Spectral properties of the Wilson-Dirac operator and random matrix theory

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Random matrix theory has been successfully applied to lattice quantum chromodynamics. In particular, a great deal of progress has been made on the understanding, numerically as well as analytically, of the spectral properties of the Wilson-Dirac operator. In this paper, we study the infrared spectrum of the Wilson-Dirac operator via random matrix theory including the three leading order a^2 correction terms that appear in the corresponding chiral Lagrangian. A derivation of the joint probability density of the eigenvalues is presented. This result is used to calculate the density of the complex eigenvalues, the density of the real eigenvalues, and the distribution of the chiralities over the real eigenvalues. A detailed discussion of these quantities shows how each low-energy constant affects the spectrum. Especially we consider the limit of small and large (which is almost the mean field limit) lattice spacing. Comparisons with Monte Carlo simulations of the random matrix theory show a perfect agreement with the analytical predictions. Furthermore we present some quantities which can be easily used for comparison of lattice data and the analytical results.

I. INTRODUCTION

The drastically increasing computational power as well as algorithmic improvements over the last decades provide us with deep insights in nonperturbative effects of quantum chromodynamics (QCD). However, the artefacts of the discretization, i.e. a finite lattice spacing, are not yet completely under control. In particular, in the past few years a large numerical [1–7] and analytical [8–14] effort was undertaken to determine the low-energy constants of the terms in the chiral Lagrangian that describe the discretization errors. It is well known that new phase structures arise such as the Aoki phase [15] and the Sharpe-Singleton scenario [16]. A direct analytical understanding of lattice QCD seems to be out of reach. Fortunately, as was already realized two decades ago, the low lying spectrum of the continuum QCD Dirac operator can be described in terms of random matrix theories (RMTs) [17,18].

Recently, RMTs were formulated to describe discretization effects for staggered [19] as well as Wilson [9,10] fermions. Although these RMTs are more complicated than the chiral random matrix theory formulated in [17,18], in the case of Wilson fermions a complete analytical solution of the RMT has been achieved [9–11,13,20–23]. Since the Wilson RMT shares the global symmetries of the Wilson-Dirac operator it will be equivalent to the corresponding (partially quenched) chiral Lagrangian in the microscopic domain (also known as the ϵ domain) [24–29].

Quite recently, there has been a breakthrough in deriving eigenvalue statistics of the infrared spectrum of the Hermitian [20] as well as the non-Hermitian [21–23] Wilson-Dirac operator. These results explain [13] why the Sharpe-Singleton scenario is only observed for the case of dynamical fermions [1,5,30–36] and not in the quenched theory [37,38] while the Aoki phase has been seen in both cases. First comparisons of the analytical predictions with lattice data show a promising agreement [4,6,7]. Good fits of the low-energy constants are expected for the distributions of individual eigenvalues [9,10,39].

Up to now, mostly the effects of W_8 [10,20,23], and quite recently also of W_6 [12,13,40], on the Dirac spectrum were studied in detail. In this article, we will discuss the effect of all three low-energy constants. Thereby we start from the Wilson RMT for the non-Hermitian Wilson-Dirac operator proposed in Ref. [9]. In Sec. II we recall this random matrix theory and its properties. Furthermore we derive the joint probability density of the eigenvalues which so far was only stated without proof in Refs. [13,22]. We also discuss the approach to the continuum limit in terms of the Dirac spectrum.

In Sec. III, we derive the level densities of D_W starting from the joint probability density. Note that due to its γ_5 Hermiticity D_W has complex eigenvalues as well as exactly real eigenvalues. Moreover, the real modes split into those corresponding to eigenvectors with positive and negative chirality. In Sec. IV, we discuss the spectrum of the quenched non-Hermitian Wilson-Dirac operator in the microscopic limit in detail. In particular the asymptotics at small and large lattice spacing is studied. The latter limit is equal to a mean field limit for some quantities which can be trivially read off.

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In Sec. V we summarize our results. In particular we present easily measurable quantities which can be used for fitting the three low-energy constants $W_{6/7/8}$ and the chiral condensate Σ. Detailed derivations are given in several appendices. The joint probability density is derived in Appendix A. Some useful integral identities are given in Appendix B and in Appendix C we perform the microscopic limit of the graded partition function that enters in the distribution of the chiralities over the real eigenvalues of D_W. Finally, some asymptotic results are derived in Appendix D.

II. WILSON RANDOM MATRIX THEORY AND ITS JOINT PROBABILITY DENSITY

In Sec. II A we introduce the random matrix theory for the infrared spectrum of the Wilson-Dirac operator and recall its most important properties. Its joint probability density is given in Sec. II B, and the continuum limit is derived in Sec. II C.

A. The random matrix ensemble

We consider the random matrix ensemble [9,10]

$$D_\mathrm{W} = \begin{pmatrix} A & W \\ -W^\dagger & B \end{pmatrix} \qquad (1)$$

distributed by the probability density

$$
\begin{aligned}
P(D_\mathrm{W}) = {} & \left(\frac{n}{2\pi a^2}\right)^{[n^2+(n+\nu)^2]/2}\left(-\frac{n}{2\pi}\right)^{n(n+\nu)} \\
& \times \exp\left[-\frac{a^2}{2}\left(\mu_\mathrm{r}^2 + \frac{n+\nu}{n}\mu_\mathrm{l}^2\right)\right] \\
& \times \exp\left[-\frac{n}{2a^2}(\mathrm{tr}A^2 + \mathrm{tr}B^2) - n\,\mathrm{tr}WW^\dagger\right] \\
& \times \exp\left[\mu_\mathrm{r}\,\mathrm{tr}A + \mu_\mathrm{l}\,\mathrm{tr}B\right]. \qquad (2)
\end{aligned}
$$

The Hermitian matrices A and B break chiral symmetry and their dimensions are $n \times n$ and $(n+\nu) \times (n+\nu)$, respectively, where ν is the index of the Dirac operator. Both μ_r and μ_l are one-dimensional real variables. The chiral RMT describing continuum QCD [17] is given by the ensemble (1) with A and B replaced by zero. The N_f flavor RMT partition function is defined by

$$Z^\nu_{N_\mathrm{f}}(m) = \int D[D_\mathrm{W}]P(D_\mathrm{W})\det^{N_\mathrm{f}}(D_\mathrm{W} + m). \qquad (3)$$

Without loss of generality we can assume $\nu \geq 0$ since the results are symmetric under $\nu \to -\nu$ together with $\mu_\mathrm{r} \leftrightarrow \mu_\mathrm{l}$.

The Gaussian integrals over the two variables μ_r and μ_l yield the two low-energy constants W_6 and W_7 [9,10]. The reason is that the integrated probability density

$$
\begin{aligned}
P(D_\mathrm{W}, W_{6/7} \neq 0) = {} & \int_{-\infty}^{\infty} P(D_\mathrm{W})\exp\left[-\frac{a^2(\mu_\mathrm{r}+\mu_\mathrm{l})^2}{16V|W_6|}\right. \\
& \left. -\frac{a^2(\mu_\mathrm{r}-\mu_\mathrm{l})^2}{16V|W_7|}\right]\frac{a^2 d\mu_\mathrm{r}d\mu_\mathrm{l}}{8\pi V\sqrt{W_6 W_7}} \qquad (4)
\end{aligned}
$$

generates the terms $(\mathrm{tr}A + \mathrm{tr}B)^2$ and $(\mathrm{tr}A - \mathrm{tr}B)^2$ which correspond to the squares of traces in the chiral Lagrangian [24–27]. In the microscopic domain the corresponding partition function for N_f fermionic flavors is then given by

$$
\begin{aligned}
Z^\nu_{N_\mathrm{f}}(\tilde{m}) = {} & \int_{U(N_\mathrm{f})} d\mu(U)\exp\left[\frac{\Sigma V}{2}\,\mathrm{tr}\tilde{m}(U + U^{-1})\right] \\
& \times \exp\left[-\tilde{a}^2 V W_6\mathrm{tr}^2(U + U^{-1})\right] \\
& \times \exp\left[-\tilde{a}^2 V W_7\,\mathrm{tr}^2(U - U^{-1})\right] \\
& \times \exp\left[-\tilde{a}^2 V W_8\,\mathrm{tr}(U^2 + U^{-2})\right]\det{}^\nu U \qquad (5)
\end{aligned}
$$

with the physical quark masses $\tilde{m} = \mathrm{diag}(\tilde{m}_1, \ldots, \tilde{m}_{N_\mathrm{f}})$, the space-time volume V, the physical lattice spacing $\tilde{a}$, and the chiral condensate Σ. The low-energy constant W_8 is generated by the term $\mathrm{tr}A^2 + \mathrm{tr}B^2$ in Eq. (2) and is *a priori* positive. We include the lattice spacing a in the standard deviation of A and B, cf. Eq. (2), out of convenience for deriving the joint probability density. We employ the sign convention of Refs. [9,10] for the low-energy constants.

The microscopic limit ($n \to \infty$) is performed in Sec. III. In this limit the rescaled lattice spacing $\hat{a}_8^2 = na^2/2 = \tilde{a}^2 V W_8$, the rescaled parameters $\hat{m}_6 = a^2(\mu_\mathrm{r} + \mu_\mathrm{l})$ and $\hat{\lambda}_7 = a^2(\mu_\mathrm{r} - \mu_\mathrm{l})$, and the rescaled eigenvalues $\hat{Z} = 2nZ = \mathrm{diag}(2nz_1, \ldots, 2nz_{2n+\nu})$ of D_W are kept fixed for $n \to \infty$. The mass $\hat{m}_6$ and axial mass $\hat{\lambda}_7$ are distributed with respect to Gaussians with variance $8\hat{a}_6^2 = -8\tilde{a}^2 V W_6$ and $8\hat{a}_7^2 = -8\tilde{a}^2 V W_7$, respectively. Note the minus sign in front of $W_{6/7}$. As was shown in Ref. [13] the opposite sign is inconsistent with the symmetries of the Wilson-Dirac operator. The notation is slightly different from what is used in the literature to get rid of the imaginary unit in $\hat{a}_6$ and $\hat{a}_7$.

The joint probability density $p(Z)$ of the eigenvalues $Z = \mathrm{diag}(z_1, \ldots, z_{2n+\nu})$ of D_W can be defined by

$$
\begin{aligned}
I[f] &= \int_{\mathbb{C}^{(2n+\nu)\times(2n+\nu)}} f(D_\mathrm{W})P(D_\mathrm{W})d[D_\mathrm{W}] \\
&= \int_{\mathbb{C}^{(2n+\nu)}} f(Z)p(Z)d[Z], \qquad (6)
\end{aligned}
$$

where f is an arbitrary $U(n, n+\nu)$ invariant function. The random matrix D_W is $\gamma_5 = \mathrm{diag}(\mathbb{1}_n, -\mathbb{1}_{n+\nu})$ Hermitian, i.e.

$$D_\mathrm{W}^\dagger = \gamma_5 D_\mathrm{W}\gamma_5. \qquad (7)$$

Hence, the eigenvalues z come in complex conjugate pairs or are exactly real. The matrix D_W has ν generic real modes and $2(n - l)$ additional real eigenvalues $(0 \leq l \leq n)$. The index l decreases by one when a complex conjugate pair enters the real axis.

B. The joint probability density of D_W

Let D_l be D_W if it can be quasidiagonalized by a noncompact unitary rotation $U \in U(n, n + \nu)$, i.e. $U\gamma_5 U^\dagger = \gamma_5$, to

$$D_l = UZ_lU^{-1} = U\begin{pmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & y_2 & 0 \\ 0 & -y_2 & x_2 & 0 \\ 0 & 0 & 0 & x_3 \end{pmatrix}U^{-1}, \quad (8)$$

where the real diagonal matrices $x_1 = \text{diag}(x_1^{(1)}, \ldots, x_{n-l}^{(1)})$, $x_2 = \text{diag}(x_1^{(2)}, \ldots, x_l^{(2)})$, $y_2 = \text{diag}(y_1^{(2)}, \ldots, y_l^{(2)})$, and $x_3 = \text{diag}(x_1^{(3)}, \ldots, x_{n+\nu-l}^{(3)})$ have the dimension $n - l$, l, l, and $n + \nu - l$, respectively. The matrices x_1 and x_3 comprise all real eigenvalues of D_l corresponding to the right-handed and left-handed modes, respectively. We refer to an eigenvector ψ of D_W as right-handed if the chirality is positive definite, i.e.

$$\langle\psi|\gamma_5|\psi\rangle > 0, \quad (9)$$

and as left-handed if the chirality is negative definite. The eigenvectors corresponding to complex eigenvalues have vanishing chirality. The complex conjugate pairs are $(z_2 = x_2 + \iota y_2, z_2^* = x_2 - \iota y_2)$. Note that it is not possible to diagonalize D_W with a $U(n, n + \nu)$ transformation with complex conjugate eigenvalues. Moreover we emphasize that almost all γ_5-Hermitian matrices can be brought to the form (8) excluding a set of measure zero.

The quasidiagonalization $D_l = UZ_lU^{-1}$ determines U up to a $U^{2n+\nu-l}(1) \times O^l(1, 1)$ transformation while the set of eigenvalues Z_l can be permuted in $l!(n-l)!(n+\nu-l)!2^l$ different ways. The factor 2^l is due to the complex conjugation of each single complex pair. The Jacobian of the transformation to eigenvalues and the coset $\mathbb{G}_l = U(n, n + \nu)/[U^{2n+\nu-l}(1) \times O^l(1, 1)]$ is given by

$$|\Delta_{2n+\nu}(Z_l)|^2, \quad (10)$$

where the Vandermonde determinant is defined as

$$\Delta_{2n+\nu}(Z) = \prod_{1\leq i<j\leq 2n+\nu}(z_i - z_j)$$
$$= (-1)^{n+\nu(\nu-1)/2}\det[z_i^{j-1}]_{1\leq i,j\leq 2n+\nu}. \quad (11)$$

The functional $I[f]$ in Eq. (6) is a sum over $n + 1$ integrations on disjoint sets, i.e.

$$I[f] = \sum_{l=0}^{n}\frac{1}{2^l(n-l)!l!(n+\nu-l)!}$$
$$\times \int_{\mathbb{R}^{\nu+2(n-l)}\times\mathbb{C}^l} f(Z_l)\left[\int_{\mathbb{G}_l} P(UZ_lU^{-1})d\mu_{\mathbb{G}_l}(U)\right]$$
$$\times |\Delta_{2n+\nu}(Z_l)|^2 d[Z_l], \quad (12)$$

where we have normalized the terms with respect to the number of possible permutations of the eigenvalues in Z_l. Thus we have for the joint probability density over all sectors of eigenvalues

$$p(Z)d[Z] = \sum_{l=0}^{n} p_l(Z_l)d[Z_l]$$
$$= \sum_{l=0}^{n}\frac{|\Delta_{2n+\nu}(Z_l)|^2 d[Z_l]}{2^l(n-l)!l!(n+\nu-l)!}$$
$$\times \int_{\mathbb{G}_l} P(UZ_lU^{-1})d\mu_{\mathbb{G}_l}(U). \quad (13)$$

Here $p_l(Z_l)$ is the joint probability density for a fixed number of complex conjugate eigenvalue pairs, namely l. The integration over U is nontrivial and will be worked out in detail in Appendix A.

In a more mathematical language the normalization factor in Eq. (13) can be understood as follows. If the permutation group of N elements is denoted by $S(N)$ while the group describing the reflection $y \to -y$ is $\mathbb{Z}_2$, the factor $2^l(n-l)!l!(n+\nu-l)!$ is the volume of the finite subgroup $S(n-l)\times S(l)\times S(n+\nu-l)\times\mathbb{Z}_2^l$ of $U(n, n+\nu)$ which correctly normalizes each summand. Originally we had to divide $U(n, n+\nu)$ by the set $U^{2n+\nu-l}(1)\times O^l(1, 1)\times S(n-l)\times S(l)\times S(n+\nu-l)\times\mathbb{Z}_2^l$ because it is the maximal subgroup whose image of the adjoint mapping commutes with Z_l. The reasoning is as follows. Let $\Sigma[Z_l] = \{UZ_lU^{-1}|U \in U(n, n+\nu)\}$ be the orbit of Z_l and $\Sigma_c[Z_l] = \{\hat{Z}_l \in \Sigma[Z_l]|[\hat{Z}_l, Z_l]_- = \hat{Z}_lZ_l - Z_l\hat{Z}_l = 0\}$ a subset of this orbit. Then all orderings in each of the three sets of eigenvalues x_1, (z_2, z_2^*), and x_3 as well as the reflections $y_j^{(2)} \to -y_j^{(2)}$ are in $\Sigma_c[Z_l]$. This subset $\Sigma_c[Z_l] \subset \Sigma[Z_l]$ can be represented by the finite group $S(n-l)\times S(l)\times S(n+\nu-l)\times\mathbb{Z}_2^l$. This group is called the Weyl group in group theory. The Lie group $U^{2n+\nu-l}(1)\times O^l(1, 1)$ acts on $\Sigma_c[Z_l]$ as the identity since it commutes with Z_l. The group $U^{2n+\nu-l}(1)$ represents $2n+\nu-l$ complex phases along the diagonal commuting with the set which consists of Z_l with a fixed l. Each noncompact orthogonal group $O(1, 1)$ reflects the invariance of a single complex conjugate eigenvalue pair under a hyperbolic transformation which is equal to a Lorentz-transformation in a $(1 + 1)$-dimensional space-time.

There are two ways to deal with the invariance under $U^{2n+\nu-l}(1)\times O^l(1,1)\times S(n-l)\times S(l)\times S(n+\nu-l)\times\mathbb{Z}_2^l$ in an integral such that we correctly weigh all points. We have either to divide $U(n, n + \nu)$ by the whole subgroup or

we integrate over a larger coset and reweight the measure by the volume of the subgroups not excluded. The ordering enforced by $\mathbf{S}(n-l) \times \mathbf{S}(l) \times \mathbf{S}(n+\nu-l) \times \mathbb{Z}_2^l$ is difficult to handle in calculations. Therefore, we have decided for a reweighting of the integration measure by $1/[(n-l)!(n+\nu-l)!l!2^l]$. However the Lie group $U^{2n+\nu-l}(1) \times O^l(1,1)$, in particular the hyperbolic subgroups, has to be excluded since its volume is infinite.

In this section as well as in Appendix A, we use the non-normalized Haar measures induced by the pseudometric

$$\mathrm{tr}dD_{\mathrm{W}}^2 = \mathrm{tr}dA^2 + \mathrm{tr}dB^2 - 2\,\mathrm{tr}dWdW^\dagger. \quad (14)$$

Therefore the measures for D_{W} and Z_l are

$$d[D_{\mathrm{W}}] = \prod_{j=1}^{n} dA_{jj} \prod_{1 \le i < j \le n} 2d\,\mathrm{Re}A_{ij}d\,\mathrm{Im}A_{ij} \prod_{j=1}^{n+\nu} dB_{jj}$$
$$\times \prod_{1 \le i < j \le n+\nu} 2d\,\mathrm{Re}B_{ij}d\,\mathrm{Im}B_{ij}$$
$$\times \prod_{\substack{1 \le i \le n \\ 1 \le j \le n+\nu}} (-2)d\,\mathrm{Re}W_{ij}d\,\mathrm{Im}W_{ij}, \quad (15)$$

$$d[Z_l] = \prod_{j=1}^{n-l} dx_j^{(1)} \prod_{j=1}^{l} 2\iota dx_j^{(2)}dy_j^{(2)} \prod_{j=1}^{n+\nu-l} dx_j^{(3)}. \quad (16)$$

The Haar measure $d\mu_{\mathbb{G}_l}$ for the coset $\mathbb{G}_l$ is also induced by $d[D_{\mathrm{W}}]$ and results from the pseudometric, i.e.

$$\mathrm{tr}dD_{\mathrm{W}}^2 = \mathrm{tr}dZ_l^2 + \mathrm{tr}[U^{-1}dU, Z_l]_-^2. \quad (17)$$

The reason for this unconventional definition is the non-normalizability of the measure $d\mu_{\mathbb{G}_l}$ because $\mathbb{G}_l$ is noncompact for $l > 0$. Hence the normalization resulting from definition (17) seems to be the most natural one, and it helps in keeping track of the normalizations.

In Appendix A we solve the coset integral (13). The first step is to linearize the quadratic terms in UZ_lU^{-1} by introducing auxiliary Gaussian integrals over additional matrices which is along the idea presented in Ref. [20]. In this way we split the integrand in a part invariant under $U(n, n + \nu)$ and a noninvariant part resulting from an external source. The group integrals appearing in these calculations are reminiscent of the Itsykson-Zuber integral. However they are over noncompact groups and, thus, much more involved than in Ref. [20]. Because of the $U(n) \times U(n + \nu)$ invariance of the probability density of D_{W}, the joint eigenvalue distribution is a symmetric function of n eigenvalues which we label by "r" and $n + \nu$ eigenvalues labeled by "l". The γ_5 Hermiticity imposes reality constraints on the eigenvalues resulting in Dirac delta functions in the joint probability density. Similar to the usual Itzykson-Zuber integral, the symmetric function of the eigenvalues turns out to be particularly simple (see Appendix A)

$$p(Z)d[Z] = c(1+a^2)^{-n(n+\nu-1/2)}a^{-n-\nu^2}\exp\left[-\frac{a^4}{4(1+a^2)}(\mu_{\mathrm{r}}-\mu_{\mathrm{l}})^2\right]\Delta_{2n+\nu}(Z)\det\begin{bmatrix}\{g_2(z_i^{(\mathrm{r})}, z_j^{(\mathrm{l})})dx_i^{(\mathrm{r})}dy_i^{(\mathrm{r})}dx_j^{(\mathrm{l})}dy_j^{(\mathrm{l})}\}_{\substack{1 \le i \le n \\ 1 \le j \le n+\nu}} \\ \{(x_j^{(\mathrm{l})})^{i-1}g_1(x_j^{(\mathrm{l})})\delta(y_j^{(\mathrm{l})})dx_j^{(\mathrm{l})}dy_j^{(\mathrm{l})}\}_{\substack{1 \le i \le \nu \\ 1 \le j \le n+\nu}}\end{bmatrix}. \quad (18)$$

The last ν rows become zero in the continuum limit resulting in ν exact zero modes (see subsection II C). At finite a they can be interpreted as broadened "zero modes." The functions in the determinant are given by

$$g_2(z_1, z_2) = g_r(x_1, x_2)\delta(y_1)\delta(y_2) + g_c(z_1)\delta(x_1 - x_2)\delta(y_1 + y_2), \quad (19)$$

$$g_r(x_1, x_2) = \exp\left[-\frac{n}{4a^2}\left(x_1 + x_2 - \frac{a^2(\mu_{\mathrm{r}} + \mu_{\mathrm{l}})}{n}\right)^2 + \frac{n}{4}(x_1 - x_2)^2\right]$$
$$\times \left[\mathrm{sign}(x_1 - x_2) - \mathrm{erf}\left[\sqrt{\frac{n(1+a^2)}{4a^2}}(x_1 - x_2) - \sqrt{\frac{a^2}{4n(1+a^2)}}(\mu_{\mathrm{r}} - \mu_{\mathrm{l}})\right]\right], \quad (20)$$

$$g_c(z) = -2\iota\mathrm{sign}(y)\exp\left[-\frac{n}{a^2}\left(x - \frac{a^2(\mu_{\mathrm{r}} + \mu_{\mathrm{l}})}{2n}\right)^2 - ny^2\right], \quad (21)$$

$$g_1(x) = \exp\left[-\frac{n}{2a^2}\left(x - \frac{a^2\mu_{\mathrm{l}}}{n}\right)^2\right]. \quad (22)$$

We employ the error function "erf" and the function "sign" which yields the sign of the argument. The constant is equal to

$$\frac{1}{c} = (-1)^{\nu(\nu-1)/2+n(n-1)/2}\left(\frac{16\pi}{n}\right)^{n/2}(2\pi)^{\nu/2}n^{-\nu^2/2-n(n+\nu)}\prod_{j=0}^{n}j!\prod_{j=0}^{n+\nu}j!, \quad (23)$$

and is essentially the volume of the coset $[U(n) \times U(n + \nu)]/[\mathbf{S}(n) \times \mathbf{S}(n + \nu)]$.

The two-point weight g_2 consists of two parts. The first term, g_r, represents a pair of real modes where one eigenvalue corresponds to a right-handed eigenvector and the other one to a left-handed one. The second term, g_c, enforces that a complex eigenvalue comes with its complex conjugate only. The function g_1 is purely Gaussian. As we will see in the next subsection, in the small a limit this will result in a distribution of the former zero modes that is broadened to the Gaussian unitary ensemble (GUE) [4,6,7,10,20,23].

For N_f dynamical quarks with quark mass m_f the joint probability density is simply given by [13]

$$p^{(N_f)}(z) = \prod_{f=1}^{N_f} \prod_{k=1}^{2n+\nu} (z_k + m_f) p(Z). \tag{24}$$

The expansion in g_c yields the joint probability density for a fixed number of complex conjugate pairs,

$$p_l(Z_l)d[Z_l] = \frac{(-1)^{(n-l)l}c(1+a^2)^{-n(n+\nu-1/2)}a^{-n-\nu^2}n!(n+\nu)!}{(n-l)!l!(n+\nu-l)!} \exp\left[-\frac{a^4}{4(1+a^2)}(\mu_r - \mu_1)^2\right]\Delta_{2n+\nu}(Z)$$

$$\times \det \begin{bmatrix} \{g_r(x_i^{(1)}, x_j^{(3)})dx_i^{(1)}dx_j^{(3)}\}_{\substack{1\le i\le n-l \\ 1\le j\le n+\nu-l}} \\ \{(x_j^{(3)})^{i-1}g_1(x_j^{(3)})dx_j^{(3)}\}_{\substack{1\le i\le \nu \\ 1\le j\le n+\nu-l}} \end{bmatrix} \prod_{j=1}^{l} g_c(z_j^{(2)})dx_j^{(2)}dy_j^{(2)}. \tag{25}$$

The factorials in the prefactor are the combinatorial factor which results from the expansion of the determinant in co-factors with l columns and l rows less. Note that they correspond to the coset of finite groups, $[\mathbf{S}(n) \times \mathbf{S}(n + \nu)]/[\mathbf{S}(n - l) \times \mathbf{S}(l) \times \mathbf{S}(n + \nu - l)]$, which naturally occurs when diagonalizing D_W in a fixed sector; see the discussion after Eq. (13).

C. The continuum limit

In this section, we take the continuum limit of the joint probability density p, i.e. $a \rightarrow 0$ at fixed z, μ_r, and μ_1. In this limit the probability density (2) of D_W trivially becomes the one of chiral RMT which is equivalent to

continuum QCD in the ϵ regime [17]. We expect that this is also the case for the joint probability density.

The small a limit of the two-point weight (19) is given by

$$g_2(z_1, z_2) \overset{a\ll 1}{=} -2\iota\,\mathrm{sign}(y_1)\sqrt{\frac{a^2\pi}{n}}\exp[-ny_1^2]\delta(x_1)$$
$$\times \delta(x_2)\delta(y_1 + y_2). \tag{26}$$

The function g_r vanishes due to the error function which cancels with the sign function. The expansion of the determinant (18) yields $(n + \nu)!/\nu!$ terms which are all the same. Thus, we have

$$\lim_{a\to 0} p(Z)d[Z] = c(-1)^{\nu(\nu-1)/2}\frac{(n+\nu)!}{\nu!}\left(-2\iota\sqrt{\frac{\pi}{n}}\right)^n \lim_{a\to 0} a^{-\nu^2}\Delta_{2n+\nu}(\iota y, -\iota y, x)\Delta_\nu(x)\prod_{j=1}^{n}\mathrm{sign}(y_j)$$

$$\times \exp[-ny_j^2]dy_j \prod_{j=1}^{\nu}\exp\left[-\frac{n}{2a^2}x_j^2\right]dx_j. \tag{27}$$

Thereby we have already evaluated the Dirac delta functions. The real part of the complex eigenvalues $z_j^{(r/l)}$, $1 \le j \le n$, and the imaginary part of $z_j^{(l)}$, $n + 1 \le j \le n + \nu$, vanish and they become the variables $\pm\iota y_j$, $1 \le j \le n$, and x_j, $1 \le j \le \nu$, respectively. Note that the random variables x scale with a while y is of order 1. Therefore the distribution of the two sets of eigenvalues factorizes into a product that can be identified as the joint probability density of a $\nu \times \nu$ dimensional GUE on the scale of a and the chiral unitary ensemble on the scale 1,

$$\lim_{a\to 0} p(Z)d[Z] = \frac{1}{(2\pi)^{\nu/2}}\left(\frac{n}{a^2}\right)^{\nu^2/2}\prod_{j=0}^{\nu}\frac{1}{j!}\Delta_\nu^2(x)\prod_{j=1}^{\nu}\exp\left[-\frac{n}{2a^2}x_j^2\right]dx_j$$

$$\times \frac{n^{n^2+\nu n}}{n!}\prod_{j=0}^{n-1}\frac{1}{(j+\nu)!j!}\Delta_n^2(y^2)\prod_{j=1}^{n}2\Theta(y_j)y_j^{2\nu+1}\exp[-ny_j^2]dy_j, \tag{28}$$

where Θ is the Heaviside distribution.

III. FROM THE JOINT PROBABILITY DENSITY TO THE LEVEL DENSITIES

The level density is obtained by integrating the joint probability density (18) over all eigenvalues of D_W except

one. We can choose to exclude an eigenvalue of $z^{(r)}$ or one of the $z^{(l)}$'s. When we exclude $z_1^{(r)}$ we have to expand the determinant (18) with respect to the first row. All resulting terms are the same and consist of a term for which $z_1^{(r)}$ is complex and a term for which $z_1^{(r)}$ is real. We thus have [23]

$$\int p(Z) \prod_{z_j \neq z_1^{(r)}} d[z_j] = \rho_r(x_1^{(r)})\delta(y_1^{(r)}) + \frac{1}{2}\rho_c(z_1^{(r)}). \quad (29)$$

When excluding $z_1^{(l)}$ and expanding the determinant (18) with respect to the first column we notice that the first n terms are the same while the remaining ν terms have to be treated separately. Again the spectral density is the sum of the density of the real modes, which are left-handed in this case, and the density of the complex modes [23]

$$\int p(Z) \prod_{z_j \neq z_1^{(l)}} d[z_j] = \rho_l(x_1^{(l)})\delta(y_1^{(l)}) + \frac{1}{2}\rho_c(z_1^{(l)}). \quad (30)$$

The level densities ρ_r and ρ_l are the densities of the real right- and left-handed modes, respectively. Interestingly the level density of the complex modes appears symmetrically in both equations. The reason is the vanishing chirality of eigenvectors corresponding to the complex eigenvalues.

Let us consider the case when excluding $z_1^{(r)}$. The Vandermonde determinant without a factor $(z_1^{(r)} - z_1^{(l)})\times$ $\prod_{k=2}^{n}(z_1^{(r)} - z_k^{(r)})(z_1^{(l)} - z_k^{(r)})\prod_{j=2}^{n+\nu}(z_1^{(r)} - z_j^{(l)})(z_1^{(l)} - z_j^{(l)})$ and the cofactor from expanding the first row of the determinant is equal to the joint probability density with one pair $(z^{(r)}, z^{(l)})$ less. The $z_1^{(l)}$ integral over this distribution together with the factor $\prod_{k=2}^{n}(z_1^{(r)} - z_k^{(r)})(z_1^{(l)} - z_k^{(r)})\times$ $\prod_{j=2}^{n+\nu}(z_1^{(r)} - z_j^{(l)})(z_1^{(l)} - z_j^{(l)})$ can be identified as the partition function with two additional flavors. We thus find

$$\rho_r(x) \propto \int_{-\infty}^{\infty} g_r(x, x')(x - x')Z_{N_f+2}^{n-1,\nu}(x, x', m_k)dx', \quad (31)$$

$$\rho_c(z) \propto g_c(z)(z - z^*)Z_{N_f+2}^{n-1,\nu}(z, z^*, m_k). \quad (32)$$

The fermionic partition function is given by

$$Z_{N_f+2}^{n-1,\nu}(z_1, z_2, m_k)$$
$$= \int \det(D_W - z_1 \mathbb{1}_{2n+\nu-2}) \det(D_W - z_2 \mathbb{1}_{2n+\nu-2})$$
$$\times \prod_{k=1}^{N_f}(D_W + m_k \mathbb{1}_{2n+\nu-2})P(D_W)d[D_W], \quad (33)$$

where D_W is given as in Eq. (1) only that n is replaced by $n - 1$. In the microscopic limit this is simply a unitary matrix integral which can be easily numerically evaluated. Note that the integral over the variables $\mu_{r/l}$ which introduces the low-energy constants $W_{6/7}$ can already be performed at this step.

Considering the exclusion of $z_1^{(l)}$ we have to expand the determinant in the joint probability density with respect to the first column resulting in a much more complicated expression

$$\rho_1(x) \propto n \int_{\mathbb{C}} d[\tilde{z}](x - \tilde{z})g_2(x, \tilde{z})Z_{N_f=2}^{n-1,\nu}(x, \tilde{z})$$
$$+ \alpha\delta(y)\sum_{p=1}^{\nu}(-1)^{\nu-p}\binom{n+\nu-1}{\nu-p}x^{p-1}g_1(x)$$
$$\times \int_{\mathbb{R}^{\nu-p}}\prod_{j=1}^{\nu-p}dx_j x_j^p g_1(x_j)\Delta_{\nu-p}(x_1, \ldots, x_{\nu-p})$$
$$\times \Delta_{\nu-p+1}(x, x_1, \ldots, x_{\nu-p})$$
$$\times Z_{N_f=\nu-p+1}^{n,p}(x, x_1, \ldots, x_{\nu-p}) \quad (34)$$

with a certain constant α which we will specify in the microscopic limit. The global proportionality constant is up to a factor n the same as the one in Eqs. (31) and (32). Again $g_2(z, z_r)$ is the sum of a term comprising the density of the complex eigenvalues and a term giving the real eigenvalue density. For the complex eigenvalue density we find the same expression as obtained by integration over $z_1^{(l)}$.

For $\nu = 1$, the density of the real eigenvalues simplifies to

$$\rho_1(x)|_{\nu=1} \propto n \int_{-\infty}^{\infty} dx_r(x_r - x)g_r(x_r, x)Z_{N_f=2}^{n-1,1}(x_r, x)$$
$$+ \alpha g_1(x)Z_{N_f=1}^{n,0}(x) \quad (35)$$

since there is no integration in the term proportional to α, cf. Eq. (34). The distribution of chirality over the real modes is the difference

$$\rho_\chi(x) = \rho_1(x) - \rho_r(x), \quad (36)$$

resulting in

$$\rho_\chi|_{\nu=1} \sim \alpha g_1(x)Z_{N_f=1}^{n,0}(x) + n\int_{-\infty}^{\infty} dx'(x' - x)(g_r(x', x)$$
$$+ g_r(x, x'))Z_{N_f=2}^{n-1,1}(x', x), \quad (37)$$

where we used that the two-flavor partition function is symmetric in x and x'. For $\mu_r = \mu_l$, the last two terms cancel resulting in a very simple expression for $\rho_\chi(x)$. Note that the integral over the second term always vanishes such that it does not contribute to the normalization of the distribution of chirality over the real modes

$$\int dx \rho_\chi(x) = \nu \quad (38)$$

which is 1 for $\nu = 1$. For $\nu = 2$ we find

$$\rho_\chi(x)|_{\nu=2} \sim \alpha x g_1(x)Z_{N_f=1}^{n,1}(x) - \alpha(n+1)g_1(x)$$
$$\times \int_{-\infty}^{\infty} dx' x'(x - x')g_1(x')Z_{N_f=2}^{n,0}(x, x')$$
$$+ n\int_{-\infty}^{\infty} dx'(x' - x)(g_r(x', x)$$
$$+ g_r(x, x'))Z_{N_f=2}^{n-1,2}(x', x). \quad (39)$$

In the microscopic limit the two-flavor partition functions can be replaced by a unitary matrix integral which still can be easily numerically evaluated including the integrals over $\hat{m}_6$ and $\hat{\lambda}_7$.

For large values of ν the expression of the distribution of chirality over the real modes obtained from expanding the determinant gets increasingly complicated. However, there is an alternative expression in terms of a supersymmetric partition function [23,41],

$$
\rho_\chi(x) \propto \lim_{\varepsilon \to 0} \mathrm{Im} \frac{\partial}{\partial J} \Big|_{J=0} \int \frac{\det (D_{\mathrm{W}} - (x + J)\mathbb{1}_{2n+\nu})}{\det (D_{\mathrm{W}} - x\mathbb{1}_{2n+\nu} - \iota\varepsilon\gamma_5)}
$$
$$
\times P(D_{\mathrm{W}})d[D_{\mathrm{W}}]. \tag{40}
$$

In the ensuing sections we will use this expression to calculate the microscopic limit of the distribution of chirality over the real modes.

A. Microscopic limit of the eigenvalue densities

The goal of this section is to derive the microscopic limit of ρ_r, ρ_χ, and ρ_c including those terms involving nonzero values of W_6 and W_7 in the chiral Lagrangian. We only give results for the quenched case. It is straightforward to include dynamical quarks but this will be worked out in a forthcoming publication. The result for the distribution of chirality over the real modes with dynamical quarks for $W_6 = W_7 = 0$ was already given in [41], and an explicit expression for the density of the complex eigenvalues in the presence of dynamical quarks and nonzero values W_6, W_7, and W_8 was derived in [13].

The microscopic limit of the spectral densities is obtained from the microscopic limit of the partition functions and the functions appearing in the joint probability density. We remind the reader that the microscopic parameters which are kept fixed for $V \to \infty$, are defined by

$$
\hat{a}_6^2 = -\tilde{a}^2 V W_6, \qquad \hat{a}_7^2 = -\tilde{a}^2 V W_7,
$$
$$
\hat{a}_8^2 = na^2/2 = \tilde{a}^2 V W_8, \qquad \hat{m}_6 = a^2(\mu_r + \mu_l),
$$
$$
\hat{\lambda}_7 = a^2(\mu_r - \mu_l), \qquad \hat{x} = 2nx. \tag{41}
$$

The microscopic limit of the probability density of $\hat{m}_6$ and $\hat{\lambda}_7$ is given by

$$
p(\hat{m}_6, \hat{\lambda}_7) = \frac{1}{16\pi\hat{a}_6\hat{a}_7} \exp\left[-\frac{\hat{m}_6^2}{16\hat{a}_6^2} - \frac{\hat{\lambda}_7^2}{16\hat{a}_7^2} \right], \tag{42}
$$

and the functions that appear in the joint probability density simplify to

$$
\hat{g}_r(\hat{x}, \hat{x}', \hat{m}_6, \hat{\lambda}_7)
$$
$$
= \exp\left[-\frac{(\hat{x} + \hat{x}' - 2\hat{m}_6)^2}{32\hat{a}_8^2} \right]
$$
$$
\times \left[\mathrm{sign}(\hat{x} - \hat{x}') - \mathrm{erf}\left[\frac{(\hat{x} - \hat{x}')/2 - \hat{\lambda}_7}{\sqrt{8}\hat{a}_8} \right] \right], \tag{43}
$$

$$
\hat{g}_c(\hat{z}) = -2\iota\mathrm{sign}(\hat{y}) \exp\left[-\frac{(\hat{x} - \hat{m}_6)^2}{8\hat{a}_8^2} \right], \tag{44}
$$

$$
\hat{g}_1(\hat{x}) = \exp\left[-\frac{(\hat{x} - \hat{m}_6 + \hat{\lambda}_7)^2}{16\hat{a}_8^2} \right]. \tag{45}
$$

The microscopic limit of the spectral densities obtained in Eqs. (31), (32), and (40) is given by

$$
\rho_r(\hat{x}) = \frac{1}{32\sqrt{2\pi}\hat{a}_8} \int_{\mathbb{R}^3} d\hat{m}_6 d\hat{\lambda}_7 d\hat{x}' p(\hat{m}_6, \hat{\lambda}_7)(\hat{x} - \hat{x}')
$$
$$
\times \hat{g}_r(\hat{x}, \hat{x}', \hat{m}_6, \hat{\lambda}_7)Z_{2/0}^\nu(\hat{x} + \hat{m}_6, \hat{x}' + \hat{m}_6, \hat{\lambda}_7, \hat{a}_8),
$$
$$
\tag{46}
$$

$$
\rho_c(\hat{z}) = \frac{\iota\hat{y}}{32\sqrt{2\pi}\hat{a}_8} \int_{\mathbb{R}^2} d\hat{m}_6 d\hat{\lambda}_7 p(\hat{m}_6, \hat{\lambda}_7)\hat{g}_c(\hat{z}, \hat{z}^*, \hat{m}_6)
$$
$$
\times Z_{2/0}^\nu(\hat{z} + \hat{m}_6, \hat{z}^* + \hat{m}_6, \hat{\lambda}_7, \hat{a}_8), \tag{47}
$$

$$
\rho_\chi(\hat{x}) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \mathrm{Im} \int d\hat{m}_6 d\hat{\lambda}_7 p(\hat{m}_6, \hat{\lambda}_7)
$$
$$
\times G_{1/1}(\hat{x} + \hat{m}_6, \hat{\lambda}_7 + \iota\varepsilon, \hat{a}_8). \tag{48}
$$

The resolvent $G_{1/1}$ follows from the graded partition function

$$
G_{1/1}(\hat{x} + \hat{m}_6, \hat{\lambda}_7 + \iota\varepsilon, \hat{a}_8)
$$
$$
= \frac{d}{d\hat{x}'} Z_{1/1}^\nu(\hat{x} + \hat{m}_6, \hat{x}' + \hat{m}_6, \hat{\lambda}_7 + \iota\varepsilon, \hat{a}_8)|_{\hat{x}'=\hat{x}}
$$
$$
= \lim_{n \to \infty} \frac{1}{2n} \int \mathrm{tr} \frac{1}{D_{\mathrm{W}} - 2n\hat{x}\mathbb{1}_{2n+\nu} - \iota\varepsilon\gamma_5} P(D_{\mathrm{W}})d[D_{\mathrm{W}}].
$$
$$
\tag{49}
$$

The two-flavor partition function is up to a constant defined by

$$
Z_{2/0}^\nu(z_1 + m_6, z_2 + m_6, \lambda_7, a)
$$
$$
\propto \int \det (z_1\mathbb{1}_{2n+\nu-2} - D_{\mathrm{W}}) \det (z_2\mathbb{1}_{2n+\nu-2} - D_{\mathrm{W}})
$$
$$
\times P(D_{\mathrm{W}})d[D_{\mathrm{W}}]. \tag{50}
$$

The microscopic limit of the two-flavor partition function follows from the chiral Lagrangian (5). In the diagonal representation of the unitary 2×2 matrix, it can be simplified by means of an Itzykson-Zuber integral and is given by

$$Z_{2/0}^\nu(\hat z_1, \hat z_2, \hat\lambda_7, \hat a_8) = \frac{1}{2\pi^2}\int d\varphi_1 d\varphi_2 \sin^2((\varphi_1 - \varphi_2)/2) e^{\iota\nu(\varphi_1+\varphi_2)}\exp\left[\iota\hat\lambda_7(\sin\varphi_1 + \sin\varphi_2) - 4\hat a_8^2(\cos^2\varphi_1 + \cos^2\varphi_2)\right]$$

$$\times \frac{\exp[\hat z_1\cos\varphi_1 + \hat z_2\cos\varphi_2] - \exp[\hat z_2\cos\varphi_1 + \hat z_1\cos\varphi_2]}{(\cos\varphi_1 - \cos\varphi_2)(\hat z_1 - \hat z_2)}. \tag{51}$$

The normalization is chosen such that we find the well-known result [42],

$$Z_{2/0}^\nu(\hat z_1, \hat z_2, \hat\lambda_7 = 0, \hat a_8 = 0)$$

$$= \frac{\hat z_1 I_{\nu+1}(\hat z_1) I_\nu(\hat z_2) - \hat z_2 I_{\nu+1}(\hat z_2) I_\nu(\hat z_1)}{\hat z_1^2 - \hat z_2^2}, \tag{52}$$

at vanishing lattice spacing, where I_ν is the modified Bessel function of the first kind.

The microscopic limit of the graded partition function follows from the chiral Lagrangian [9] which can be written as an integral over a $(1/1) \times (1/1)$ supermatrix [41]

$$U = \begin{bmatrix} e^\vartheta & \eta^* \\ \eta & e^{\iota\varphi} \end{bmatrix}, \qquad \vartheta \in \mathbb{R}, \qquad \varphi \in [0, 2\pi], \tag{53}$$

with η and η^* two independent Grassmann variables, see Refs. [43–47] for the supersymmetry method in random matrix theory. Let the normalization of the integration over the Grassmann variables be

$$\int \eta^*\eta \, d\eta \, d\eta^* = \frac{1}{2\pi}. \tag{54}$$

Then the graded partition function is

$$Z_{1/1}^\nu(\hat z_1, \hat z_2, \hat\lambda_7 \pm \iota\varepsilon, \hat a_8)$$

$$= \int \frac{\iota d e^{\iota\varphi}}{2\pi} d e^\vartheta d\eta \, d\eta^* \text{Sdet}^\nu U \exp[-\hat a_8^2 \text{Str}(U^2 + U^{-2})]$$

$$\times \exp\left[\pm\frac{\iota}{2}\text{Str}\hat Z(U - U^{-1}) - \left(\varepsilon \pm \frac{\iota\hat\lambda_7}{2}\right)\text{Str}(U + U^{-1})\right], \tag{55}$$

where $\hat Z = \text{diag}(\hat z_1, \hat z_2)$ and the normalization adjusted by the continuum limit [48,49]

$$Z_{1/1}^\nu(\hat z_1, \hat z_2, \hat\lambda_7 = 0, \hat a_8 = 0)$$

$$= \hat z_1 K_{\nu+1}(\hat z_1) I_\nu(\hat z_2) - \hat z_2 I_{\nu+1}(\hat z_2) K_\nu(\hat z_1). \tag{56}$$

The function K_ν is the modified Bessel function of the second kind.

There are various ways to calculate the integral (55). One possibility is a brute force evaluation of the Grassmann integrals as in [9,40]. Then the Gaussian integrals over $\hat m_6$ and $\hat\lambda_7$ can be performed analytically leaving us with a nonsingular two-dimensional integral. A second possibility would be to rewrite the integrals as in [41]. Then we end up with a two-dimensional singular integral (see Appendix C) which can be evaluated numerically with some effort. The third way to evaluate the integral is a variation of the method in [41] and results in a one-dimensional integral and a sum over Bessel functions that can be easily numerically evaluated (see Sec. IV C).

IV. THE EIGENVALUE DENSITIES AND THEIR PROPERTIES

To illustrate the effect of nonzero $\hat a_6$ and $\hat a_7$ we first discuss the case $\hat a_8 = 0$. For the general case, with $\hat a_8$ also nonzero, we will discuss the density of the additional real eigenvalues, the density of the complex eigenvalues, and finally the distribution of chirality over the real eigenvalues of D_W.

A. Spectrum of D_W for $\hat a_8 = 0$

The low-energy constants $\hat a_6$ and $\hat a_7$ are introduced through the addition of the Gaussian stochastic variables $\hat m_6 + \hat\lambda_7\gamma_5$ to $D_W|_{a=0}$ resulting in the massive Dirac operator[1]

$$D = D_W|_{a=0} + (\hat m + \hat m_6)\mathbb{1} + \hat\lambda_7\gamma_5. \tag{57}$$

For $\hat a_8 = 0$ the Dirac operator $D_W|_{a=0}$ is anti-Hermitian, and the eigenvalues of $D_W(\hat\lambda_7, \hat m_6) = D - \hat m$ are given by

$$\hat z_\pm = \hat m_6 \pm \iota\sqrt{\lambda_W^2 - \hat\lambda_7^2}, \tag{58}$$

where $\iota\lambda_W$ is an eigenvalue of $D_W|_{a=0}$. The density of the eigenvalues of D is obtained after integrating over the Gaussian distribution of $\hat m_6$ and $\hat\lambda_7$.

As can be seen from Eq. (58), in case $\hat a_6 = \hat a_8 = 0$ and $\hat a_7 \neq 0$, the eigenvalues of D are either purely imaginary or purely real depending on whether $\hat\lambda_7$ is smaller or larger than λ_W, respectively. Paired imaginary eigenvalues penetrate the real axis only through the origin when varying $\hat\lambda_7$, see Fig. 1. Introducing a nonzero W_6, broadens the spectrum by a Gaussian parallel to the real axis but nothing crucial happens because $\hat m_6$ is just an additive constant to the eigenvalues, cf. Fig. 1.

In the continuum the low lying spectral density of the quenched Dirac operator is given by [17]

$$\rho_{\text{cont}}(\hat z) = \delta(\hat x)\left[\nu\delta(\hat y) + \frac{|\hat y|}{2}(J_\nu^2(\hat y) - J_{\nu-1}(\hat y)J_{\nu+1}(\hat y))\right]$$

$$= \delta(\hat x)[\nu\delta(\hat y) + \rho_{\text{NZ}}(\hat y)]. \tag{59}$$

The function J_ν is the Bessel function of the first kind. The level density ρ_{NZ} describes the density of the generic nonzero eigenvalues only.

For nonzero $W_{6/7}$ the distribution of the zero modes represented by the Dirac delta functions in Eq. (59) is broadened by a Gaussian, i.e.

---

[1]Notice that in this subsection the terms proportional to $\hat m_6$ and $\hat\lambda_7$ are explicitly included in the Dirac operator rather than in the probability distribution as in the earlier sections. Moreover the operators D and D_W are already multiplied with ΣV such that we consider the dimensionless, rescaled spectrum of the Dirac operator.
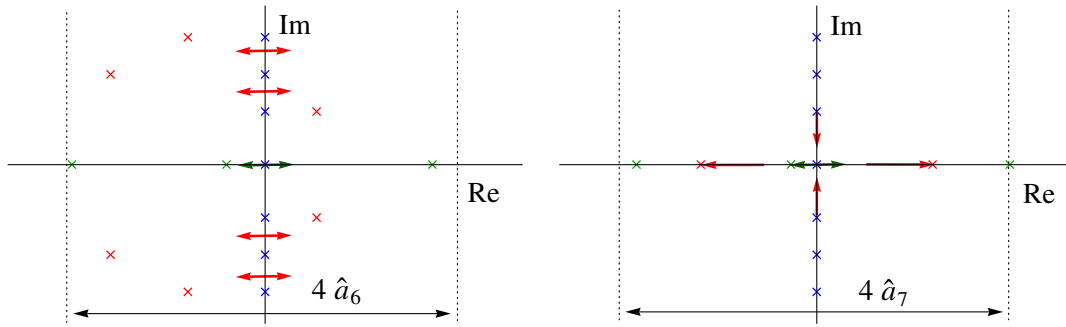
FIG. 1 (color online). Schematic plots of the effects of W_6 (left) and of W_7 (right). The low-energy constant W_6 broadens the spectrum parallel to the real axis according to a Gaussian with width $4\hat{a}_6 = 4\sqrt{-VW_6\tilde{a}^2}$, but does not change the continuum spectrum in a significant way. When W_7 is switched on and $W_6 = 0$ the purely imaginary eigenvalues invade the real axis through the origin and only the real (green crosses along the real axis) are broadened by a Gaussian with width $4\hat{a}_7 = 4\sqrt{-VW_7\tilde{a}^2}$.

$$\rho_\chi(\hat{z}, \hat{a}_8 = 0) = \frac{\nu}{\sqrt{16\pi(\hat{a}_6^2 + \hat{a}_7^2)}} \exp\left[-\frac{\hat{x}^2}{16(\hat{a}_6^2 + \hat{a}_7^2)}\right]. \tag{60}$$

Complex modes have vanishing chirality and do not contribute to the distribution of chirality over the real modes. Additional pairs of real modes also do not contribute to ρ_χ. The reason is the symmetric integration of $\hat{\lambda}_7$ over the real axis. The eigenvalues remain the same under the change $\hat{\lambda}_7 \to -\hat{\lambda}_7$; see Eq. (58). However the corresponding eigenvectors interchange the sign of the chirality which can be seen by the symmetry relation

$$D_W(\hat{\lambda}_7, \hat{m}_6) = -\gamma_5 D_W(-\hat{\lambda}_7, -\hat{m}_6)\gamma_5. \tag{61}$$

Thus the normalized eigenfunctions ($\langle\psi_\pm|\psi_\pm\rangle = 1$) corresponding to the eigenvalues $\hat{z}_\pm$, i.e.

$$D_W(\hat{\lambda}_7, \hat{m}_6)|\psi_\pm\rangle = \hat{z}_\pm|\psi_\pm\rangle, \tag{62}$$

also fulfills the identity

$$D_W(-\hat{\lambda}_7, -\hat{m}_6)\gamma_5|\psi_\pm\rangle = -\hat{z}_\pm\gamma_5|\psi_\pm\rangle. \tag{63}$$

Since the quark mass $\hat{m}_6$ enters with unity we have also

$$D_W(-\hat{\lambda}_7, \hat{m}_6)\gamma_5|\psi_\pm\rangle = \hat{z}_\mp\gamma_5|\psi_\pm\rangle. \tag{64}$$

The wave functions $\gamma_5|\psi_\pm\rangle$ share the same chirality with $|\psi_\pm\rangle$. Moreover $|\psi_+\rangle$ and $|\psi_-\rangle$ have opposite chirality because the pair of eigenvalues $\hat{z}_\pm$ is assumed to be real and their difference $|\hat{z}_+ - \hat{z}_-|$ nonzero. This can be seen by the eigenvalue equations

$$D_W(\hat{\lambda}_7, \hat{m}_6 = 0)|\psi_\pm\rangle = \pm\sqrt{\hat{\lambda}_7^2 - \lambda_W^2}|\psi_\pm\rangle,$$
$$\langle\psi_\pm|D_W(-\hat{\lambda}_7, \hat{m}_6 = 0) = \langle\gamma_5 D_W(-\hat{\lambda}_7, \hat{m}_6 = 0)\gamma_5\psi_\pm|$$
$$= \mp\sqrt{\hat{\lambda}_7^2 - \lambda_W^2}\langle\psi_\pm|. \tag{65}$$

In the second equation we used the γ_5 Hermiticity of D_W. We multiply the first equation with $\langle\psi_\pm|$ and the second with $|\psi_\pm\rangle$ and employ the normalization of the eigenmodes such that we find

$$\langle\psi_\pm|D_W(\hat{\lambda}_7, \hat{m}_6 = 0)|\psi_\pm\rangle = \pm\sqrt{\hat{\lambda}_7^2 - \lambda_W^2},$$
$$\langle\psi_\pm|D_W(-\hat{\lambda}_7, \hat{m}_6 = 0)|\psi_\pm\rangle = \mp\sqrt{\hat{\lambda}_7^2 - \lambda_W^2}. \tag{66}$$

We subtract the second line from the first and use the identity $D_W(\hat{\lambda}_7, \hat{m}_6 = 0) - D_W(-\hat{\lambda}_7, \hat{m}_6 = 0) = 2\hat{\lambda}_7\gamma_5$, i.e.

$$\hat{\lambda}_7\langle\psi_\pm|\gamma_5|\psi_\pm\rangle = \pm\sqrt{\hat{\lambda}_7^2 - \lambda_W^2}, \tag{67}$$

which indeed shows the opposite chirality of $|\psi_+\rangle$ and $|\psi_-\rangle$. Thus $|\psi_+\rangle$ and $\gamma_5|\psi_-\rangle$ have opposite sign of chirality but their corresponding eigenvalues are the same. Therefore the average of their chiralities at a specific eigenvalue vanishes.

The density of the complex eigenvalues can be obtained by integrating over those λ_W fulfilling the condition $|\lambda_W| > |\hat{\lambda}_7|$. After averaging over $\hat{m}_6$ and $\hat{\lambda}_7$ we find

$$\rho_c(\hat{z} = \hat{x} + i\hat{y}, \hat{a}_8 = 0) = \frac{\exp[-\hat{x}^2/(16\hat{a}_6^2)]}{16\pi|\hat{a}_6\hat{a}_7|} \int_{\mathbb{R}^2} \rho_{NZ}(\lambda_W) \exp\left[-\frac{\hat{\lambda}_7^2}{16\hat{a}_7^2}\right]\delta(\sqrt{\lambda_W^2 - \hat{\lambda}_7^2} - |\hat{y}|)\Theta(|\lambda_W| - |\hat{\lambda}_7|)d\lambda_W d\hat{\lambda}_7$$
$$= \frac{\exp[-\hat{x}^2/(16\hat{a}_6^2)]}{4\pi|\hat{a}_6\hat{a}_7|} \int_{|\hat{y}|}^\infty \frac{|\hat{y}|\rho_{NZ}(\lambda_W)d\lambda_W}{\sqrt{\lambda_W^2 - \hat{y}^2}} \exp\left[\frac{\lambda_W^2 - \hat{y}^2}{16\hat{a}_7^2}\right]. \tag{68}$$

The original continuum result is smoothened by a distribution with a Gaussian tail. The oscillations in the microscopic spectral density dampen due to a nonzero W_7 similar to the effect of a nonzero value W_8; cf. Ref. [23]. We also expect a loss

of the height of the first eigenvalue distributions around the origin. Pairs of eigenvalues are moving from the imaginary axis into the real axis and thus lowering their probability density on the imaginary axis. The density ρ_c for nonzero $\hat{a}_8$ will be discussed in full detail in Sec. IV B 2.

The density of the additional real modes can be obtained by integrating the continuum distribution, ρ_{NZ} over $|\lambda_W| < |\lambda_7|$ analogous to the complex case. We find

$$\rho_r(\hat{x}, W_8 = 0) = \frac{1}{16\pi|\hat{a}_6\hat{a}_7|} \int_{\mathbb{R}^3} \rho_{NZ}(\lambda_W) \exp\left[-\frac{\hat{m}_6^2}{16\hat{a}_6^2} - \frac{\hat{\lambda}_7^2}{16\hat{a}_7^2}\right] \delta\left(\sqrt{\hat{\lambda}_7^2 - \lambda_W^2} - |\hat{m}_6 - \hat{x}|\right)\Theta(|\hat{\lambda}_7| - |\lambda_W|)d\lambda_W d\hat{\lambda}_7 d\hat{m}_6$$

$$= \int_{\mathbb{R}^2} \frac{|\hat{m}_6|d\hat{m}_6 d\lambda_W}{8\pi|\hat{a}_6\hat{a}_7|\sqrt{\lambda_W^2 + \hat{m}_6^2}} \rho_{NZ}(\lambda_W) \exp\left[-\frac{\lambda_W^2 + \hat{m}_6^2}{16\hat{a}_7^2} - \frac{(\hat{m}_6 + \hat{x})^2}{16\hat{a}_6^2}\right]. \tag{69}$$

The number of additional real modes given by the integral of $\rho_r(\hat{x})$ over $\hat{x}$ only depends on $\hat{a}_7$, as it should be since $\hat{m}_6$ is just an additive constant to the eigenvalues. Moreover ρ_r will inherit the oscillatory behavior of ρ_{NZ} although most of it will be damped by the Gaussian cutoff. The mixture of this effect with the effect of a nonzero W_8 is highly nontrivial, but we expect that, at small lattice spacings, we can separate both contributions. For a sufficiently small value of $\hat{a}_6$ the behavior of $\rho_r(\hat{x})$ for $\hat{x} \to 0$ is given by $\rho_r(\hat{x}) = \tilde{c}|\hat{x}| + \cdots$ with $\tilde{c} > 0$ for vanishing W_8 and, thus, $\rho_r(\hat{x}) = c_0 + c_1\hat{x}^2 + \cdots$ with $c_0, c_1 > 0$ for nonzero W_8. Hence, we will see a soft repulsion of the additional real eigenvalues from the origin which still allows real eigenvalues to be zero.

The discussion of the real modes for nonzero $\hat{a}_8$ as well is given in Sec. IV B 1.

B. Eigenvalue densities for nonzero values of W_6, W_7, and W_8

In this subsection all three low-energy constants are nonzero. As in the previous subsection, we will consider

the density of the real eigenvalues of D_W, the density of the complex eigenvalues of D_W, and the distribution of the chiralities over the real eigenvalues of D_W. The expressions for these distributions were already given in Sec. III, but in this section we further simplify them and calculate the asymptotic expressions for large and small values of $\hat{a}$.

1. Density of the additional real modes

The quenched eigenvalue density of the additional real modes is given by Eq. (46). The Gaussian average over the variables $\hat{m}_6$ and $\hat{\lambda}_7$ can be worked out analytically. The result is given by (see Appendix B for integrals that were used to obtain this result)

$$\rho_r(\hat{x}) = \frac{1}{16\pi^2} \int_{[0,2\pi]^2} d\varphi_1 d\varphi_2 \sin^2\left[\frac{\varphi_1 - \varphi_2}{2}\right]$$

$$\times e^{\iota\nu(\varphi_1 + \varphi_2)} \frac{\tilde{k}(\hat{x}, \varphi_1, \varphi_2) - \tilde{k}(\hat{x}, \varphi_2, \varphi_1)}{\cos\varphi_2 - \cos\varphi_1} \tag{70}$$

with

$$\tilde{k}(\hat{x}, \varphi_1, \varphi_2) = \exp\left[4\hat{a}_6^2(\cos\varphi_1 - \cos\varphi_2)^2 - 4\hat{a}_7^2(\sin\varphi_1 + \sin\varphi_2)^2\right]$$

$$\times \exp\left[4\hat{a}_8^2\left(\cos\varphi_1 - \frac{\hat{x}}{8\hat{a}_8^2}\right)^2 - 4\hat{a}_8^2\left(\cos\varphi_2 - \frac{\hat{x}}{8\hat{a}_8^2}\right)^2\right]\left[\text{erf}\left[\frac{\hat{x} - 8(\hat{a}_6^2 + \hat{a}_8^2)\cos\varphi_1 + 8\hat{a}_6^2\cos\varphi_2}{\sqrt{8(\hat{a}_8^2 + 2\hat{a}_6^2)}}\right]\right.$$

$$+ \text{erf}\left[\frac{8(\hat{a}_6^2 + \hat{a}_8^2)\cos\varphi_1 - 8\hat{a}_6^2\cos\varphi_2 - 8\iota\hat{a}_7^2\sin\varphi_1 - 8\iota\hat{a}_7^2\sin\varphi_2 - \hat{x}}{\sqrt{16(\hat{a}_8^2 + \hat{a}_6^2 + \hat{a}_7^2)}}\right]\right]. \tag{71}$$

The effect of each low-energy constant on ρ_r is shown in Fig. 2.

At small lattice spacing, $\hat{a} \ll 1$, the density ρ_r has support on the scale of $\hat{a}$. In particular it is given by derivatives of a specific function, i.e.

$$\rho_r(\hat{x}) \overset{\hat{a}\ll 1}{=} \frac{1}{4}\left(\frac{1}{(\nu!)^2}\frac{\partial^{2\nu}}{\partial t_1^\nu \partial t_2^\nu} - \frac{1}{(\nu-1)!(\nu+1)!}\frac{\partial^{2\nu}}{\partial t_1^{\nu-1}\partial t_2^{\nu+1}}\right)\Bigg|_{t_1=t_2=0} \frac{\hat{k}(\hat{x}, t_1, t_2) - \hat{k}(\hat{x}, t_2, t_1)}{t_2 - t_1}, \tag{72}$$
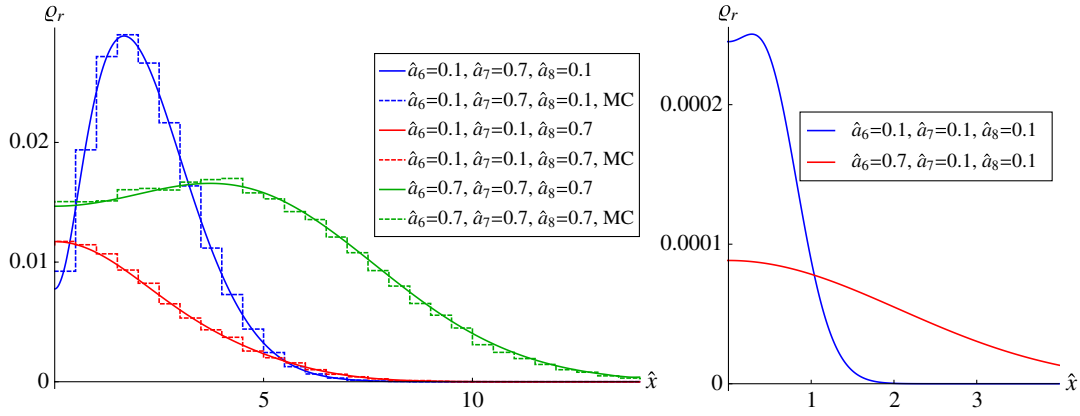
where

FIG. 2 (color online). The density of additional real modes is shown for various parameters $\hat{a}_{6/7/8}$. The analytical results (solid curves) agree with the Monte Carlo simulations of the random matrix theory (histogram [MC] with bin size 0.5 and with different ensemble and matrix sizes such that statistics are about 1%–5%) for $\nu = 1$. We plot only the positive real axis since $\rho_{\rm r}$ is symmetric. Notice that the height of the two curves for $\hat{a}_7 = \hat{a}_8 = 0.1$ (right) are 2 orders smaller than the height of the other curves (left) and because of bad statistics we have not performed simulations for this case. Notice the soft repulsion of the additional real modes from the origin at large $\hat{a}_7 = \sqrt{-VW_7}\tilde{a}$ as discussed in the introductory section. The parameter $\hat{a}_6 = \sqrt{-VW_6}\tilde{a}$ smoothens the distribution.

$$\hat{k}(\hat{x}, t_1, t_2) = \exp\left[\hat{a}_6^2(t_1 - t_2)^2 + \hat{a}_7^2(t_1 + t_2)^2 + \hat{a}_8^2\left(t_1 - \frac{\hat{x}}{4\hat{a}_8^2}\right)^2 - \hat{a}_8^2\left(t_2 - \frac{\hat{x}}{4\hat{a}_8^2}\right)^2\right]$$

$$\times \left[\operatorname{erf}\left[\frac{\hat{x} - 4(\hat{a}_6^2 + \hat{a}_8^2)t_1 + 4\hat{a}_6^2 t_2}{\sqrt{8(\hat{a}_8^2 + 2\hat{a}_6^2)}}\right] + \operatorname{erf}\left[\frac{4(\hat{a}_6^2 + \hat{a}_7^2 + \hat{a}_8^2)t_1 - 4(\hat{a}_6^2 - \hat{a}_7^2)t_2 - \hat{x}}{\sqrt{16(\hat{a}_8^2 + \hat{a}_6^2 + \hat{a}_7^2)}}\right]\right]. \quad (73)$$

The error functions guarantee a Gaussian tail on the scale of $\hat{a}$. Furthermore, the height of the density is of order $\hat{a}^{2\nu+1}$. Hence, additional real modes are strongly suppressed for $\nu > 0$ and the important contributions only result from $\nu = 0$. This behavior becomes clearer for the expression of the average number of the additional real modes. This quantity directly follows from the result (71),

$$N_{\rm add} = 2\int_{-\infty}^{\infty}\rho_{\rm r}(\hat{x})d\hat{x} = \int_0^{2\pi}\frac{d\Phi}{4\pi}\cos[2\nu\Phi]\frac{1 - \exp[-(4\hat{a}_8^2 + 8\hat{a}_7^2)\sin^2\Phi]I_0[(4\hat{a}_8^2 - 8\hat{a}_7^2)\sin^2\Phi]}{\sin^2\Phi}$$

$$= \sum_{n=\nu+1}^{\infty}\sum_{j=0}^{\lfloor n/2\rfloor}(-1)^{\nu-1+n}\frac{(2n-2)!(\hat{a}_8^2 - 2\hat{a}_7^2)^{2j}(\hat{a}_8^2 + 2\hat{a}_7^2)^{n-2j}}{2^{2j-1}\Gamma(n-\nu)\Gamma(n+\nu)\Gamma(n-2j+1)(j!)^2}, \quad (74)$$

where the symbol $\lfloor n/2\rfloor$ denotes the largest integer smaller than or equal to $n/2$.

The average number of the real modes does not depend on the low-energy constant $W_6 = -\hat{a}_6^2/(\tilde{a}^2 V)$ because this constant induces overall fluctuations of the Dirac spectrum parallel to the $\hat{x}$ axis.

The asymptotics of $N_{\rm add}$ at small and large lattice spacing is given by

$$N_{\rm add} = \begin{cases} \sum_{j=0}^{\lfloor \nu/2\rfloor}\dfrac{(\hat{a}_8^2 - 2\hat{a}_7^2)^{2j}(\hat{a}_8^2 + 2\hat{a}_7^2)^{\nu-2j+1}}{2^{2j-1}\Gamma(\nu-2j+2)(j!)^2} \propto \hat{a}^{2\nu+2}, & \hat{a} \ll 1, \\[1.5em] \sqrt{\dfrac{64\hat{a}_7^2}{\pi^3}}E\left(\sqrt{1 - \dfrac{\hat{a}_8^2}{2\hat{a}_7^2}}\right) \propto \hat{a}, & \hat{a} \gg 1. \end{cases}$$

$$(75)$$

See Appendix D 1 for a derivation. The function E is the elliptic integral of the second kind, i.e

$$E(x) = \int_0^{\pi/2}\sqrt{1 - x^2\sin^2\varphi}\,d\varphi. \quad (76)$$

In Ref. [23] this result was derived for $\hat{a}_6 = \hat{a}_7 = 0$. Notice that for large lattice spacings the number of additional real modes increases linearly with $\hat{a}$ and is independent of ν.

The average number of additional real modes can be used to fix the low-energy constants from lattice simulations. For $\nu = 0$, a sufficient number of eigenvalues [50] can be generated to keep the statistical error small. For $\nu = 0$ and $\nu = 1$ the average number of additional real modes is given by

$$N_{\rm add}^{\nu=0} \overset{\hat{a}\ll 1}{=} 2(\hat{a}_8^2 + 2\hat{a}_7^2) = 2V\tilde{a}^2(W_8 - 2W_7), \quad (77)$$

$$N_{\rm add}^{\nu=1} \overset{\hat{a}\ll 1}{=} (\hat{a}_8^2 + 2\hat{a}_7^2)^2 + \frac{1}{2}(\hat{a}_8^2 - 2\hat{a}_7^2)^2$$

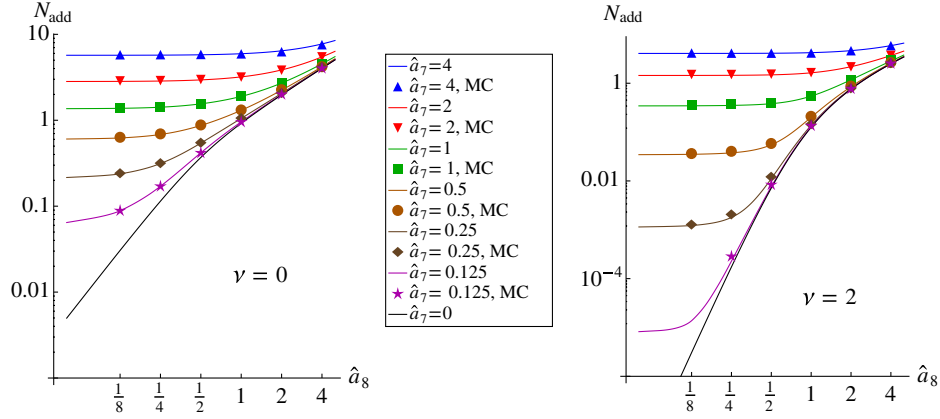$$= V^2\tilde{a}^4\left[(W_8 - 2W_7)^2 + \frac{1}{2}(W_8 + 2W_7)^2\right]. \quad (78)$$

FIG. 3 (color online). Log-log plots of N_{add} as a function of $\hat{a}_8 = \sqrt{VW_8 \tilde{a}^2}$ for $\nu = 0$ (left) and $\nu = 2$ (right). The analytical results (solid curves) are compared to Monte Carlo simulations of RMT (symbols; ensemble and matrix size varies such that the statical error is about 1%–5%). Notice that W_6 has no effect on N_{add}. The saturation around zero is due to a nonzero value of $\hat{a}_7 = \sqrt{-VW_7 \tilde{a}^2}$. For $\hat{a}_7 = 0$ (lowest curves) the average number of additional real modes behaves like $\hat{a}_8^{2\nu+2}$; see Ref. [23].

These simple relations can be used to fit lattice data at small lattice spacing. In Fig. 3 we illustrate the behavior of N_{add} by a log-log plot.

The density ρ_r takes a much simpler form at large lattice spacing. Then, the integrals can be evaluated by a saddle point approximation resulting in the expression (see Appendix D 2)

$$\rho_r(\hat{x}) \overset{\hat{a} \gg 1}{=} \begin{cases} \dfrac{1}{8\pi^2 \hat{a}_7 \hat{a}_6} \displaystyle\int_0^\infty d\tilde{x} \cosh\left(\dfrac{\tilde{x}\hat{x}}{8\hat{a}_6^2}\right) K_0\left(\dfrac{\tilde{x}^2}{32\hat{a}_7^2}\right) \tilde{x} \exp\left[-\dfrac{\tilde{x}^2}{32\hat{a}_7^2} - \dfrac{\tilde{x}^2 + \hat{x}^2}{16\hat{a}_6^2}\right], & \hat{a}_8 = 0, \\[16pt] \dfrac{\Theta(8\hat{a}_8^2 - |\hat{x}|)}{2(2\pi)^{3/2}\hat{a}_8^2}\sqrt{\hat{a}_8^2 + 2\hat{a}_7^2 \dfrac{\hat{x}^2}{(8\hat{a}_8^2)^2 - \hat{x}^2}}, & \hat{a}_8 \neq 0. \end{cases} \tag{79}$$

Notice that we have square root singularities at the two edges of the support if both $\hat{a}_7 \neq 0$ and $\hat{a}_8 \neq 0$; cf. Fig. 4. So the effect of the low-energy constant W_7 is different than what we would have expected naively.

2. Density of the complex eigenvalues

The expression for the density of the complex eigenvalues given in Eq. (47) can be simplified by performing the integral of $\hat{m}_6$ and $\hat{\lambda}_7$ resulting in

$$\rho_c(\hat{z}) = \frac{|\hat{y}|}{2(2\pi)^{5/2}\sqrt{\hat{a}_8^2 + 2\hat{a}_6^2}} \int_{[0,2\pi]^2} d\varphi_1 d\varphi_2 \sin^2\left[\frac{\varphi_1 - \varphi_2}{2}\right] \cos\left[\nu(\varphi_1 + \varphi_2)\right] \text{sinc}[\hat{y}(\cos\varphi_1 - \cos\varphi_2)]$$

$$\times \exp\left[-4\hat{a}_8^2\left(\left(\cos\varphi_1 - \frac{\hat{x}}{8\hat{a}_8^2}\right)^2 + \left(\cos\varphi_2 - \frac{\hat{x}}{8\hat{a}_8^2}\right)^2\right)\right]$$

$$\times \exp\left[\frac{4\hat{a}_6^2\hat{a}_8^2}{\hat{a}_8^2 + 2\hat{a}_6^2}\left(\cos\varphi_1 + \cos\varphi_2 - \frac{\hat{x}}{4\hat{a}_8^2}\right)^2 - 4\hat{a}_7^2(\sin\varphi_1 + \sin\varphi_2)^2\right]. \tag{80}$$

The function $\text{sinc}(x) = \sin x/x$ is the *sinus cardinalis*. This result reduces to the expressions obtained in Ref. [23] for $\hat{a}_6 = \hat{a}_7 = 0$.

To compare to numerical simulations it is useful to consider the projection of the complex modes onto the imaginary axis. The result for the projected eigenvalue density can be simplified to

$$\rho_{cp}(\hat{y}) = \int_{-\infty}^\infty \rho_c(\hat{x} + \iota\hat{y})d\hat{x}$$

$$= \frac{|\hat{y}|}{(2\pi)^2} \int_{[0,2\pi]^2} d\varphi_1 d\varphi_2 \sin^2\left[\frac{\varphi_1 - \varphi_2}{2}\right] \text{sinc}[\hat{y}(\cos\varphi_1 - \cos\varphi_2)] \cos\left[\nu(\varphi_1 + \varphi_2)\right]$$

$$\times \exp\left[-2\hat{a}_8^2(\cos\varphi_1 - \cos\varphi_2)^2 - 4\hat{a}_7^2(\sin\varphi_1 + \sin\varphi_2)^2\right]. \tag{81}$$
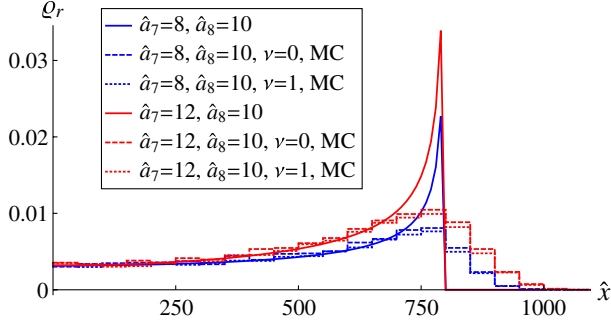
FIG. 4 (color online). At large lattice spacing the density of additional real modes develops square root singularities at the boundaries. The analytical results at $\hat{a} \rightarrow \infty$ (solid curves) are compared to Monte Carlo simulations at nonzero, but large lattice spacing (histogram [MC], with bin size 50, $\hat{a}_6 = \sqrt{-VW_6\tilde{a}^2} = 0.01$ and $n = 2000$ for an ensemble of 1000 matrices). Due to the finite matrix size and the finite lattice spacing, ρ_r has a tail which drops off much faster than the size of the support. The low-energy constant $\hat{a}_8 = \sqrt{VW_8\tilde{a}^2}$ is chosen equal to 10. Therefore the boundary is at $\hat{x} = 800$ which is confirmed by the Monte Carlo simulations. The dependence on W_6 and ν is completely lost.

Again this function is independent of W_6 as was the case for N_{add}. The reason is that the Gaussian broadening with respect to the mass $\hat{m}_6$ is absorbed by the integral over the real axis. At small lattice spacing ρ_{cp} approaches the continuum result ρ_{NZ} given in Eq. (59) (see Fig. 5). Therefore it is still a good quantity to determine the chiral condensate Σ from lattice simulations. In Fig. 5, we compare the projected spectral density (solid curves) with numerical
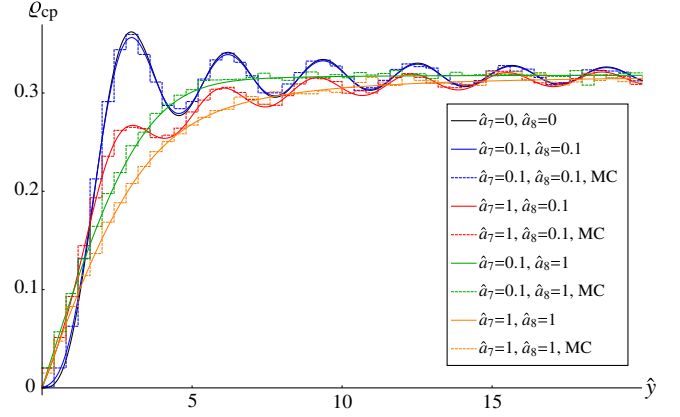

FIG. 5 (color online). Comparison of the analytical result (solid curves) and Monte Carlo simulations of the random matrix theory (histogram [MC] with bin size equal to 0.4 and varying ensemble size and matrix size such that the statistical error is about 1%–5%) for the density of the complex eigenvalues projected onto the imaginary axis. The index of the Wilson-Dirac operator is $\nu = 1$ for all curves. Notice that $\hat{a}_6 = \sqrt{-VW_6}\tilde{a}$ does not affect this density. The comparison of $\hat{a}_7 = \hat{a}_8 = 0.1$ with the continuum result (black, thick curve) shows that ρ_{cp} is still a good quantity to extract the chiral condensate Σ at small lattice spacing.

results from an ensemble of random matrices (histograms). The spectral density at a couple of lattice spacings away from the origin can be used to determine the chiral condensate according to the Banks-Casher formula.

At small lattice spacing, ρ_c factorizes into a Gaussian distribution of the real part of the eigenvalues and of the level density of the continuum limit,

$$\rho_c(\hat{z}) \overset{\hat{a} \ll 1}{=} \frac{|\hat{y}|}{2(2\pi)^{5/2}\sqrt{\hat{a}_8^2 + 2\hat{a}_6^2}} \exp\left[-\frac{\hat{x}^2}{8(\hat{a}_8^2 + 2\hat{a}_6^2)}\right] \int_{[0,2\pi]^2} d\varphi_1 d\varphi_2 \sin^2\left[\frac{\varphi_1 - \varphi_2}{2}\right] \cos[\nu(\varphi_1 + \varphi_2)]\text{sinc}[\hat{y}(\cos\varphi_1 - \cos\varphi_2)]$$

$$= \frac{1}{\sqrt{8\pi(\hat{a}_8^2 + 2\hat{a}_6^2)}} \exp\left[-\frac{\hat{x}^2}{8(\hat{a}_8^2 + 2\hat{a}_6^2)}\right] \rho_{\text{NZ}}(\hat{y}). \tag{82}$$

Therefore the support of ρ_c along the real axis is on the scale $\hat{a}$ while it is of order 1 along the imaginary axis. It also follows from perturbation theory in the non-Hermitian part of the Dirac operator that the first order correction to the continuum result is a Gaussian broadening perpendicular to the imaginary axis. The width of the Gaussian can be used to determine the combination $\hat{a}_8^2 + 2\hat{a}_6^2 = V\tilde{a}^2(W_8 - 2W_6)$ from fitting the results to lattice simulations. Since most of the eigenvalues of D_W occur in complex conjugate pairs at small lattice spacing, it is expected to have a relatively small statistical error in this limit. A further reduction of the statistical error can be achieved by integrating the spectral density over $\hat{y}$ up to the Thouless energy (see Ref. [51] for a definition of the Thouless energy in QCD).

The behavior drastically changes in the limit of large lattice spacing. Then the density reads (see Appendix D 3)

$$\rho_c(\hat{z}) = \begin{cases} \dfrac{\Theta(8\hat{a}_8^2 - |\hat{x}|)}{16\pi\hat{a}_8^2} \text{erf}\left[\dfrac{|\hat{y}|}{\sqrt{8\hat{a}_8^2}}\sqrt{\dfrac{(8\hat{a}_8^2)^2 - \hat{x}^2}{(8\hat{a}_8^2)^2 - (1 - 2\hat{a}_7^2/\hat{a}_8^2)\hat{x}^2}}\right], & \hat{a}_8 > 0, \\[20pt] \dfrac{|\hat{y}|}{16\pi^2|\hat{a}_6\hat{a}_7|} \exp\left[-\dfrac{\hat{x}^2}{16\hat{a}_6^2} + \dfrac{\hat{y}^2}{32\hat{a}_7^2}\right] K_0\left(\dfrac{\hat{y}^2}{32\hat{a}_7^2}\right), & \hat{a}_8 = 0. \end{cases} \tag{83}$$

There is no dependence on ν, and in the case of $\hat{a}_8 > 0$, the result does not depend on $\hat{a}_6$ as well and becomes a strip of width $16\hat{a}_8^2$ along the imaginary axis. To have any structure, the imaginary part of the eigenvalues has to be of order $\hat{a}$. In the mean field limit, where $|\hat{y}|/\sqrt{8\hat{a}_8^2} \gg 1$, ρ_c is equal to $1/(16\pi\hat{a}_8^2)$ on a strip of width $16\hat{a}_8^2$. Hence, the low-energy constants $W_{6/7}$, do not alter the mean field limit of ρ_c; cf. Ref. [23]. This was already observed in Ref. [13].

The effect of $\hat{a}_6$ is an overall Gaussian fluctuation perpendicular to the strip of the eigenvalues, and for $\hat{a}_8 = 0$, when there is no strip, only the Gaussian fluctuations remain. The second case of Eq. (83) can also be obtained from Eq. (68) since for large $\hat{y}$, ρ_{NZ} is equal to $1/\pi$.

C. The distribution of chirality over the real eigenvalues

The distribution of chirality over the real eigenvalues given in Eq. (48) is an expression in terms of the graded partition function $Z_{1/1}^{\nu}$ and the partition function of two fermionic flavors, $Z_{2/0}^{\nu}$, which is evaluated in Appendix C. Including the integrals over $\hat{m}_6$ and $\hat{\lambda}_7$ we obtain from Eq. (C7)

$$\rho_\chi(\hat{x}) = \frac{(-1)^\nu}{(16\pi)^{3/2}\hat{a}_8^2|\hat{a}_7|} \int_{-\infty}^{\infty} d\hat{\lambda}_7 \int_{\mathbb{R}^2} \frac{ds_1 ds_2}{s_1 - \iota s_2} (\iota s_2 + \hat{\lambda}_7)^\nu (s_1 - \hat{\lambda}_7)^\nu \exp\left[-\frac{1}{16\hat{a}_8^2}((s_1 - \hat{x})^2 + (s_2 + \iota\hat{x})^2) \right.$$
$$\left. + \frac{\hat{a}_6^2}{16\hat{a}_8^4}(s_1 - \iota s_2)^2 - \frac{\hat{\lambda}_7^2}{16\hat{a}_7^2} \right] \left[\frac{\delta^{(\nu-1)}(s_1 + \hat{\lambda}_7)}{(\nu-1)!(s_1 - \hat{\lambda}_7)^\nu} \left(\frac{s_1^2 - \hat{\lambda}_7^2}{s_2^2 + \hat{\lambda}_7^2}\right)^{\nu/2} Z_{1/1}^{\nu}\left(\sqrt{s_1^2 - \hat{\lambda}_7^2}, \iota\sqrt{s_2^2 + \hat{\lambda}_7^2}; \hat{a} = 0\right) \right.$$
$$\left. - \text{sign}(\hat{\lambda}_7)\Theta(|\hat{\lambda}_7| - |s_1|)(s_1^2 + s_2^2) \frac{Z_{2/0}^{\nu}\left(\sqrt{s_1^2 - \hat{\lambda}_7^2}, \iota\sqrt{s_2^2 + \hat{\lambda}_7^2}; \hat{a} = 0\right)}{[(s_1^2 - \hat{\lambda}_7^2)(s_2^2 + \hat{\lambda}_7^2)]^{\nu/2}} \right]. \tag{84}$$

We recognize the two terms that were obtained in Eqs. (37) and (39) from the expansion in the first column of the determinant in the joint probability density.

Equation (84) is a complicated expression which is quite difficult to numerically evaluate. However, it is possible to derive an alternative expression in terms of an integral over the supersymmetric coset manifold $U \in \text{Gl}(1/1)/\text{U}(1/1)$. We start from the equality

$$\int_{-\infty}^{\infty} \exp\left[-\frac{\hat{\lambda}_7^2}{16\hat{a}_7^2} - \frac{\iota\hat{\lambda}_7}{2}\text{Str}(U + U^{-1}) \right] d\hat{\lambda}_7 = 4\sqrt{\pi}\hat{a}_7 \exp[-\hat{a}_7^2 \text{Str}^2(U + U^{-1})]$$
$$= \exp[4\hat{a}_7^2(\text{S}\det U + \text{S}\det U^{-1} - 2)] \int_{-\infty}^{\infty} \exp\left[-\frac{\hat{\lambda}_7^2}{16\hat{a}_7^2} - \frac{\iota\hat{\lambda}_7}{2}\text{Str}(U - U^{-1}) \right] d\hat{\lambda}_7$$
$$= \sum_{j=-\infty}^{\infty} I_j(8\hat{a}_7^2)\text{S}\det^j U e^{-8\hat{a}_7^2} \int_{-\infty}^{\infty} \exp\left[-\frac{\hat{\lambda}_7^2}{16\hat{a}_7^2} - \frac{\iota\hat{\lambda}_7}{2}\text{Str}(U - U^{-1}) \right] d\hat{\lambda}_7, \tag{85}$$

based on an identity for the Gl(1/1)/U(1/1) graded unitary matrices,

$$\text{Str}^2(U + U^{-1}) = 8 - 4(\text{S}\det U + \text{S}\det U^{-1}) + \text{Str}^2(U - U^{-1}), \tag{86}$$

and the expansion of the generating function for the modified Bessel functions of the first kind, I_j,

$$\exp\left[x\left(t + \frac{1}{t}\right) \right] = \sum_{j=-\infty}^{\infty} I_j(2x)t^j. \tag{87}$$

This allows us to absorb $\hat{m}_6$ and $\hat{\lambda}_7$ by a shift of the eigenvalues of the auxiliary supermatrix σ introduced to linearize the terms quadratic in U. The integral over U can now be identified as a graded 1/1 partition function at $\hat{a} = 0$ and we obtain the result

$$\rho_\chi(\hat{x}) = \frac{\exp(-8\hat{a}_7^2)}{16\pi\hat{a}_8^2} \sum_{j=1}^{\infty} (I_{j-\nu}(8\hat{a}_7^2) - I_{j+\nu}(8\hat{a}_7^2))$$
$$\times \int_{\mathbb{R}^2} \exp\left[-\frac{1}{16\hat{a}_8^2}((s_1 - \hat{x})^2 + (s_2 + \iota\hat{x})^2) \right.$$
$$\left. + \frac{\hat{a}_6^2 + \hat{a}_7^2}{16\hat{a}_8^4}(s_1 - \iota s_2)^2 \right] \frac{(-|s_1|)^j \delta^{(j-1)}(s_1)}{(j-1)!}$$
$$\times Z_{1/1}^{j}(|s_1|, \iota s_2; \hat{a} = 0) \frac{ds_1 ds_2}{s_1 - \iota s_2}. \tag{88}$$

Notice that the $j = 0$ term does not contribute to the distribution of chirality over the real modes because of the symmetry of the modified Bessel function $I_\nu = I_{-\nu}$. The derivatives of the Dirac delta function originate from the $\text{Im}[1/(s_1 - \iota\epsilon)^j]$ term.
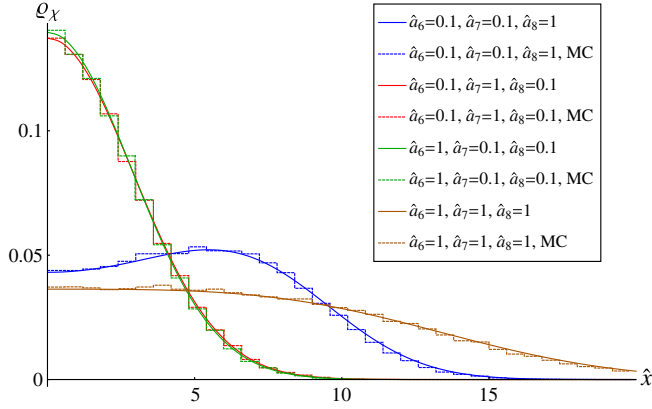
FIG. 6 (color online). The analytical result (solid curves) for ϱ_χ is compared to Monte Carlo simulations of RMT (histogram [MC] with bin size 0.6 and varying ensemble and matrix size such that the statistical error is about 1%–5%) for $\nu = 1$. We plotted only the positive real axis since the distribution is symmetric around the origin. At small $\hat{a}_8 = \sqrt{VW_8}\tilde{a}^2$ the distributions for $(\hat{a}_6, \hat{a}_7) = (\sqrt{-VW_6}\tilde{a}^2, \sqrt{-VW_7}\tilde{a}^2) = (1, 0.1), (0.1, 1)$ are almost the same Gaussian as the analytical result predicts. At large $\hat{a}_8$ the maximum reflects the predicted square root singularity which starts to build up. We have not included the case $\hat{a}_{6/7/8} = 0.1$ since it exceeds the other curves by a factor of 10 to 100.

The representation (88) is effectively a one-dimensional integral due to the Dirac delta-function. Please notice that Eq. (88) reduces to Eq. (60) for $\hat{a}_8 = 0$. Two plots, Fig. 6 ($\nu = 1$) and Fig. 7 ($\nu = 2$), illustrate the effect of each low-energy constant $\hat{a}_{6/7/8}$ on the distribution ρ_χ.
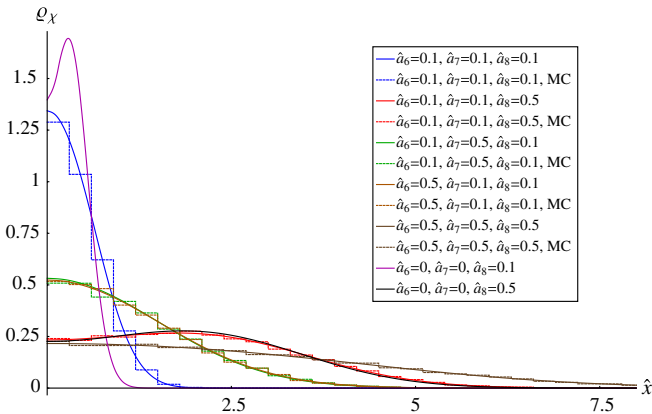

FIG. 7 (color online). We compare the analytical result of ρ_χ (solid curves) with Monte Carlo simulations of RMT (histogram [MC] with bin size 0.6 and with varying ensemble and matrix size such that the statistical error is about 1%–5%) for $\nu = 2$. Again we only plotted the positive real-axis because ρ_χ is symmetric in the quenched theory. Two curves with $W_{6/7} = 0$ and $W_8 = 0.1, 0.5$ (highest, purple and thick, black curve) are added to emphasize that the two peaks (ρ_χ has to be reflected at the origin) can be strongly suppressed by nonzero $W_{6/7}$ although they are only of the same order as W_8. Recall that the two peaks are relics of a 2×2 GUE which is formed by W_8.

For $\hat{a}_7 = 0$ and $\nu = 1$ one can derive a more compact result in a straightforward way starting from the expression (37). In this case the two-point weight for two real eigenvalues $g_r(x_1, x_2)$ is antisymmetric in its two arguments; see Eq. (20). Then the integral in Eq. (37) involving $Z_{2/0}^1$ is absent. Employing the representation of the one-flavor partition function as a unitary integral, see Eq. (5), we perform the integral over $\hat{m}_6$. Thus, $\rho_\chi(\hat{x})|_{\nu=1}$ can be expressed as

$$\rho_\chi(\hat{x})|_{\nu=1} = \frac{1}{\sqrt{16\pi(\hat{a}_8^2 + \hat{a}_6^2)}} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \exp\left[\frac{(\hat{x} + 8\hat{a}_8^2 \sin\theta)^2}{\hat{a}_8^2 + \hat{a}_6^2}\right].$$

(89)

Let us come back to the general result (88). At small lattice spacing, $0 < \hat{a} \ll 1$, the distribution ρ_χ as well as the integration variables $s_{1/2}$ are of order $\hat{a}$. Since $I_j(8\hat{a}_7^2) \propto \hat{a}_7^{2j}$, the leading order term is given by $j = \nu$ in the sum over j. Thus we have

$$\rho_\chi(\hat{x}) \stackrel{\hat{a} \ll 1}{=} \frac{1}{16\pi\hat{a}_8^2} \int_{\mathbb{R}^2} \exp\left[-\frac{1}{16\hat{a}_8^2}((s_1 - \hat{x})^2 + (s_2 + \iota\hat{x})^2) \right.$$
$$\left. + \frac{\hat{a}_6^2 + \hat{a}_7^2}{16\hat{a}_8^4}(s_1 - \iota s_2)^2\right] \frac{(-|s_1|)^\nu \delta^{(\nu-1)}(s_1)}{(\nu - 1)!}$$
$$\times Z_{1/1}^\nu(s_1, \iota s_2; \hat{a} = 0) \frac{ds_1 ds_2}{s_1 - \iota s_2}.$$

(90)

In the small $\hat{a}$ limit we can replace $Z_{1/1}^\nu(s_1, \iota s_2; \hat{a} = 0) \rightarrow (\iota s_2/|s_1|)^\nu$. The result becomes a polynomial in $\hat{x}^2$ times a Gaussian of width $\sqrt{32(\hat{a}_8^2 + \hat{a}_6^2 + \hat{a}_7^2)}$. Notice that the polynomial is not the one of a GUE anymore as in the case of $\hat{a}_6 = \hat{a}_7 = 0$ [10]. For $\nu = 1$, ρ_χ is a pure Gaussian,

$$\rho_\chi^{\nu=1}(\hat{x}) \stackrel{\hat{a} \ll 1}{=} \frac{1}{\sqrt{16\pi(\hat{a}_8^2 + \hat{a}_6^2 + \hat{a}_7^2)}} \exp\left[-\frac{\hat{x}^2}{16(\hat{a}_8^2 + \hat{a}_6^2 + \hat{a}_7^2)}\right],$$

(91)

and for $\nu = 2$ it is given by

$$\rho_\chi^{\nu=2}(\hat{x}) \stackrel{\hat{a} \ll 1}{=} \frac{1}{\sqrt{16\pi(\hat{a}_8^2 + \hat{a}_6^2 + \hat{a}_7^2)^3}}$$
$$\times \left[\hat{a}_8^2 + 2(\hat{a}_6^2 + \hat{a}_7^2) + \frac{\hat{a}_8^2}{8(\hat{a}_8^2 + \hat{a}_6^2 + \hat{a}_7^2)}\hat{x}^2\right]$$
$$\times \exp\left[-\frac{\hat{x}^2}{16(\hat{a}_8^2 + \hat{a}_6^2 + \hat{a}_7^2)}\right].$$

(92)

At small lattice spacing, ρ_χ only depends on the combinations $\hat{a}_8^2$ and $(\hat{a}_6^2 + \hat{a}_7^2)$. Therefore it is in principle possible to determine the two following combinations of low-energy constants, W_8 and $W_6 + W_7$, by fitting ρ_χ to lattice results. For example the second moment (variance) of ρ_χ given by

$$\frac{1}{\nu}\int_{-\infty}^{\infty}\rho_\chi(\hat{x})\hat{x}^2 d\hat{x} \overset{\hat{a}\ll 1}{=} 8(\nu\hat{a}_8^2 + \hat{a}_6^2 + \hat{a}_7^2)$$

$$= 8V\tilde{a}^2(\nu W_8 - W_6 - W_7), \qquad \nu > 0, \tag{93}$$

at small lattice spacing can be used to fit the combinations $\nu W_8 - W_6 - W_7$. The statistical error in this quantity scales with the inverse square root of the number of configurations with the index ν. The ensemble of configurations generated in Ref. [7] yields a statistical error of about 2%–3%. The statistics can be drastically increased by performing a fit of the variance of ρ_χ to a linear function in the index ν; cf. Eq. (93). The slope is then determined by W_8 and the offset by $W_6 + W_7$ yielding two important quantities.

In Appendix D 4 we calculate ρ_χ in the limit of large lattice spacing. Then the distribution of chirality over the real eigenvalues has a support on the scale of $\hat{a}^2$. The function ρ_χ reads

$$\rho_\chi(\hat{x}) \overset{\hat{a}\gg 1}{=} \begin{cases} \dfrac{\nu}{\pi}\dfrac{\Theta(8\hat{a}_8^2 - |\hat{x}|)}{\sqrt{(8\hat{a}_8^2)^2 - \hat{x}^2}}, & \hat{a}_8 > 0, \\[4mm] \dfrac{\nu}{\sqrt{16\pi(\hat{a}_6^2 + \hat{a}_7^2)}}\exp\left[-\dfrac{\hat{x}^2}{16(\hat{a}_6^2 + \hat{a}_7^2)}\right], & \hat{a}_8 = 0. \end{cases} \tag{94}$$

Interestingly, the low-energy constants $W_{6/7}$ have no effect on the behavior of ρ_χ in this limit if $\hat{a}_8 \neq 0$ which is completely different in comparison to ρ_r and ρ_c. The square root singularities at the boundary of the support are unexpected and were already mentioned in Ref. [23].

V. CONCLUSIONS

Starting from RMT for the Wilson-Dirac operator, we have derived the microscopic limit of the spectral density and the distribution of the chiralities over the Dirac spectrum. We have focused on the quenched theory, but all arguments can be simply extended to dynamical Wilson fermions. Wilson RMT is equivalent to the ϵ limit of the Wilson chiral Lagrangian and describes the Wilson QCD partition function and Dirac spectra in this limit. The starting point of our analytical calculations is the joint probability density of the random matrix ensemble for the non-Hermitian Wilson-Dirac operator D_W. This density was first obtained in Ref. [23], but a detailed derivation is given in this paper; see Appendix A.

More importantly, we studied in detail the effect of the three low-energy constants, $W_{6/7/8}$, on the quenched microscopic level density of the complex eigenvalues, the additional real eigenvalues, and the distribution of chirality over the real eigenvalues. In terms of the effect on the spectrum of D_W, the low-energy constants W_6 and W_7 are structurally different from W_8. The first two can be interpreted in terms of "collective" fluctuations of the eigenvalues, whereas a

nonzero W_8 induces stochastic interactions between all modes, particularly those with different chiralities. Therefore, the effect of a nonzero W_6 and W_7 at $W_8 = 0$ is just a Gaussian broadening of the Dirac spectrum on the scale of $\hat{a}$. When $a^2VW_8 \gg 1$ the interactions between the modes result in a strip of Dirac eigenvalues in the complex plane with real part inside the interval $[-8VW_8\tilde{a}^2, 8VW_8\tilde{a}^2]$. The structure along the imaginary axis is on the scale $\hat{a}$. As was already discussed in Ref. [13], in the mean field limit, the lattice spacing $\tilde{a}^2V$ and the eigenvalues $V\tilde{z}$ fixed, this structure becomes a boxlike strip with hard edges at the boundary of the support and with height $1/(16\pi VW_8\tilde{a}^2)$.

We also discussed the limit of small lattice spacing, i.e. the limit $|VW_{6/7/8}|\tilde{a}^2 \ll 1$. In practice, this limit is already reached when $|VW_{6/7/8}|\tilde{a}^2 \lesssim 0.1$. Such values can be indeed achieved via clover improvement as discussed in Ref. [7]. In the small $\hat{a}$ limit we have identified several quantities that are suitable to fit the four low-energy constants, $W_{6/7/8}$ and Σ, to lattice simulations and our analytical results.

Several promising quantities (applicable *only* at small lattice spacing) are the following.

(i) According to the Banks-Casher formula we have

$$\Delta \overset{\hat{a}\ll 1}{=} \frac{\pi}{\Sigma V} \tag{95}$$

for the average spacing Δ of the imaginary part of the eigenvalues several eigenvalue spacings from the origin.

(ii) The average number of the additional real modes for $\nu = 0$:

$$N_{\text{add}}^{\nu=0} \overset{\hat{a}\ll 1}{=} 2V\tilde{a}^2(W_8 - 2W_7). \tag{96}$$

(iii) The width of the Gaussian shaped strip of complex eigenvalues:

$$\frac{\sigma^2}{\Delta^2} \overset{\hat{a}\ll 1}{=} \frac{4}{\pi^2}\tilde{a}^2V(W_8 - 2W_6). \tag{97}$$

(iv) The variance of the distribution of chirality over the real eigenvalues:

$$\frac{\langle\tilde{x}^2\rangle_{\rho_\chi}}{\Delta^2} \overset{\hat{a}\ll 1}{=} \frac{8}{\pi^2}V\tilde{a}^2(\nu W_8 - W_6 - W_7), \quad \nu > 0. \tag{98}$$

These quantities are easily accessible in lattice simulations. We believe they will lead to an improvement of the fits performed in Refs. [4,6,7]. Note that ρ_χ is close to the density of the real eigenvalues in the limit of small lattice spacing (again we mean by this $|VW_{6/7/8}|\tilde{a} \approx 0.1$ and smaller). This statement is not true in the limit of large lattice spacing where the density of the additional real modes dominates the density of the real eigenvalues.

The relations (95)–(98) are an over-determined set for the low-energy constants $W_{6/7/8}$ and Σ^2 and are only consistent if we have relations between these quantities. This can be seen by writing the relations as

$$\tilde{a}^2 V \begin{bmatrix} 0 & -2 & 1 \\ -2 & 0 & 1 \\ -1 & -1 & 1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} W_6 \\ W_7 \\ W_8 \end{bmatrix} = \frac{\pi^2}{8} \begin{bmatrix} 4N_{\mathrm{add}}^{\nu=0}/\pi^2 \\ 2\sigma^2/\Delta^2 \\ \langle \tilde{x}^2 \rangle_{\rho_\chi}^{\nu=1}/\Delta^2 \\ \langle \tilde{x}^2 \rangle_{\rho_\chi}^{\nu=2}/\Delta^2 \end{bmatrix}. \quad (99)$$

The first three relations are linearly dependent, but none of the other triplets are. We thus have the consistency relation

$$\frac{\langle \tilde{x}^2 \rangle_{\rho_\chi}^{\nu=1}}{\Delta^2} = \frac{\sigma^2}{\Delta^2} + \frac{2}{\pi^2} N_{\mathrm{add}}^{\nu=0}. \quad (100)$$

There are more relations like Eqs. (95)–(98) which can be derived from our analytical results. The only assumption is a sufficiently small lattice spacing.

The value of W_8 follows immediately from the ν dependence of $\langle \tilde{x}^2 \rangle_{\rho_\chi}$. If there are additional real modes, it cannot be that W_7 and W_8 are both equal to zero. In Ref. [7] it was found $W_8 = 0$ (with clover improvement) and results were fitted as a function of W_6 with $W_7 = 0$. Our prediction is that the number of additional real modes is zero and it would be interesting if the authors of Ref. [7] could confirm that.

The nontrivial effect of W_7 on the quenched spectrum was a surprise for us. In Ref. [13] it was argued that W_7 does not affect the phase structure of the Dirac spectrum. Indeed, we found that the complex eigenvalue density only shows a weak dependence on W_7, and actually becomes W_7 independent in the small $\tilde{a}^2 V$ limit. Such a dependence on W_7 can be found in the large $\tilde{a}^2 V$ limit but vanishes again in the thermodynamic limit. Since in the thermodynamic limit the number of real eigenvalues is suppressed as $1/\sqrt{V}$ with respect to the number of complex eigenvalues, W_7 will not affect the phase structure of the partition function. However, a nonzero value of W_7 significantly changes the density of the real eigenvalues. In particular, in the large $\hat{a}$ limit, we find a square root singularity at the boundary of the support of the additional real eigenvalues if $W_7 \neq 0$, while it is a uniform density for $W_7 = 0$; see Ref. [23]. Nevertheless, we expect in the case of dynamical fermions that the discussion of Ref. [13] also applies to the real spectrum of D_{W}.

ACKNOWLEDGMENTS

APPENDIX A: DERIVATION OF THE JOINT PROBABILITY DENSITY

In this appendix, we derive the joint probability density in three steps. In Appendix A 1, following the derivation for the joint probability density of the Hermitian Dirac operator [20], we introduce an auxiliary Gaussian integral such that we obtain a Harish-Chandra-Itzykson-Zuber-like integral that mixes two different types of variables. In Appendix A 2 this problem is reduced to a Harish-Chandra-Itzykson-Zuber-like integral considered in a bigger framework. We derive an educated guess which fulfills a set of differential equations and a boundary value problem. The asymptotics of the integral for large arguments serves as the boundary. In Appendix A 2 b we perform a stationary phase approximation which already yields the full solution implying that the semiclassical approach is exact and the Duistermaat-Heckman localization theorem [52] applies. In the last step we plug the result of Appendix A 2 into the original problem, see Appendix A 2 c, and integrate over the remaining variables to arrive at the result for the joint probability density given in the main text.

1. Introducing auxiliary Gaussian integrals

We consider the functional $I[f]$, see Eq. (12), with an integrable test-function f invariant under $\mathrm{U}(n, n + \nu)$. The idea is to rewrite the exponent of the probability density $P(D_{\mathrm{W}})$ as the sum of a $\mathrm{U}(n, n + \nu)$ invariant term $\mathrm{Tr} D_{\mathrm{W}}^2$ and a symmetry breaking term which is linear in D_{W}. This is achieved by introducing two Gaussian distributed Hermitian matrices S_{r} and S_{l} with dimensions $n \times n$ and $(n + \nu) \times (n + \nu)$, respectively, i.e.

$$I[f] = (2\pi\sqrt{1 + a^2})^{-n^2 - (n+\nu)^2} \left(-\frac{n}{2\pi} \right)^{n(n+\nu)}$$

$$\times \exp\left[-\frac{a^2}{2} \left(\mu_{\mathrm{r}}^2 + \frac{n+\nu}{n} \mu_{\mathrm{l}}^2 \right) \right] \int d[D_{\mathrm{W}}] f(D_{\mathrm{W}})$$

$$\times \int d[S_{\mathrm{r}}, S_{\mathrm{l}}] \exp\left[\frac{n}{2} \mathrm{tr} D_{\mathrm{W}}^2 + \iota \, \mathrm{tr} D_{\mathrm{W}} \mathrm{diag}(S_{\mathrm{r}}, S_{\mathrm{l}}) \right]$$

$$\times \exp\left[-\frac{a^2}{2n(1 + a^2)} (\mathrm{tr}(S_{\mathrm{r}} + \iota \mu_{\mathrm{r}} \mathbb{1}_n)^2 \right.$$

$$\left. + \mathrm{tr}(S_{\mathrm{l}} + \iota \mu_{\mathrm{l}} \mathbb{1}_{n+\nu})^2) \right]. \quad (A1)$$

The matrix $\mathrm{diag}(S_{\mathrm{r}}, S_{\mathrm{l}})$ is a block-diagonal matrix with S_{r} and S_{l} on the diagonal blocks. The measure for $S_{\mathrm{r/l}}$ is

$$d[S_{\mathrm{r}}, S_{\mathrm{l}}] = \prod_{j=1}^n dS_{jj}^{(\mathrm{r})} \prod_{1 \le i < j \le n} 2d\,\mathrm{Re} S_{ij}^{(\mathrm{r})} d\,\mathrm{Im} S_{ij}^{(\mathrm{r})} \prod_{j=1}^{n+\nu} dS_{jj}^{(\mathrm{l})}$$

$$\times \prod_{1 \le i < j \le n+\nu} 2d\,\mathrm{Re} S_{ij}^{(\mathrm{l})} d\,\mathrm{Im} S_{ij}^{(\mathrm{l})}. \quad (A2)$$

Then the noncompact unitary matrix diagonalizing D_{W} only appears quadratically in the exponent. Notice that we have to integrate first over the Hermitian matrices $S_{\mathrm{r/l}}$ and have to be careful when interchanging integrals with integrals over D_{W}. Obviously the integrations over the eigenvalues of D_{W} are divergent without performing the $S_{\mathrm{r/l}}$ integrals first and cannot be interchanged with

these integrals. Also the coset integrals over $\mathbb{G}_l = \mathrm{U}(n, n + \nu)/[\mathrm{U}^{2n+\nu-l}(1) \times \mathrm{O}^l(1, 1)]$, cf. Eq. (12), are not absolutely convergent. However we can understand them in a weak way and, below, we will find Dirac delta functions resulting from the noncompact integrals.

Diagonalizing the matrices $D_l = UZ_lU^{-1}$ and $S_{r/l} = V_{r/l}s_{r/l}V_{r/l}^\dagger$ with $s_r = \mathrm{diag}(s_1^{(r)}, ..., s_n^{(r)})$ and $s_l = \mathrm{diag}(s_1^{(l)}, ..., s_{n+\nu}^{(l)})$ we can absorb the integrals over V_r and V_l in the $U \in \mathbb{G}_l$ integral. Then we end up with the integral

$$I[f] = \frac{C}{n!(n+\nu)!} \sum_{l=0}^n \frac{1}{2^l(n-l)!l!(n+\nu-l)!} \int_{\mathbb{R}^{\nu+2(n-l)}\times\mathbb{C}^l} d[Z_l]|\Delta_{2n+\nu}(Z_l)|^2 \int_{\mathbb{R}^{2n+\nu}} d[s_r, s_l]\Delta_n^2(s_r)\Delta_{n+\nu}^2(s_l)f(Z_l)$$

$$\times \exp\left[\frac{n}{2}\mathrm{tr}Z_l^2 - \frac{a^2}{2n(1+a^2)}(\mathrm{tr}(s_r + \imath\mu_r\mathbb{1}_n)^2 + \mathrm{tr}(s_l + \imath\mu_l\mathbb{1}_{n+\nu})^2)\right]\int_{\mathbb{G}_l} \exp[\imath\mathrm{tr}UZ_lU^{-1}\mathrm{diag}(s_r, s_l)]d\mu_{\mathbb{G}_l}(U) \quad \text{(A3)}$$

and the normalization constant

$$C = \left(-\frac{n}{2\pi}\right)^{n(n+\nu)} \prod_{j=0}^{n-1} \frac{(2\pi)^j \exp[-a^2\mu_r^2/2n]}{j!(2\pi\sqrt{1+a^2})^{2j+1}}$$

$$\times \prod_{j=0}^{n+\nu-1} \frac{(2\pi)^j \exp[-a^2\mu_l^2/2n]}{j!(2\pi\sqrt{1+a^2})^{2j+1}}. \quad \text{(A4)}$$

See Sec. II B for a discussion of the prefactors in the sum.

2. The Harish-Chandra-Itzykson-Zuber integral over the noncompact coset $\mathbb{G}_l$

In the next step we calculate the integral

$$I_l(Z_l, s) = \int_{\mathbb{G}_l} \exp[\imath\mathrm{tr}UZ_lU^{-1}s]d\mu_{\mathbb{G}_l}(U). \quad \text{(A5)}$$

with $s = \mathrm{diag}(s_r, s_l)$. For $l = 0$ this integral was derived in Ref. [53].

We calculate this integral by determining a complete set of functions and expanding the integral for asymptotically large s in this set. In this limit it can be calculated by a stationary phase approximation. It turns out that this integral, as is the case with the usual Harish-Chandra-Itzykson-Zuber integral, is semiclassically exact.

a. Noncompact Harish-Chandra-Itzykson-Zuber integral

Let us consider the noncompact integral

$$I_l(Z_l, Z'_{l'}) = \int_{\mathbb{G}_l} \exp[\imath\mathrm{tr}UZ_lU^{-1}Z'_{l'}]d\mu_{\mathbb{G}_l}(U) \quad \text{(A6)}$$

in a bigger framework where $Z'_{l'}$ is a quasidiagonal matrix with l' complex conjugate eigenvalue pairs. The integral is invariant under the Weyl group $\mathbf{S}(n - l) \times \mathbf{S}(l) \times \mathbf{S}(n + \nu - l) \times \mathbb{Z}_2^l$ in Z_l. To make the integral well defined we have to assume that $l \geq l'$ otherwise the integral is divergent since the noncompact subgroup $\mathrm{O}^{l'-l}(1, 1) \subset \mathbb{G}_l$ commutes with $Z'_{l'}$.

The integral (A6) should be contrasted with the well-known compact Harish-Chandra-Itzykson-Zuber integral [54,55]

$$\mathcal{I}^{\mathrm{com}}(X, X') = \int_{\mathrm{U}(2n+\nu)} \exp[\imath\mathrm{tr}UXU^{-1}X']d\mu_{\mathrm{U}(2n+\nu)}(U)$$

$$= \frac{(-2\pi\imath)^{\nu(\nu-1)/2}}{\Delta_\nu(x)\Delta_\nu(x')} \det[\exp(\imath x_i x'_j)]_{1\leq i,j\leq\nu} \quad \text{(A7)}$$

with Weyl group $\mathbf{S}(2n + \nu)$. Moreover the compact case is symmetric when interchanging X with X'. This symmetry is broken in Z_l and $Z'_{l'}$ due to the coset $\mathbb{G}_l$.

For a γ_5-Hermitian matrix V with eigenvalues Z_l, we can rewrite the integral (A6) as

$$I_l(Z_l, Z'_{l'}) = I_l(V, Z'_{l'})$$

$$= \int_{\mathbb{G}_l} \exp[\imath\,\mathrm{tr}UVU^{-1}Z'_{l'}]d\mu_{\mathbb{G}_l}(U). \quad \text{(A8)}$$

This trivially satisfies the Sekigushi-like differential equation [56,57]

$$\det\left(\frac{\partial}{\partial V_{kl}} + u\mathbb{1}_{2n+\nu}\right)I_l(V, Z'_{l'})$$

$$= \det(\imath Z'_{l'} + u\mathbb{1}_{2n+\nu})I_l(V, Z'_{l'}) \quad \text{for all } u \in \mathbb{C}. \quad \text{(A9)}$$

This equation is written in terms of the independent matrix elements of V and, hence, is independent of the fact to which sector l the matrix V can be quasidiagonalized.

We would like to rewrite Eq. (A9) in terms of derivatives with respect to the eigenvalues [58]. Because of the coefficients that enter after applying the chain rule when changing coordinates, the derivatives do not commute and a direct evaluation of the determinant is cumbersome. Therefore we will calculate $I_l(Z_l, Z'_{l'})$ in an indirect way. We will do this by constructing a complete set of $\mathbf{S}(n - l) \times \mathbf{S}(l) \times \mathbf{S}(n + \nu - l) \times \mathbb{Z}_2^l$ symmetric functions in the space of the $\{Z_l\}$ with the $\{Z'_{l'}\}$ as quantum numbers which have to be $\mathbf{S}(n-l') \times \mathbf{S}(l') \times \mathbf{S}(n+\nu-l') \times \mathbb{Z}_2^{l'}$ symmetric. Then we expand $I_l(Z_l, Z'_{l'})$ in this set of functions and determine the coefficients for asymptotic large $\{Z_l\}$ where the integral can be evaluated by a stationary phase approximation.

To determine the complete set of functions, we start from the usual Harish-Chandra-Itzykson-Zuber integral over the compact group $\mathrm{U}(2n + \nu)$. This integral is well

known and satisfies the Sekigushi-like differential equation [56,57] with

$$\frac{1}{\Delta_{2n+\nu}(X)} \det\left(\frac{\partial}{\partial X} + u\mathbb{1}_{2n+\nu}\right)\Delta_{2n+\nu}(X)\mathcal{I}^{\mathrm{com}}(X, X')$$
$$= \det(\imath X' + u\mathbb{1}_{2n+\nu})\mathcal{I}^{\mathrm{com}}(X, X') \qquad (A10)$$

in terms of the $(2n + \nu)$ real eigenvalues $X = \mathrm{diag}(x_1, \ldots, x_{2n+\nu})$ with

$$\det\left(\frac{\partial}{\partial X} + u\mathbb{1}_{2n+\nu}\right) = \prod_{j=1}^{2n+\nu}\left(\frac{\partial}{\partial x_j} + u\right). \qquad (A11)$$

The expansion in powers of u gives the complete set of $2n + \nu$ independent Casimir operators on the Cartan subspace of $\mathrm{U}(2n + \nu)$, so that the Sekigushi equation determines a complete set of functions $\mathcal{I}_l(Z_l, Z'_{l'})$ up to the Weyl group. Since the noncompact group $\mathrm{U}(n + \nu, n)$ shares the same complexified Lie algebra as $\mathrm{U}(2n + \nu)$ the Casimir operators are the same, i.e. the corresponding operator for $\mathrm{U}(n + \nu, n)$ to the one in Eq. (A10) is

$$D_{Z_l}(u) = \frac{1}{\Delta_{2n+\nu}(Z_l)} \det\left(\frac{\partial}{\partial Z_l} + u\mathbb{1}_{2n+\nu}\right)\Delta_{2n+\nu}(Z_l) \quad (A12)$$

with

$$\det\left(\frac{\partial}{\partial Z_l} + u\mathbb{1}_{2n+\nu}\right) = \prod_{j=1}^{n-l}\left(\frac{\partial}{\partial x_j^{(1)}} + u\right) \prod_{j=1}^{n+\nu-l}\left(\frac{\partial}{\partial x_j^{(3)}} + u\right)$$
$$\times \prod_{j=1}^{l}\left(\frac{\partial}{\partial z_j^{(2)}} + u\right)\left(\frac{\partial}{\partial z_j^{(2)*}} + u\right). \qquad (A13)$$

In the compact case, the Sekigushi-like equation (A10) follows from Eq. (A9) by transforming the equation in terms of the eigenvalues and eigenvectors of $V = UXU^{-1}$; see Ref. [57]. The only difference in the noncompact case is that the parameters of U as well as some of the eigenvalues x become complex, but the algebraic manipulations to obtain the Sekigushi-like differential equation in terms of eigenvalues remain the same. Let f be an integrable test function on the Cartan-subset $\mathbb{R}^{2n+\nu-2l'} \times \mathbb{C}^{l'}$. Then the noncompact integral (A6) satisfies the weak Sekigushi-like equation

$$D_{Z_l}(u) \int_{\mathbb{R}^{2(n-l')+\nu} \times \mathbb{C}^{l'}} d[Z'_{l'}]f(Z'_{l'})\mathcal{I}_l(Z_l, Z'_{l'})$$
$$= \int_{\mathbb{R}^{2(n-l')+\nu} \times \mathbb{C}^{l'}} d[Z'_{l'}]f(Z'_{l'})\det(\imath Z'_{l'} + u\mathbb{1}_{2n+\nu})\mathcal{I}_l(Z_l, Z'_{l'}), \qquad (A14)$$

and solutions of this equation yield a complete set of functions for the noncompact case as well. The only difference is the corresponding Weyl group. The completeness can be seen because we can generate any polynomial of order $k \in \mathbb{N}_0$ (the non-negative integers)

in $Z'_{l'}$ symmetric under $\mathbf{S}(n - l') \times \mathbf{S}(l') \times \mathbf{S}(n + \nu - l') \times \mathbb{Z}_2^{l'}$ via the differential operator $\prod_{j=1}^{k} D_{Z_l}(u_j)$. Since those polynomials are dense in the space of $\mathbf{S}(n - l') \times \mathbf{S}(l') \times \mathbf{S}(n + \nu - l') \times \mathbb{Z}_2^{l'}$ invariant functions, it immediately follows that if a function is in the kernel of $D_{Z_l}(u)$ for all u it is zero, i.e.

$$D_{Z_l}(u)F(Z_l) = 0 \quad \forall u \in \mathbb{C} \Leftrightarrow F(Z_l) = 0. \qquad (A15)$$

Therefore if we found a solution for Eq. (A14) for an arbitrary test-function f we found $\mathcal{I}_l(Z_l, Z'_{l'})$ up to the normalization which can be fixed in the large $\mathrm{tr}Z_l Z_l^\dagger$ limit.

Some important remarks about Eq. (A14) are in order. The Vandermonde determinant $\Delta_{2n+\nu}(Z_l)$ enters in a trivial way in the operator $D_{Z_l}(u)$ and the remaining operator has plane waves as eigenfunctions which indeed build a complete set of functions. Thus a good ansatz of $\mathcal{I}_l(Z_l, Z'_{l'})$ is

$$\mathcal{I}_l(Z_l, Z'_{l'}) = \frac{1}{\Delta_{2n+\nu}(Z_l)} \prod_{j=1}^{l} \frac{y_j^{(2)}}{|y_j^{(2)}|} \sum_{\omega \in \mathbf{S}} c_\omega^{(ll')}(Z'_{l'})$$
$$\times \exp\left[\imath \, \mathrm{tr}\Pi_\omega Z_l \Pi_\omega^{-1} \hat{Z}'_{l'}\right], \qquad (A16)$$

where the coefficients $c_\omega^{(ll')}(Z'_{l'})$ have to be determined. The factors $y_j^{(2)}/|y_j^{(2)}|$ guarantee the invariance under complex conjugation of each complex eigenvalue pair of Z_l. We sum over the permutation group ω and Π_ω is its standard representation in terms of $(2n + \nu) \times (2n + \nu)$ matrices. The $\mathbf{S}(n - l) \times \mathbf{S}(l) \times \mathbf{S}(n + \nu - l) \times \mathbb{Z}_2^l$ invariance in Z_l and the $\mathbf{S}(n - l') \times \mathbf{S}(l') \times \mathbf{S}(n + \nu - l') \times \mathbb{Z}_2^{l'}$ invariance in $Z'_{l'}$ carry over to the coefficients $c_\omega^{(ll')}(Z'_{l'})$. Hence, we can reduce all coefficients to coefficients independent of ω,

$$\mathcal{I}_l(Z_l, Z'_{l'}) = \frac{1}{\Delta_{2n+\nu}(Z_l)} \prod_{j=1}^{l} \frac{y_j^{(2)}}{|y_j^{(2)}|}$$
$$\times \sum_{\substack{\omega \in \mathbf{S}(n-l) \times \mathbf{S}(l) \times \mathbf{S}(n+\nu-l) \times \mathbf{Z}_2^l \\ \omega' \in \mathbf{S}(n-l') \times \mathbf{S}(l') \times \mathbf{S}(n+\nu-l') \times \mathbf{Z}_2^{l'}}} \mathrm{sign}\omega c^{(ll')}(Z'_{l'\omega'})$$
$$\times \exp\left[\imath \, \mathrm{tr}Z_{l\omega} Z'_{l'\omega'}\right] \qquad (A17)$$

where we employ the abbreviation

$$Z_{l\omega} = \Pi_\omega Z_l \Pi_\omega^{-1} \quad \text{and} \quad Z'_{l'\omega'} = \Pi_{\omega'} Z'_{l'} \Pi_{\omega'}^{-1}. \qquad (A18)$$

The sign of elements in the group $\mathbb{Z}_2$ generating the complex conjugation of single complex conjugated pairs is always $+1$. Moreover, any element in the permutation group $\mathbf{S}(l)$ is an even permutation since it interchanges a complex conjugate pair with another one and, thus, always yields a positive sign. Hence the sign of the permutation ω is the product of the sign of the permutations in $\mathbf{S}(n - l)$ and in $\mathbf{S}(n + \nu - l)$.

Solving the weak Sekigushi-like equation (A14) for the general case $l \neq l'$ is quite complicated but as we will show below for $l = l'$ the ansatz

$I_l(Z_l, Z'_l)$

$$= \frac{(-2\pi\imath)^{(2n+\nu)(2n+\nu-1)/2}}{\Delta_{2n+\nu}(Z_l)\Delta_{2n+\nu}(Z'_l)} \det[\exp(\imath x_i^{(1)} x_j'^{(1)})]_{1\leq i,j\leq n-l}$$

$$\times \mathrm{perm}\Bigg[\frac{y_i^{(2)} y_j'^{(2)}}{|y_i^{(2)} y_j'^{(2)}|} (\exp[2\imath\,\mathrm{Re}z_i^{(2)} z_j'^{(2)}]$$

$$+ \exp[2\imath\,\mathrm{Re}z_i^{(2)*} z_j'^{(2)}])\Bigg]_{1\leq i,j\leq l}$$

$$\times \det[\exp(\imath x_i^{(3)} x_j'^{(3)})]_{1\leq i,j\leq n+\nu-l}, \qquad (A19)$$

i.e. $c^{(ll)}(Z'_{l\omega'}) \propto (\prod_{j=1}^{l'} y_j'^{(2)}/|y_j'^{(2)}|)/\Delta_{2n+\nu}(Z'_{l\omega'})$, does the job. Note that we have again the symmetry when interchanging Z_l with Z'_l since both matrices are in the Cartan subspace corresponding to $\mathbb{G}_l$. The constant can be fixed by a stationary phase approximation when taking $\mathrm{tr}Z_l Z_l^\dagger \to \infty$. The function "perm" is the permanent which is defined analogously to the determinant but without the sign function in the sum over the permutations. It arises because the Vandermonde determinants are even under the interchange of a complex pair with another one, i.e. it is the **S**(l) invariance of the corresponding Weyl group. It can be explicitly shown that the ansatz (A19) satisfies the completeness relation in the space of functions on $\mathbb{R}^{\nu+2(n-l)} \times \mathbb{C}^l$ invariant under **S**($n - l$) × **S**(l) × **S**($n + \nu - l$) × $\mathbb{Z}_2^l$ and with the measure $|\Delta_{2n+\nu}(Z_l)|^2 d[Z_l]$, i.e.

$$\int_{\mathbb{R}^{\nu+2(n-l)} \times \mathbb{C}^l} I_l(Z_l, Z'_l) I_l(Z''_l, Z_l) |\Delta_{2n+\nu}(Z_l)|^2 d[Z_l]$$

$$\propto \frac{1}{\Delta_{2n+\nu}(Z'_l)\Delta_{2n+\nu}(Z''_l)} \det[\delta(x_i'^{(1)} - x_j''^{(1)})]_{1\leq i,j\leq n-l}$$

$$\times \mathrm{perm}\Bigg[\frac{y_i'^{(2)} y_j''^{(2)}}{|y_i'^{(2)} y_j''^{(2)}|} \delta(|y_i'^{(2)}| - |y_j''^{(2)}|)$$

$$\times \delta(x_i'^{(2)} - x_j''^{(2)})\Bigg]_{1\leq i,j\leq l}$$

$$\times \det[\delta(x_i'^{(3)} - x_j''^{(3)})]_{1\leq i,j\leq n+\nu-l}. \qquad (A20)$$

Therefore, for given $l' = l$ and $Z'_{l'}$ the ansatz (A19) for $I_l(Z_l, Z'_l)$ is the unique solution of the Sekiguchi-like equation (A14). One has only to show that the global prefactor is correct; see A 2 b.

What happens in the general case $l \neq l'$? The ansatz (A17) can only fulfill the Sekigushi-like differential equation (A14) if we assume that the coefficient $c^{(ll')}(Z'_{l'\omega'})$ restricts the matrix $Z'_{l'}$ to a matrix in the sector with l complex conjugate eigenvalue pairs (notice that Z_l has the representation given in Eq. (8)). This is only possible on the boundary of the Cartan subsets $\mathbb{R}^{2(n-l)+\nu} \times \mathbb{C}^l$ and $\mathbb{R}^{2(n-l')+\nu} \times \mathbb{C}^{l'}$, i.e. the coefficient has to be proportional to Dirac delta functions

$$c^{(ll')}(Z'_{l'\omega'}) \propto \prod_{j=1}^{l-l'} \delta(x_{\omega'(n-l+j)}'^{(1)} - x_{\omega'(j)}'^{(3)}). \qquad (A21)$$

The reason for this originates in the fact that not all complex pairs of Z_l can couple with a complex eigenvalue pair in $Z'_{l'}$ and, hence, $\mathrm{tr}Z_{l\omega}Z'_{l'\omega'}$ does not depend on the combinations $x_{\omega'(n-l+j)}'^{(1)} - x_{\omega'(j)}'^{(3)}$. Therefore we would miss it in the determinant $\det(Z'_{l'} + u\mathbb{1}_{2n+\nu})$ generated by the differential operator $D_{Z_l}(u)$. To cure this we have to understand $I_l(Z_l, Z'_{l'})$ as a distribution where the Dirac delta functions set these missing terms to zero. In A 2 b we show that the promising ansatz

$$I_l(Z_l, Z'_{l'}) = \frac{c^{(ll')}}{\Delta_{2n+\nu}(Z_l)\Delta_{2n+\nu}(Z'_{l'})} \prod_{j=1}^l \frac{y_j^{(2)}}{|y_j^{(2)}|} \prod_{j=1}^{l'} \frac{y_j'^{(2)}}{|y_j'^{(2)}|}$$

$$\times \sum_{\substack{\omega\in\mathbb{S}(n-l)\times\mathbb{S}(l)\times\mathbb{S}(n+\nu-l)\times\mathbb{Z}_2^l \\ \omega'\in\mathbb{S}(n-l')\times\mathbb{S}(l')\times\mathbb{S}(n+\nu-l')\times\mathbb{Z}_2^{l'}}} \mathrm{sign}\omega\omega'$$

$$\times \exp(\imath\,\mathrm{tr}Z_{l\omega}Z'_{l'\omega'}) \prod_{j=1}^{|l-l'|} (x_{\omega'(n-l+j)}'^{(1)} - x_{\omega'(j)}'^{(3)})$$

$$\times \delta(x_{\omega'(n-l+j)}'^{(1)} - x_{\omega'(j)}'^{(3)}) \qquad (A22)$$

is indeed the correct result.

Note that the ansatz (A22) agrees with the solution (A19) for the case $l = l'$. Furthermore one can easily verify that it also solves the weak Sekiguchi-like differential equation (A14). Indeed, the ansatz is trivially invariant under the two Weyl groups **S**($n-l$)×**S**(l)×**S**($n+\nu-l$)× $\mathbb{Z}_2^l$ and **S**($n - l'$)×**S**(l')×**S**($n + \nu - l'$)× $\mathbb{Z}_2^{l'}$ due to the sum. The global prefactor $1/\Delta_{2n+\nu}(Z'_{l'})$ reflects the singularities when an eigenvalue in $x'^{(1)}$ agrees with one in $x'^{(3)}$ as well as a complex eigenvalue pair in $x'^{(2)}$ degenerates with another eigenvalue in $Z'_{l'}$, namely then $Z'_{l'}$ commutes with some noncompact subgroups in $\mathbb{G}_l$. Hereby the eigenvalues which have to degenerate via the Dirac delta functions are excluded.

In the next section we calculate the global coefficients in Eq. (A22). For this we consider the stationary phase approximation which fixes this coefficient.

b. The stationary phase approximation of $I_l(Z_l, Z'_{l'})$

Let us introduce a scalar parameter t as a small parameter in the integral $I_l(t^{-1}Z_l, Z'_{l'})$ as a bookkeeping device for the expansion around the saddle points. Taking $t \to 0$ the group integral (A6) can be evaluated by a stationary phase approximation. The saddle point equation is given by

$$\mathrm{tr}dUU^{-1}[UZ_l U^{-1}, Z'_{l'}]_- = 0. \qquad (A23)$$

If $l \neq l'$ this equation cannot be satisfied in all directions. The reason is that the quasidiagonal matrix $Z'_{l'}$ will never

commute with a γ_5-Hermitian matrix with exactly $l \neq l'$ complex conjugate eigenvalue pairs since UZ_lU^{-1} can be at most quasidiagonalized by $\mathrm{U}(n, n + \nu)$ and generically $[Z_l, Z'_{l'}]_- \neq 0$. This means that we can only expand the sub-Lie-algebra $\mathrm{o}^{l-l'}(1/1)$ to the linear order while the remaining massive modes are expanded to the second order. The extrema are given by

$$U_0 = \Pi'\Phi\Pi \in \mathbb{G}_l \qquad (A24)$$

where the permutations are

$$\Pi \in \mathbf{S}(n - l) \times [\mathbf{S}(l)/[\mathbf{S}(l') \times \mathbf{S}(l - l')]] \times \mathbf{S}(n + \nu - l),$$
$$\Pi' \in [\mathbf{S}(n - l')/\mathbf{S}(n - l)] \times \mathbf{S}(l')$$
$$\times [\mathbf{S}(n + \nu - l')/\mathbf{S}(n + \nu - l)] \times \mathbb{Z}^{l'}, \qquad (A25)$$

and a block-diagonal matrix

$$\Phi = \begin{bmatrix} \mathbb{1}_{n-l} & 0 & 0 & 0 & 0 & 0 \\ 0 & \exp[\imath\widehat{\Phi}] & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbb{1}_{l'} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1}_{l'} & 0 & 0 \\ 0 & 0 & 0 & 0 & \exp[-\imath\widehat{\Phi}] & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbb{1}_{n+\nu-l} \end{bmatrix}, \qquad (A26)$$

where the diagonal matrix of angles is $\widehat{\Phi} = \mathrm{diag}(\varphi_1, \ldots, \varphi_{l-l'}) \in [0, \pi]^{l-l'}$. The matrix Φ describes the set $\mathrm{U}^{l-l'}(1)$ ($l - l'$ unit circles in the complex plane) which commutes with $Z'_{l'}$ and is a subgroup of $\mathbb{G}_l$. Note that other rotations commuting with $Z'_{l'}$ are already divided out in $\mathbb{G}_l$. The matrix of phases already comprises the complex conjugation of the complex eigenvalues represented by the finite group $\mathbb{Z}_2^{l-l'}$; choosing $\varphi_j = \pi/2$ switches the sign of the imaginary part y'_j. However we have to introduce the complex conjugation for those complex conjugated pairs in $Z'_{l'}$ which couple with pairs in Z_l; cf. the group $\mathbb{Z}^{l'}$ in Π'.

The expansion of U reads

$$U = \Pi'\Phi\left(\mathbb{1}_{2n+\nu} - tH_1 - \sqrt{t}H_2 + \frac{t}{2}H_2^2\right)\Pi. \qquad (A27)$$

We employ the notation (A18) for the action of $\omega \in \mathbf{S}(n-l) \times \mathbf{S}(l) \times \mathbf{S}(n+\nu-l) \times \mathbb{Z}^l$ and $\omega' \in \mathbf{S}(n-l') \times \mathbf{S}(l') \times \mathbf{S}(n+\nu-l') \times \mathbb{Z}^{l'}$ on the matrices $Z_{l\omega}$ and $Z'_{l'\omega'}$, respectively. Note that the matrix Φ commutes with $Z'_{l'\omega'}$ for any ω' and, hence, only yields an overall prefactor

$\pi^{l-l'}$. The matrix H_1 spans the Lie algebra $\mathrm{o}^{l-l'}(1, 1)$ and is embedded as

$$H_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & h & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ with } h = \mathrm{diag}\,(h_1, \ldots, h_{l-l'}) \in \mathbb{R}^{l-l'}. \qquad (A28)$$

The matrix H_2 is in the tangent space of the coset $\mathbb{G}_l/[\mathrm{U}^{l-l'}(1) \times \mathrm{O}^{l-l'}(1, 1)] = \mathrm{U}(n, n + \nu)/[\mathrm{U}^{2n+\nu-2l+l'}(1) \times \mathrm{O}^{l'}(1, 1) \times \mathrm{U}^{l-l'}(1, 1)]$ and has the form

$$H_2 = \begin{pmatrix} H_{11} & H_{12} & H_{13} & H_{14} & H_{15} & H_{16} \\ -H_{12}^\dagger & H_{22} & H_{23} & H_{24} & H_{25} & H_{26} \\ -H_{13}^\dagger & -H_{23}^\dagger & H_{33} & H_{34} & H_{35} & H_{36} \\ H_{14}^\dagger & H_{24}^\dagger & H_{34}^\dagger & H_{44} & H_{45} & H_{46} \\ H_{15}^\dagger & H_{25}^\dagger & H_{35}^\dagger & -H_{45}^\dagger & H_{55} & H_{56} \\ H_{16}^\dagger & H_{26}^\dagger & H_{36}^\dagger & -H_{46}^\dagger & -H_{56}^\dagger & H_{66} \end{pmatrix}, \qquad (A29)$$

where H_{11}, H_{22}, H_{55}, and H_{66} are anti-Hermitian matrices without diagonal elements since they are divided out in the coset $\mathbb{G}_l$ or are lost to Φ. The two matrices H_{33} and H_{44} are anti-Hermitian matrices whose diagonal elements are the same with opposite sign which is also because of the subgroup we divide out in $\mathbb{G}_l$. The matrices H_{12}, H_{13}, H_{14}, H_{15}, H_{16}, H_{23}, H_{24}, H_{26}, H_{35}, H_{36}, H_{45}, H_{46}, and H_{56} are arbitrary complex matrices. Since we have to remove the degrees of freedom already included in H_1 and in the subgroups quotient out in $\mathbb{G}_l$ the matrix H_{25} is a complex matrix with all $l - l'$ diagonal elements removed and H_{34} is a complex matrix whose diagonal entries are real. The sizes of the blocks of H_1 and H_2 correspond to the sizes shown in the diagonal matrix of phases Φ; see Eq. (A26). The double lines in the matrix (A29) shall show the decomposition of Z_l in its real and complex eigenvalues whereas the single lines represent the decomposition for $Z'_{l'}$.

The exponent in the coset integral (A6) takes the form

$$\mathrm{tr}UZ_lU^{-1}Z'_{l'} = \mathrm{tr}Z_{l\omega}Z'_{l'\omega'} - t\,\mathrm{tr}[Z_{l\omega}, Z'_{l'\omega'}]_-H_1$$
$$- \frac{t}{2}\,\mathrm{tr}[Z_{l\omega}, H_2]_-[Z'_{l'\omega'}, H_2]_-. \qquad (A30)$$

The measure for H_1 and H_2 is the induced Haar measure, i.e.

$$\mathrm{tr}[U^{-1}dU, Z_{l\omega}]_-^2 = \mathrm{tr}[\Phi^\dagger d\Phi, Z_{l\omega}]_-^2 + t^2\,\mathrm{tr}[dH_1, Z_{l\omega}]_-^2$$
$$+ t\,\mathrm{tr}[dH_2, Z_{l\omega}]_-^2 \qquad (A31)$$

which gives

$$d\mu_{\mathbb{G}}(U) = t^{(2n+\nu)(2n+\nu-1)/2}d[H_1]d[\Phi]d[H_2]$$

$$= (-1)^{n(n+\nu)}\left(\frac{2}{\iota}\right)^{l'}\prod_{k=1}^{l-l'}\frac{4t}{\iota}d\varphi_k dh_k$$

$$\times \prod_{j,i}2td\operatorname{Re}(H_2)_{ij}d\operatorname{Im}(H_2)_{ij}. \tag{A32}$$

The product over the two indices i and j is over all independent matrix elements of H_2.

We emphasize again that the integrand in $I_l(t^{-1}Z_l, Z'_{l'})$ does not depend on Φ making this integration trivial and yielding the prefactor $\pi^{l-l'}$. The integral over H_1 yields

the $l - l'$ Dirac delta functions mentioned in Eq. (A21), i.e. it yields

$$(2\pi)^{l-l'}\prod_{j=1}^{l-l'}\delta(2y^{(2)}_{\omega(j)}[x'^{(1)}_{\omega'(n-l+j)} - x'^{(3)}_{\omega'(j)}])$$

$$= \prod_{j=1}^{l-l'}\frac{\pi\delta(x'^{(1)}_{\omega'(n-l+j)} - x'^{(3)}_{\omega'(j)})}{|y^{(2)}_{\omega(j)}|}. \tag{A33}$$

Notice that the other term in the expansion of the Dirac delta function does not contribute because of the order of the integrations [59].

The integrals over H_2 are simple Gaussian integrals resulting in the main result of this section,

$$I_l(t^{-1}Z_l, Z'_{l'}) = \frac{(-2\pi\iota)^{l-l'}}{l'!(l-l')!(n-l)!(n+\nu-l)!2^l}\frac{(-2\pi\iota)^{(2n+\nu)(2n+\nu-1)/2}}{\Delta_{2n+\nu}(t^{-1}Z_l)\Delta_{2n+\nu}(Z'_{l'})}\prod_{j=1}^{l}\frac{y^{(2)}_j}{|y^{(2)}_j|}\prod_{j=1}^{l'}\frac{y'^{(2)}_j}{|y'^{(2)}_j|}$$

$$\times \sum_{\substack{\omega\in\mathbb{S}(n-l)\times\mathbb{S}(l)\times\mathbb{S}(n+\nu-l)\times\mathbb{Z}_2^l \\ \omega'\in\mathbb{S}(n-l')\times\mathbb{S}(l')\times\mathbb{S}(n+\nu-l')\times\mathbb{Z}_2^{l'}}}\operatorname{sign}\omega\omega'\exp\left(\frac{\iota}{t}\operatorname{tr}Z_{l\omega}Z'_{l'\omega'}\right)\prod_{j=1}^{|l-l'|}(x'^{(1)}_{\omega'(n-l+j)} - x'^{(3)}_{\omega'(j)})\delta(x'^{(1)}_{\omega'(n-l+j)} - x'^{(3)}_{\omega'(j)}). \tag{A34}$$

The overall coefficient $c^{(ll')}$ in Eq. (A22) can be easily read off. Thereby the numerator of the first factor results from the integral over H_1 and is related to the $l - l'$ Dirac delta functions. The denominator is the volume of the finite group $\mathbb{S}(n-l)\times\mathbb{S}(l')\times\mathbb{S}(l-l')\times\mathbb{S}(n+\nu-l)\times\mathbb{Z}_2^l$ which we extend to summing over the full Weyl groups for Z_l and $Z'_{l'}$. We recall that the sum over permutations in $\mathbb{S}(l)$ and $\mathbb{S}(l')$ describe the interchange of complex pairs which are even permutations because we interchange both z_k and z_k^* with another pair. The numerator of the term with the Vandermonde determinants essentially results from the Gaussian integrals and always appears independent of how many complex pairs Z_l and $Z'_{l'}$ have. The factors of t^{-1} appear as prefactors of Z_l and can be omitted again since they have done their job as bookkeeping device.

Let us summarize what we have found. Comparing the result (A34) with the Z_l dependence of the ansatz $I(Z_l, Z'_l)$

given in Eq. (A22), we observe that they are exactly the same. This implies that the asymptotic large Z_l result for the integral (A6) is actually equal to the exact result. We conclude that the noncompact Harish-Chandra-Itzykson-Zuber integral is semiclassically exact and seems to fulfill the conditions of the Duistermaat-Heckman theorem [52].

Let us consider two particular cases. For $l = l'$ we sum over all permutations in $\mathbb{S}(l)$ which yields the permanent in Eq. (A20), whereas the sum over permutations in $\mathbb{S}(n+\nu-l)$ and $\mathbb{S}(n-l)$ gives determinants and, thus, agrees. The special case $n = 0$ yields the original Harish-Chandra-Itzykson-Zuber integral [54,55]; see Eq. (A7).

c. The joint probability density

We explicitly write out Z_l and apply Eq. (A34) for $Z'_{l'} = s$. Then, we find for our original noncompact group integral

$$I_l(Z_l, s) = \frac{(-2\pi\iota)^{(2n+\nu)(2n+\nu-1)/2}}{(n-l)!l!(n+\nu-l)!}\frac{(-2\pi\iota)^l}{\Delta_{2n+\nu}(Z_l)\Delta_{2n+\nu}(s)}\sum_{\substack{\omega'\in\mathbf{S}(n-l)\times\mathbf{S}(l)\times\mathbf{S}(n+\nu-l) \\ \omega\in\mathbf{S}(n)\times\mathbf{S}(n+\nu)}}\operatorname{sign}(\omega\omega')\prod_{j=1}^{n-l}\exp\left(\iota x^{(1)}_{\omega'(j)}s^{(\mathrm{r})}_{\omega(j)}\right)$$

$$\times \prod_{j=1}^{n+\nu-l}\exp\left(\iota x^{(3)}_{\omega'(j)}s^{(1)}_{\omega(l+j)}\right)\prod_{j=1}^{l}\frac{y^{(2)}_{\omega'(j)}}{|y^{(2)}_{\omega'(j)}|}(s^{(\mathrm{r})}_{\omega(n-l+j)} - s^{(1)}_{\omega(j)})\delta(s^{(\mathrm{r})}_{\omega(n-l+j)} - s^{(1)}_{\omega(j)})\exp\left(\iota x^{(2)}_{\omega'(j)}(s^{(\mathrm{r})}_{\omega(n-l+j)} + s^{(1)}_{\omega(j)})\right). \tag{A35}$$

Now we are ready to integrate over s.

We plug Eq. (A35) into the integral (A3). The sum over the permutations can be absorbed by the integral due to relabeling resulting in

$$I[f] = C \sum_{l=0}^{n} \frac{(-2\pi\iota)^{(2n+\nu)(2n+\nu-1)/2+l}}{2^l (n-l)! l! (n+\nu-l)!} \int_{\mathbb{R}^{\nu+2(n-l)} \times \mathbb{C}^l} d[Z_l] \Delta_{2n+\nu}(Z_l^*) f(Z_l) \prod_{j=1}^{l} \frac{y_j^{(2)}}{|y_j^{(2)}|} \int_{\mathbb{R}^{2n+\nu}} d[s_r, s_l] \frac{\Delta_n^2(s_r) \Delta_{n+\nu}^2(s_l)}{\Delta_{2n+\nu}(s)}$$

$$\times \prod_{j=1}^{n-l} \exp\left[\frac{n}{2}(x_j^{(1)})^2 + \iota x_j^{(1)} s_j^{(r)} - \frac{a^2}{2n(1+a^2)}(s_j^{(r)} + \iota\mu_r)^2 \right] \prod_{j=1}^{n+\nu-l} \exp\left[\frac{n}{2}(x_j^{(3)})^2 + \iota x_j^{(3)} s_{l+j}^{(l)} - \frac{a^2}{2n(1+a^2)}(s_{l+j}^{(l)} + \iota\mu_l)^2 \right]$$

$$\times \prod_{j=1}^{l} (s_{n-l+j}^{(r)} - s_j^{(l)}) \delta(s_{n-l+j}^{(r)} - s_j^{(l)}) \exp\left[\frac{a^2}{4n(1+a^2)}(\mu_r - \mu_l)^2 \right]$$

$$\times \exp\left[n((x_j^{(2)})^2 - (y_j^{(2)})^2) + 2\iota x_j^{(2)} s_j^{(l)} - \frac{a^2}{n(1+a^2)}\left(s_j^{(l)} + \iota\frac{\mu_r + \mu_l}{2}\right)^2 \right]. \tag{A36}$$

The quotient of the Vandermonde determinants is

$$\frac{\Delta_n^2(s_r) \Delta_{n+\nu}^2(s_l)}{\Delta_{2n+\nu}(s)} = (-1)^{n(n-1)/2 + \nu(\nu-1)/2} \det\left[\begin{array}{c} \left\{ \dfrac{1}{s_i^{(r)} - s_j^{(l)}} \right\}_{\substack{1 \le i \le n \\ 1 \le j \le n+\nu}} \\[2ex] \left\{ (s_j^{(l)})^{i-1} \right\}_{\substack{1 \le i \le \nu \\ 1 \le j \le n+\nu}} \end{array} \right]. \tag{A37}$$

This determinant also appears in the supersymmetry method of RMT [57,60] and is a square root of a Berezinian (the supersymmetric analogue of the Jacobian).

Expanding the determinant (A37) in the first l columns not all terms will survive. Only those terms which cancel the prefactor of the Dirac delta functions do not vanish. The integration over $\text{diag}(s_{n-l+1}^{(r)}, \ldots, s_n^{(r)}, s_1^{(l)}, \ldots, s_l^{(l)})$ yields

$$I[f] = C \sum_{l=0}^{n} \frac{(-2\pi\iota)^{(2n+\nu)(2n+\nu-1)/2+l} (-1)^{n(n-1)/2 + \nu(\nu-1)/2 + (n+l)l}}{2^l (n-l)! l! (n+\nu-l)!} \int_{\mathbb{R}^{\nu+2(n-l)} \times \mathbb{C}^l} d[Z_l] \Delta_{2n+\nu}(Z_l^*) f(Z_l) \int_{\mathbb{R}^{\nu+2(n-l)}} d[s_r, s_l]$$

$$\times \det\left[\begin{array}{c} \left\{ \dfrac{1}{s_i^{(r)} - s_{l+j}^{(l)}} \right\}_{\substack{1 \le i \le n-l \\ 1 \le j \le n+\nu-l}} \\[2ex] \left\{ (s_{l+j}^{(l)})^{i-1} \right\}_{\substack{1 \le i \le \nu \\ 1 \le j \le n+\nu-l}} \end{array} \right] \prod_{j=1}^{n-l} \exp\left[\frac{n}{2}(x_j^{(1)})^2 + \iota x_j^{(1)} s_j^{(r)} - \frac{a^2}{2n(1+a^2)}(s_j^{(r)} + \iota\mu_r)^2 \right]$$

$$\times \prod_{j=1}^{n+\nu-l} \exp\left[\frac{n}{2}(x_j^{(3)})^2 + \iota x_j^{(3)} s_{l+j}^{(l)} - \frac{a^2}{2n(1+a^2)}(s_{l+j}^{(l)} + \iota\mu_l)^2 \right] \prod_{j=1}^{l} \sqrt{\frac{n\pi(1+a^2)}{a^2}} \frac{y_j^{(2)}}{|y_j^{(2)}|} \exp\left[\frac{a^2}{4n(1+a^2)}(\mu_r - \mu_l)^2 \right]$$

$$\times \exp\left[-\frac{n}{a^2}(x_j^{(2)})^2 - n(y_j^{(2)})^2 + x_j^{(2)}(\mu_r + \mu_l) \right]. \tag{A38}$$

The other exponential functions as well as the remaining integrations over s_r and s_l can be pulled into the determinant. The integrals in the ν bottom rows yield harmonic oscillator wave function. These can be reordered into monomials times a Gaussian. This results in

$$I[f] = C \sum_{l=0}^{n} \frac{(-2\pi\iota)^{(2n+\nu)(2n+\nu-1)/2+l} (-1)^{n(n-1)/2 + \nu(\nu-1)/2 + (n+l)l}}{2^l (n-l)! l! (n+\nu-l)!} (2\pi)^{\nu/2} \iota^{\nu(\nu-1)/2} \left(\frac{n(1+a^2)}{a^2}\right)^{\nu^2/2}$$

$$\times \int_{\mathbb{R}^{\nu+2(n-l)} \mathbb{C}^l} d[Z_l] \Delta_{2n+\nu}(Z_l^*) f(Z_l) \det\left[\begin{array}{c} \left\{ \tilde{G}(x_i^{(1)}, x_j^{(3)}) \right\}_{\substack{1 \le i \le n-l \\ 1 \le j \le n+\nu-l}} \\[2ex] \left\{ (x_j^{(3)})^{i-1} \exp\left[-\frac{n}{2a^2}(x_j^{(3)})^2 + \mu_l x_j^{(3)} \right] \right\}_{\substack{1 \le i \le \nu \\ 1 \le j \le n+\nu-l}} \end{array} \right]$$

$$\times \prod_{j=1}^{l} \sqrt{\frac{n\pi(1+a^2)}{a^2}} \frac{y_j^{(2)}}{|y_j^{(2)}|} \exp\left[\frac{a^2}{4n(1+a^2)}(\mu_r - \mu_l)^2 - \frac{n}{a^2}(x_j^{(2)})^2 - n(y_j^{(2)})^2 + x_j^{(2)}(\mu_r + \mu_l) \right]. \tag{A39}$$

What remains is to simplify the function

$$\tilde{G}(x_i^{(1)}, x_j^{(3)}) = \int_{\mathbb{R}^2} ds_r ds_l \frac{\exp[x_i^{(1)}\mu_r + x_j^{(3)}\mu_l]}{s_r - s_l + \imath n(1 + a^2)(x_i^{(1)} - x_j^{(3)})/a^2}$$

$$\times \exp\left[-\frac{n}{2a^2}((x_i^{(1)})^2 + (x_j^{(3)})^2) - \frac{a^2}{2n(1 + a^2)}((s_r + \imath\mu_r)^2 + (s_l + \imath\mu_l)^2)\right]. \quad (A40)$$

We use the difference $x_i^{(1)} - x_j^{(3)}$ as a regularization of the integral. This works because generically this difference is not equal to zero. Then we can express the denominator as an exponential function. Let $\beta = (x_i^{(1)} - x_j^{(3)})/|x_i^{(1)} - x_j^{(3)}|$ be the sign of this difference. The integral (A40) can be written as

$$\tilde{G}(x_i^{(1)}, x_j^{(3)}) = \frac{\beta}{\imath} \exp\left[x_i^{(1)}\mu_r + x_j^{(3)}\mu_l - \frac{n}{2a^2}((x_i^{(1)})^2 + (x_j^{(3)})^2)\right]$$

$$\times \int_0^\infty dt \int_{\mathbb{R}^2} ds_r ds_l \exp\left[-\frac{n(1 + a^2)}{a^2}|x_i^{(1)} - x_j^{(3)}|t + \imath\beta(s_r - s_l)t\right]$$

$$\times \exp\left[-\frac{a^2}{2n(1 + a^2)}((s_r + \imath\mu_r)^2 + (s_l + \imath\mu_l)^2)\right]$$

$$= \frac{-2\pi\imath n(1 + a^2)}{a^2}\beta \exp\left[x_i^{(1)}\mu_r + x_j^{(3)}\mu_l - \frac{n}{2a^2}((x_i^{(1)})^2 + (x_j^{(3)})^2)\right]$$

$$\times \int_0^\infty \exp\left[-\frac{n(1 + a^2)}{a^2}t^2 + \left(\beta(\mu_r - \mu_l) - \frac{n(1 + a^2)}{a^2}|x_i^{(1)} - x_j^{(3)}|\right)t\right]dt$$

$$= -\pi\imath\sqrt{\frac{\pi n(1 + a^2)}{a^2}}\beta \exp\left[-\frac{n}{4a^2}(x_i^{(1)} + x_j^{(3)})^2 + \frac{n}{4}(x_i^{(1)} - x_j^{(3)})^2\right]$$

$$\times \exp\left[\frac{1}{2}(x_i^{(1)} + x_j^{(3)})(\mu_r + \mu_l) + \frac{a^2}{4n(1 + a^2)}(\mu_r - \mu_l)^2\right]$$

$$\times \operatorname{erfc}\left[\sqrt{\frac{n(1 + a^2)}{4a^2}}|x_i^{(1)} - x_j^{(3)}| - \beta\sqrt{\frac{a^2}{4n(1 + a^2)}}(\mu_r - \mu_l)\right]. \quad (A41)$$

Plugging this result into Eq. (A39) we get the joint probability density for a fixed number of real eigenvalues given in Eq. (25). Moreover one can perform the sum over l to find the joint probability density of all eigenvalues given in Eq. (18).

APPENDIX B: TWO USEFUL INTEGRAL IDENTITIES

In this appendix we evaluate two integrals that have been used to simplify the expression for ρ_r and ρ_c.

1. Convolution of a Gaussian with an error function

Let $\operatorname{Re}\gamma^2 > -1$. We consider the integral

$$I(\alpha, \gamma) = \int_{\mathbb{R}} \exp[-(x + \alpha)^2]\operatorname{erf}(\gamma x)dx. \quad (B1)$$

The solution can be obtained by constructing an initial value problem. Since the Gaussian is symmetric and the error function antisymmetric around the origin we have

$$I(0, \gamma) = 0. \quad (B2)$$

The derivative is

$$\partial_\alpha I(\alpha, \gamma) = \int_{\mathbb{R}} \operatorname{erf}(\gamma x)\partial_x \exp[-(x + \alpha)^2]dx$$

$$= -\frac{2\gamma}{\sqrt{\pi}} \int_{\mathbb{R}} \exp[-(x + \alpha)^2 - \gamma^2 x^2]dx$$

$$= -\frac{2\gamma}{\sqrt{\gamma^2 + 1}} \exp\left[-\frac{\gamma^2\alpha^2}{\gamma^2 + 1}\right]. \quad (B3)$$

Integrating the derivative from 0 to α we find the desired result

$$\int_{\mathbb{R}} \exp[-(x + \alpha)^2]\operatorname{erf}(\gamma x)dx = -\sqrt{\pi}\operatorname{erf}\left(\frac{\gamma\alpha}{\sqrt{\gamma^2 + 1}}\right). \quad (B4)$$

This integral is needed to simplify the term (43).

Another integral identity which is used for the derivation of the level density of the real eigenvalues with positive chirality is given by

$$\int_{\mathbb{R}^2} \exp\left(-\alpha_1 x_1^2 - \alpha_2 x_2^2 + \beta_1 x_1 + \beta_2 x_2\right)$$

$$\times \operatorname{erf}\left(\frac{x_1 + \delta x_2}{\gamma} + \epsilon\right) dx_1 dx_2$$

$$= \frac{\pi}{\sqrt{\alpha_1 \alpha_2}} \exp\left[\frac{1}{4}\left(\frac{\beta_1^2}{\alpha_1} + \frac{\beta_2^2}{\alpha_2}\right)\right]$$

$$\times \operatorname{erf}\left(\frac{\alpha_2 \gamma \beta_1 + \alpha_1 \gamma \delta \beta_2 + 2\alpha_1 \alpha_2 \gamma^2 \epsilon}{2\sqrt{\alpha_1 \alpha_2 \gamma^2 (\alpha_1 \alpha_2 \gamma^2 + \alpha_1 \delta^2 + \alpha_2)}}\right). \quad (B5)$$

This identity is a direct consequence of the identity (B4). The constants α_i (with $\operatorname{Re}\alpha_i > 0$), β_i, $\gamma \neq 0$, δ, and ϵ are arbitrary.

2. Convolution of a Gaussian with a *sinus cardinalis*

The second integral enters in the simplification of the asymptotic behavior of ρ_c. It is the convolution integral

$$\tilde{I}(\alpha, \gamma) = \int_{\mathbb{R}} dx \exp\left[-(x + \alpha)^2\right]\operatorname{sinc}(\gamma x). \quad (B6)$$

To evaluate this integral we introduce an auxiliary integral to obtain a Fourier transform of a Gaussian, i.e.

$$\tilde{I}(\alpha, \gamma) = \frac{1}{\gamma}\int_0^\gamma d\tilde{\gamma}\int_{\mathbb{R}} dx \exp\left[-(x + \alpha)^2\right]\cos(\tilde{\gamma}x). \quad (B7)$$

First we integrate over x and then over $\tilde{\gamma}$ to obtain an expression in terms of error functions,

$$\tilde{I}(\alpha, \gamma) = \frac{\pi}{\gamma}\exp(-\alpha^2)\operatorname{Re}\operatorname{erf}\left(\frac{\gamma}{2} + \iota\alpha\right). \quad (B8)$$

APPENDIX C: THE $Z_{1/1}^\nu$-PARTITION FUNCTION

In this appendix we evaluate the partition function $Z_{1/1}^\nu$ which enters in the expression for the distribution of the chiralities over the real eigenvalues of D_W. The derivation below is along the lines given in Ref. [41].

We employ the parametrization (53) to evaluate

$$\lim_{\varepsilon \to 0} \operatorname{Im} \int \frac{\det(D_W - \hat{z}_1/(2n)\mathbb{1}_{2n+\nu})}{\det(D_W - \hat{x}_2/(2n)\mathbb{1}_{2n+\nu} \mp \iota\varepsilon\gamma_5)} P(D_W)d[D_W]$$

$$\stackrel{n \gg 1}{=} -\lim_{\varepsilon \to 0} \operatorname{Im} \int \frac{de^{\iota\varphi}}{2\pi\iota} de^\vartheta d\eta d\eta^* \operatorname{S}\det{}^\nu U \exp\left[-\hat{a}_8^2 \operatorname{Str}(U^2 + U^{-2})\right]$$

$$\times \exp\left[\pm\frac{\iota}{2}\operatorname{Str}\operatorname{diag}(\hat{x}_2 - \hat{m}_6, \hat{z}_1 - \hat{m}_6)(U - U^{-1}) - \left(\varepsilon \pm \frac{\iota\hat{\lambda}_7}{2}\right)\operatorname{Str}(U + U^{-1})\right]. \quad (C1)$$

We employ the same trick as in Ref. [41] to linearize the exponent in U and U^{-1} by introducing an auxiliary Gaussian integral over a supermatrix, i.e.

$$\exp\left[-\hat{a}_8^2 \operatorname{Str}(U^2 + U^{-2})\right] = \int d[\sigma]\exp\left[-\frac{1}{16\hat{a}_8^2}\operatorname{Str}\sigma^2 \pm \frac{\iota}{2}\operatorname{Str}\sigma(U - U^{-1})\right] \quad (C2)$$

with

$$\sigma = \begin{bmatrix} \sigma_1 & \eta_\sigma \\ \eta_\sigma^* & \iota\sigma_2 \end{bmatrix} \quad \text{and} \quad d[\sigma] = d\sigma_1 d\sigma_2 d\eta_\sigma d\eta_\sigma^*. \quad (C3)$$

After plugging Eq. (C2) in Eq. (C1) we diagonalize $\sigma = V\operatorname{diag}(s_1, s_2)V^\dagger$ and integrate over $V \in \mathrm{U}(1/1)$. We obtain

$$\lim_{\varepsilon \to 0} \operatorname{Im} \int \frac{\det(D_W - \hat{z}_1/(2n)\mathbb{1}_{2n+\nu})}{\det(D_W - \hat{x}_2/(2n)\mathbb{1}_{2n+\nu} \mp \iota\varepsilon\gamma_5)} P(D_W)d[D_W]$$

$$\stackrel{n \gg 1}{=} \frac{1}{16\pi\hat{a}_8^2}(\hat{z}_1 - \hat{x}_2)\lim_{\varepsilon \to 0} \operatorname{Im} \int \frac{ds_1 ds_2}{s_1 - \iota s_2}\left(\frac{s_1 - \hat{\lambda}_7 \pm \iota\varepsilon}{s_1 + \hat{\lambda}_7 \mp \iota\varepsilon}\frac{\iota s_2 + \hat{\lambda}_7 \mp \iota\varepsilon}{\iota s_2 - \hat{\lambda}_7 \pm \iota\varepsilon}\right)^{\nu/2}$$

$$\times \exp\left[-\frac{1}{16\hat{a}_8^2}((s_1 - \hat{x}_2 + \hat{m}_6)^2 - (s_2 + \iota\hat{z}_1 - \iota\hat{m}_6)^2)\right]Z_{1/1}^\nu\left(\sqrt{s_1^2 - (\hat{\lambda}_7 \mp \iota\varepsilon)^2}, \iota\sqrt{s_2^2 + (\hat{\lambda}_7 \mp \iota\varepsilon)^2}; \hat{a} = 0\right), \quad (C4)$$

which expresses the partition function at nonzero lattice spacing in terms of an integral over the partition function with one bosonic and one fermionic flavor at zero lattice spacing (56).

The resolvent $G_{1/1}$ is given by the derivative with respect to $\hat{z}_1$; see Eq. (49). To obtain a nonzero result we necessarily have to differentiate the prefactor $(\hat{z}_1 - \hat{x}_2)$. The distribution of the chiralities of the real eigenvalues of D_W follows from the imaginary part of the resolvent. The Efetov-Wegner term [61,62] appearing after diagonalizing σ is the normalization $Z_{1/1}^\nu(1, 1) = 1$ and vanishes when taking the imaginary part.

Two terms contribute to the imaginary part of the resolvent. First, the imaginary part of

$$\frac{1}{\pi}\lim_{\varepsilon\to 0}\text{Im}\left[\frac{1}{(s_1+\hat{\lambda}_7-\imath\varepsilon)^\nu}\right]=\frac{(-1)^{\nu-1}}{(\nu-1)!}\delta^{(\nu)}(s_1+\hat{\lambda}_7)$$ (C5)

is the νth derivative of the Dirac delta function. Second, when $|s_1|<|\hat{\lambda}_7|$, the imaginary part arising from the logarithmic contribution of $K_\nu(z)$, i.e.

$$K_\nu(z)=(-1)^{\nu+1}I_\nu(z)\log z+\frac{1}{z^\nu}\sum_{k=0}^{\infty}a_k z^k,$$ (C6)

also contributes to the imaginary part of the resolvent. The Bessel functions of the imaginary part of $Z_{1/1}^\nu(x_1,x_2,\hat{a}=0)$ combine into the two-flavor partition function $Z_{2/0}^\nu(x_1,x_2,\hat{a}=0)$. Adding both contributions we arrive at the result

$$\lim_{\varepsilon\to 0}\text{Im}\int\frac{\det(D_W-z_1\mathbb{1}_{2n+\nu})}{\det(D_W-x_2\mathbb{1}_{2n+\nu}\mp\imath\varepsilon\gamma_5)}P(D_W)d[D_W]$$

$$\stackrel{n\gg 1}{=}\frac{1}{16\pi\hat{a}_8^2}(\hat{z}_1-\hat{x}_2)\,\text{Im}\lim_{\varepsilon\to 0}\int\frac{ds_1 ds_2}{s_1-\imath s_2}(\imath s_2+\hat{\lambda}_7)^\nu(s_1+\hat{\lambda}_7)^\nu\exp\left[-\frac{1}{16\hat{a}_8^2}((s_1-\hat{x}_2+\hat{m}_6)^2-(s_2+\imath\hat{z}_1-\imath\hat{m}_6)^2)\right]$$

$$\times\left[\frac{\delta^{(\nu-1)}(s_1+\hat{\lambda}_7)}{(\nu-1)!(s_1-\hat{\lambda}_7)^\nu}\left(\frac{s_1^2-\hat{\lambda}_7^2}{s_2^2+\hat{\lambda}_7^2}\right)^{\nu/2}Z_{1/1}^\nu\left(\sqrt{s_1^2-\hat{\lambda}_7^2},\imath\sqrt{s_2^2+\hat{\lambda}_7^2};\hat{a}=0\right)\right.$$

$$\left.-\,\text{sign}(\hat{\lambda}_7)\Theta(|\hat{\lambda}_7|-|s_1|)(s_1^2+s_2^2)\frac{Z_{2/0}^\nu\left(\sqrt{s_1^2-\hat{\lambda}_7^2},\imath\sqrt{s_2^2+\hat{\lambda}_7^2};\hat{a}=0\right)}{[(s_1^2-\hat{\lambda}_7^2)(s_2^2+\hat{\lambda}_7^2)]^{\nu/2}}\right]$$ (C7)

yielding Eq. (84).

APPENDIX D: DERIVATIONS OF THE ASYMPTOTIC RESULTS GIVEN IN SEC. IV

The derivation of asymptotic limits of the spectral density can be quite nontrivial because of cancellations of the leading contributions so that a naive saddle point approximation cannot be used. In the subsections below, we derive asymptotic expressions for the average number of additional real modes (Appendix D 1), the level density of the right-handed modes (Appendix D 2), and the level density of the complex modes (Appendix D 3). In Appendix D 4 we consider the distribution of chirality over the real modes.

1. The average number of additional real modes

The limit of small lattice spacing is obvious and will not be discussed here. At large lattice spacing we rewrite Eq. (74) as

$$N_{\text{add}}=\int_{[0,2\pi]^2}\frac{d\Phi d\varphi}{8\pi^2}\cos[2\nu\Phi]\frac{1-\exp[-8(\hat{a}_8^2\sin^2\varphi+2\hat{a}_7^2\cos^2\varphi)\sin^2\Phi]}{\sin^2\Phi}.$$ (D1)

Since $\hat{a}_{7/8}$ are large we expand the angle Φ around the origin, in particular

$$\Phi=\frac{\delta\Phi}{\sqrt{\hat{a}_8^2\sin^2\varphi+2\hat{a}_7^2\cos^2\varphi}}\ll 1.$$ (D2)

Note that we have two equivalent saddle points at 0 and at π. We thus have

$$N_{\text{add}}=\int_{\mathbb{R}\times[0,2\pi]}\frac{d\delta\Phi d\varphi}{4\pi^2}\frac{1-\exp[-8\delta\Phi^2]}{\delta\Phi^2}\sqrt{\hat{a}_8^2\sin^2\varphi+2\hat{a}_7^2\cos^2\varphi}.$$ (D3)

The integral over $\delta\Phi$ is equal to $\sqrt{32\pi}$, and the integral over φ is the elliptic integral of the second kind. Hence we obtain the result (75).

2. The density of the additional real modes

We have two different cases for the behavior of ρ_r at large lattice spacing. To derive the large $\hat{a}$ asymptotics in the case $\hat{a}_8^2 = 0$ we rewrite Eq. (70) as a group integral, i.e.

$$\rho_r(\hat{x}) = \frac{1}{2^{15/2}\pi\hat{a}_7\sqrt{\hat{a}_8^2 + 2\hat{a}_6^2}} \int_{\mathbb{R}^2} d\hat{\lambda}_7 d\tilde{x} \int_{U(2)} d\mu(U) \det{}^\nu U \exp\left[-\frac{\hat{\lambda}_7^2}{16\hat{a}_7^2}\right]\left[\text{sign}(\tilde{x} - \hat{x}) - \text{erf}\left(\frac{\tilde{x} - \hat{x} + 2\hat{\lambda}_7}{\sqrt{32\hat{a}_8^2}}\right)\right](\tilde{x} - \hat{x})$$

$$\times \exp\left[-\hat{a}_8^2 \operatorname{tr}\left(U + U^{-1} - \frac{\tilde{x} + \hat{x}}{8\hat{a}_8^2}\mathbb{1}_2\right)^2 + \frac{\hat{a}_6^2\hat{a}_8^2}{\hat{a}_8^2 + 2\hat{a}_6^2}\operatorname{tr}^2\left(U + U^{-1} - \frac{\tilde{x} + \hat{x}}{8\hat{a}_8^2}\mathbb{1}_2\right)\right]$$

$$\times \exp\left[\frac{\tilde{x} - \hat{x}}{4}\operatorname{tr}\operatorname{diag}(1, -1)(U + U^{-1}) + \frac{\hat{\lambda}_7}{2}\operatorname{tr}(U - U^{-1})\right]. \tag{D4}$$

For $\hat{a}_8^2 = 0$ Eq. (D4) simplifies to

$$\rho_r(\hat{x}) = \frac{1}{256\pi\hat{a}_7\hat{a}_6}\int_{\mathbb{R}^2} d\hat{\lambda}_7 d\tilde{x}\int_{U(2)} d\mu(U)\det{}^\nu U \exp\left[-\frac{\hat{\lambda}_7^2}{16\hat{a}_7^2}\right][\text{sign}(\tilde{x} - \hat{x}) - \text{sign}(\tilde{x} - \hat{x} + 2\hat{\lambda}_7)](\tilde{x} - \hat{x})$$

$$\times \exp\left[-\frac{(\tilde{x} + \hat{x})^2}{64\hat{a}_6^2} + \frac{\tilde{x} - \hat{x}}{4}\operatorname{tr}\operatorname{diag}(1, -1)(U + U^{-1}) + \frac{\hat{\lambda}_7}{2}\operatorname{tr}(U - U^{-1})\right]$$

$$= \frac{1}{32\pi\hat{a}_7\hat{a}_6}\int_{\mathbb{R}} d\tilde{x}\left(\Theta(\tilde{x})\int_{-\infty}^{-\tilde{x}} - \Theta(-\tilde{x})\int_{-\tilde{x}}^{\infty}\right)d\hat{\lambda}_7\int_{U(2)} d\mu(U)\det{}^\nu U\tilde{x}$$

$$\times \exp\left[-\frac{\hat{\lambda}_7^2}{16\hat{a}_7^2} - \frac{(\tilde{x} + \hat{x})^2}{16\hat{a}_6^2} - \frac{\sqrt{\hat{\lambda}_7^2 - \tilde{x}^2}}{2}\operatorname{tr}(U - U^{-1})\right]. \tag{D5}$$

For the second equality we substituted $\tilde{x} \to 2\tilde{x} + \hat{x}$ and replaced the sign functions by the integration domains of $\hat{\lambda}_7$. Moreover we used the fact that the group integral only depends on $\sqrt{\hat{\lambda}_7^2 - \tilde{x}^2}$.

The saddle point equation of the U integral in Eq. (D5) gives four saddle points,

$$U = \pm\iota\mathbb{1}_2, \quad\text{and}\quad U = \pm\iota\operatorname{diag}(1, -1). \tag{D6}$$

The saddle points which are proportional to unity are algebraically suppressed while the contribution of the other two saddle points is the same. We thus find

$$\rho_r(\hat{x}) = \frac{1}{8\pi^2\hat{a}_7\hat{a}_6}\int_0^\infty d\tilde{x}\int_{\tilde{x}}^\infty d\hat{\lambda}_7\frac{\tilde{x}}{\sqrt{\hat{\lambda}_7^2 - \tilde{x}^2}}$$

$$\times \cosh\left(\frac{\tilde{x}\hat{x}}{8\hat{a}_6^2}\right)\exp\left[-\frac{\hat{\lambda}_7^2}{16\hat{a}_7^2} - \frac{\tilde{x}^2 + \hat{x}^2}{16\hat{a}_6^2}\right]. \tag{D7}$$

After substituting $\hat{\lambda}_7 \to \tilde{x}\cosh\vartheta$ the integral over ϑ yields the first case of Eq. (79).

For $\hat{a}_8 \neq 0$ we again start with Eq. (70). The integration over the two error functions, see Eq. (71), makes it difficult to evaluate the result directly, particularly when $\hat{a}_7 \neq 0$. As long as $\hat{a}_7$ is finite, the second error function does not yield anything apart from giving a Gaussian cutoff to the integral. The imaginary part of the argument of the second error function shows strong oscillations resulting in cancellations. These oscillations also impede a numerical evaluation of the integrals for large lattice spacing.

Let $\hat{a}_6 = 0$ to begin with. A nonzero value of $\hat{a}_6$ can be introduced later by a convolution with a Gaussian in $\hat{x}$. To obtain the correct contribution from the first term we consider a slight modification of ρ_r,

$$I(X, \alpha) = \int_{[0,2\pi]^2} d\varphi_1 d\varphi_2 \sin^2\left[\frac{\varphi_1 - \varphi_2}{2}\right]e^{\iota\nu(\varphi_1 + \varphi_2)}\frac{\hat{k}(X, \varphi_1, \varphi_2) - \hat{k}(X, \varphi_2, \varphi_1)}{\cos\varphi_2 - \cos\varphi_1} \tag{D8}$$

with

$$\hat{k}(X, \varphi_1, \varphi_2) = \exp[4\hat{a}_8^2(\cos\varphi_1 - X)^2 - 4\hat{a}_8^2(\cos\varphi_2 - X)^2 - 4\hat{a}_7^2(\sin\varphi_1 + \sin\varphi_2)^2]$$

$$\times [\text{erf}[\sqrt{8}\hat{a}_8(X - \cos\varphi_1)] + \text{erf}[2\hat{a}_8\alpha(\cos\varphi_1 - X)]]. \tag{D9}$$

The variable X plays the role of $\hat{x}/(8\hat{a}_8^2)$. The error function with the constant α replaces the second error function in Eq. (71) and is of order one in the limit $\hat{a} \to \infty$. It regularizes the integral and its contribution will be removed at the end. However it has to fulfill some constraints to guarantee the existence of the saddle points

$$\varphi_1^{(0)}, \varphi_2^{(0)} \in \{\pm \arccos X\} \quad \text{with} \quad X \in [-1, 1]. \tag{D10}$$

Nevertheless these saddle points are independent of α. The saddle point $\varphi_1^{(0)} = \varphi_2^{(0)}$ is algebraically suppressed in comparison to $\varphi_1^{(0)} = -\varphi_2^{(0)}$ due to the $\sin^2$ factor in the measure. Expanding about the saddle points yields

$$I(X, \alpha) \propto \frac{\Theta(1 - |X|)}{\hat{a}_8} \int_{\mathbb{R}^2} \frac{d\delta\varphi_1 d\delta\varphi_2}{\delta\varphi_1 + \delta\varphi_2} \exp\left[-\frac{\hat{a}_7^2}{\hat{a}_8^2} \gamma^2 (\delta\varphi_1 + \delta\varphi_2)^2\right] [\exp(\delta\varphi_1^2 - \delta\varphi_2^2)(\text{erf}(\sqrt{2}\delta\varphi_1) - \text{erf}(\alpha\delta\varphi_1))$$
$$+ \exp(\delta\varphi_2^2 - \delta\varphi_1^2)(\text{erf}(\sqrt{2}\delta\varphi_2) - \text{erf}(\alpha\delta\varphi_2))] \tag{D11}$$

with

$$\gamma(X) = \frac{X}{\sqrt{1 - X^2}}. \tag{D12}$$

In the next step we change the coordinates to center-of-mass-relative coordinates, i.e. $\Phi = \delta\varphi_1 + \delta\varphi_2$ and $\Delta\varphi = \delta\varphi_1 - \delta\varphi_2$, and find

$$I(X, \alpha) \propto \frac{\Theta(1 - |X|)}{\hat{a}_8} \int_{\mathbb{R}^2} \frac{d\Phi d\Delta\varphi}{\Phi} \exp\left[-\frac{\hat{a}_7^2}{\hat{a}_8^2} \gamma^2(X)\Phi^2\right] \left[\exp(\Phi\Delta\varphi)\left(\text{erf}\left(\frac{\Phi + \Delta\varphi}{\sqrt{2}}\right) - \text{erf}\left(\frac{\alpha}{2}(\Phi + \Delta\varphi)\right)\right)\right.$$
$$\left. + \exp(-\Phi\Delta\varphi)\left(\text{erf}\left(\frac{\Phi - \Delta\varphi}{\sqrt{2}}\right) - \text{erf}\left(\frac{\alpha}{2}(\Phi - \Delta\varphi)\right)\right)\right]. \tag{D13}$$

We perform an integration by parts in $\Delta\varphi$ yielding Gaussian integrals in $\Delta\varphi$ which evaluate to

$$I(X, \alpha) \propto \frac{\Theta(1 - |X|)}{\hat{a}_8} \int_{\mathbb{R}} \frac{d\Phi}{\Phi^2} \exp\left[-\frac{\hat{a}_7^2}{\hat{a}_8^2} \gamma^2(X)\Phi^2\right] \left[\exp\left(-\frac{\Phi^2}{2}\right) - \exp\left(-\frac{1 + \alpha^2}{\alpha^2}\Phi^2\right)\right]. \tag{D14}$$

The $1/\Phi^2$ term can be exponentiated by introducing an auxiliary integral and the resulting Gaussian over Φ can be performed. We obtain

$$I(X, \alpha) \propto \Theta(1 - |X|) \int_{(1+\alpha^2)/\alpha^2}^{1/2} \frac{dt}{\sqrt{\hat{a}_7^2\gamma^2(X) + \hat{a}_8^2 t}} \propto \frac{\Theta(1 - |X|)}{\hat{a}_8^2} \left(\sqrt{\hat{a}_7^2\gamma^2(X) + \frac{\hat{a}_8^2}{2}} - \sqrt{\hat{a}_7^2\gamma^2(X) + \frac{1 + \alpha^2}{\alpha^2}\hat{a}_8^2}\right). \tag{D15}$$

The contribution of the artificial term depending on α can be readily read off, but it fixes the integral only up to an additive constant. This constant can be determined by integrating the result over $\hat{x}$ which has to agree with the large $\hat{a}$ limit of N_{add}; cf. Eq. (75). It turns out that this constant is equal to zero. The overall constant is also obtained by comparing to N_{add}.

The convolution with the Gaussian distribution generating $\hat{a}_6$ does not give something new in the limit of large lattice spacing. The width of this Gaussian scales with $\hat{a}$ while the density ρ_{r} has support on $\hat{a}^2$, so that it becomes a Dirac delta function in the large $\hat{a}$ limit.

3. The density of the complex eigenvalues

Let $\hat{a}_8 > 0$ and $\hat{a} \gg 1$. Then we perform a saddle point approximation of Eq. (80) in the integration variables $\varphi_{1/2}$. The saddle points are given by

$$\varphi_1^{(0)} = -\varphi_2^{(0)} = \pm \arccos\left(\frac{\hat{x}}{8\hat{a}_8^2}\right) \quad \text{with} \quad \hat{x} \in [-8\hat{a}_8^2, 8\hat{a}_8^2]. \tag{D16}$$

We have also the saddle points $\varphi_1^{(0)} = \varphi_2^{(0)}$ if $\hat{a}_7 = 0$. However they are algebraically suppressed due to the Haar measure. Notice the two saddle points in Eq. (D16) yield the same contribution. After the integration over the massive modes about the saddle point we find the first case of Eq. (83). In the calculation we used the convolution integral derived in Appendix B 2.

Let us now look at the case with $\hat{a}_8 = 0$. Then we have

$$\rho_c(\hat{z}) \overset{\hat{a} \gg 1}{=} \frac{|\hat{y}|}{4\pi^2 \sqrt{16\pi\hat{a}_6^2} \sqrt{16\pi\hat{a}_7^2}} \exp\left[-\frac{\hat{x}^2}{16\hat{a}_6^2}\right] \int_{\mathbb{R}} d\hat{\lambda}_7 \exp\left[-\frac{\hat{\lambda}_7^2}{16\hat{a}_7^2}\right]$$

$$\times \int_{[0,2\pi]^2} d\varphi_1 d\varphi_2 \sin^2\left[\frac{\varphi_1 - \varphi_2}{2}\right] \cos[\nu(\varphi_1 + \varphi_2)] \text{sinc}[\hat{y}(\cos\varphi_1 - \cos\varphi_2)] \exp[\iota\hat{\lambda}_7(\sin\varphi_1 + \sin\varphi_2)]. \quad (D17)$$

The integrals over the angles can be rewritten as a group integral over U(2),

$$\int_{[0,2\pi]^2} d\varphi_1 d\varphi_2 \sin^2\left[\frac{\varphi_1 - \varphi_2}{2}\right] e^{\nu\iota(\varphi_1 + \varphi_2)} \text{sinc}[\hat{y}(\cos\varphi_1 - \cos\varphi_2)] \exp[\iota\hat{\lambda}_7(\sin\varphi_1 + \sin\varphi_2)]$$

$$= 2\pi^2 \int_{U(2)} d\mu(U) \det{}^\nu U \exp\left[\frac{1}{2} \text{tr}(\Lambda U - \Lambda^* U^\dagger)\right], \quad (D18)$$

with $\Lambda = \text{diag}(\hat{\lambda}_7 + \iota\hat{y}, \hat{\lambda}_7 - \iota\hat{y})$. This integral only depends on the quantity $\sqrt{\hat{\lambda}_7^2 + \hat{y}^2}$ because the angle of the combined complex variable $\hat{\lambda} + \iota\hat{y}$ can be absorbed into U, i.e.

$$\int_{U(2)} d\mu(U) \det{}^\nu U \exp\left[\frac{1}{2} \text{tr}(\Lambda U - \Lambda^* U^\dagger)\right] = \int_{U(2)} d\mu(U) \det{}^\nu U \exp\left[\frac{\sqrt{\hat{\lambda}_7^2 + \hat{y}^2}}{2} \text{tr}(U - U^\dagger)\right]. \quad (D19)$$

The variable $\hat{y}$ as well as the integration variable $\hat{\lambda}_7$ are of the order $\hat{a}$. Therefore we can perform a saddle point approximation and end up with

$$\rho_c(\hat{z}) \overset{\hat{a} \gg 1}{=} \frac{|\hat{y}|}{\pi \sqrt{16\pi\hat{a}_6^2} \sqrt{16\pi\hat{a}_7^2}} \exp\left[-\frac{\hat{x}^2}{16\hat{a}_6^2}\right] \int_{\mathbb{R}} d\hat{\lambda}_7 \frac{\exp[-\hat{\lambda}_7^2/(16\hat{a}_7^2)]}{\sqrt{\hat{\lambda}_7^2 + \hat{y}^2}}, \quad (D20)$$

resulting in the second case of Eq. (83).

4. The distribution of chirality over the real eigenvalues

In this appendix we derive the large $\hat{a}$ limit of $\rho_\chi(\hat{x})$ for $\hat{a}_8 > 0$ given in Eq. (94). The case $\hat{a}_8 = 0$ reduces $\rho_\chi(\hat{x})$ to the result (60) and will not be discussed in this section. We set $\hat{a}_{6/7} = 0$ to begin with and introduce them later on.

The best way to obtain the asymptotics for large lattice spacing is to start with Eq. (C1) with $\hat{m}_6 = \hat{\lambda}_7 = \varepsilon = 0$. The integral does not need a regularization since the $\hat{a}_8$ term guarantees the convergence. We also omit the sign in front of the linear trace terms in the Lagrangian because we can change $U \to -U$.

In the first step we substitute $\eta \to e^{\iota\varphi}\eta$ and $\eta^* \to e^{\vartheta}\eta^*$. Then the measure is $d\varphi d\vartheta d\eta d\eta^*$ and the parametrization of U is given by

$$U = \begin{bmatrix} e^\vartheta & 0 \\ 0 & e^{\iota\varphi} \end{bmatrix} \begin{bmatrix} 1 & \eta^* \\ \eta & 1 \end{bmatrix}, \qquad U^{-1} = \begin{bmatrix} 1 + \eta^*\eta & -\eta^* \\ -\eta & 1 - \eta^*\eta \end{bmatrix} \begin{bmatrix} e^{-\vartheta} & 0 \\ 0 & e^{-\iota\varphi} \end{bmatrix}. \quad (D21)$$

There are two saddle points in the variables ϑ and φ, i.e.

$$e^{\vartheta_0} = -\frac{\iota\hat{x}_2}{8\hat{a}_8^2} + \sqrt{1 - \left(\frac{\hat{x}_2}{8\hat{a}_8^2}\right)^2}, \qquad e^{\iota\varphi_0} = -\frac{\iota\hat{z}_1}{8\hat{a}_8^2} + L\sqrt{1 - \left(\frac{\hat{z}_1}{8\hat{a}_8^2}\right)^2} \quad (D22)$$

with $L = \pm 1$. Moreover, the variables $\hat{z}_1$, $\hat{x}_2$ have to be in the interval $[-8\hat{a}_8^2, 8\hat{a}_8^2]$ or else the contributions will be exponentially suppressed. We have no second saddle point for the variable ϑ since the real part of the exponential has to be positive definite. Other saddle points which can be reached by shifting φ and ϑ by $2\pi\iota$ independently are forbidden since they are not accessible in the limit $\hat{a}_8 \to \infty$. Notice that the saddle point solutions (D22) are phases, i.e. $|e^{\vartheta_0}| = |e^{\iota\varphi_0}| = 1$.

In the second step we expand the integration variables

$$e^{\vartheta} = e^{\vartheta_0}\left(1 + \frac{\delta\vartheta}{\sqrt{(8\hat{a}_8^2)^2 - \hat{x}_2^2}}\right), \qquad e^{\iota\varphi} = e^{\iota\varphi_0}\left(1 + \frac{\iota\delta\varphi}{\sqrt{(8\hat{a}_8^2)^2 - \hat{z}_1^2}}\right). \tag{D23}$$

All terms in front of the exponential as well as of the Grassmann variables are replaced by the saddle point solutions ϑ_0 and φ_0. The resulting Gaussian integrals over the variables $\delta\vartheta$ and $\delta\varphi$ yield

$$\mathrm{Im}\int d\mu(U)\mathrm{Sdet}^{\nu}U\exp\left[-\hat{a}_8^2\,\mathrm{Str}(U-U^{-1})^2 + \frac{\iota}{2}\,\mathrm{Str}\,\mathrm{diag}(\hat{x}_2,\hat{z}_1)(U-U^{-1})\right]$$

$$\propto \sum_{L\in\{\pm 1\}}\mathrm{Im}\frac{\exp[\nu(\vartheta_0 - \iota\varphi_0)]}{\sqrt{(8\hat{a}_8^2)^2 - \hat{x}_2^2}\sqrt{(8\hat{a}_8^2)^2 - \hat{z}_1^2}}\exp\left[-\frac{\hat{x}_2^2 - \hat{z}_1^2}{16\hat{a}_8^2}\right]$$

$$\times \int d\eta d\eta^*(1 - \eta^*\eta)^{\nu}\exp[-2\hat{a}_8^2(e^{\vartheta_0} + e^{-\iota\varphi_0})(e^{-\vartheta_0} + e^{\iota\varphi_0})\eta^*\eta]. \tag{D24}$$

After the integration over the Grassmann variables we have two terms, one is of order one, and the other one of order $\hat{a}_8^2$ which exceeds the first term for $\hat{a}_8 \gg 1$. Hence we end up with

$$\mathrm{Im}\int d\mu(U)\mathrm{Sdet}^{\nu}U\exp\left[-\hat{a}_8^2\,\mathrm{Str}(U-U^{-1})^2 + \frac{\iota}{2}\,\mathrm{Str}\,\mathrm{diag}(\hat{x}_2,\hat{z}_1)(U-U^{-1})\right]$$

$$\propto \sum_{L\in\{\pm 1\}}\mathrm{Im}\frac{1}{\sqrt{(8\hat{a}_8^2)^2 - \hat{x}_2^2}\sqrt{(8\hat{a}_8^2)^2 - \hat{z}_1^2}}\left(\frac{-\iota\hat{x}_2 + \sqrt{(8\hat{a}_8^2)^2 - \hat{x}_2^2}}{-\iota\hat{z}_1 + L\sqrt{(8\hat{a}_8^2)^2 - \hat{z}_1^2}}\right)^{\nu}$$

$$\times \left[\left(\frac{\hat{z}_1 - \hat{x}_2}{8\hat{a}_8^2}\right)^2 + \left(\sqrt{1 - \left(\frac{\hat{x}_2}{8\hat{a}_8^2}\right)^2} + L\sqrt{1 - \left(\frac{\hat{z}_1}{8\hat{a}_8^2}\right)^2}\right)^2\right]\exp\left[-\frac{\hat{x}_2^2 - \hat{z}_1^2}{16\hat{a}_8^2}\right]. \tag{D25}$$

Notice that both saddle points, $L\in\{\pm 1\}$, give a contribution for independent variables $\hat{z}_1$ and $\hat{x}_2$. To obtain the resolvent we differentiate this expression with respect to $\hat{z}_1$ and put $\hat{z}_1 = \hat{x}_1$ afterwards. The first term between the large brackets and the second term for $L = -1$ are quadratic in $\hat{z}_1 - \hat{x}_2$ and do not contribute to the resolvent. For $L = +1$ we obtain

$$\mathrm{Im}\partial_{\hat{z}_1}|_{\hat{z}_1=\hat{x}_2}\int d\mu(U)\mathrm{Sdet}^{\nu}U\exp\left[-\hat{a}_8^2\,\mathrm{Str}(U-U^{-1})^2 + \frac{\iota}{2}\,\mathrm{Str}\,\mathrm{diag}(\hat{x}_2,\hat{z}_1)(U-U^{-1})\right] \propto \nu\frac{\Theta(8\hat{a}_8^2 - |\hat{x}_2|)}{2\hat{a}_8^2\sqrt{64\hat{a}_8^4 - \hat{x}_2^2}}. \tag{D26}$$

This limit yields the square root singularity. The normalization of ρ_{χ} to ν yields an overall normalization constant of $2\hat{a}_8^2/\pi$.

The effect of $\hat{a}_6$ is introduced by the integral

$$\frac{1}{4\hat{a}_6\sqrt{\pi}}\int_{\mathbb{R}}\exp\left[-\frac{\hat{m}_6^2}{16\hat{a}_6^2}\right]\rho_{\chi}(\hat{x} - \hat{m}_6)|_{\hat{a}_6=0}d\hat{m}_6 = \frac{1}{4\hat{a}_6\sqrt{\pi}}\int_{\mathbb{R}}\exp\left[-\frac{\hat{m}_6^2}{16\hat{a}_6^2}\right]\frac{\Theta(8\hat{a}_8^2 - |\hat{x} - \hat{m}_6|)}{\pi\sqrt{(8\hat{a}_8^2)^2 - (\hat{x} - \hat{m}_6)^2}}d\hat{m}_6$$

$$= \frac{1}{4\hat{a}_6\pi^{3/2}}\int_0^{\pi}\exp\left[-\frac{4\hat{a}_8^4}{\hat{a}_6^2}\left(\cos\varphi + \frac{\hat{x}}{8\hat{a}_8^2}\right)^2\right]d\varphi. \tag{D27}$$

In the large $\hat{a}$ limit this evaluates to

$$\frac{1}{4\hat{a}_6\sqrt{\pi}}\int_{\mathbb{R}}\exp\left[-\frac{\hat{m}_6^2}{16\hat{a}_6^2}\right]\frac{\Theta(8\hat{a}_8^2 - |\hat{x} - \hat{m}_6|)}{\pi\sqrt{(8\hat{a}_8^2)^2 - (\hat{x} - \hat{m}_6)^2}}d\hat{m}_6 \stackrel{\hat{a}\gg 1}{=} \nu\frac{\Theta(8\hat{a}_8^2 - |\hat{x}|)}{\pi\sqrt{64\hat{a}_8^4 - \hat{x}^2}}, \tag{D28}$$

which is exactly the same Heaviside distribution with the square root singularities in the interval $[-8\hat{a}_8^2, 8\hat{a}_8^2]$ of Eq. (D26). The introduction of $\hat{a}_7$ follows from Eq. (88). We have to replace $\hat{a}_6^2 \rightarrow \hat{a}_6^2 + \hat{a}_7^2$ and sum the result over the index j with the

prefactor $\exp(-8\hat{a}_7^2)[I_{j-\nu}(8\hat{a}_7^2) - I_{j-\nu}(8\hat{a}_7^2)]$. The intermediate result (D28) is independent of $\hat{a}_7$ and linear in the index; in the sum this index is j. The sum over j can be performed according to

$$\sum_{j=1}^{\infty} j(I_{j-\nu}(8\hat{a}_7^2) - I_{j+\nu}(8\hat{a}_7^2)) = \nu \exp(8\hat{a}_7^2) \tag{D29}$$

resulting in the asymptotic result (94).

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