} = 1.200936955$. This means that obtaining the charge ratio due to the $SU(3)$ rather than there being no factor with $SO(5)$ [$\approx$ Spin(5)]—not even a new Z_2 to divide out when we already have done so using $SU(2)$—we gain 20% in the goal quantity. The only 4.343315332% advantage of the semi-simple SO(5) over the SU(3) when combined to $SU(2) \times$ Spin(5) and $SU(2) \times SU(3)$, respectively, is thus rather easily overshadowed by the effect of the e_A/e_F from the SU(3), which is of the order of 20% in the goal quantity.

After the inclusion of the Abelian charge type of ratio we found that the final advantage of the standard model group $S(U(2)\times U(3)) = (U(1)\times SU(2)\times SU(3))/Z_6$ compared to the group with the $SU(3)$ replaced by the competing Spin(5) $\approx SO(5)$, namely $U(1) \times SU(2) \times \text{Spin}(5)/Z_2$ is $(\frac{2.957824511}{2.54602555} - 1) * 100\% = 16.174188\%$

C. Some property of our goal quantity

We must have in mind the property of our "goal quantity" due to its logarithmic averaging that taking a repeated cross product of whatever group with itself necessarily leads to groups with the same goal quantity as the one multiplied up. Thus if the standard model group wins, then at the same time any number of crossings of the standard model with itself will stand even and share first place with the standard model alone.

We above essentially had the discussion that lead to the standard model winning except that we did not sufficiently carefully compare groups with different numbers of simple group factors. For instance, the obvious and very serious competitor to the standard model is simply $U(2) = U(1) \times SU(2)/Z_2$, which obtains the goal quantity

$$\text{“}C_A/C_F \text{ replacement for } U(2)\text{“} = \left(2^2 * \left(\frac{8}{3}\right)^3\right)^{\frac{1}{4}} = 2.951151786. \tag{32}$$

This is truly an exceedingly close run to the standard model, but the standard model wins over even $U(2)$ on the fourth cipher. Indeed the advantage of the standard model group over the so closely competing $U(2)$ (which would physically be that there were no strong interactions, but only the Weinberg Salam Glashow model) is by $(\frac{2.957824511}{2.951151786} - 1) * 100\% = .2261058\%$ The contribution from the Abelian invariant subgroup $U(1)$, namely the "charge ratio," is so important that we might look for the winning group by first taking that into account. We may therefore look for possibilities for the group with simple group factors with one combination of center groups at a time. For example, we could among the simple group combinations with one having Z_2 and one having Z_3, say that the $SU(2) = A_1$ will be best to use among the ones with Z_2, while $SU(3) = A_2$ will be "best" among the ones with Z_3.

We namely notice that for the same center of a Lie group with a simple Lie algebra different such simple Lie algebras will play the same role with respect to the division out of the center and the charges for the Abelian group(s).

One may see rather easily that involving the more complicated center groups in the simple Lie algebras shall hardly pay.

If you seek, as would be best, a Z_k center with k being prime with respect to the other k values, say 2 and 3, we get up to $k = 5$ and $SU(5)$ already has the high dimension 24 and would largely reduced away almost the effect of even a factor 5^2; in fact $(5^2)^{1/24} = 1.143529836$.

Thus we may look at the series and expect that the winner must be there:

$$\text{“}C_A/C_F \text{ replacement for } U(1)\text{”} = (1)^1 = 1, \tag{33}$$

$$\text{“}C_A/C_F \text{ replacement for } (U(1) \times SU(2))/Z_2\text{”} = (2^2 * (8/3)^3)^{\frac{1}{4}} \tag{34}$$

$$= 2.951151786, \tag{35}$$

$$\text{“}C_A/C_F \text{ replacement for } (U(1) \times SU(2) \times SU(3))/Z_6\text{”} = (6^2 * (8/3)^3 * (9/4)^8)^{\frac{1}{12}} \tag{36}$$

$$= 2.957824511, \tag{37}$$

$$\text{“}C_A/C_F \text{ replacement for } (U(1) \times SU(2) \times SU(3) \times SU(5))/Z_{30}\text{“} = \tag{38}$$

$$(6^2 * (8/3)^3 * (9/4)^8 * (25/12)^{24})^{\frac{1}{36}} = 2.341513375. \tag{39}$$

We see that in this series of the most promising candidates with given centers of the covering groups for the simple Lie algebras the standard model lies at the (flat) maximum.

The reader can check in detail and get help by studying our earlier work [1], and see that indeed the standard model wins our game with its value 2.957824511, sharply followed by the group $U(2)$, which achieves 2.951151786 (for its silver medal).

We think it is remarkable that such a relatively simple proposal for a goal quantity as our slightly *ad hoc* extended essentially Dynkin index C_A/C_F, the precise definition of which is largely determined from requirements of not depending too much on the notation choice, leads to just the gauge group that Nature has chosen. One should think that there is truly something in it. By this statement that it is largely fixed by independence of notation, we mean that it had to be a ratio of quadratic Casimirs, if it shall be given by quadratic Casimirs at all; otherwise it would depend on the normalization of the quadratic Casimir, which would make it much more complicated to define. Our just called "*ad hoc*" extension to the inclusion of $U(1)$ cross product factors is really very analogous to the C_A/C_F, so indeed it is not such a seriously arbitrary choice.

We even have a speculative physical mechanism behind it, which might be later replaced by some other version.

III. COMPETITION AMONG LORENTZ GROUPS ON C_A/C_F AND THE LIKE

The main new point of the present article is to present why we have just 3 + 1 (or say just $d = 4$) space-time dimensions. This explanation is that treating the Lorentz or better Poincare group as "the gauge group for, say, general relativity" and using the small *ad hoc* procedures to be suggested in subsection III A, the experimentally realized $d = 4$ gets singled out as having the largest "goal quantity" for the "gauge group." Here this "goal quantity" is taken to be the same one as the one that singled out—see Sec. II or Bennett and me [1]—the standard model group by requiring it to be maximal.

A. Development of goal quantities

We shall, however, slightly develop the goal quantity used above for getting the standard model gauge group singled out, because we have to (i) choose which group should be considered the "gauge group" relevant for general relativity on which to apply our previous game, (ii) the Poincare group which is the best suggestion is for our purpose slightly unpleasant because it does not have nice compact representations of finite dimension like, e.g., the standard model group had.

Indeed we seek to get a statement that the experimental number of dimensions just maximizes some quantity, that is a relatively simple function of the group structure of say the Lorentz group, and which we then call a "goal quantity."

Let me therefore list some of the first approximation simplified proposals which we suggest for this goal quantity. But this is for the dimension a two step procedure: (i) we first use the proposals in our previous article [1] to give a number—a goal quantity—for any Lie group. (ii) we have to specify on which group we shall take and use the procedure of previous work; shall it be the Lorentz group, its covering group or somehow an attempt with the Poincare group? Here are four successive proposals:

(i) Just take the Lorentz group and calculate for that the Dynkin index [13] or rather the quantity which we already used as goal quantity in the previous article [1] C_A/C_F. This gets especially simple for the (except for dimension $d = 2$ or smaller) semisimple Lorentz groups (simple in the mathematical sense of not having any invariant nontrivial subgroup; semisimple: no Abelian invariant subgroup); since the Lorentz group shall "have simple Lie algebra" to apply the Dynkin index related ratio C_A/C_F without further specifications, though the global structure of the Lorentz group is not fixed until we assign it a meaning we really have in mind. For simple groups we can ignore the minor corrections invented for the improvement in the case of an Abelian component present in the potential gauge group.

(ii) We supplement in a somewhat *ad hoc* way the above point, i.e., C_A/C_F by taking its $\frac{d+1}{d-1}$th power. The idea behind this proposal is that we think of the Poincare group instead of as under (i) only on the Lorentz group part, though still in a crude way. This means we think of a group, which is the Poincare group, except that for simplicity we ignore that the translation generators do not commute with the Lorentz group part. Then we assign in accordance with the *ad hoc* rule used in [1] the Abelian sub-Lie-algebra a formal replacement 1 for the ratio of the quadratic Casimirs C_A/C_f: i.e., we put "$C_A/C_F|$"$_{\text{Abelean formal}} = 1$. Next we construct an "average" averaged in a logarithmic way (meaning that we average the logarithms and then exponentiate again) weighted with the dimension of the Lie groups over all the dimensions of the Poincare Lie group. Since the dimension of the Lorentz group for d dimensional space-time is $\frac{d(d-1)}{2}$ while the Poincare group has dimension $\frac{d(d-1)}{2} + d = \frac{d(d+1)}{2}$ the logarithmic averaging means that we get

$$\exp\left(\frac{\frac{d(d-1)}{2}\ln(C_A/C_F)|_{\text{Lorentz}} + \ln(1) * d}{d(d+1)/2}\right) = (C_A/C_F)|_{\text{Lorentz}}^{\frac{d(d-1)}{2}/\frac{d(d+1)}{2}} = (C_A/C_F)|_{\text{Lorentz}}^{\frac{d-1}{d+1}} \quad (40)$$

That is to say we shall make a certain *ad hoc* partial inclusion of the Abelian dimensions in the Poincare groups.

To be concrete, we here propose to say crudely: Let the Poincare group have of course d "Abelian" generators or dimensions. Let the dimension of the Lorentz group be $d_{\text{Lor}} = d(d-1)/2$; then the total dimension of the Poincare group is $d_{\mathrm{Poi}} = d + d_{\mathrm{Lor}} = d(d+1)/2$. If we crudely followed the idea of weighting proposed in the previous article [1] as if the d "Abelian" generators were just simple cross product factors—and not as they really are: not quite usual by not commuting with the Lorentz generators—then since we formally are from this previous article suggested to use the as if number 1 for the Abelian groups, we should use the quantity

$$(C_A/C_F)|_{\mathrm{Lor}}^{\frac{d_{\mathrm{Lor}}}{d_{\mathrm{Poi}}}} = (C_A/C_F)|_{\mathrm{Lor}}^{\frac{d-1}{d+1}} \tag{41}$$

as the goal quantity.

(iii) We could improve the above proposals for goal quantities (i) or (ii) by including into the quadratic Casimir C_A for the adjoint representation the contributions from the translation generating generators, so as to define a quadratic Casimir for the whole Poincare group. This would mean that in order to calculate our goal quantity, we would do as above but

$$\text{Replace}: C_A \rightarrow C_A + C_V, \tag{42}$$

where C_V is the vector representation quadratic Casimir, meaning the representation under which the translation generators transform under the Lorentz group. Since in the below equations we denoted "no fermions" have taken the "small representation" F to be this vector representation V, this replacement means that we replace the goal quantity ratio C_A/C_F like this:

$$\mathbf{(S)O(d)}, \quad \text{“no spinors”}: \tag{43}$$

$$C_A/C_F = C_A/C_V \rightarrow (C_A + C_V)/C_F = C_A/C_F + 1 \tag{44}$$

$$\mathbf{S\,pin(d)}, \quad \text{“with spinors”}: \tag{45}$$

$$C_A/C_F \rightarrow (C_A + C_V)/C_F \tag{46}$$

$$= C_A/C_F + (C_A/C_V)^{-1}(C_A/C_F) \tag{47}$$

$$= (1 + (C_A/C_F)|_{\text{no spinors}}^{-1})C_A/C_F. \tag{48}$$

Let me stress though that this proposal is not quite "fair" in as far as it is based on the Poincare group, while the representations considered are not faithful with respect to the whole Poincare group, but only with respect to the Lorentz group.

(iv) To make the above proposal a bit more "fair" we should at least say: Since we considered a representation which was only faithful with respect to the Lorentz subgroup of the Poincare group we should at least correct the quadratic Casimir—expected crudely to be "proportional" to the number of dimensions of the (Lie)group—by a factor $\frac{d+1}{d-1}$ being the ratio of the dimension of the Poincare (Lie)group, $d + d(d-1)/2$ to that of actually faithfully represented Lorentz group $d(d-1)/2$. That is to say we should, before forming the ratio of the improved C_A meaning $C_A + C_V$ (as calculated under (iii) to C_F, replace this C_F by $\frac{d+1}{d-1} * C_F$, i.e., we perform the replacement:

$$C_F \rightarrow C_F * \frac{d(d-1)/2 + d}{d(d-2)/2} = C_F * \frac{d+1}{d-1}. \tag{49}$$

Inserted into $(C_A + C_V)/C_F$ from (iii), we obtain for the more "fair" approximate "goal quantity"

$$\text{“goal quantity”}|_{\text{no spinor}} = (C_A/C_F + 1) * \frac{d-1}{d+1} \tag{50}$$

$$\text{“goal quantity”}|_{\text{w. spinor}} = (1 + (C_A/C_F)|_{\text{no spinor}}^{-1}) * C_A/C_F * \frac{d-1}{d+1} \tag{51}$$

This proposal (iv) should then at least be crudely balanced with respect to how many dimensions that are represented faithfully.

B. Calculation of goal quantities

Let us now begin listing the values of these "goal quantities" for the Lorentz groups for the various numbers d which the dimension of space-time might take on.

In the first table we give the "goal quantity" and the in order to go crudely towards the Poincare group "goal quantity."

C. Discussion of Table I

Motivated either by the fact that we have spinor transforming particles in nature—namely the fermions—or because the goal numbers for the spinor groups are the biggest anyway (most competitive), we should think of the Lorentz group as the spinor group and therefore in the above table read the Spin(d) entrances rather than the $(S)O(d)$-entrances:

Concentrating on the Spin(d)-entrances we then find that with the proposal (i) of Sec. III A the dimensions d = 3 and d = 4 stand even with the same goal number 8/3 = 2.6667. But note that at least the experimental dimension 4 already is in the sample of the "winners" with the simple choice of (i), meaning that we only consider the genuine Lorentz group while totally ignoring the Abelian part of the Poincare group.

Next when we go to the slightly more complicated version of a goal quantity, namely (ii), we get the separation between also d = 3 and d = 4, and it is the d = 4

TABLE I. We evaluate for different dimensions d of the Minkowski space-time—for simplicity here replaced by the Euclideanized d dimensional space-time, but that makes no difference for our calculation here—the first two goal quantities proposed, (i) and (ii) in Sec. III A written in, respectively, 5th and 7th columns. Because of the ambiguity of the global structure of the Lorentz group, the group in d dimension may be either O(d) [essentially SO(d) if we do not include parity] or Spinor(d) if we use the covering group. Therefore, we have for each value of the dimension d two items corresponding to these two global extensions of the Lie algebra of the Lorentz group, and they are denoted by "no spinors" and "with spinors," respectively.

Dimension	Lorentz group	Spinor or not	Rank	Ratio C_A/C_F max a)	$\frac{d_{\rm Lor}}{d_{\rm Poi}}$	$(C_A/C_F)^{\frac{d_{\rm Lor}}{d_{\rm Poi}}}$ max b)
d = 1	discrete		0	⋯	0	indefinite 0^0
d = 2[a]	(S)O(2) = U(1)	no spinor	1	-(formally 1)	1/3	-(formally 1)
	U(1)	with spinor	1	-(formally 2)	1/3	-(formally $2^{1/3} = 1.26$)
d = 3	(S)O(3)	no spinor	1	1	1/2	1
	Spin(3) = SU(2)	with spinor	1	8/3 = 2.6667	1/2	$\sqrt{8/3} = 1.632993162$
d = 4	(S)O(4)	no spinor	2	4/3 = 1.3333	3/5	$(4/3)^{3/5} = 1.188401639$
	Spin(4) = $SU(2) \times SU(2)$	with spin	2	8/3 = 2.6667	3/5	$(8/3)^{3/5} = 1.801280051$
d = 5	(S)O(5)	no spinor	2	3/2 = 1.5	2/3	$(3/2)^{2/3} = 1.310370697$
	Spin(5)	with spinor	2	12/5 = 2.4	2/3	$(12/5)^{2/3} = 1.792561899$
d = 6	(S)O(6)	no spinor	3	8/5 = 1.6	5/7	$(8/5)^{5/7} = 1.398942897$
	Spin(6) = SU(4)	with spinor	3	32/15 = 2.1333	5/7	$(32/15)^{5/7} = 1.718074304$
d = 7	(S)O(7)	no spinor	3	13/7 = 1.8571	3/4	$(13/7)^{3/4} = 1.590867407$
	Spin(7)	with spinor	3	40/21 = 1.9048	3/4	$(40/21)^{3/4} = 1.621363987$
d = 8	(S)O(8)	no spinor	4	12/7 = 1.7143	7/9	$(12/7)^{7/9} = 1.520774129$
	Spin(8)	with spinor	4	12/7 = 1.7143	7/9	$(12/7)^{7/9} = 1.520774129$
d = 9	(S)O(9)	no spinor	4	7/4 = 1.75	4/5	$(7/4)^{4/5} = 1.564697681$
	Spin(9)	with spinor	4	14/9 = 1.5556	4/5	$(14/9)^{4/5} = 1.423994858$
d = 10	(S)O(10)	no spinor	5	16/9 = 1.7778	9/11	$(16/9)^{9/11} = 1.601198613$
	Spin(10)	with spinor	5	64/45 = 1.4222	9/11	$(64/45)^{9/11} = 1.33399805$
d = 11	(S)O(11)	no spinor	5	9/5 = 1.8	5/6	$(9/5)^{5/6} = 1.632026054$
	Spin(11)	with spinor	5	72/55 = 1.3091	5/6	$(72/55)^{5/6} = 1.251626758$
d = 12	(S)O(12)	no spinor	6	44/23 = 1.9130	11/13	$(44/23)^{11/13} = 1.731340775$
	Spin(12)	with spinor	6	40/33 = 1.2121	11/13	$(40/33)^{11/13} = 1.176773318$
d = 13	(S)O(13)	no spinor	6	25/13 = 1.9231	6/7	$(25/13)^{6/7} 1.75156277$
	Spin(13)	with spinor	6	44/39 = 1.1282	6/7	$(44/39)^{6/7} = 1.108929813$
d = 14	(S)O(14)	no spinor	7	24/13 = 1.8461	13/15	$(24/13)^{13/15} = 1.701239682$
	Spin(14)	with spinor	7	104/105 = 0.9905	13/15	$(104/105)^{13/15} = 0.991740772$
d odd	(S)O(d)	no spinor	n = (d − 1)/2	2 − 1/n = 2 − 2(d − 1)	$\frac{d-1}{d+1}$	$(2 - \frac{1}{d-1})^{\frac{d-1}{d+1}}$
	Spin(d)	with spinor	n = (d − 1)/2	$\frac{8(2n-1)}{n(2n+1)} = \frac{16(d-2)}{d(d-1)}$	$\frac{d-1}{d+1}$	$(\frac{16(d-2)}{d(d-1)})^{\frac{d-1}{d+1}}$
d even	(S)O(d)	no spin	n = d/2	$\frac{4(n-1)}{2n-1} = \frac{2(d-2)}{d-1}$	$\frac{d-1}{d+1}$	$(\frac{2(d-1)}{d-1})^{\frac{d-1}{d+1}}$
	Spin(d)	with spinor	n = d/2	$\frac{16(d-2)}{d(d-1)}$	$\frac{d-1}{d+1}$	$(\frac{16(d-2)}{d(d-1)})^{\frac{d-1}{d+1}}$
d → ∞		no spinor	c → ∞	→ 2	→ 1	→ 2
		with spinor	→ ∞	→ 0	→ 1	→ 0

[a]The case d = 2 is special because the Lorentz group is Abelian $U(1)$ for d = 2, and we must apply the formal extension definition $C_A/C_F = 1$ for $U(1)$ from our previous work [1], and even include an extra factor connected with dividing out a subgroup of the center, or even better say that the formal quadratic Casimir shall behave like the charge squared for the $U(1)$.

dimension that "wins," because we get for d = 3 only 1.6330, while we for d = 4 we obtain 1.8013. Thus in this approximate treatment of the Abelian part also being included the "little" difference between the two schemes (i) and (ii) leads to giving the d = 4 case—the experimental case—the little push forward making the experimental dimension d = 4 be the only winner.

We see from the Table II, for simplicity made only for the most competitive case of "with spinor" in the terminology the foregoing table, that with column (iii) goal numbers actually it is d = 3 rather than the experimental dimension d = 4 that "wins." That is to say that the number in the fifth column–called the (iii)-quantity or max c)–in the Table II takes the largest value for the

TABLE II. We have put the goal numbers for the third proposal (iii) in which I (in a bit more detail) seek to make an analogon to the number used in Ref. [1] in which we studied the gauge group of the standard model. The purpose of (iii) is to approximate using the Poincare group a bit more detailed, but still not by making a true representation of the Poincare group. That is, it is still not truly the Poincare group that we represent faithfully, but only the Lorentz group, or here in the table only the covering group Spin(d) of the Lorentz group. However, I include in the column marked "c., max c)" in the quadratic Casimir C_A of the Lorentz group an extra term coming from the structure constants describing the noncommutativity of the Lorentz group generators with the translation generators C_V so as to replace C_A in the starting expression of ours C_A/C_F by $C_A + C_V$. In the column marked "d., max d)" we correct the ratio to be more "fair" by counting at least that because of the truly faithfully represented part of the Poincare group in the representations that I use has only dimension $d(d-1)/2$ (it is namely only the Lorentz group). The full Poincare group—which was already used (iii) but also in (iv) with the improved C_A being $C_A + C_V$—is $d(d-1)/2 + d = d(d+1)/2$. The correction is crudely made by the dimension ratio dim (Lorentz)/ dim (Poincare) $= (d-1)/(d+1)$ given in the next to last column.

Dimension	Lorentz group (covering group)	Ratio C_A/C_F for spinor	Ratio C_A/C_V as "no spinor"	(iii)-quantity max c)	$\frac{d-1}{d+1}$	(iv)-quantity max d)
2[a]	U(1)	-(formally 2)	-(formally 1)	4	$1/3$	$4/3 = 1.33$
3	spin(3)	$\frac{8}{3} = 2.6667$	1	$\frac{16}{3} = 5.3333$	$\frac{2}{4} = .5$	$\frac{8}{3} = 2.6667$
4	$\mathrm{Spin}(4) = SU(2) \times SU(2)$	$\frac{8}{3} = 2.6667$	$\frac{4}{3}$	$\frac{14}{3} = 4.6667$	$\frac{3}{5} = .6$	$\frac{14}{5} = 2.8$
5	Spin(5)	$\frac{12}{5} = 2.4$	$\frac{3}{2} = 1.5$	4	$\frac{4}{6} = .667$	$\frac{8}{3} = 2.6667$
6	Spin(6)	$\frac{32}{15}$	$\frac{8}{5} = 1.6$	$\frac{52}{15} = 3.4667$	$\frac{5}{7} = .714$	$\frac{52}{21} = 2.4762$
d odd	Spin(d)	$\frac{8(2n-1)}{n(2n+1)} = \frac{16(d-2)}{d(d-1)}$	$2 - 1/n = 2 - 2/(d-1)$	$\frac{8(3d-5)}{d(d-1)}$	$\frac{d-1}{d+1}$	$\frac{8(3d-5)}{d(d+1)}$
d even	Spin(d)	$\frac{16(d-2)}{d(d-1)}$	$\frac{4(n-1)}{2n-1} = \frac{2d-4}{d-1}$	$\frac{8(3d-5)}{d(d-1)}$	$\frac{d-1}{d+1}$	$\frac{8(3d-5)}{d(d+1)}$
d odd $\to \infty$	Spin(d)	$\approx 16/d$	$\to 2$	$\approx 24/d$	$\to 1$	$\approx 24/d \to 0$
d even $\to \infty$	Spin(d)	$\approx 16/d$	$\to 2$	$\approx 24/d$	$\to 1$	$\approx 24/d \to 0$

[a]The case d = 2 is only getting its C_A/C_F rather formally by seeking to roughly use the rules of our previous article [1] because the Lorentz group is Abelian.

dimension being d = 3. That would have meant that if these numbers were used our prediction would have been that *space-time* should have three dimension and thus not agreed with experiment. However, this series of numbers (iii) or c is not truly "fair" in as far as one has effectively used only the Lorentz group in the denominator C_F but at least crudely the full Poincare group in the numerator $C_A + C_V$. Thus in order to avoid a simple wrong expected variation of a quadratic Casimir with the dimensionality of the Lie group, we should at least correct the denominator C_F by multiplying it by the ratio of the dimension of the Poincare Lie group over that of the Lorentz Lie group, $(d+1)/(d-1)$. When we make this "fairness correction" at least crudely getting no obvious wrong Lie-group-dimension-dependent factor in, then the dimension d = 4 becomes (again) the winner. In fact we get for d = 4 (the experimental dimension) the goal quantity in column (iv) equal to $14/5 = 2.8$ while accidentally the two neighboring dimensions d = 3 and d = 5 both get instead $8/3 = 2.66667$, which is less.

Notice that it is a rather smoothly peaked curve with the peak near the experimental dimension 4, so that the latter becomes the winner among integers, but it is only by a tiny bit it wins. That is to be expected from the smoothness of the variation of the goal number with the dimension d. This smallness of the excess making the d = 4 be the winner of course makes the uncertainty bigger and my "crude" corrections rather than exactly calculating some well-defined quantity is thus not so convincing. The accuracy may be good enough and the simplicity of the proposed goal quantities sufficient to make it at least highly suggestive, that the coincidence of the winning dimension and the experimental one means that we are on the right track.

IV. CONCLUSION

We have found that a couple of very reasonable specifications of what the extension of our previous [1] quantity should be to be maximized to obtain the standard model gauge group leads to that the maximization of the generalized quantity gives as the "winner" the dimension d = 4 as is empirically the dimension. That is to say we have found a possible explanation for why we have just 4 (meaning 3 + 1) dimensions of space-time.

In fact we have extended the main idea of claiming the maximization of essentially the (to the Dynkin index related) group dependent quantity C_A/C_F (with C_A and C_F being the quadratic Casimirs for, respectively, the adjoint representation C_A and for that (essentially) faithful representation F chosen so as to maximize the ratio C_A/C_F.) to lead to the experimentally realized group (the standard model group). It should be admitted that the victory of exactly the standard model group was dependent on our slightly *ad hoc* treatment of the Abelian invariant subgroup—i.e., the $U(1)$—needed because of the ratio C_A/C_F being rather meaningless *a priori* for an Abelian group. We must also admit that in Ref. [1] we snuck in a dependence on the group rather than only Lie algebra by considering the volume of the group which depends on the identification of center elements, properties being revealed phenomenologically via the representations of the gauge group (except for the case of $d = 2$ this detail is though not relevant for the Lorentz group and thus the dimension). In the review II we saw that the standard model group came out as the group with the highest goal quantity. This should be considered a very remarkable victory for our type of scheme because there are a lot of groups which *a priori* could have been the gauge group relevant for nature.

By extension, we consider the Lorentz group or even the Poincare group instead of the gauge group of our previous work Ref. [1] or Sec. II. Then of course our quantity C_A/C_F or slight modifications/"improvements" of it—enumerated (a), (b), (c), (d)—will depend on the dimension d of space- time. The dimension d gives of course a different Lorentz group for each value of d. We then inserted this d-dependent Lorentz group instead of the gauge group which was studied in last paper [1]. The various modifications, (a), (b), (c), (d), shall be considered attempts to use the Poincare group instead of the Lorentz group, but rather than truly doing that, make some approximate treatment as if crudely using the Poincare group.

The results of the search for the dimension having the largest "goal-quantity," using various proposals for the exact form such as (i) meaning C_F/C_F simply, are the following:

(a) The simple quantity C_A/C_F for the Lorentz group with the same formal assignment for the Abelian group as used in [1], here making $d = 2$ noncompetitive(but at least having a score formally), leads to $d = 3$ and $d = 4$ standing even, both scoring the same number $C_A/C_F = 8/3$.

(b) Making a crude correction to consider instead the quantity $(C_A/C_F)^{(d-1)/(d+1)}$ leads to the experimental dimension of space-time $d = 4$ getting the largest score. The meaning of this slight modification of (a) is that we make an attempt to take the group to replace the gauge group in our previous paper [1] to be the Poincare group rather than the Lorentz group. We, however, only make a crude attempt in that direction. Since the Poincare group has the translation subgroups, which are by themselves Abelian, we naturally tend to use the formal version—just like in Ref. [1]—to assign a factor 1 to the Abelian groups. Then we average in the logarithm our goal numbers for the various factors into which the group falls weighting with the dimension in the Lie algebra. The inclusion of the Poincare group is not done in a fully correct way though in as far as we only consider the faithful representations of the Lorentz group and only extend a bit speculatively to weight as if we had the Poincare group.

(c) Still thinking of crudely using the Poincare group rather than the Lorentz group, we proposed to still take a representation F only of the Lorentz group, but evaluating the quadratic Casimir for the Poincare group, although that sounds not quite "fair." The quadratic Casimir we used here under (c) for "the Poincare group" were taken to $C_A + C_V$, where V denotes the vector representation and thus C_V its quadratic Casimir. In this "unfair" game the dimension for space-time $d = 3 = 2 + 1$ got the highest score. So our hoped for victory of the experimental dimension failed in this "unfair" proposal. But since I stress the "unfairness" of this proposal, we should not take this proposal seriously.

(d) This last proposal in the present article is a crude attempt to at least correct for the fact that the ratio of the dimensions of the Poincare and the Lorentz Lie groups is space-time dimension d dependent. That is to say, we argue that the quadratic Casimir C_F for the representation F of the Lorentz group should at least be scaled so as to correspond to a representation of the Poincare group by being multiplied by the ratio of the Lie group dimensions of the Poincare group relative to that of the Lorentz group, $\frac{d(d-1)/2+d}{d(d-1)/2} = \frac{d+1}{d-1}$. That is to say we perform the crude correction of replacing

$$C_F \to C_F * \frac{d+1}{d-1}. \tag{52}$$

Since the quantity C_F occurs in the denominator of the quantity $(C_F + C_V)/C_F$ maximized under (c) of course this quantity is scaled the opposite way, and the goal quantity in this proposal (d) is taken as

$$\text{“goal quantity } d.\text{”} = \frac{(C_A + C_V)(d-1)}{C_F(d+1)}. \tag{53}$$

Now the result becomes that the experimental dimension $d = 4$ has the largest value for the goal quantity d., in as far as it gets

$$\frac{(C_A + C_V)(d-1)}{C_F(d+1)}\bigg|_{d=4} = \frac{14}{5} = 2.8, \tag{54}$$

while by accident the two neighboring space-time dimensions 3 and 5 score only $\frac{8}{3} = 2.6667$. So indeed the experimental space-time dimension 4 won the most developed suggestion (d).

This means that apart from the "unfair" proposal (c), all the four proposals here have the space-time dimension $d = 4$ realized in nature obtain a largest "goal quantity" among the winners! In (a) $d = 3$ and $d = 4$ share the winner place, but in the two other "fair" proposals (b) and (d) it is indeed space-time dimension $d = 4$, the experimental one, that gives the highest "goal quantity."

Taking this result seriously, and not as being just accidental coincidence or a result of inventive construction, it must tell us about the reason for that the dimension became just d = 4. We must look for what is the spirit behind the proposals above, so as to obtain an answer to "Why did we get just d = 4 space-time dimensions?" This "spirit" behind these proposals is set up to select the experimental dimension d = 4. It turns out to be that the group—e.g., the Poincare or Lorentz group or the gauge group—should be representable in a way where the matrices or other objects on which the group is represented are relatively slowly varying under the group.

We may, if taking this "slowness" of the motion of the representative in the representation with the group element seriously, seek to invent a model behind the 4-dimensional space-time and the standard model gauge group that could explain that slowness. One possible such explanation could be that the fundamental physics model or theory is truly "random" and that without the symmetries we seek to explain. Then "by accident" there appears approximately some symmetries, and we here hope for some Lorentz invariance symmetry. Now we dream that there may be some way in which such an approximate symmetry can automatically become exact in practice. We, Førster, Ninomiya and me [7] (see also Damgaard *et al.* [12]), have actually argued that gauge symmetry with electrodynamics (and Yang Mills theory) as an example can occur in a whole phase in practice giving precisely the massless photon in that phase. Thus we can speculate (for symmetries that can somehow be considered gauge symmetry, as can the Lorentz symmetry in general relativity) that such symmetries appear in practice as exact provided that they are there approximately at first [17,18]. But now the crux of the matter is that if a symmetry is represented by slowly moving matrices, for example, then one must expect that statistically it would be easier to get the symmetry approximately by accident. If it were such that the fundamental theory could be considered random and only obtaining some symmetries by accident—at first approximately, but perhaps made exact by some mechanism [7,12,17,18]—then we could consider the practically random Lagrange or action as taking random values for regions of some (small) size in the value space for the representation of the group which gives the transformation properties of the fields or degrees of freedom under the group in question. Now when a group is represented by a representation, which in some sense is the represented matrix or field, and these fields or matrices move slowly for an appropriately normalized motion of the group element represented, then one can vary the group element a lot before one varies the representation field much. But this means that one needs less good luck to get an accidental symmetry the slower the representation moves, because the displacement inside the group (itself) corresponding to one of the (small) size regions (over which we assume essential constancy of the action) becomes bigger the slower the representation motion rate.

The crucial point should be that one would, with the in some sense random action, have a better chance to obtain by accident a certain symmetry, when this symmetry is represented on the fields or degrees of freedom by a "slowly moving representation,"so such a symmetry would more often occur by accident, if one thinks this random action way.

So when our various "goal quantities" favor the experimentally found gauge group and the dimension of space-time, it means that the groups realized in Nature are the ones that have the optimal chance to come out of a random action model. This is because these goal quantities that are large means that the representation motion is slow.

So the message from the gauge group and the dimension is that such a random action philosophy is one possible mechanism behind the choice by nature of the gauge groups and dimension.

The idea that there have been a lot of random attempts of groups to be tested off is reminiscent of the idea of a gaugeglass [19], which means the action is random quenched randomly locally, but that the gauge group is given from the start; however, the spirit is similar.

A priori one should speculate about possible other physical machineries that could explain that precisely our type of "goal quantities" should point to a realized gauge group and dimension of space-time; but at first it seems that the random action type of model allowing symmetries of the type with highest goal quantities is a good idea and very likely something like that could be the reason behind Natures choice of the gauge group (of the standard model) and of the dimension.

In any case we have found a surprisingly simple principle—the maximization of our rather closely related "goal quantities"—leading to both the gauge group of the standard model and the dimension of space-time being 4.

Let me stress that the present work and that of Ref. [1]—finding a goal quantity leading to the realized groups—is an attempt to ask in a phenomenological way whether there is some signal in the details of the presently by phenomenology supported theory that successively can give us hint(s) about the more fundamental theory behind the presently working standard model with its gauge group $S(U(2) \times U(3))$ and the seeming dimension 4.

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APPENDIX

1. Why the standard model group

The first question attacked by the present series of articles, namely this and the previous work [1] is "Why did Nature choose just the standard model group $S(U(2) \times U(3))$, it could seemingly have chosen among a lot of Lie groups?" Historically this standard model group has appeared as pieces arising from different types of interactions, one $U(1)$-subgroup came from electromagnetic interactions, built in a complicated way into what as a group stands as $U(2)$ (namely $SU(2) \times U(1)$ Lie-algebra-wise) and contains also the weak nuclear forces. Finally the QCD describing strong interaction connected to the $SU(3)$ part of the standard gauge group were added. The standard model is, so to speak, by phenomenology found piecewise: sub-algebra for sub-algebra. It is first, afterwards, or at least by inclusion of further possible pieces of the gauge group that speculations of grand unification models of various types have appeared.

The specification of some appropriate grand unified theory (GUT) model [20], say as the simplest and most promising $SU(5)$, together with some breaking scheme, makes up an explanation for the standard model gauge Lie algebra and also easily for the gauge group—which we take to be implemented as a restriction on which representations are allowed, so that we can indeed claim a by phenomenology accessible element of knowledge—being the one realized in Nature. Truly a major part of the success of the GUT $SU(5)$ model is that the representations of the $SU(5)$ gauge group are automatically representations of the subgroup of the $SU(5)$ with the Lie algebra of the standard model, and this subgroup is just the $S(U(2) \times U(3))$, so that the GUT $SU(5)$ precisely can explain the same restrictions on the allowed representations as can the Lie group $S(U(2) \times U(3))$. If we can explain this group structure and not only the Lie algebra structure of the standard model group, then we would have less need for grand unification, because it would mean obtaining similar predictions for the representations of say quarks and leptons under the gauge group. It is simply so that $S(U(2) \times U(3)) \subset SU(5)$ while, e.g., the group $U(1) \times SU(2) \times SU(3)$ is not a subgroup of $SU(5)$. Therefore the restrictions on the representation from $SU(5)$ interpreted as containing the standard model after some breaking leads to the restrictions of the subgroup $S(U(2) \times U(3))$. Other grand unified groups that have success typically contain $SU(5)$ as a subgroup and thus can reproduce the same representation restrictions corresponding to the standard model group $S(U(2) \times U(3))$. Especially we can mention the $SO(10)$-group, which comes out of [21]. In principle a different model behind the standard model is the flipped $SU(5)$ [22].

Often super string theories would go through some of these grand unification type models, but one can also construct models going to the standard model without going via the unifying groups.

I think one can say that the possibilities are so many that one should admit that it is only if you somehow have already gotten to know the grand unified group, that there is much predictive power as to what group we get in practice. Otherwise the resulting group could be so many different possible ones that there is not much predictive power in these models.

So only if one has a prediction of the unified group in a model should we consider the standard model group explained by grand unification.

In this respect the unification of spin and charge model of Norma Mankoc Borstnik *et al.* [21] is better by leading to $SO(N)$ groups just from the spirit of it. To get just $SO(10)$ as is needed for getting the standard model the dimension $13 + 1$ should be put in, and so after all the prediction of the standard model does not come quite without seemingly *ad hoc* numbers being put in. We think of the number 14 for the space-time dimension.

The type of explanation for the standard model group, which we were after in the present series of works is rather to seek to characterize the group by properties defined for the abstract Lie group, and then postulate some number-valued group characterization, which we should guess so cleverly that it will specify just the wanted standard model group, to be arranged to be maximal by Nature.

One attempt of this type was our work, Niels Brene *et al.* [2] in which we define a concept of "skewness" in a quantitative way, so that one obtains a number for each Lie group [we never got the idea completely developed to specify how many $U(1)$ invariant subgroups there should be, but I think we essentially get there being just one $U(1)$ factor]. If we assumed (to help the project a bit) that we should only consider the possible gauge groups with just one $U(1)$-factor and construct a measure for the degree of symmetry as the logarithm of the number of outer automorphisms o divided by the rank of the Lie algebra r

$$\text{“symmetry”} = \frac{\ln(o)}{r}, \tag{A1}$$

we get the standard model group to have the minimal value of this quantity "symmetry." In this sense the standard model group is among the most "skew" in the sense of being the least symmetric.

The subject of the present series of papers [1] and the present article is a different approach of the same character as the just mentioned "skewness" characterization of the standard model group. However, our present attempt to characterize the gauge group is by a different quantity from the "symmetry" quantity; it is a new attempt. It must, however, be admitted that there appears one overlapping element in the two different quantities to be extremized: some of the potential outer automorphisms that are to be counted in o can be gotten away with by dividing out of the group a subgroup of the center. Actually such a division out of a subgroup of the center seems to be a rather characteristic property of the standard model group so that it is quite helpful if a quantity to determine the standard model group has a strong dependence on the appearance of an effect of such a division out of a subgroup of the center, so as to favor a gauge group with a lot of division out of the center. This division out is not relevant in the very first version of our present proposal, namely the ratio C_A/C_F of the quadratic Casimir of the adjoint representation A, denoted C_A to the quadratic Casimir of that faithful representation F having the smallest quadratic Casimir C_F. However, when we began to help the construction of our quantity by specifying the details concerning the Abelian $U(1)$ components in the Lie group for which our quadratic Casimir ratio is not *a priori* well defined, we managed to—one could almost sa y—"sneak in" the division out, so that we by the details of defining what to do when we have $U(1)$ factors get the groups with a complicated division out of the center get favored to win the game and become the gauge group chosen by nature.

In this way our present proposal for what nature has chosen to maximize and our older proposal of maximizing "skewness" are not completely different because they have an overlap by both favoring the complicate division out of the center. But apart from that they look superficially very different. Thus our present proposal is quite new after all.

I shall review the successive small improvements in finding partly phenomenologically our final suggestion for the quantity or the game that should specify the gauge group to be chosen in Sec. II below.

2. History of explaining 3 + 1 = 4 dimensions

As said previously, the main purpose of the present article is to use the idea from our earlier article [1] to explain why Nature should have chosen just 4 (meaning 3 + 1) space-time dimensions. We explain in this article a new, relative to earlier attempts way to explain why we shall just have four dimensions:

One of the earlier attempts is my own [6] starting the idea of "random dynamics" by pointing out that in a non-Lorentz invariant theory—being a quantum field theory in which neither rotational nor boost invariance is present, but only translational invariance—one finds generically that assuming an appropriate Fermi surface an effective Lorentz invariance with 3 + 1 dimensions appears automatically In this sense I claimed to derive under very general assumptions—as almost unavoidable—the appearance of both Lorentz and thereby rotational invariance and of just the right number of dimensions, $4 = 3 + 1$. The success of this dimension-post-diction [6] was for me the introduction to a long series of works seeking to derive from almost nothing or from a random theory—not obeying many of the usual principles—which I gave the name "random dynamics" [6–10], many of the known physical laws. Really we may consider the present work as an alternative attempt to derive the dimension of space-time, much in a random dynamics way, in as far as we ended up suggesting the philosophy that it would be most likely to get just the experimental dimension by accident. (If successful then of course the present work would be a second derivation of 3 + 1 dimensions in random dynamics).

Also Max Tegmark has derived 3 + 1 dimensions from a similar random dynamics-like philosophy of "all mathematics being realized"[3]. Max Tegmark considers the differential equations for the time development of fields so as to guarantee equations with predictivity, as well as the stability of atoms. From his arguments, his figure shows that the field equation would be elliptic and thus unpredictable for d = 1, too simple for d = 2 and d = 3, and unstable meaning unstable atoms say for d = 5 or more. The latter point goes back to Ehrenfest in 1917 [4], who argued that neither atoms nor planetary systems could be stable in more than four space-time dimensions.

Also known is the story that in, say, two spatial dimensions (corresponding to $3 = 2 + 1$ space-time dimensions) an animal, such as ourselves, having an intestinal channel would fall apart into two pieces. Thus by an anthropic principle, $3 = 2 + 1$ should not be possible if we are to exist.

According to a review of anthropic questions by Gordon Kane [5]:

"One aspect of our universe we want to understand is the fact that we live in three space dimensions. There is an anthropic explanation. It was realized about a century ago [4] that planetary orbits are not stable in four or more space dimensions, so planets would not orbit a sun long enough for life to originate. For the same reason atoms are not stable in four or more space dimensions. And in two or one space dimensions there can be neither blood flow nor large numbers of neuron connections. Thus interesting life can only exist in three dimensions. Alternatively, it may be that we can derive the fact that we live in three dimensions, because the unique ground state of the relevant string theory turns out to have three large dimensions (plus perhaps some small ones we are not normally aware of). Or string theory may have many states with three space dimensions, and all of them may give universes that contain life."

Further one has considered the renormalizability of quantum field theories not being possible for higher than 4 dimensions, except for the scalar ϕ^3 coupling theory, which is not good [23].

In theories, which like string theories or Norma Mankoc Borstnik's model [21] are Kaluza-Klein-like, the question of understanding the effective dimension for long distances being 3 space plus 1 time dimension would *a priori* mean an understanding of why precisely there is that number of extra dimensions being somehow "compactified" that just three space dimensions survive as essentially flat and extended. In super-string theory the consistency requires fundamentally $9 + 1$ dimensions of space-time. If one takes it that it is needed that the compact space described by the extra dimensions appearing as dimensions must be a Calabi Yau space, then since the latter has 6 dimensions the observed or flat dimensions must make up $(9 + 1) - (6 + 0) = 3 + 1$. So the combination of the superstring with the requirement of using Calabi-Yau compactification do indeed explain why we have just the experimental number of dimensions [24,25].

In [26] you find:

"Now to make contact with our 4-dimensional world we need to compactify the 10-dimensional superstring theory on a 6-dimensional compact manifold. Needless to say, the Kaluza Klein picture described above becomes a bit more complicated. One way could simply be to put the extra 6 dimensions on 6 circles, which is just a 6-dimensional Torus. As it turns out this would preserve too much supersymmetry. It is believed that some supersymmetry exists in our 4-dimensional world at an energy scale above 1 TeV (this is the focus of much of the current and future research at the highest energy accelerators around the world!). To preserve the minimal amount of supersymmetry, N = 1 in 4 dimensions, we need to compactify on a special kind of 6-manifold called a Calabi-Yau manifold."

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