

Relating neutrino masses and mixings by discrete symmetries

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Lepton mixing can originate from the breaking of a flavor symmetry in different ways in the neutrino and the charged lepton sector. We propose an extension of this framework that allows us to connect the mixing parameters with masses and, more precisely, with certain types of degeneracy of the neutrino mass spectrum. We obtain relations between the mixing parameters for the cases of partial degeneracy, $m_1 = m_2$, and complete degeneracy, $m_1 = m_2 = m_3$. These relations determine also the Majorana phases. It is shown that relatively small corrections to these lowest-order results can produce the required mass splitting and modify the mixing without significant changes of the other symmetry results.

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I. INTRODUCTION

Lepton mixing can be a consequence of the breaking of a flavor symmetry G_f into different *residual symmetries*, G_ν and G_ℓ , for the neutrino and charged lepton mass matrices, respectively [1]. To a large extent, this approach was motivated by the peculiar values of the lepton mixing parameters that seemed to be well approximated by the so-called tribimaximal (TBM) mixing [2]. TBM mixing turned out to be difficult to connect with the known ratios of masses of the charged leptons and neutrinos, and therefore, in this approach, the masses of neutrinos and charged leptons “decouple” from mixing. In other words, TBM mixing implies the form invariance of the mass matrix—a situation in which mixing is determined by symmetry alone. Models that reproduced the TBM mixing and other interesting mixing patterns were built along these lines [3–7]. Independent physics (additional symmetries) was assumed to be responsible for the hierarchy of the masses of leptons. Indirect relations between mixing and the mass spectrum appear in some specific models, as a result of model structure and particle content.

The separate description of mixing and mass hierarchies (ratios of masses) with different physics involved looks unsatisfactory. Indeed, the following are true:

- (1) In general, masses and mixing have the same origins following formally from the diagonalization of the mass matrices. They are generated by the same type of Yukawa couplings and so should be somehow related.
- (2) In the quark sector, various relations between the mass ratios and mixing parameters have been uncovered, with the Gatto–Sartori–Tonin relation [8] being the most appealing one. The latter can be explained by an Abelian flavor symmetry, e.g., in the Froggatt–Nielsen approach [9]. There were some attempts to use discrete symmetries to produce

relations between masses and mixing (see, e.g., Ref. [10]). In a number of recent models with discrete flavor symmetries, the sum rules for neutrino masses (linear relations between masses or their inverse quantities) have been realized (see, e.g., Ref. [11]). However, these relations do not depend on mixing and turn out to be consequences of specific restricted model contents and vacuum alignments and do not follow from residual symmetries of the neutrino mass matrix.

- (3) The now-established relatively large 1-3 mixing and the indications of significant deviations of the 2-3 mixing from maximal [12] rule out the exact TBM mixing and therefore cast doubts about the explanation of mixing separated from masses through non-Abelian symmetries. At the same time, it was shown in Refs. [13–16] that flavor symmetries can still accommodate the recent results on the 1-3 and 2-3 mixings.

In Refs. [14,15], we proposed a formalism for “symmetry building,” which relies on the aforementioned partial breaking of G_f into two subgroups, G_ℓ and G_ν . We used it to obtain relations between the mixing parameters without explicit reference to any particular model. It was shown that consequences of symmetries for mixing, that is, relations between the mixing parameters or elements of the mixing matrix (at least at the lowest order) can be obtained immediately once residual symmetries (transformations) of the neutrino and charged lepton mass matrices are known. These symmetry group relations can be obtained without model building and explicit construction of the mass matrices and their diagonalizations. Essentially, it is only assumed that the model is constructed and it leads to mass matrices with given symmetry properties. The relations in Refs. [14,15] can be viewed as a tool of symmetry or/and model building. Once the required relations and the corresponding residual symmetries are identified, one can come back and construct the corresponding complete flavor symmetry and model. After a model is constructed, one can compute corrections to zero-order structures. In general, this formalism does not allow us to compute the latter model-dependent corrections.

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In this paper, we further develop this formalism, which in Refs. [14,15] was elaborated for mixing only, in order to also include neutrino masses. Consequently, we obtain relations between the mixing parameters and certain types of the neutrino mass spectrum.

The crucial point of the explanation of mixing decoupled from masses was to use, as G_ν , the symmetry of a generic neutrino mass matrix M_ν with arbitrary eigenvalues. For Majorana neutrinos, the maximal symmetry of M_ν is $\mathbf{Z}_2 \times \mathbf{Z}_2$, as can be seen immediately in the basis in which the mass matrix is diagonal.¹ This generic symmetry does not constrain the masses and therefore leads to the decoupling of mixing. Hence, for the symmetry to predict both masses and mixings, G_ν should be enlarged in such a way that the invariance of the mass matrix is satisfied only for certain mass spectra.

In this paper, we focus on residual symmetries, G_ν , that lead to equalities of the neutrino masses. Indeed, unitary symmetry transformations can lead either to equalities of the masses or to zero values of masses.² In this connection, we will explore two possibilities: (i) two degenerate neutrinos, i.e., equality of two masses, or (ii) three degenerate neutrinos. The first case can be considered as the lowest-order approximation to spectra with both normal and inverted mass hierarchies. Then, corrections are required that lead to splitting between the masses. In the case of normal mass hierarchy with two vanishing masses, the corrections should generate the mass of at least the second neutrino.

The paper is organized as follows. In Sec. II, we describe the model-independent method for symmetry building, applied here to the case of the specific neutrino mass spectrum. In Sec. III, we consider the case of partial degeneracy (equality of two masses). We derive the relations between mixing parameters, which also include the Majorana phases. In Sec. IV it is shown that relatively small corrections to the partially degenerate neutrino mass matrix can account for the mass splitting and the measured mixing angles. In Sec. V, we derive constraints on the mixing and phases in the case of the completely degenerate spectrum. A discussion and conclusions are presented in Sec. VI.

II. SYMMETRY RELATIONS FOR MIXING AND MASSES

We assume that neutrinos are Majorana particles. Working in the flavor basis, the mass terms of the lepton sector of the Lagrangian read

$$\mathcal{L}_{\text{mass}} = \bar{E}_R m_\ell \ell_L + \frac{1}{2} \bar{\nu}^c_{fL} M_{\nu U} \nu_{fL} + \text{H.c.}, \quad (1)$$

where ν_{fL} , ℓ_L , and E_R are the leptonic fields: $\nu_{fL} \equiv (\nu_e, \nu_\mu, \nu_\tau)_L^T$, $\ell_L \equiv (e, \mu, \tau)_L^T$, $E_R \equiv (e, \mu, \tau)_R^T$,

¹A complete scan of the groups with order less than 1536 and $G_\nu = \mathbf{Z}_2 \times \mathbf{Z}_2$ was performed in Ref. [17].

²The possibility of the symmetry leading to vanishing masses was considered in Ref. [18].

and $m_\ell \equiv \text{diag}\{m_e, m_\mu, m_\tau\}$. The flavor neutrino states are related to the mass eigenstates, $\nu_L \equiv (\nu_1, \nu_2, \nu_3)_L^T$, by the Pontecorvo–Maki–Nakagawa–Sakata mixing matrix: $\nu_{fL} = U_{\text{PMNS}} \nu_L$. Then, the neutrino mass matrix in the flavor basis, $M_{\nu U}$, can be expressed via the diagonal mass matrix of the neutrino mass eigenvalues, $m_\nu \equiv \text{diag}\{m_1, m_2, m_3\}$, and U_{PMNS} as

$$M_{\nu U} = U_{\text{PMNS}}^* m_\nu U_{\text{PMNS}}^\dagger. \quad (2)$$

Let \mathcal{G}_ℓ and $\mathcal{G}_{\nu U}$ be the groups of symmetry transformations that leave invariant the charged lepton and neutrino mass terms in Eq. (1). The residual flavor symmetries in the charged lepton and neutrino sectors are, in general, finite subgroups of these: $G_\ell \subset \mathcal{G}_\ell$ and $G_\nu \subset \mathcal{G}_{\nu U}$. We proceed to identify G_ν and G_ℓ systematically.

For the charged leptons, we have $\mathcal{G}_\ell \equiv U(1)^3$ corresponding to the electron, muon, and tau lepton numbers. As in our previous papers [14,15], we assume that $G_\ell = \mathbf{Z}_m$. The fact that \mathbf{Z}_m is generated by one element implies that it leads to a minimal number of constraints on the mixing matrix. Analysis of bigger groups, which lead to stricter conditions U_{PMNS} , is beyond the scope of this paper.

A representation of G_ℓ is given by the matrix T such that ℓ_L and E_R transform as

$$\ell \rightarrow T \ell_L, \quad E_R \rightarrow T E_R, \quad (3)$$

where

$$T \equiv \text{diag}\{e^{i\phi_e}, e^{i\phi_\mu}, e^{i\phi_\tau}\} \quad (4)$$

and

$$\phi_\alpha \equiv 2\pi \frac{k_\alpha}{m}, \quad \alpha = e, \mu, \tau. \quad (5)$$

Invariance of the charged lepton mass matrix m_ℓ under T means that the following equality holds: $T m_\ell T^\dagger = m_\ell$. According to Eqs. (4) and (5), T satisfies the condition $T^m = \mathbb{1}$. It is enough to consider a subgroup of $SU(3)$ as the flavor group. This restriction simplifies the considerations without having any impact in the results of the paper. One can show that the additional $U(1)$ of $U(3)$ can be factored out and does not induce any constraint on mixing. Thus, we impose the equality

$$\phi_e + \phi_\mu + \phi_\tau = 0$$

or, equivalently, $k_\tau = -k_e - k_\mu$, that ensures $\text{Det}[T] = 1$.

Considering now the symmetry group \mathcal{G}_ν of the Majorana mass term of neutrinos, we explore the possibility of approximate degeneracy of the mass spectrum. In the neutrino mass basis, the invariance of the mass matrix under the transformation

$$\nu_L \rightarrow S \nu_L, \quad (6)$$

where S belongs to the group \mathcal{G}_ν , means that

$$S^T m_\nu S = m_\nu. \quad (7)$$

If S satisfies Eq. (7), the corresponding matrix S_U that leaves $M_{\nu U}$ invariant, i.e.,

$$S_U^T M_{\nu U} S_U = M_{\nu U},$$

can be found by switching to the flavor basis:

$$S_U = U_{\text{PMNS}} S U_{\text{PMNS}}^\dagger. \quad (8)$$

Hence, the group $\mathcal{G}_{\nu U}$ is obtained from \mathcal{G}_ν by applying a similarity transformation with U_{PMNS} to all elements of \mathcal{G}_ν . The residual symmetry G_ν is a discrete subgroup of $\mathcal{G}_{\nu U}$.

Three cases can be distinguished for \mathcal{G}_ν with increasing symmetry that corresponds to an increasing degree of degeneracy of m_ν .

- (A) No degeneracy: $\mathcal{G}_\nu = \mathbf{Z}_2 \otimes \mathbf{Z}_2$

The diagonal neutrino mass matrix, m_ν , with arbitrary eigenvalues is invariant under the transformations (6) with

$$S_1 = \text{diag}\{1, -1, -1\}, \quad S_2 = \text{diag}\{-1, 1, -1\}, \quad (9)$$

and $S_3 = S_1 S_2$. This case was analyzed in Refs. [14,15], and $G_\nu = \mathbf{Z}_2$ or $G_\nu = \mathbf{Z}_2 \otimes \mathbf{Z}_2$ are possible.

- (B) Two degenerate neutrinos: $\mathcal{G}_\nu = SO(2) \otimes \mathbf{Z}_2$

In addition to the matrices in Eq. (9), the neutrino mass matrix is invariant under rotations of the plane of mass degeneracy.³ Taking into account the measured neutrino mass differences, the approximate degeneracy must be between m_1 and m_2 . Therefore, we take

$$m_{\nu 3} \simeq \begin{pmatrix} m & & \\ & m & \\ & & m' \end{pmatrix}, \quad (10)$$

which is invariant under the transformation

$$S \equiv S_\zeta = \begin{pmatrix} c_\zeta & -s_\zeta & \\ s_\zeta & c_\zeta & \\ & & 1 \end{pmatrix}, \quad (11)$$

where $c_\zeta \equiv \cos \zeta$, $s_\zeta \equiv \sin \zeta$. Analogously to the case of charged leptons, we impose $G_\nu = \mathbf{Z}_n$ so that $S_\zeta^n = \mathbb{1}$ and $\zeta = 2\pi l/n$. In the flavor basis, Eq. (8), we must also have

³A model for two degenerate neutrinos based on the group D_N —a discrete subgroup of $O(2)$ —has been proposed in Ref. [19], in which a relation that connects θ_{13} , the electron mass, and the CP phases was obtained. The approach in Ref. [19] differs from ours in that D_N is the symmetry of whole leptonic sector, introduced to explain small observed parameters that describe masses and mixing.

$$S_U^n = \mathbb{1}.$$

- (C) Three degenerate neutrinos: $\mathcal{G}_\nu = SO(3)$
In this case, we have

$$m_\nu \simeq m \mathbb{1}, \quad (12)$$

and S can be any orthogonal 3×3 matrix. We note that, according to the Euler rotation theorem, any three-dimensional (3D) rotation is a one-dimensional rotation around a certain axis. Thus, any matrix S that is a symmetry of m_ν can be written as

$$S \equiv S_{\zeta O} = O S_\zeta O^T, \quad (13)$$

where O is an orthogonal matrix. The Euler rotation theorem implies that a basis in the neutrino sector can be always selected such that the matrix S in this basis has the form S_ζ . Turning the argument around, imposing only a \mathbf{Z}_n symmetry generated by $S_{\zeta O}$ is not enough to force the three neutrinos to be degenerate. The most general mass matrix that is left invariant by a \mathbf{Z}_n subgroup of $O(3)$ has only two equal eigenvalues. Thus, if G_ν imposes full degeneracy of the neutrino mass matrix, it must be one of the non-Abelian subgroups of $O(3)$ with 3D representations, i.e., \mathbf{A}_4 , \mathbf{S}_4 , or \mathbf{A}_5 .⁴

As it was shown in Ref. [14], the relations between mixing matrix elements follow from the condition that the symmetry transformations of the charged leptons and neutrinos in the flavor basis belong to the same discrete group G_f . That is, the product

$$W_U \equiv S_U T \quad (14)$$

must also belong to G_f . Furthermore, since G_f is finite, there must exist an integer p such that

$$W_U^p = (S_U T)^p = \mathbb{1}. \quad (15)$$

The relations

$$S_U^n = T^m = W_U^p = \mathbb{1} \quad (16)$$

form a presentation of G_f and define the von Dyck group $D(n, m, p)$.

Equation (15) is a constraint on the mixing matrix [14]. To see this, notice that the eigenvalues of W_U are three p th roots of unity, $\lambda_1^{(p)}$, $\lambda_2^{(p)}$, and $\lambda_3^{(p)}$. Defining

$$a \equiv \text{Tr}[W_U] = \lambda_1^{(p)} + \lambda_2^{(p)} + \lambda_3^{(p)},$$

we have from Eq. (15) that

$$\text{Tr}[U_{\text{PMNS}} S U_{\text{PMNS}}^\dagger T] \equiv \text{Tr}[W] = a. \quad (17)$$

⁴For an early \mathbf{A}_4 model predicting a nearly degenerate neutrino spectrum, see Ref. [20].

Since the p th roots of unity are a finite set, the rhs of this equation takes values from a finite set of known complex numbers. For known a , and given S and T , Eq. (17) is a complex condition that the entries of U_{PMNS} must satisfy.

Although we will proceed below in all generality, the case analysis of n , m , and p is significantly reduced after the following consideration. It is a known fact [14] that in order for the von Dyck group to be finite, one of n , m , or p must be equal to 2. In Refs. [14,15], we took $n = 2$ consistent with $G_\nu = \mathbf{Z}_2$. However, to enforce degeneracy in the neutrino mass matrix, it must be $n \geq 3$. Assuming also that all charged leptons have different charges under T , i.e., $m \geq 3$, we obtain that due to the finiteness of the group, it must be $p = 2$.

Equation (15) can then be written as

$$W_U^2 = (U_{\text{PMNS}} S U_{\text{PMNS}}^\dagger T)^2 = \mathbb{1},$$

$$U_{\text{PMNS}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & e^{-i\delta}s_{13} \\ -s_{12}c_{23} - e^{i\delta}c_{12}s_{23}s_{13} & c_{12}c_{23} - e^{i\delta}s_{12}s_{23}s_{13} & s_{23}c_{13} \\ s_{12}s_{23} - e^{i\delta}c_{12}c_{23}s_{13} & -c_{12}s_{23} - e^{i\delta}s_{12}c_{23}s_{13} & c_{23}c_{13} \end{pmatrix} \Gamma_M, \quad (20)$$

where U_{ij} are the matrices of rotations in the ij planes on the angles θ_{ij} ,

$$\Gamma_\delta \equiv \text{diag}\{1, 1, e^{i\delta}\}, \quad \Gamma_M \equiv \text{diag}\{1, e^{i\kappa}, e^{i\lambda}\}, \quad (21)$$

and $c_{12} \equiv \cos \theta_{12}$, $s_{12} \equiv \sin \theta_{12}$, etc.

III. CONSTRAINTS ON MIXING FOR THE PARTIALLY DEGENERATE SPECTRUM

For the partially degenerate spectrum, the neutrino mass matrix and the corresponding symmetry are given by Eqs. (10) and (11), respectively, with $\zeta = 2\pi l/n$. The matrix m_ν can be a good lowest-order approximation to both normal and inverted mass hierarchies. Corrections could then produce small splitting between the degenerate states and modify mixing angles when needed.

Setting $S = S_\zeta$, we have from Eq. (18) the symmetry relation

$$\text{Tr}[S_U T] = \text{Tr}[U_{\text{PMNS}} S_\zeta U_{\text{PMNS}}^\dagger T] = -1.$$

Explicit computation of S_U [see Eq. (8)] gives

$$(S_U)_{\alpha\alpha} = c_\zeta + 2s_\zeta^2 |U_{\alpha 3}|^2 + i2s_\zeta \text{Im}(U_{\alpha 2} U_{\alpha 1}^*), \quad (22)$$

where $s_{\zeta/2} \equiv \sin(\zeta/2)$. It is convenient to introduce the real and imaginary parts of $(S_U)_{\alpha\alpha} = R_\alpha + iI_\alpha$:

$$R_\alpha = c_\zeta + 2s_\zeta^2 |U_{\alpha 3}|^2, \quad I_\alpha = 2s_\zeta \text{Im}(U_{\alpha 2} U_{\alpha 1}^*). \quad (23)$$

Notice that the index 3 in the real part of $(S_U)_{\alpha\alpha}$ is related to the nondegenerate third mass eigenstate. According to Eq. (17), the trace of W_U equals

and the eigenvalues of W_U must be equal to $+1$ or -1 . Moreover, taking into account that $\det[W_U] = 1$, the eigenvalues of W_U must be $\{1, -1, -1\}$, if W_U is not trivial. Hence, we obtain

$$a = \text{Tr}[W_U] = \text{Tr}[U_{\text{PMNS}} S U_{\text{PMNS}}^\dagger T] = -1. \quad (18)$$

The condition in Eq. (18) is appropriate when a residual symmetry in the neutrino sector forces the neutrino mass matrix to be of the form B or C.

In what follows, we find explicitly the constraints imposed on U_{PMNS} and compare them with experimental data. For U_{PMNS} , we will use the standard parametrization given by

$$U_{\text{PMNS}} = U_{23}(\theta_{23}) \Gamma_\delta U_{13}(\theta_{13}) \Gamma_\delta^* U_{12}(\theta_{12}) \Gamma_M \quad (19)$$

$$\text{Tr}(W_U) = (S_U)_{ee} e^{i\phi_e} + (S_U)_{\mu\mu} e^{i\phi_\mu} + (S_U)_{\tau\tau} e^{i\phi_\tau} = a,$$

and, consequently, from the real and imaginary parts of this equation, we obtain using Eq. (23)

$$\sum_\alpha (R_\alpha \cos \phi_\alpha - I_\alpha \sin \phi_\alpha) = \text{Re}[a], \quad \alpha = e, \mu, \tau, \quad (24)$$

$$\sum_\alpha (R_\alpha \sin \phi_\alpha - I_\alpha \cos \phi_\alpha) = \text{Im}[a], \quad \alpha = e, \mu, \tau. \quad (25)$$

These are the conditions imposed on mixing by the symmetry in the case of partial degeneracy. Explicit equations for the mixing angles and phases in Eq. (20) can be found by substituting R_α and I_α from Eq. (23) in Eqs. (24) and (25).

As an example, we consider the case in which $\phi_e = 0$, $\phi_\mu = -\phi_\tau \equiv \psi$, so that the charged lepton transformation matrix has the form

$$T = \text{diag}\{1, e^{i\psi}, e^{-i\psi}\}. \quad (26)$$

Equations (24) and (25) are then reduced to

$$R_e + c_\psi (R_\mu + R_\tau) + s_\psi (I_\tau - I_\mu) = \text{Re}[a], \quad (27)$$

$$I_e + c_\psi (I_\mu + I_\tau) + s_\psi (R_\mu - R_\tau) = \text{Im}[a], \quad (28)$$

and inserting Eq. (23) in Eqs. (27) and (28), we obtain

$$|U_{e3}|^2 - 2x \text{Im}[U_{\mu 2} U_{\mu 1}^* - U_{\tau 2} U_{\tau 1}^*] + x^2 = \frac{1 + \text{Re}[a]}{4s_{\psi/2}^2 s_{\zeta/2}^2}, \quad (29)$$

$$2 \operatorname{Im}[U_{e2}U_{e1}^*] - y(|U_{\tau3}|^2 - |U_{\mu3}|^2) = \frac{\operatorname{Im}[a]}{s_\zeta}. \quad (30)$$

Here, we have introduced parameters

$$x \equiv \cot \frac{\psi}{2} \cot \frac{\zeta}{2}, \quad y \equiv \cot \frac{\psi}{2} \tan \frac{\zeta}{2}, \quad (31)$$

which depend only on the group properties. From Eq. (31), $xy = \cot^2 \frac{\psi}{2}$ and $x/y = \cot^2 \frac{\zeta}{2}$, and, consequently, x and y should have the same sign.

Equations (29) and (30) can be immediately generalized to the cases in which the lepton that has zero charge under T is the muon or the tau. The general equations are

$$|U_{\alpha3}|^2 - 2x \operatorname{Im}[U_{\beta1}U_{\beta2}^* - U_{\gamma1}U_{\gamma2}^*] + x^2 = \frac{1 + \operatorname{Re}[a]}{4s_{\psi/2}^2 s_{\zeta/2}^2}, \quad (32)$$

$$2 \operatorname{Im}[U_{\alpha1}U_{\alpha2}^*] - y(|U_{\gamma3}|^2 - |U_{\beta3}|^2) = \frac{\operatorname{Im}[a]}{s_\zeta}. \quad (33)$$

Here, (α, β, γ) is (e, μ, τ) or any other combination with a cyclic permutation of these flavor indices. Equations (29) and (30) correspond to the case $\alpha = e$. Notice that Eqs. (32) and (33) represent yet another generalization of the results of Ref. [14], which can be reproduced by setting $\zeta = \pi$.

When there is partial degeneracy in the neutrino mass matrix, only $a = -1$ [see Eq. (18)] leads to finite groups. The explicit conditions on mixing imposed by the symmetry are then reduced to

$$(|U_{\alpha3}| \mp x)^2 = 2x (\operatorname{Im}[U_{\beta1}U_{\beta2}^* - U_{\gamma1}U_{\gamma2}^*] \mp |U_{\alpha3}|), \quad (34)$$

$$2 \operatorname{Im}[U_{\alpha1}U_{\alpha2}^*] = y(|U_{\gamma3}|^2 - |U_{\beta3}|^2). \quad (35)$$

The set of solutions of Eqs. (34) and (35) is very restricted. To show this, we will use the standard parametrization, Eq. (20), for $|U_{\alpha i}|^2$ and consider for definiteness the case $\alpha = e$. Notice, nonetheless, that our results do not lose generality since, for any choice of α , there exists a parametrization such that the equations have the form to be discussed below.

We distinguish two cases, $x > 0$ and $x < 0$, which imply $y > 0$ and $y < 0$, respectively. In the standard parametrization and for $|U_{e3}| = \sin \theta_{13} \neq 0$, Eqs. (34) and (35) can be written as

$$(\sin \theta_{13} \mp x)^2 = 2x(A_1 + A_2), \quad (36)$$

$$\sin 2\theta_{12} \sin \kappa = y \cos 2\theta_{23}. \quad (37)$$

Here, κ is the Majorana phase defined in Eq. (20), and the minus (plus) sign corresponds to $x > 0$ (< 0). A_1 and A_2 are given by

$$A_1 = -\frac{1}{2}(1 + \sin^2 \theta_{13}) \cos 2\theta_{23} \sin 2\theta_{12} \sin \kappa, \quad (38)$$

$$A_2 = \sin \theta_{13} [(\cos \kappa \sin \delta - \cos \delta \sin \kappa \cos 2\theta_{12}) \times \sin 2\theta_{23} \mp 1]. \quad (39)$$

Substituting $\cos 2\theta_{23}$ from Eq. (37) into Eq. (38), we obtain

$$A_1 = -\frac{1}{2y}(1 + \sin^2 \theta_{13}) \sin^2 2\theta_{12} \sin^2 \kappa, \quad (40)$$

so that $A_1 \leq 0$ for $x > 0$ and $A_1 \geq 0$ for $x < 0$ (recall that x and y have the same sign). Since

$$|\cos \kappa \sin \delta - \cos \delta \sin \kappa \cos 2\theta_{12}| \leq 1,$$

from Eq. (39), we have $A_2 \leq 0$ for $x > 0$ and $A_2 \geq 0$ for $x < 0$. We can combine these two results:

$$\begin{aligned} A_i &\leq 0, & x > 0, \\ A_i &\geq 0, & x < 0, \end{aligned} \quad (i = 1, 2). \quad (41)$$

Let us consider the case $x > 0$ and therefore $y > 0$. Since both A_i are negative, see Eq. (41), the equality in Eq. (36) can be satisfied only if

$$\sin \theta_{13} = x, \quad A_1 = 0, \quad A_2 = 0. \quad (42)$$

Then, according to Eq. (40), there are two types of solutions of equation $A_1 = 0$:

- (1) $\kappa = 0$. In this case, we find from Eq. (39) $A_2 = \sin \theta_{13} (\sin \delta \sin 2\theta_{23} - 1) = 0$, which gives $\delta = \pi/2$ and $\theta_{23} = \pi/4$. There are no constraints on θ_{12} .
- (2) $\theta_{12} = 0(\pi/2)$. Then, from Eq. (39), we have $A_2 = \sin \theta_{13} [\sin(\delta \mp \kappa) \sin 2\theta_{23} - 1] = 0$, which is satisfied if $\theta_{23} = \pi/4$ and $\delta \mp \kappa = \pi/2$. Here, the minus (plus) sign corresponds to $\theta_{12} = 0$ ($\pi/2$). Thus, only one combination of the two phases is determined in this case.

In particular, we stress that there exists a mixing matrix that satisfies the constraints for every value of θ_{12} , which is in this sense undetermined.

Similarly, one can consider the case of $x < 0$ that leads to $\sin \theta_{13} = -x$ and changes the signs of the phases. So, for both signs of x , the first class of solutions can be written as

$$\sin \theta_{13} = \pm x, \quad \theta_{23} = \frac{\pi}{4}, \quad \delta = \pm \frac{\pi}{2}, \quad \kappa = 0, \quad (43)$$

where θ_{12} and the second Majorana phase χ are undetermined. This type of solution can provide a good first approximation to the mixing matrix.

For the second type, we can take $\theta_{12} = 0$ for definiteness. Then, the solution is given by

$$\begin{aligned} \sin \theta_{13} &= \pm x, & \theta_{23} &= \frac{\pi}{4}, \\ \theta_{12} &= 0, & \delta - \kappa &= \pm \frac{\pi}{2}. \end{aligned} \quad (44)$$

Clearly, the vanishing value of θ_{12} (or $\theta_{12} = \pi/2$) contradicts observation. However, since $m_1 = m_2$, small perturbations that produce splitting can also generate large 1-2 mixing (see Sec. IV).

There is a simple interpretation of the two cases considered above in terms of the U_{PMNS} in Eq. (24) and the neutrino mass matrix in the flavor basis, Eq. (2). If $\kappa = 0$, then $\Gamma_M = \text{diag}(1, 1, e^{i\chi})$, so that U_{12} and Γ_M commute. After permutation of these matrices in Eq. (2), U_{12} turns out to be attached to the degenerate mass matrix. Consequently, it can be omitted, and we obtain

$$U_{\text{PMNS}} = U_{23}\Gamma_\delta U_{13}\Gamma_\delta^*\Gamma_M. \quad (45)$$

It is evident that the same result, Eq. (45), can be obtained for the second case in which $\theta_{12} = 0(\pi/2)$. Thus, the two solutions, we have found, correspond to the elimination of the U_{12} matrix.

The two types of solutions can be represented in the form of immediate relations between the mixing parameters and mass ratios. In the first case, Eq. (43), we have

$$\sin^2 2\theta_{23} = \pm \sin \delta = \cos \kappa = \frac{m_1}{m_2} = 1, \quad (46)$$

and in the second one,

$$\sin^2 2\theta_{23} = \cos 2\theta_{12} = \pm \sin(\delta - \kappa) = \frac{m_1}{m_2} = 1. \quad (47)$$

In both cases, θ_{13} is not related to masses. Although the relations (46) and (47) are obtained *post factum*, their appearance is not accidental; symmetry that led to certain values of mixing angles and phases encodes information about masses (mass spectrum).

It is not hard to understand why θ_{12} should be undetermined in the solution of type 1. Indeed, for the partially degenerate spectrum, we have the additional freedom to perform an arbitrary rotation in the 1-2 plane, $O_{12} = O_{12}(\theta)$. In this case, the mixing matrix has the general form

$$U_{\text{PMNS}} = U_{23}\Gamma_\delta U_{13}\Gamma_\delta^* U_{12}\Gamma_M O_{12}.$$

If $\kappa = 0$, as is the case in the type 1 solution, then $\Gamma_M = \text{diag}\{1, 1, e^{i\chi}\}$, so that $\Gamma_M O_{12} = O_{12}\Gamma_M$. Consequently, the additional 1-2 rotation is reduced to

$$U_{12}(\theta_{12}) \rightarrow U_{12}(\theta_{12} + \theta),$$

where θ is arbitrary and therefore the 1-2 mixing is undetermined. On the other hand, for the type 2 solution, κ can be different from zero. Thus, the freedom to redefine θ_{12} no longer exists, and it is natural that a precise value for θ_{12} is predicted, $\theta_{12} = 0$ or $\pi/2$ in this case.

The solutions we obtained correspond to zero values of the left- and right-handed parts of Eqs. (34) and (35) separately. They can be written in a parametrization-independent form as

$$|U_{\alpha 3}| = \pm x, \quad (48)$$

$$\text{Im}[U_{\beta 1}U_{\beta 2}^* - U_{\gamma 1}U_{\gamma 2}^*] = \pm x, \quad (49)$$

$$\text{Im}[U_{\alpha 1}U_{\alpha 2}^*] = 0, \quad (50)$$

$$|U_{\beta 3}|^2 = |U_{\gamma 3}|^2, \quad (51)$$

being valid for any choice of flavor index α . The relation (51) leaves only $\alpha = e$ for a plausible explanation of the experimental data.

Let us compute the group parameter x , which determines θ_{13} [see Eq. (43)]. The combinations of numbers $(m, n) = (3, 3), (3, 4), (4, 3), (3, 5),$ and $(5, 3)$, which determine the angles $\psi = 2\pi k/m$ and $\zeta = 2\pi l/n$, exhaust the finite von Dyck groups. We find that the experimental value of θ_{13} is best approximated by choosing $(m, n) = (5, 3)$ or $(3, 5)$, which corresponds to the group \mathbf{A}_5 . In this case, for $k = 2$, $l = 1$ we obtain

$$\sin \theta_{13} = \cot \frac{\pi}{3} \cot \frac{2\pi}{5} = \sqrt{\frac{1}{3} \left(1 - \frac{2}{\sqrt{5}}\right)} \approx 0.187. \quad (52)$$

This value, $\theta_{13} \approx 11^\circ$, is a good first approximation to the measured one [12].

As another example, we take $(m, n) = (3, 3)$ with $k = 1$ and $l = 1$. This corresponds to an \mathbf{A}_4 group and leads to

$$\sin \theta_{13} = \cot^2 \frac{\pi}{3} = \frac{1}{3}, \quad (53)$$

which has interesting theoretical implications for the case of complete degeneracy, as we see in Sec. V, despite being rather far from the experimental value.

IV. CORRECTIONS TO SYMMETRY RESULTS

The mixing and mass splitting we have obtained in the previous sections as consequences of symmetry do not agree with experimental data. In particular, the 1-2 mass splitting is zero; the 1-2 mixing is zero, $\pi/2$, or undefined; and the 2-3 mixing is maximal, which is somewhat disfavored by present data. In what follows, we will show that actually the obtained mass and mixing patterns can be considered as zero-order structures. For this, we prove that relatively small corrections to *the neutrino mass matrix* (and not to the mixing) can fix the problems listed above.

For definiteness, we will consider the effect of a perturbation on the first solution, Eqs. (43) and (45)—the second solution can be considered similarly. In this case, the mixing matrix is given by

$$U_{\text{PMNS}}^0 = U_{23}(45^\circ)\Gamma_{\pi/2}U_{13}\Gamma_{-\pi/2} \\ = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}c_{13} & 0 & -i\sqrt{2}s_{13} \\ -is_{13} & 1 & c_{13} \\ -is_{13} & -1 & c_{13} \end{pmatrix}. \quad (54)$$

In what follows, it will be convenient to consider the Majorana phase attached to the third mass eigenvalue. That is, we start with a zeroth-order mass matrix,

$$m_{\nu d}^0 \equiv \text{diag}\{m, m, m'_\chi\}, \quad (55)$$

with

$$m'_\chi \equiv m' e^{-i2\chi}.$$

Let us introduce a perturbation of Eq. (55),

$$m_\nu^0 = U_{\text{PMNS}}^{0*} m_d U_{\text{PMNS}}^{0\dagger} = m \begin{pmatrix} 1 - s_{13}^2(1+r) & i\frac{1}{\sqrt{2}}s_{13}c_{13}(1+r) & i\frac{1}{\sqrt{2}}s_{13}c_{13}(1+r) \\ \cdots & \frac{c_{13}^2}{2}(1+r) & \frac{c_{13}^2}{2}(1+r) - 1 \\ \cdots & \cdots & \frac{c_{13}^2}{2}(1+r) \end{pmatrix} \quad (58)$$

and

$$r \equiv \frac{m'}{m} e^{-i2\chi}. \quad (59)$$

The matrix in Eq. (58) has the features that correspond to maximal θ_{23} and vanishing θ_{12} , i.e., $m_{e\mu}^0 = m_{e\tau}^0$ and $m_{\mu\mu}^0 = m_{\tau\tau}^0$. The partial degeneracy is encoded in a more complicated relation between the elements: $m_{\mu\mu}^0(m_{ee}^0 - m_{\mu\mu}^0 + m_{\mu\tau}^0) = m_{e\tau}^0{}^2$. Violation of these equalities leads to generation of the 1-2 mixing and splitting as well as deviation of the 2-3 mixing from the maximal.

The matrix of corrections δm_ν in Eq. (57) can be written as

$$\delta m_\nu = \epsilon V \times V^T, \quad (60)$$

where V is the second column of the PMNS matrix:

$$V^T \equiv \left\{ s_{12}c_{13}, \frac{1}{\sqrt{2}}(c_{12} + is_{12}s_{13}), \frac{1}{\sqrt{2}}(-c_{12} + is_{12}s_{13}) \right\}.$$

Here, we left all the parameters unchanged except for the introduction of nonzero 1-2 mixing. From Eq. (56), we obtain

$$\epsilon = \sqrt{\Delta m_{21}^2 + m^2} - m. \quad (61)$$

In the case of strong normal mass hierarchy, $m \ll \epsilon$ and $\epsilon = \sqrt{\Delta m_{21}^2}$. On the other hand, for a strongly degenerate spectrum, we have $\epsilon = \Delta m_{21}^2/2m$. The latter expression is also obtained in the case of strong inverted mass hierarchy when $m \approx \sqrt{\Delta m_{31}^2}$. In this case,

$$m_{\nu d} = m_{\nu d}^0 + \delta m_{\nu d},$$

where $\delta m_{\nu d}$ is assumed to take a simple form,

$$\delta m_{\nu d} \equiv \text{diag}(0, \epsilon, 0).$$

This perturbation yields the 1-2 mass splitting

$$\Delta m_{21}^2 = 2m\epsilon + \epsilon^2 \quad (56)$$

and makes the 1-2 rotation physical. Using Eqs. (2) and (54), we obtain in the flavor basis

$$m_\nu = m_\nu^0 + \delta m_\nu, \quad (57)$$

where the zeroth-order matrix

$$\frac{\epsilon}{m} \approx \frac{\Delta m_{21}^2}{2\Delta m_{31}^2} \approx 1.6 \times 10^{-2}.$$

Comparing the zeroth-order values of the elements of the mass matrix, Eq. (58), with the corrections in Eq. (60), we arrive at the following conclusions:

- (1) For the ee element, $\delta m_{ee}/m_{ee}^0 \approx 2$ for the normal mass hierarchy when $m \ll \epsilon$. The ratio goes below 0.4 when $m^2 \geq \Delta m_{21}^2$.
- (2) For the off-diagonal elements in the case of normal mass hierarchy, we obtain $\delta m_{e\mu}/m_{e\mu}^0 \sim 1/2s_{13}$, $\sqrt{\Delta m_{21}^2/\Delta m_{31}^2} \sim 0.4$ for $m \ll \epsilon$. If $m^2 \geq \Delta m_{21}^2$, the ratio is less than 0.15.
- (3) In the case of inverted mass hierarchy for $r \ll 1$, the corrections equal $\delta m_{ee}/m_{ee}^0 \sim \epsilon/3m \approx 5 \times 10^{-3}$ and $\delta m_{e\mu}/m_{e\mu}^0 = \epsilon/(m \sin 2\theta_{13}) \approx 5 \times 10^{-2}$. The corrections for the elements of the $\mu - \tau$ block are of the order ϵ/m .

Thus, except for the ee elements in the case of strong normal mass hierarchy, the relative corrections to the mass matrix required to generate 1-2 mass splitting and 1-2 mixing are small, less than 0.2. At the same time, other parameters—the masses, 1-2 and 1-3 mixing, and the CP phase—can remain unchanged. The latter, however, implies correlations among the corrections to different elements of the mass matrix, which might be difficult to achieve.

If generic corrections of order $\epsilon \sim 0.2m$ are introduced in the mass matrix, all the mass and mixing parameters will be modified. Let us prove that these modifications can be small. For this, we will take the simple perturbation matrix

$$\delta m_{\text{dem}} = \frac{\epsilon}{3} \Gamma_{\pi} D \Gamma_{\pi} = \frac{\epsilon}{3} \begin{pmatrix} 1 & 1 & -1 \\ \cdots & 1 & -1 \\ \cdots & \cdots & 1 \end{pmatrix}, \quad (62)$$

where D is the democratic matrix with all elements being 1 and $\Gamma_{\pi} = \text{diag}(1, 1, -1)$.⁵

Let us compute the masses and mixing parameters for $m'_{\nu} = m_{\nu}^0 + \delta m_{\text{dem}}$, where m_{ν}^0 is given in Eq. (58). For the mass eigenvalues m'_i of m'_{ν} , and neglecting contributions of order s_{13}^2 , we obtain

$$m'_1 = m, \quad m'_2 = m + \frac{1}{3}(2 + c_{13}^2) \simeq m + \epsilon, \quad m'_3 \simeq m'. \quad (63)$$

To find the corresponding mixing angles, we first make the zeroth-order rotation in Eq. (45), which yields

$$U_{\text{PMNS}}^{0T} (m_{\nu}^0 + \delta m_{\text{dem}}) U_{\text{PMNS}}^0 = \frac{\epsilon}{3} \begin{pmatrix} c_{13}^2 + 3m/\epsilon & \sqrt{2}c_{13} & -is_{13}c_{13} \\ \cdots & 2 + 3m/\epsilon & -i\sqrt{2}s_{13} \\ \cdots & \cdots & -s_{13}^2 + 3mr/\epsilon \end{pmatrix}. \quad (64)$$

The matrix above can be subsequently diagonalized by a rotation,

$$U' = \Gamma_{\pi/2} U'_{13} U'_{23} U'_{12} \Gamma_{-\pi/2}. \quad (65)$$

Up to order s_{13}^2 corrections and other small angles corrections, this gives $\sin^2 \theta'_{12} \approx 1/3$ in good agreement with the data. Furthermore, if we assume for simplicity that $\chi = 0$ so that r is real, and multiply Eq. (64) by $\Gamma_{\pi/2}$, which follows from Eq. (65), we obtain

$$\tan \theta'_{13} \approx -\frac{\epsilon}{3m'} s_{13} c_{13} \leq s_{13} \frac{1}{3} \sqrt{\frac{\Delta m_{21}^2}{2\Delta m_{31}^2}} \sim 0.05 s_{13},$$

and

$$\tan \theta'_{23} \approx -\frac{1}{\sqrt{2}} \frac{2s_{13}}{3} \frac{\epsilon}{m'} \sim \frac{0.02}{\sqrt{2}},$$

i.e., less than 2%. Thus, the PMNS matrix including corrections can be written as

$$\begin{aligned} U_{\text{PMNS}} &= U_{\text{PMNS}}^0 U' \\ &= U_{23}(45^\circ) \Gamma_{\pi/2} U_{13}(\theta_{13} + \theta'_{13}) U'_{23} U'_{12} \Gamma_{-\pi/2} \\ &= U_{\text{PMNS}}^0(\theta_{13} + \theta'_{13}) \Gamma_{\pi/2} U'_{23} U'_{12} \Gamma_{-\pi/2}. \end{aligned} \quad (66)$$

The PMNS matrix is determined by Eq. (66) up to a phase matrix, which can be attached from the right, and we will use this to reduce Eq. (66) to standard parametrization form. We can now compute the elements of the matrix in Eq. (66) explicitly and identify them with the elements of the mixing matrix in the standard parametrization

⁵This matrix is close to the correction matrix in Eq. (60) and can be motivated by symmetry arguments.

(subscripts s). The $e2$ element equals $c_{13}^s s_{12}^s = c_{13} s_{12}^s - s_{13} c_{12} s_{23}^s$; that is, the correction to the equality $\theta_{12}^s = \theta'_{12}$ is of the order $s_{13} s_{23}^s$. To determine other angles and the Dirac CP phase, it is enough to consider the third column of Eq. (66),

$$(U_{\text{PMNS}})_{\alpha 3}^T = (-is_{13} c_{23}^s e^{-i\pi/2}, \rho e^{i\xi}, \rho e^{-i\xi}), \quad (67)$$

where

$$\rho = \frac{1}{\sqrt{2}} \sqrt{c_{13}^2 c_{23}^2 + s_{23}^2} \approx \frac{1}{\sqrt{2}}, \quad \tan \xi = -\frac{1}{c_{13}} \tan \theta'_{23} \quad (68)$$

or $\xi \approx -\theta'_{23}$. The phase of the $U_{\mu 3}$ element in Eq. (67) can be removed (as it should be in the standard parametrization) by acting on the right-hand side of Eq. (66) with the additional phase matrix $\text{diag}(1, 1, e^{-i\xi})$. This means that the CP phase is modified to $\delta = \pi/2 + \xi$.

Thus, we have shown that a simple correction matrix can generate an acceptable 1-2 mixing, the required mass splitting, and produces only small (few percent) corrections to the other mixings and to the CP -violation phase.

V. CONSTRAINTS ON MIXING FOR THE COMPLETELY DEGENERATE SPECTRUM

As we remarked in Sec. II, $M_{\nu U}$ can be forced to be completely degenerate, if a non-Abelian discrete subgroup of $O(3)$ with three-dimensional representations is imposed as G_{ν} . The possible groups are thus restricted to \mathbf{A}_4 , \mathbf{S}_4 , and \mathbf{A}_5 . These can be generated by two matrices, S_{ξ} and P , which satisfy

$$S_{\xi}^n = P^2 = (S_{\xi} P)^r = \mathbb{1}. \quad (69)$$

We take a basis for the neutrinos such that S_{ξ} is given by Eq. (11). The second matrix, P , can be represented as

$$P = O^T P_D O, \quad (70)$$

where

$$P_D = \text{diag}\{1, -1, -1\}, \quad (71)$$

and $O = O(\phi_{12}, \phi_{13}, \phi_{23})$ is a generic orthogonal matrix of rotations on the angles ϕ_{ij} .

In the charged lepton sector, we take, as before, $G_{\ell} = \mathbf{Z}_m$. The generator T must now satisfy conditions like Eq. (15) with both S_U and $P_U = U_{\text{PMNS}} P U_{\text{PMNS}}^{\dagger}$. Hence, the complete presentation for the flavor group G_f is given by

$$S_U^n = T^m = P_U^2 = \mathbb{1}, \quad (S_U P_U)^r = (S_{\xi} P)^r = \mathbb{1}, \quad (72)$$

$$(S_U T)^2 = (P_U T)^q = \mathbb{1}. \quad (73)$$

Notice that this presentation does not guarantee that G_f is finite. Following the same argument exploited in case B, we obtain that Eqs. (72) and (73) impose a set of conditions on matrices U_{PMNS} and O ,

$$\text{Tr}[U_{\text{PMNS}} S_\zeta U_{\text{PMNS}}^\dagger T] = -1, \quad (74)$$

$$\text{Tr}[O S_\zeta O^T P_D] = a_r, \quad (75)$$

$$\text{Tr}[U_{\text{PMNS}} O^T P_D O U_{\text{PMNS}}^\dagger T] = a_q, \quad (76)$$

where $a_r(a_q)$ is the sum of three r th (q th) roots of unity. The solutions of Eq. (74), which coincides with condition of the previous case, are given in Eqs. (43) and (44). Two other equations are new; Eq. (75) is the one for the matrix O , instead of U_{PMNS} , and it can be solved in a similar way. Using a parametrization for matrix O similar to Eq. (20) with vanishing CP phases, we get

$$\sin^2 \phi_{13} = \frac{a_r + 1}{2(1 + \cos \zeta)}, \quad (77)$$

where ϕ_{13} is the angle in O equivalent to θ_{13} in Eq. (20). Substituting Eqs. (43) and (77) in Eq. (76), we obtain a new equation for the remaining parameters of U_{PMNS} that either has no solution—and the group representation in question does not exist—or fixes the Majorana phase χ .

For the values of the parameters in Eqs. (43) and (52), Eq. (76) has no solutions. For the pattern with the 1-3 mixing from Eq. (53), Eq. (76) does have a solution if $r = 3$ and $q = 3$. For these values of r and q , the flavor group is $G_\nu = \mathbf{A}_4$. We obtain for the second Majorana phase, χ ,

$$\chi = \frac{3\pi}{2}. \quad (78)$$

A few comments are in order. It is easy to check that T can be written as a combination of P_U , and S_U , so that it is not an independent generator. Since $G_f = \mathbf{A}_4$, this theory corresponds to a case in which the flavor group G_f remains *unbroken* in the neutrino sector while it is broken to a \mathbf{Z}_3 subgroup in the charged lepton sector.

Out of six parameters that appear in U_{PMNS} , three are unphysical in the fully degenerate case [21]. This seems to be in contradiction with the fact that we have determined five parameters $\{\theta_{13}, \theta_{23}, \delta, \kappa, \chi\}$ by means of the symmetry. Actually, some of these parameters have been fixed, not by the symmetry but by our choice of basis. Indeed, to perform the analysis, we assumed that the group $G_f = \mathbf{A}_4$ included the 1-2 rotation S_ζ as one of the generators. However, for fully degenerate neutrinos, rotations around any axis could serve as symmetries of the neutrino mass matrix. Hence, if S_U , P_U , and T satisfy Eqs. (74) and (75) for some U_{PMNS} , then also T and the new matrices S'_U and P'_U defined as

$$S'_U = V S_U V^T, \quad P'_U = V P_U V^T \quad (79)$$

satisfy Eqs. (74) and (75) for a mixing matrix U'_{PMNS} given by

$$U'_{\text{PMNS}} = U_{\text{PMNS}} V^T. \quad (80)$$

Here, V is any orthogonal matrix. Thus, the mixing parameters found are written in basis-dependent form. One can only say that there exists a basis in which the U_{PMNS} angles and phases have the values in Eqs. (43) and (78). In general, according to Eq. (80), three parameters are unphysical out of the six that appear in U_{PMNS} in the standard parametrization.

The basis-independent physical quantities are combinations of the elements of U_{PMNS} that are invariant under orthogonal rotations of the neutrino fields and the usual phase redefinitions of leptons. These functions are nothing else but the absolute values of the elements of the matrix $\mathcal{U} = U_{\text{PMNS}}^T U_{\text{PMNS}}$. It is easy to see that since \mathcal{U} is symmetric and unitary, only three out of the nine elements $|\mathcal{U}_{ij}|$ are independent as expected according to the analysis above. Furthermore, the matrix \mathcal{U} is proportional to the mass matrix in the flavor basis, which has physical meaning; e.g., its ee element determines the amplitude on neutrinoless double-beta decay.

VI. CONCLUSIONS

In this paper, we further developed the formalism of the symmetry building in such a way that it includes both mixing parameters and neutrino masses. More precisely, the formalism connects partially and completely degenerate neutrino spectra with the mixing angles and CP phases. These are the only possibilities (along with zero mass) that can be obtained as consequences of the unitary residual symmetries.

The case of partial degeneracy, $m_1 = m_2$, follows when a \mathbf{Z}_n subgroup of $SO(2)$ with $n \geq 3$ is preserved in the neutrino sector. It can be a good lowest-order approximation to the spectrum of normal (inverted) mass hierarchy. This case is very restrictive, leading to four conditions on the mixing parameters. For $m_1 = m_2$, we have found two types of solutions with four mixing parameters fixed. Both solutions show maximal 2-3 mixing and 1-3 mixing determined directly by the group parameters. They differ by the values of the 1-2 mixing and CP -violation phases. In the first solution, θ_{12} is undefined, but both phases are fixed: $\delta = \pi/2$ and $\kappa = 0$. The second solution has θ_{12} equal to zero or $\pi/2$ and one condition on the phases: $\delta \mp \kappa = \pi/2$. In the case that gives the best approximation to the measured values, the symmetry group is \mathbf{A}_5 , and we obtain $\sin \theta_{13} = 0.187$.

These solutions should be considered as a lowest-order approximation. Relatively small corrections can produce the mass splitting and fix θ_{12} in one case and generate θ_{12} in another. Corrections may also give better agreement of the 1-3 and 2-3 mixings with observations. We show that in the first case, the corrections proportional to the ‘‘democratic’’ matrix can produce the 1-2 mass splitting and mixing in agreement with observations while giving rise to very small corrections to the other mixing parameters and CP phases.

A completely degenerate spectrum is achieved if the residual symmetry in the neutrino sector is either \mathbf{A}_4 , \mathbf{S}_4 , or \mathbf{A}_5 . In this case, U_{PMNS} has only three physical parameters, all of which are determined by the symmetry. In our formalism, this is made explicit by the fact that, in a particular basis, all the angles and CP phases of the mixing matrix—except for θ_{12} , which remains undefined—are fixed.

The values of the charged lepton masses are not involved in this consideration. In fact, the inclusion of charged leptons may produce corrections that will make the scheme with degeneracy to be viable. At the same time, it will be

probably difficult to immediately extend this consideration to the quark sector and treat two light families as being degenerate in the first approximation.

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