PHYSICAL REVIEW D **88,** 094021 (2013)

Quark number susceptibilities at high temperatures

A. Bazavov,[1] H.-T. Ding,[1,2] P. Hegde,[3] F. Karsch,[1,3] C. Miao,[4] Swagato Mukherjee,[1] P. Petreczky,[1] C. Schmidt,[3] and A. Velytsky[1]

[1]*Physics Department, Brookhaven National Laboratory, Upton, New York 11973, USA*
[2]*Physics Department, Columbia University, New York, New York 10027, USA*
[3]*Fakultät für Physik, Universität Bielefeld, D-33615 Bielefeld, Germany*
[4]*Institute of Nuclear Physics, Johannes Gutenberg-Universität Mainz, Johann-Joachim-Becher-Weg 45, D-55099 Mainz, Germany*
(Received 4 October 2013; published 26 November 2013)

We calculate second and fourth order quark number susceptibilities for 2 + 1 flavor QCD in the high temperature region. In our study, we use two improved staggered fermion formulations, namely, the highly improved staggered quark formulation and the so-called p4 formulation, as well as several lattice spacings. Second order quark number susceptibilities are calculated using both improved staggered fermion formulations, and we show that in the continuum limit the two formulations give consistent results. The fourth order quark number susceptibilities are studied only using the p4 formulation and at nonzero lattice spacings. We compare our results on quark number susceptibilities with recent weak coupling calculations and find that these agree reasonably well with the lattice calculations within the estimated uncertainties.

DOI: 10.1103/PhysRevD.88.094021 PACS numbers: 12.38.−t, 12.38.Gc

I. INTRODUCTION

At high temperatures, strongly interacting matter undergoes a deconfinement transition to a new state, where thermodynamics can be described in terms of quark and gluonic degrees of freedom. Studying the properties of this new state of matter is the subject of a large effort in lattice QCD, see Refs. [1,2]. This is in part due to the fact that nonperturbative effects could be important even at very high temperatures due to infrared problems [3]. Therefore, it is important to clarify using lattice QCD calculations at what temperatures the deconfined medium can be described as weakly interacting quark-gluon gas. This is especially important in light of the recent experimental findings showing that the matter created in ultrarelativistic heavy ion collisions behaves like a strongly coupled liquid [4].

Fluctuations of conserved charges are known to be sensitive probes of deconfinement and suitable for testing the weakly (or strongly) interacting nature of the deconfined medium. They are defined as the derivatives of the pressure with respect to the corresponding chemical potentials. Fluctuations of conserved charges are expected to be exponentially small in the low temperature region where the conserved charges are carried by massive hadrons. However, they are not suppressed at high temperatures, where the dominant degrees of freedom are light quarks. Therefore, fluctuations of conserved charges are good probes of deconfinement.

In 2 + 1 flavor QCD, there are three chemical potentials corresponding to baryon number, electric charge, and strangeness, or equivalently to u, d, and s quark chemical potentials. Since at high temperatures the dominant degrees of freedom are quarks and gluons, the quark number basis provides a natural way to study the fluctuations. In this paper, we study second and fourth order quark number fluctuations, also known as quark number susceptibilities, defined as

$$\chi_{2n}^q = \frac{\partial^{2n}(p/T^4)}{\partial(\mu_q/T)^{2n}}\bigg|_{\mu_q=0}, \qquad q = u, d, s, \quad n = 1, 2. \quad (1)$$

Though in our calculations the u and d quark masses are degenerate, the corresponding chemical potential is always assumed to be different, i.e., we consider single flavor susceptibilities. Often in weak coupling calculations the chemical potential for different degenerate quark flavors is considered to be the same. In this case, one effectively calculates the baryon number susceptibilities (up to normalization factors) that are different from the single flavor quark number susceptibilities defined in Eq. (1). In this paper, we are interested in the high temperature behavior of the quark number susceptibilities and comparison of the lattice results with weak coupling calculations. To better control the continuum extrapolation, we used two different improved staggered quark formulations in our calculations, namely, the so-called p4 action [5] and the highly improved staggered quark (HISQ) action [6].

Fluctuations of conserved charges have been studied in lattice QCD for many years and confirm the expected temperature dependence discussed above. Second order quark number susceptibilities have been studied by several groups [7–14]. Fourth order fluctuations have also been studied using the so-called p4 improved staggered fermion action [15–17]. More recently, they have been studied using HISQ and stout actions [18–21].

At sufficiently high temperatures, one should be able to describe quark number susceptibilities using weak coupling techniques. Lattice QCD calculations of the quark number susceptibilities thus can provide useful tests for the range of validity of these weak coupling approaches. Second order quark number susceptibilities have been calculated using resummed perturbation theory [22–24] as well as the dimensionally reduced effective theory [25,26] for some time. The fourth order quark number susceptibilities have been first calculated in perturbation theory in Ref. [27], and very recently have also been studied using resummed perturbation theory [28–31]. In this paper, we make precision tests of the applicability of various weak coupling calculation techniques by performing continuum extrapolated lattice QCD calculations of second order quark number susceptibilities at even higher temperatures. The analysis of various second and fourth order susceptibilities [19] suggests that the deconfined medium becomes weakly interacting for $T > 300$ MeV.

We extend our earlier calculations of diagonal quark number susceptibilities in $2+1$ flavor QCD to higher temperatures and compare them with perturbative calculations. The rest of the paper is organized as follows. In Sec. II we present the details of the lattice simulations. Section III is dedicated to the discussion of the cutoff effects of quark number susceptibilities in the free theory. Our main numerical results are summarized in Sec. IV. In Sec. V we compare our lattice results with the results of weak coupling calculations. Finally, Sec. VI presents our conclusions. Some preliminary results have been presented in conference proceedings [32,33].

II. DETAILS OF THE LATTICE SIMULATIONS

We performed calculations in $2+1$ flavor QCD using p4 and HISQ actions at the physical value of the strange quark mass m_s. The gauge configurations have been generated using the rational hybrid Monte Carlo algorithm [34]. The lattice spacing has been fixed using the r_1 and r_0 scales defined in terms of the static potential

$$r^2 \frac{dV}{dr}\bigg|_{r_x} = C_x, \quad x = 0, 1, \tag{2}$$

where $C_0 = 1.65$ and $C_1 = 1.0$. The parameter r_0 is also known as the Sommer scale [35]. As in Ref. [36], we use the values $r_0 = 0.468$ fm and $r_1 = 0.3106$ fm. The lattice spacing in units of r_0 and r_1 as a function of the bare gauge coupling was given in Ref. [37] for the p4 action and in Ref. [36] for the HISQ action. The quark number susceptibilities can be expressed in terms of the quark matrix and the corresponding formulas were given in Refs. [15,16,38]. The necessary operators are evaluated using the random noise method (see Ref. [38] for details).

We calculated second order quark number susceptibilities using the HISQ action on lattices with temporal extent $N_\tau = 4$, 6, 8, 10, 12, and 16, and aspect ratio $N_\sigma/N_\tau = 4$, with N_σ denoting the spatial extent of the lattice. For light quark masses, we used $m_l = m_s/20$ corresponding to a pion mass of 160 MeV in the continuum limit [36]. Our calculations covered a wide temperature range from 200 to about 950 MeV (1400 MeV for $N_\tau = 4$). We used 20 random noise vectors to evaluate the operators needed for quark number susceptibilities for $N_\tau = 4$ and 6 lattices, 50 random noise vectors for $N_\tau = 8$ lattices, 100 random noise vectors for $N_\tau = 10$ lattices, and 200 random noise vectors for $N_\tau = 12$. We accumulated 3000 to 8000 molecular dynamics trajectories of unit length for each temperature value.

For the p4 action, the calculations have been performed on $32^3 \times N_\tau$ lattices with temporal extent $N_\tau = 6$, 8, and 12. The light quark mass in this calculation was $m_l = m_s/10$ corresponding to a pion mass of 220 MeV in the continuum limit [37]. For $N_\tau = 6$ and 8 lattices, we accumulated between 10,000 and 25,000 trajectories, while for $N_\tau = 12$ we accumulated between 3000 and 10,000 trajectories. The length of the molecular dynamic trajectory here was 0.5. On $N_\tau = 6$ and $N_\tau = 8$ lattices, we calculated quark number susceptibilities both for the strange and light quarks. We typically used 196 random vectors in these calculations. For $N_\tau = 12$ in the light quark sector, we only calculated second order quark number susceptibilities. Here we used 96 random vectors to evaluate the necessary operators. In the strange quark sector, both second and fourth order quark number susceptibilities have been evaluated and 196 random vectors have been used. Note that in the studied temperature range, the slightly larger than physical light quark masses do not have any significant effect [39,40]. In fact, at these high temperatures, the light quark masses in temperature units are negligibly small and can be considered to be zero.

III. QUARK NUMBER SUSCEPTIBILITIES IN THE FREE THEORY

As a first step toward understanding the cutoff dependence of quark number susceptibilities and making continuum extrapolations at high temperature, let us review their cutoff dependence in the noninteracting massless theory. Since we consider the massless case, the index q in the quark number susceptibilities will be omitted. The cutoff dependence of quark number susceptibilities has been studied in Refs. [8,15,41]. The p4 action and the HISQ action both contain a three-link term in the Dirac operator and completely eliminate $\mathcal{O}(1/N_\tau^2)$ discretization errors at tree level. In the case of the p4 action, bended three-link terms are used, while the HISQ action has a straight three-link path known as the Naik term [42]. The different types of the three-link terms not only eliminate the $\mathcal{O}(1/N_\tau^2)$ discretization effects but also have the same corrections at order $1/N_\tau^4$ [41,43]. The differences between the Naik action and the p4 action appear at higher orders of $1/N_\tau$ [41,43]. In Ref. [41], the N_τ dependence of quark number susceptibilities was calculated analytically up to $\mathcal{O}(1/N_\tau^6)$,

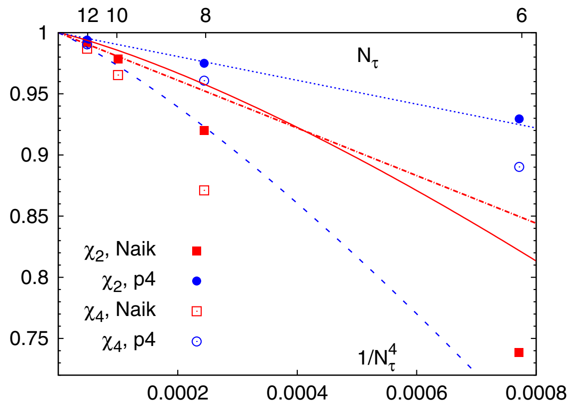

FIG. 1 (color online). Second (filled symbols) and fourth order (open symbols) quark number susceptibilities in the free theory as functions of N_τ^{-4} normalized by the corresponding continuum Stefan-Boltzmann values. Also shown in the figure are the free theory results expanded in $1/N_\tau$ up to $\mathcal{O}(N_\tau^6)$ [41]. These are shown as dotted, solid, dashed, and dashed-dotted lines corresponding to χ_2 and p4 action, χ_2 and Naik action, χ_4 and p4 action, and χ_4 and Naik action, respectively.

$$\begin{aligned}
\chi_2 &= 1 + a_{24}\frac{\pi^4}{N_\tau^4} + a_{26}\frac{\pi^6}{N_\tau^6},\\
\chi_4 &= \frac{6}{\pi^2}\left(1 + a_{44}\frac{\pi^4}{N_\tau^4} + a_{46}\frac{\pi^6}{N_\tau^6}\right),\\
a_{24} &= -\frac{93}{70}, \qquad a_{44} = -\frac{21}{10},\\
a_{26} &= -\frac{381}{70}, \qquad a_{46} = \frac{62}{945}, \qquad \text{Naik}\\
a_{26} &= \frac{127}{3150}, \qquad a_{46} = -\frac{62}{7}, \qquad \text{p4.}
\end{aligned} \tag{3}$$

Note, however, that numerically higher order terms in $1/N_\tau$ may be important for $N_\tau \le 8$. As one can see from the above equation, there is a large coefficient in front of the $1/N_\tau^6$ term in χ_2 for the Naik action and a small one for the p4 action. For χ_4, the situation is just the opposite. In Fig. 1, we show the complete results on the cutoff dependence of χ_2 and χ_4 and contrast it with the analytic results. For second order quark number susceptibility and p4 action, the analytic result expanded up to order $1/N_\tau^6$ describes the complete cutoff dependence fairly accurately. In all other cases, the truncated analytic results give only a qualitative description of the complete result. The complete result indicates a quite small cutoff dependence for both χ_2 and χ_4 in the case of the p4 action. It should be stressed that the free theory results in general cannot be used for a quantitative description of the cutoff dependence of the lattice data, as higher order perturbative corrections are important and could be significant [43]. In particular, there are cutoff effects proportional to g^{2n}/N_τ^2 both for the p4 and HISQ actions. Yet the analysis of the cutoff dependence in the free theory provides an important starting point for the discussion of the cutoff effects in the high temperature lattice calculations.

IV. RESULTS

In this section we present our numerical results for χ_2^q and χ_4^q and discuss the cutoff effects in quark number susceptibilities, as well as the details of the continuum extrapolations. Our main result is summarized in Fig. 3. Readers not interested in technical details are advised to skip the following text till the end of Sec. IV B where this result is discussed.

A. Numerical results on second order quark number susceptibilities

We start the discussion of our numerical results with the second order light and strange quark number susceptibilities. In Fig. 2, we show the numerical results for the HISQ and p4 actions in the high temperature region, $T > 200$ MeV. In the continuum limit, the quark number susceptibilities approach unity at very high temperatures. The strange quark number susceptibilities approach the high temperature limit slower than the light quark number susceptibilities. The difference between the light and strange quark number susceptibilities becomes small above temperatures of 400 MeV and negligible for $T > 600$ MeV. The difference between the light and strange quark number susceptibilities can be understood in terms of the strange quark mass for $T > 250$ MeV, but at lower temperatures it is significantly larger than the expected suppression due to the relatively large strange quark mass, $m_s \simeq 90$ MeV, see the discussion in Ref. [17]. The cutoff effects are relatively small for $T < 300$ MeV but become significant above that temperature.

In the case of the HISQ action, there is a qualitative change in the behavior of the cutoff effects for $T > 300$ MeV; i.e., the ordering of quark number susceptibilities calculated at different N_τ qualitatively starts to follow expectations based on the systematics seen for the free theory. The continuum limit seems to be approached from below. The free theory result shown as horizontal lines in Fig. 2 shows larger cutoff dependence than the numerical data. This is expected to some extent; from the analysis of the high temperature limit of pure gauge theories [44], it is known that cutoff effects in the interacting theory typically are about a factor 2 smaller than in the free theory. Interestingly enough though, at the highest temperature, the $N_\tau = 4$ data point is very close to the lattice ideal gas value.

For the p4 action, the pattern of the cutoff dependence observed in the free theory is not seen in the interacting case for the entire temperature range explored by us. As discussed in Sec. III, for improved actions the quark number susceptibilities in the free theory approach the continuum limit from below. Our p4 lattice data, on the other hand, seem to approach the continuum limit from above.

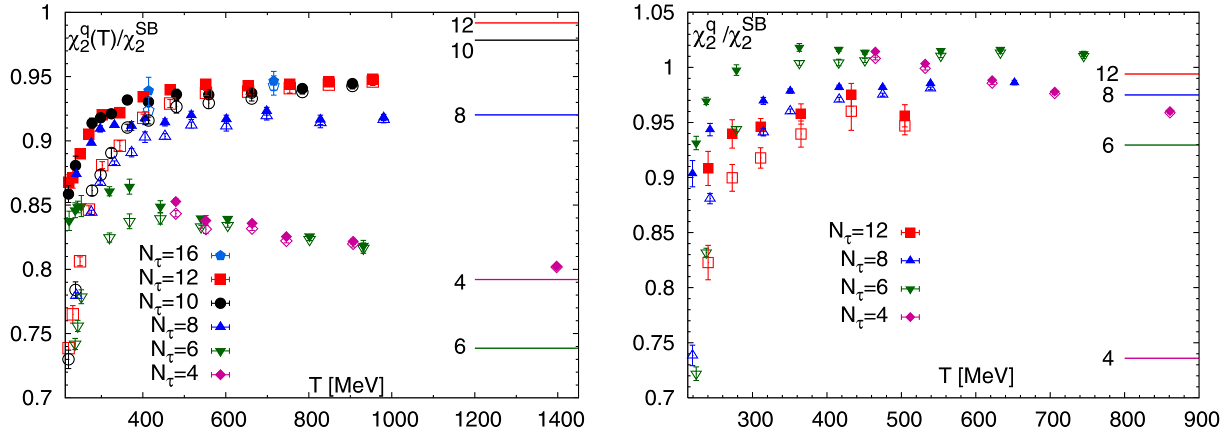

FIG. 2 (color online). Second order quark number susceptibilities calculated on different lattices with the HISQ action (left) and p4 action (right) as functions of the temperatures. The filled symbols correspond to light quark number susceptibilities, while open symbols correspond to strange quark number susceptibilities. The horizontal lines refer to the values of the quark number susceptibilities in the free theory corresponding to different temporal extent N_τ. The free theory result for $N_\tau = 4$ and the p4 action has been shifted upwards by 0.2 for better visibility. Note that $\chi_2^{SB} = 1$.

This implies that cutoff effects proportional to α_s and higher orders in the coupling constant are significant, contrary to the case of the HISQ action. The main difference between the p4 and HISQ action that could be responsible for the difference in the cutoff behavior is the use of smeared gauge fields in the latter. The use of smeared gauge fields is known to reduce cutoff effects in higher order perturbative corrections in lattice calculations [45].

B. Continuum extrapolation of second order quark number susceptibilities

Using the lattice results for quark number susceptibilities at different N_τ, we perform continuum extrapolations. First, for each N_τ we interpolate the lattice results using smooth splines. The errors on the interpolated values were estimated using the bootstrap method. We used the R package for this analysis [46]. Using the interpolations, we obtain the values of the quark number susceptibilities at the same set of temperatures. Finally, we perform continuum extrapolations at this set of temperatures. The $N_\tau = 4$ data have not been used in the continuum extrapolations.

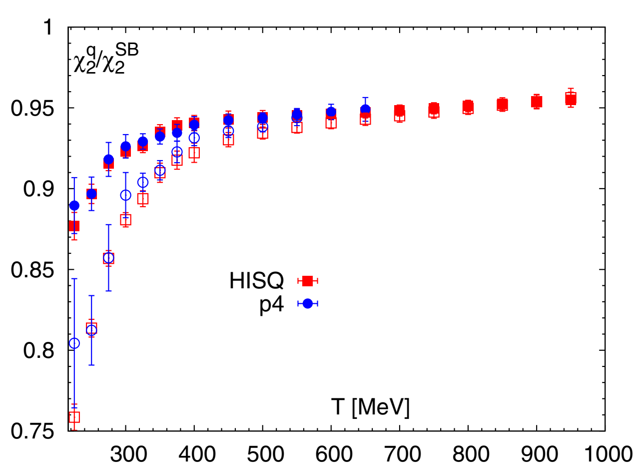

FIG. 3 (color online). Second order light (filled symbols) and strange (open symbols) quark number susceptibilities as functions of the temperature in the continuum limit. Note that $\chi_2^{SB} = 1$.

In the case of the HISQ action, we consider the temperature interval from 225 to 950 MeV for each N_τ, with the step of 25 MeV for $T \leq 400$ MeV and the step 50 MeV for larger temperatures. Our extrapolations for HISQ are motivated by the leading order N_τ dependence of quark number susceptibilities in the free theory. Namely, we performed continuum extrapolations using the following form:

$$\chi_2^q(N_\tau) = a + b/N_\tau^4 + c/N_\tau^6. \quad (4)$$

We also performed extrapolations using the simpler $a + b/N_\tau^4$ form and data for $N_\tau \geq 8$ only. The two fits gave identical results within the estimated errors. Furthermore, we used the complete tree-level result for the N_τ dependence of the quark number susceptibility to perform the continuum extrapolation; i.e., we fitted the data for each temperature with $a + c \cdot (\chi_2^{q,\text{free}}(N_\tau) - \chi_2^{SB})$. This gives extrapolated values for χ_2^q that are systematically lower than the above fits but still agree within errors. For the coefficient c, we typically find values around 0.6; i.e., the cutoff effects in the interacting theory are 40% smaller than in the free field limit. Since beyond tree level there are also discretization errors proportional to $1/N_\tau^2$, we also performed extrapolations using $a + c \cdot (\chi_2^{q,\text{free}}(N_\tau) - \chi_2^{SB}) + d/N_\tau^2$. These extrapolations give results that agree within errors with the extrapolations obtained using the ansatz $a + b/N_\tau^4 + c/N_\tau^6$, though they are systematically higher. The coefficient d turns out to be negative, and at the highest temperatures it is compatible with zero. Thus, it is possible

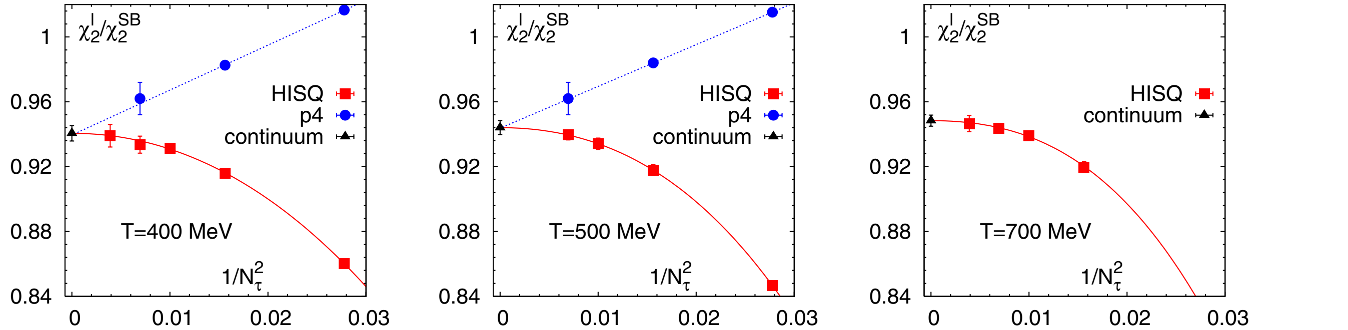

FIG. 4 (color online). The N_τ dependence of χ_2^l obtained with the HISQ and p4 actions for $T = 400$ (left), 500 (middle), and 700 MeV (right, HISQ only). The lines correspond to the fit of N_τ dependence based on Eq. (4) and $1/N_\tau^2$ form for the HISQ and p4 action, respectively. The triangles correspond to the continuum value obtained using the HISQ action. The $N_\tau = 16$ data have not been included in the fit.

that the $1/N_\tau^2$ term just mimics cutoff effects proportional to $1/N_\tau^n$, $n \geq 4$ at higher order in the weak coupling expansion. It turns out that the differences between the mean values of χ_2^q obtained by using the above two fits and the fit that uses Eq. (4) are smaller or equal to the statistical errors of the three-parameter fit given by Eq. (4). In other words, the systematic errors estimated as the differences of the above three fits are smaller or of the same size as the statistical errors of that fit. Therefore, we use the extrapolation based on Eq. (4) and its statistical errors as our final continuum result for HISQ.

For the p4 action, the continuum limit is approached from above, contrary to the free field expectations. This implies that the dominant cutoff effects come from higher orders in the weak coupling expansion and scale like $1/N_\tau^2$. Therefore, we performed the continuum extrapolation using the simple constant plus $1/N_\tau^2$ form. We also tried to add an $1/N_\tau^4$ term in the fit when doing the continuum extrapolations. It turns out that the inclusion of such a term did not change the result within the errors. Moreover, the coefficient of the $1/N_\tau^4$ term was 4–10 times smaller than in the free theory. This confirms our assumption that the dominant cutoff effects in the case of the p4 action go like $1/N_\tau^2$.

The continuum extrapolated quark number susceptibilities are shown in Fig. 3 for both p4 and HISQ actions. Overall, the p4 results and HISQ results agree quite well. Note that the agreement between the p4 results and HISQ results is particularly good for $T > 400$ MeV. This is remarkable in view of the different nature of the cutoff effects for the HISQ and p4 actions. To better illustrate this point, in Fig. 4 we show the N_τ dependence of χ_2^l for the p4 and HISQ action together with the continuum extrapolations based on the $1/N_\tau^2$ form and Eq. (4), respectively, for two temperatures $T = 400$ MeV and $T = 500$ MeV. In the figure, we also show the N_τ dependence of the HISQ results for $T = 700$ MeV together with the fit based on Eq. (4). Note that the $N_\tau = 16$ HISQ data point was not included in the continuum extrapolations but happens to lie on the fitted curve.

For the p4 action, the cutoff effects at order $1/N_\tau^4$ and higher seem to be very small, and the dominant cutoff effects are proportional to $1/N_\tau^2$ with a positive coefficient. For the HISQ action, there is no indication for such a term in the data, and if it is put in the ansatz for the extrapolations, the corresponding coefficient turns out to be negative. The cutoff dependence, thus, is well described by the free theory modified by a multiplicative factor. The continuum extrapolated values for χ_2^q obtained using the HISQ action are given in Table I. Remarkably, the continuum extrapolated χ_2^q results at high temperatures deviate from the massless ideal gas limit only by 5%. We also have compared our results with recent continuum extrapolated results obtained by using the stout action [13]. For χ_2^s, our result agrees with the stout results within errors up to

TABLE I. Continuum extrapolated values for the second order light and strange quark number susceptibilities obtained using the HISQ action.

T (MeV)	χ_2^l	χ_2^s
225	0.8768(85)	0.7585(81)
250	0.8968(60)	0.8137(55)
275	0.9156(44)	0.8568(48)
300	0.9230(38)	0.8807(44)
325	0.9263(40)	0.8939(49)
350	0.9352(44)	0.9098(59)
375	0.9393(46)	0.9176(56)
400	0.9406(47)	0.9222(60)
450	0.9428(52)	0.9303(44)
500	0.9440(43)	0.9344(37)
550	0.9452(39)	0.9379(35)
600	0.9462(36)	0.9407(36)
650	0.9472(35)	0.9430(38)
700	0.9484(34)	0.9451(40)
750	0.9497(35)	0.9474(39)
800	0.9511(35)	0.9497(38)
850	0.9526(37)	0.9516(38)
900	0.9539(41)	0.9538(45)
950	0.9551(47)	0.9563(58)

350 MeV. For 350 MeV $< T <$ 400 MeV, the stout results are lower by 1.2–1.5 standard deviations. For χ_2^l, our results agree with the stout results only up to 260 MeV. Above that temperature, the stout results are lower than ours by 2 standard deviations.

C. Fourth order quark number susceptibilities

As mentioned already in Sec. II, for the p4 action we also calculated the strange and light fourth order quark number susceptibilities for $N_\tau = 6$ and 8. The calculation of the fourth order light quark number susceptibility is more demanding than the calculation of the corresponding strange quark number susceptibility. Therefore, for $N_\tau =$ 12 we calculated the fourth order quark number susceptibility only in the strange quark sector. Our numerical results for χ_4^q normalized by the corresponding massless ideal gas (SB) value are shown in Fig. 5. The cutoff dependence of χ_4^q is qualitatively the same as for χ_2^q; namely, the continuum limit is approached from above, contrary to the free theory results shown in Fig. 5 as horizontal lines. As discussed above, the difference between the light and strange quark number susceptibilities is expected to be small for $T >$ 400 MeV and our numerical data clearly show this. In fact, with the exception of the $N_\tau = 8$ data point at the lowest temperature, the difference between the light and strange fourth order quark number susceptibilities is of the same order or smaller than the statistical errors. For χ_4^q, the deviations from the ideal gas value seem to be significantly larger than for χ_2^q at temperatures 350 MeV $< T <$ 500 MeV and increase with increasing N_τ.

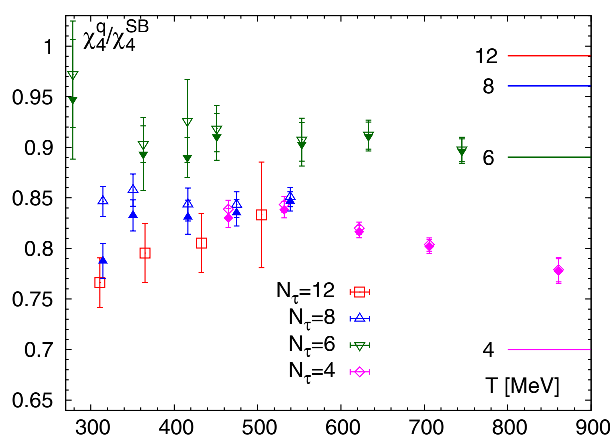

FIG. 5 (color online). Fourth order quark number susceptibilities calculated with the p4 action and normalized by the corresponding Stefan-Boltzmann value. The horizontal lines correspond to the free field value of χ_4^q/χ_4^{SB} in the massless limit. The free field result for $N_\tau = 4$ has been shifted upwards by 0.44 for better visibility. The open symbols correspond to the strange quark, while the filled symbols correspond to the light quarks. Note that $\chi_4^{SB} = 6/\pi^2$.

Given the large statistical errors of the $N_\tau = 12$ lattice data, it is at present difficult to perform a reliable continuum extrapolation for χ_4^s. However, the N_τ dependence of χ_4^s for 350 MeV $< T <$ 500 MeV is compatible with the $1/N_\tau^2$ behavior of the cutoff effects, and such an extrapolation would result in the value of $\chi_4^s/\chi_4^{q,SB}$ around 0.77; i.e., the deviations from the massless ideal gas limit for χ_4 could be almost 4 times larger than for the second order quark number susceptibilities. Looking at the data shown in Fig. 5, it is possible that the ordering of $N_\tau = 8$ and $N_\tau = 12$ data will change for $T >$ 550 MeV, becoming qualitatively compatible with the free theory expectations. Therefore, it would be very important to extend the lattice calculations of fourth order susceptibility to higher temperatures.

V. COMPARISON OF THE LATTICE AND THE WEAK COUPLING RESULTS

Let us finally compare our lattice findings with results from weak coupling calculations. As already discussed in Sec. I, weak coupling results have existed for the second order quark number susceptibility for some time. Recently, resummed perturbative calculations for the fourth order quark number fluctuations also became available [28–31]. In these studies, also the second order quark number susceptibilities have been considered.

In Fig. 6, we show the comparison of our continuum extrapolated data for χ_2^l with the perturbative calculations in the dimensionally reduced effective field theory [29].[1] The scale in the dimensionally reduced calculations was fixed using the criteria of the fastest apparent convergence for the gauge coupling of the dimensionally reduced effective theory [47] and was varied around this optimal value by a factor of 2. The width of the band shown in Fig. 6 corresponds to this variation of the scale as well as to the uncertainty in the value of the gauge coupling [29]. As mentioned in Sec. I, in the case of more than one quark flavor, one has to distinguish between the case where different quark flavors couple to different or the same chemical potential. In the latter case, there are contributions from off-diagonal quark number susceptibilities χ_{11}^{ij}, $i \neq j$, $i = u, d, s$ which are absent in the single flavor quark number susceptibilities defined in Eq. (1).

The resummed perturbative calculations in Refs. [23,30,31] consider the case of three degenerate massless quark flavors with equal chemical potential. Therefore, the corresponding results cannot be directly compared with our lattice calculations of χ_2^l. However,

[1] We compare here with the updated version of the 3D resummed perturbative calculation shown in Ref. [29]. We thank S. Mogliacci and A. Vuorinen for bringing to our attention a problem with an earlier version of the perturbative calculation for χ_2^l and sending us the corrected version of it prior to publication.

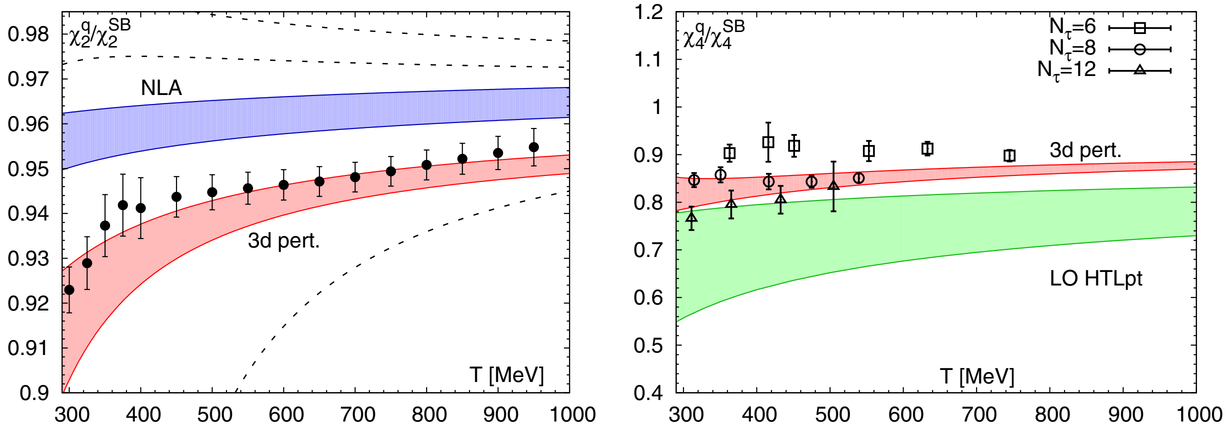

FIG. 6 (color online). The second order quark number susceptibility (left) and fourth order quark number susceptibility (right) calculated on the lattice and in resummed perturbation theory. For the second order quark number susceptibilities, we show the results obtained in dimensionally reduced 3D effective theory [29], the results of NLA calculations [23], and of three-loop HTLpt (dashed lines) [31] corresponding to different renormalization scales $\mu = \pi T$, $2\pi T$, and $4\pi T$ (bottom to top). For the fourth order quark number susceptibilities, we show the results obtained in reduced 3D effective theory and the one-loop HTLpt [29]. The ideal massless quark gas susceptibilities are $\chi_2^{SB} = 1$ and $\chi_4^{SB} = 6/\pi^2$.

we also estimated the second order off-diagonal quark number susceptibilities χ_{11}^{ij} and find that their contribution is about 1% or less in the temperature range considered here. Furthermore, since χ_{11}^{ij} has very large statistical errors and is only marginally different from zero, its inclusion would not change the comparison of our lattice data and the results of the resummed perturbative calculations. Therefore, in Fig. 6 we also compare our results with resummed perturbative calculations in so-called next-to-leading log approximation (NLA) [23] shown as the blue band and to the three-loop hard thermal loop perturbation theory (HTLpt) [31]. The width of the blue band reflects the scale uncertainties of the NLA calculations and corresponds to the variation of the renormalization scale from $\mu = \pi T$ to $\mu = 4\pi T$. The dashed lines shown in Fig. 6 correspond to the three-loop HTLpt results for the renormalization scale $\mu = \pi T$, $2\pi T$, and $4\pi T$.

As one can see from Fig. 6, the resummed perturbative calculation based on the dimensionally reduced effective theory reproduces the lattice results reasonably well, while the calculation performed in NLA approximation is above the lattice data by a few percent. The three-loop HTLpt results also agree with the lattice results, given their quite large-scale uncertainty. Note, however, that the one-loop HTLpt results of Refs. [29,30] are below the lattice data. Furthermore, the calculations of Ref. [30] consider the case of three degenerate quarks with equal chemical potential, while Ref. [29] considers single flavor quark number susceptibilities. However, both calculations yield very similar results. This further justifies the comparison of our lattice results with the results of Refs. [23,31].

The comparison of the lattice and weak coupling results for the fourth order quark number susceptibility normalized to the Stefan-Boltzmann limit is also shown in Fig. 6. Unlike for the second order quark number susceptibilities, the contribution from the off-diagonal terms is significant. Here we only show the comparison of the lattice results for χ_4^s with resummed calculations within the effective 3D theory and LO HTLpt calculations from Ref. [29] that considers single flavor quark number susceptibilities. The width of the bands again correspond to the scale uncertainty of the perturbative calculations. For the calculations within the dimensionally reduced effective theory, the uncertainty band was estimated in the same manner as for the second order susceptibility (see above). The uncertainty of the LO HTLpt calculations of Ref. [29] comes from the scale uncertainty as well as the uncertainty of the gauge coupling. Using the lattice data for χ_4^s for the comparison with the perturbative results is justified, as the effects of the nonvanishing strange quark mass are smaller than the current errors in lattice calculations. Within errors, the $N_\tau = 8$ and 12 lattice results for χ_4^s are compatible with the perturbative calculations. However, as already discussed in the previous section, for temperatures below 400 MeV there is a clear tendency for χ_4^s to decrease with increasing N_τ. Thus, in the continuum limit, χ_4^s may turn out to be below the above-mentioned perturbative results, at least for $T < 400$ MeV. If we assume a $1/N_\tau^2$ behavior of the cutoff effects for χ_4^s (which seems to be correct for χ_2^s) the continuum limit would be below the perturbative result. Of course, to see if this is indeed the case, calculations with the HISQ action for the fourth order susceptibility will be required. Let us mention that the preliminary continuum estimate of the fourth order quark number susceptibility obtained with the stout action is also below the resummed 3D perturbative result [48].

The fourth order baryon number susceptibility has been calculated in three-loop HTLpt [31], and the corresponding results show agreement with the available lattice data for $T < 400$ MeV. As discussed above, the fourth order baryon number susceptibility differs significantly from the fourth order quark number susceptibility by off-diagonal contributions. Therefore, to further validate the perturbative calculations, it will be important to also calculate off-diagonal susceptibilities, which we plan to do in the near future.

VI. CONCLUSIONS

We have calculated second and fourth order quark number susceptibilities for $T > 200$ MeV in lattice QCD using two different improved staggered fermion formulations: the p4 and HISQ actions. We performed continuum extrapolations for the second order quark number susceptibilities. While the cutoff dependence of the quark number susceptibilities is quite different for the HISQ and p4 actions, we obtain consistent results in the continuum limit. This makes us confident that the continuum extrapolations are under control. The detailed study of the cutoff effects in the quark number susceptibilities at high temperatures provides valuable information for analyzing the cutoff dependence of the pressure and other thermodynamic quantities, where the numerical calculations are much more involved due to the need of proper vacuum subtractions. In particular, we find indications that the cutoff effects in the temperature interval 400 MeV $< T <$ 950 MeV are 40% smaller than in the free theory.

Lattice calculations of quark number susceptibilities provide stringent constraints on the applicability of resummed perturbative calculations at high temperature. We performed a detailed comparison with the available perturbative results and find agreement with them given the uncertainties of the latter. To further constrain the reliability of the perturbative results, it will be important to perform continuum extrapolations for the fourth order quark number susceptibilities as well as to study off-diagonal quark number susceptibilities.

ACKNOWLEDGMENTS

This work was supported by the U.S. Department of Energy under Contract No. DE-AC02-98CH10886. The numerical computations have been carried out on the QCDOC computer of the RIKEN-BNL research center, on the QCDOC computer and the PC clusters of the USQCD Collaboration in FNAL, the BlueGene/L computer at the New York Center for Computational Sciences (NYCCS), and in NERSC. P. P. is also grateful to the Department of Atomic Physics of Eötvös University, for its hospitality, where this manuscript was partly finalized and to N. Haque, S. Mogliacci, M. Mustafa, A. Rebhan, M. Strickland, and A. Vuorinen for correspondence.

[1] P. Petreczky, J. Phys. G **39**, 093002 (2012).
[2] O. Philipsen, Prog. Part. Nucl. Phys. **70**, 55 (2013).
[3] A. D. Linde, Phys. Lett. **96B**, 289 (1980).
[4] B. Muller and J. L. Nagle, Annu. Rev. Nucl. Part. Sci. **56**, 93 (2006).
[5] F. Karsch, E. Laermann, and A. Peikert, Nucl. Phys. **B605**, 579 (2001).
[6] E. Follana, Q. Mason, C. Davies, K. Hornbostel, G. Lepage, J. Shigemitsu, H. Trottier, and K. Wong (HPQCD Collaboration and UKQCD Collaboration), Phys. Rev. D **75**, 054502 (2007).
[7] R. V. Gavai and S. Gupta, Phys. Rev. D **67**, 034501 (2003).
[8] C. Bernard, T. Burch, C. DeTar, J. Osborn, S. Gottlieb, E. Gregory, D. Toussaint, U. Heller, and R. Sugar (MILC Collaboration), Phys. Rev. D **71**, 034504 (2005).
[9] Y. Aoki, Z. Fodor, S. Katz, and K. Szabo, Phys. Lett. B **643**, 46 (2006).
[10] Y. Aoki, S. Borsányi, S. Dürr, Z. Fodor, S. D. Katz, S. Krieg, and K. Szabo, J. High Energy Phys. 06 (2009) 088.
[11] S. Borsányi, Z. Fodor, C. Hoelbling, S. D. Katz, S. Krieg, C. Ratti, and K. K. Szabó (Wuppertal-Budapest Collaboration), J. High Energy Phys. 09 (2010) 073.
[12] A. Bazavov and P. Petreczky (HotQCD Collaboration), J. Phys. Conf. Ser. **230**, 012014 (2010).
[13] S. Borsányi, Z. Fodor, S. D. Katz, S. Krieg, C. Ratti, and K. Szabó, J. High Energy Phys. 01 (2012) 138.
[14] A. Bazavov *et al.* (HotQCD Collaboration), Phys. Rev. D **86**, 034509 (2012).
[15] C. Allton, S. Ejiri, S. Hands, O. Kaczmarek, F. Karsch, E. Laermann, and C. Schmidt, Phys. Rev. D **68**, 014507 (2003).
[16] C. R. Allton, M. Döring, S. Ejiri, S. J. Hands, O. Kaczmarek, F. Karsch, E. Laermann, and K. Redlich, Phys. Rev. D **71**, 054508 (2005).
[17] M. Cheng *et al.*, Phys. Rev. D **79**, 074505 (2009).
[18] A. Bazavov *et al.*, Phys. Rev. Lett. **109**, 192302 (2012).
[19] A. Bazavov *et al.*, Phys. Rev. Lett. **111**, 082301 (2013).
[20] R. Bellwied, S. Borsanyi, Z. Fodor, S. D. Katz, and C. Ratti, arXiv:1305.6297 [Phys. Rev. Lett. (to be published)].
[21] S. Borsányi, Z. Fodor, S. D. Katz, S. Krieg, C. Ratti, and K. K. Szabó, Phys. Rev. Lett. **111**, 062005 (2013).
[22] J. Blaizot, E. Iancu, and A. Rebhan, Phys. Lett. B **523**, 143 (2001).
[23] A. Rebhan, *Strong and Electroweak Matter 2002* (World Scientific, Singapore, 2003).
[24] N. Haque and M. G. Mustafa, arXiv:1007.2076.
[25] A. Vuorinen, Phys. Rev. D **67**, 074032 (2003).

[26] A. Hietanen and K. Rummukainen, J. High Energy Phys. 04 (2008) 078.
[27] A. Vuorinen, Phys. Rev. D **68**, 054017 (2003).
[28] J. O. Andersen, S. Mogliacci, N. Su, and A. Vuorinen, Phys. Rev. D **87**, 074003 (2013).
[29] S. Mogliacci, J. O. Andersen, M. Strickland, N. Su, and A. Vuorinen, arXiv:1307.8098.
[30] N. Haque, M. G. Mustafa, and M. Strickland, J. High Energy Phys. 07 (2013) 184.
[31] N. Haque, J. O. Andersen, M. G. Mustafa, M. Strickland, and N. Su, arXiv:1309.3968.
[32] P. Petreczky, P. Hegde, and A. Velytsky (RBC-Bielefeld Collaboration), Proc. Sci., LAT2009 (2009) 159 [arXiv:0911.0196].
[33] A. Bazavov, H. Ding, and P. Petreczky, J. Phys. Conf. Ser. **389**, 012017 (2012).
[34] M. Clark, A. Kennedy, and Z. Sroczynski, Nucl. Phys. B, Proc. Suppl. **140**, 835 (2005).
[35] R. Sommer, Nucl. Phys. **B411**, 839 (1994).
[36] A. Bazavov *et al.*, Phys. Rev. D **85**, 054503 (2012).
[37] M. Cheng *et al.*, Phys. Rev. D **77**, 014511 (2008).
[38] C. Allton, S. Ejiri, S. Hands, O. Kaczmarek, F. Karsch, E. Laermann, Ch. Schmidt, and L. Scorzato, Phys. Rev. D **66**, 074507 (2002).
[39] M. Cheng *et al.*, Phys. Rev. D **81**, 054504 (2010).
[40] S. Borsányi, G. Endrődi, Z. Fodor, A. Jakovác, S. D. Katz, S. Krieg, C. Ratti, and K. K. Szabó, J. High Energy Phys. 11 (2010) 077.
[41] P. Hegde, F. Karsch, E. Laermann, and S. Shcheredin, Eur. Phys. J. C **55**, 423 (2008).
[42] S. Naik, Nucl. Phys. **B316**, 238 (1989).
[43] U. M. Heller, F. Karsch, and B. Sturm, Phys. Rev. D **60**, 114502 (1999).
[44] G. Boyd, J. Engels, F. Karsch, E. Laermann, C. Legeland, M. Lütgemeier, and B. Petersson, Nucl. Phys. **B469**, 419 (1996).
[45] A. Hasenfratz, Nucl. Phys. B, Proc. Suppl. **119**, 131 (2003).
[46] http://www.r project.org/.
[47] K. Kajantie, M. Laine, K. Rummukainen, and M. E. Shaposhnikov, Nucl. Phys. **B503**, 357 (1997).
[48] S. Borsanyi, Nucl. Phys. **A904–905**, 270c (2013).