Magnetic focusing in atomic, nuclear, and hadronic processes

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(Received 16 September 2013; published 5 November 2013)

The processes with oppositely charged spinor particles in initial and/or final states in homogeneous magnetic field B are subject to focusing effects in their relative motion, which yield the amplifying factors in probabilities growing as eB. In addition the increasing energy of some Landau levels influences the phase space. As a result, some processes in the proper spin states can be enlarged as $\sim \frac{eB}{\kappa^2}$, where κ^2 is the characteristic phase space factor available for perpendicular to **B** motion. Several examples, including neutron β decay, positronium decay, and e^+e^- pair production, are quantitatively considered.

DOI: 10.1103/PhysRevD.88.093001 PACS numbers: 13.40.−f

I. INTRODUCTION

The motion of charged particles in magnetic field (m.f.) is a standard topic of textbooks [1–3], and the behavior of atomic and nuclear systems in the framework of QED is extensively studied [4,5]. Recently the hadronic systems in m.f. have attracted a lot of attention as well [6–16]. In particular, the role of m.f. in chiral symmetry breaking was stressed both analytically [6] and on the lattice [7] (see [8] for a review and references), and the corresponding phenomenon was coined magnetic catalysis.

The dynamical origin of magnetic catalysis in QCD was studied recently in [8–17] and shown to be an example of a more general phenomenon—the magnetic focusing, which assembles together particles of opposite charges.

The study of relativistic QCD systems (quarks, gluons, hadrons) in m.f. has made it necessary to create and exploit the relativistic formalism, based on the path integrals with interaction as in Wilson loops, which allows us to write down simple form Hamiltonians incorporating electromagnetic and strong interactions [9].

This formalism was used recently to calculate spectra of mesons in m.f. [10–12], including the Nambu-Goldstone mesons [13] and meson magnetic moments [14].

In the course of these studies it was found that m.f. plays a very important role in "assembling" opposite charges near one another, i.e., the "focusing effect," contributing to hyperfine (hf) splitting $\sim|\psi(0)|^2$, growing as eB in hydrogen [15], as well as in relativistic $q\bar{q}$ systems [16], making it necessary to introduce the smearing effect to consider hf as a perturbation. Moreover, it was found in [17], that the characteristic growth of quark condensate $|\langle\bar{q}q\rangle|$ with eB is again due to the fact that it is proportional to $|\psi(0)|^2 \sim eB$, i.e., the focusing inside the $q\bar{q}$ pair.

It is clear that the focusing mechanism is of a general character and should show up in all cases, where such a factor $|\psi(0)|^2$ for the wave function of relative coordinates of two oppositely charged particles appears. From the general scattering theory [18] it was shown that this factor (for orbital momentum zero) always appears whenever the reaction has two strongly different ranges, $r_{\rm ext} \gg r_{\rm int}$

$$dw = |\psi^{(f)}_{\rm ext}(r_{\rm int})|^2 dw_{\rm int} |\psi^{(i)}_{\rm ext}(r_{\rm int})|^2, \tag{1}$$

where superscripts f, i refer to final, initial states. These effects of initial state interaction or final state interaction (FSI) were carefully studied for the combination of Coulomb and nuclear forces [19].

In the case of m.f., the following two differences appear:

(1) m.f. induces a two-dimensional discrete spectrum in external motion, hence the sum over the spectrum should enter in (1) instead of a simple factor $|\psi_{\rm ext}(0)|^2$.

(2) The masses of this discrete spectrum are generally growing with eB and strongly influence the available phase space, making in some cases the process impossible.

However, for some lowest Landau levels (LLL),

$$E_{n_\perp} = \sqrt{m^2 + (2n_\perp + 1 - \sigma)eB + p_z^2}, \tag{2}$$

with $n_\perp = 0$, where spin (magnetic moment) contribution $\sigma = 1$ exactly cancels the radial motion, $E_0 = \sqrt{m^2 + p_z^2}$, one can gain in the resulting energy and phase space. One obtains, for the creation of a pair, the factor of growth

$$\rho(eB) = \frac{w(eB)}{w(0)} = \frac{eB}{\kappa^2}, \tag{3}$$

where κ^2 is the characteristic phase space available for the perpendicular relative motion of two charges.

Our subsequent discussion in Sec. II will be on using the relativistic Hamiltonians in m.f. and the resulting eigenfunctions and energies, obtained in [9–14] and applicable in QED and QCD, augmented by the appropriate interaction terms. We shall proceed in Sec. III with the simple example of e^+e^- production by γ in some reaction, and then compare it with the pair creation in the constant electric field.

In Sec. IV we turn to the three-body final state and consider the neutron β decay in m.f.

Other possible systems are discussed in Sec. V. In Sec. VI we give a short summary and prospectives.

The Appendix contains a short derivation of Eq. (1) using Jost solutions for external interaction.

II. RELATIVISTIC AND NONRELATIVISTIC DYNAMICS IN STRONG MAGNETIC FIELD

Our final goal is to demonstrate how m.f. changes the relative motion of oppositely charged particles and leads to the enhancement of the corresponding wave function at small distances, thus yielding the amplification factor for annihilation or production of such particles—the phenomenon of magnetic focusing.

To this end we are writing the relativistic Hamiltonian for a pair of particles with charges $e_1 = -e_2 \equiv e$, which was already derived and exploited in [10–12] and rederived in the framework of the new path integral representation in [9]. For particles with masses m_1, m_2 in m.f. $\mathbf{B}$ along the z axis, the Hamiltonian has the form (after the proper pseudomomentum factorization [9–11])

$$H = \frac{\mathbf{P}^2}{2(\omega_1+\omega_2)} + \frac{\boldsymbol{\pi}^2}{2\tilde{\omega}} + \frac{1}{2\tilde{\omega}}\frac{e^2}{4}(\mathbf{B}\times\boldsymbol{\eta})^2 + \sum_{i=1,2}\frac{m_i^2+\omega_i^2 - e_i\boldsymbol{\sigma}_i\mathbf{B}}{2\omega_i} + \hat{V}. \tag{4}$$

Here $\boldsymbol{\eta} = \mathbf{r}_1 - \mathbf{r}_2$, $\boldsymbol{\pi} = \frac{\partial}{i\partial\boldsymbol{\eta}} = \frac{\omega_1\mathbf{k}_1 - \omega_2\mathbf{k}_2}{\omega_1+\omega_2}$, $\tilde{\omega} = \frac{\omega_1\omega_2}{\omega_1+\omega_2}$, and $\hat{V}$ is the sum of all interaction terms, including photon or gluon exchange $V_{\rm Coulomb} \equiv V_c(\boldsymbol{\eta})$, confining interaction $V_{\rm conf}$ for quarks, and spin-dependent and self-energy corrections (for details see [16]).

The eigenfunction $\Psi(\omega_1,\omega_2)$ and eigenvalues $M_j \equiv M_{n_\perp,n_z}(\omega_1,\omega_2)$ depend on ω_1, ω_2, and the actual energy eigenvalue $M_j^{(0)}$ is obtained from $M_j(\omega_1,\omega_2)$ by the stationary value procedure,

$$M_j^{(0)} = M_j(\omega_1^{(0)},\omega_2^{(2)}), \qquad \frac{\partial M_j(\omega_1,\omega_2)}{\partial\omega_i}\bigg|_{\omega_i=\omega_i^{(0)}} = 0. \tag{5}$$

This scheme is discussed in detail in [9].

For nonrelativistic approximation the dominant terms are $\frac{m_i^2+\omega_i^2}{2\omega_i}$, which automatically give $\omega_i^{(0)} = m_i$.

For strong m.f., when one can neglect $\hat{V}$ in (4), i.e., for $eB \gg \sigma$ in hadron systems and $eB \gg (m_e\alpha)^2$ in atomic systems, one immediately obtains the c.m. values of $M_j^{(0)}$,

$$M_j^{(0)} = \sqrt{m_1^2+\pi_z^2+eB(2n_\perp+1-\sigma_{1z})} + \sqrt{m_2^2+\pi_z^2+eB(2n_\perp+1+\sigma_{2z})}, \tag{6}$$

and the eigenfunction for $n_\perp = 0$ is (neglecting the Coulomb interaction)

$$\Psi(z,\boldsymbol{\eta}_\perp) = \frac{e^{i\pi_z z}}{\sqrt{L}}\varphi_{n_\perp}(\boldsymbol{\eta}_\perp), \qquad \varphi_{n_\perp}(\boldsymbol{\eta}_\perp) = \frac{e^{-\frac{\eta_\perp^2}{2r_\perp^2}}}{\sqrt{\pi}r_\perp}, \qquad r_\perp = \sqrt{\frac{2}{eB}}, \qquad \varphi_0^2(0) = \frac{eB}{2\pi}. \tag{7}$$

Note that $\varphi_0^2(0)$ grows linearly with eB, the property, which is the basis for the magnetic focusing phenomenon. As will be seen, another property is important: the energy eigenvalue $M^{(0)}_{n_\perp=0}(\sigma_{1z}=+1,\sigma_{2z}=-1)$ does not grow with (and does not depend on) eB. Thus the LLL with $n_\perp = 0$ and $\sigma_{1z} = -\sigma_{2z} = 1$ (we shall call it the "zero level") ensures the enhancement of probability of production or annihilation of the pair without increase of the spectrum, which would otherwise stop the process.

III. THE ELECTRON-POSITRON PAIR PRODUCTION IN M.F.

Consider the process $A+B\to C+(e^+e^-)$, where e^+e^- is produced by a virtual photon. The amplitude for the process without m.f. can be written as

$$\mathcal{M} = C_\mu\frac{1}{Q^2}(\bar\psi\gamma_\mu\psi), \qquad \psi = \frac{u}{\sqrt{V_3}}e^{ik_+x}, \qquad \bar\psi = \frac{\bar u e^{-ik_-x}}{\sqrt{V_3}}, \tag{8}$$

and one should have in mind that both e^+ and e^- are created at one point, x. The probability can be written as

$$dw = \left|C_\mu\frac{(\bar u\gamma_\mu u)}{Q^2}\right|^2\frac{d^3k_+d^3k_-}{(2\pi)^6}\delta^{(4)}(Q-k^+-k^-)(2\pi)^4 = \left|C_\mu\frac{(\bar u\gamma_\mu u)}{Q^2}\right|^2\frac{d^3k_+}{(2\pi)^2}\delta\left(Q_0-2\sqrt{(\mathbf{k}^+)^2+m_e^2}\right), \tag{9}$$

where we have assumed that $\mathbf{Q} = 0$.

We now turn to the case of nonzero m.f. Now the energy of e^+e^- in m.f. $\mathbf{B}$ can be written, using (6), as

$$E_{n_\perp}(\pi_z,B) = \sqrt{m_e^2+\pi_z^2+eB(2n_\perp+1-\sigma_{+z})} + \sqrt{m_e^2+\pi_z^2+eB(2n_\perp+1+\sigma_{-z})}, \tag{10}$$

where $\pi_z = \frac{k_{+z}-k_{-z}}{2} = k_{+z}$ and σ_{+z} and σ_{-z} are doubled spin projections of e^+ and e^-, respectively, and we take into account that $e_+ = -e_- \equiv e$.

For the probability one can write

$$dw = \left|C_\mu\frac{(\bar u\gamma_\mu u)}{Q^2}\right|^2\frac{d\pi_z}{2\pi}\sum_{n_\perp=0}\delta(Q_0-E_{n_\perp}(\pi_z))\varphi_{n_\perp}^2(0), \tag{11}$$

and again $\varphi_0^2(0) = \frac{eB}{2\pi}$, as in (7).

It is important that for the lowest energy state at $eB\to\infty$, both terms $(1-\sigma_{1z})$ and $(1+\sigma_{2z})$ should vanish, in which case the term with $n_\perp = 0$ always survives in

(11), since E_0 does not contain eB. Looking at the structure $\bar{u}\gamma_\mu u$ and taking into account that $u_+ = C\bar{u}$, $C = \gamma_2\gamma_4$, one comes to the conclusion, that for $\gamma_\mu = \gamma_3$ the cancellation mentioned above is possible, while for $\gamma_\mu = \gamma_1, \gamma_2, \gamma_4$ both σ_{+z} and σ_{-z} have the same sign. Therefore for $eB \to \infty$, only the term with γ_3 survives and one obtains the behavior

$$dw \cong \frac{eB}{2\pi}\left| C_3 \frac{(\bar{u}\gamma_3 u)}{Q^2}\right|^2 \frac{d\pi_z}{2\pi}\delta(Q_0 - E_0(\pi_z)), \qquad (12)$$

which demonstrates the linear growth of the pair production with eB.

In the standard calculation of the e^+e^- pair production in m.f. (see, e.g., Sec. 91 of [3], and original papers [20]), the probability was calculated using crossing relations with the process $e^- \to e^- + \gamma$ in m.f. and therefore the magnetic focusing effect was not taken into account.

It is interesting to apply our results to the process of e^+e^- pair creation in the constant electric field (in Minkovski space-time).

In the proper-time formalism [21], (see also Chap. 4 of [4]), the pair probability is

$$w(x) = \mathrm{Re}\,\mathrm{tr}\int_0^\infty \frac{ds}{s} e^{-is(m^2 - i\varepsilon)}\langle x|(e^{is\hat{D}^2} - e^{is\hat{\partial}^2})|x\rangle \qquad (13)$$

and $D_\mu = \partial_\mu - ieA_\mu(x)$, $A_3(x) = -Et$, $t = x_0$, $\mathbf{A}_\perp = \frac{1}{2}(\mathbf{B} \times \mathbf{x})$, when $\mathbf{B}$ is along the z axis.

According to our discussion above, only in this situation, when $\mathbf{E}\|\mathbf{B}$, does the factor γ_3 in $\hat{A}$ generate zero levels and the magnetic focusing produce the factor $\frac{eB}{2\pi}$ for each pair without increasing the effective mass of the e^+e^- pair. The explicit form of the pair-production rate per volume per time was found in [22] for parallel $\mathbf{E}$ and $\mathbf{B}$ (remember γ_3 reasoning above),

$$w = \frac{(eE)(eB)}{(2\pi)^2}\coth\left(\frac{\pi B}{E}\right)\exp\left(-\frac{\pi m^2}{eE}\right), \qquad (14)$$

where for $\pi B \gg E$ one can see a linear growth of w with increasing eB. See more on pair creation in [4].

Now comparing (9) and (11) one finds the correspondence

$$\frac{d^2\pi_\perp}{(2\pi)^2} \to \sum_{n_\perp=0}^{n_\perp(\max)} \varphi^2_{n_\perp}(0). \qquad (15)$$

At this point one can compare our results with the statistical weight argument suggested in [1,2], where the substitution should be performed with inclusion of the magnetic field

$$\int \frac{d^3p}{(2\pi)^3} f(E) \to \frac{|e|B}{2\pi}\frac{dp_z}{2\pi}\sum_{n_\perp=0}^{\infty}(2\delta_{n_\perp o})f(E(B)), \qquad (16)$$

and the sum over spin projections is assumed. One can see a close correspondence between (11) and (16); however, (16) has to be attributed to any final particle, and only quasiclassical arguments are used for (16) in [1,2].

It is interesting how the form (11) goes over into (9) when $eB \to 0$. To this end we write, generalizing $\varphi^2_{n\perp}(0)$ to $|\varphi_{n\perp}(r_{\mathrm{int}})|^2$,

$$\varphi_{n_\perp}(r_{\mathrm{int}}) = \int \tilde{\varphi}_{n_\perp}(\boldsymbol{\pi})\frac{d^2\boldsymbol{\pi}}{(2\pi)^2}\exp(i\boldsymbol{\pi}\mathbf{r}_{\mathrm{int}})$$

and

$$\sum_{n_\perp=0}^{n_\perp(\max)} |\varphi_{n_\perp}(r_{\mathrm{int}})|^2 = \sum_{n_\perp=0}^{n_\perp(\max)} \iint \frac{d^2\boldsymbol{\pi}}{(2\pi)^2}\frac{d^2\boldsymbol{\pi}'}{(2\pi)^2}\tilde{\varphi}_{n_\perp}(\boldsymbol{\pi})\tilde{\varphi}^*_{n_\perp}(\boldsymbol{\pi}')e^{i(\boldsymbol{\pi}-\boldsymbol{\pi}')\mathbf{r}_{\mathrm{int}}}. \qquad (17)$$

For large $n_\perp(\max) \gg 1$ one can use the property of completeness of the set $\{\tilde{\varphi}_{n_\perp}(\boldsymbol{\pi})\}$, which yields

$$\sum_{n_\perp=0}^{\infty}\tilde{\varphi}_{n_\perp}(\boldsymbol{\pi})\tilde{\varphi}^*_{n_\perp}(\boldsymbol{\pi}') = (2\pi)^2\delta^{(2)}(\boldsymbol{\pi} - \boldsymbol{\pi}') \qquad (18)$$

and

$$\sum_{n_\perp=0}^{n_\perp(\max)} |\varphi_{n_\perp}(r_{\mathrm{int}})|^2 \cong \int_0^{\boldsymbol{\pi}(\max)}\frac{d^2\boldsymbol{\pi}}{(2\pi)^2}. \qquad (19)$$

In this way we are proving the correspondence (15) and (13), for any r_{int}; note, however, that we have accounted only for the final state interaction (FSI) in m.f. and have shown that it can be equivalently written in the form of a modified statistical weight. However, in the sum over $n_\perp$, the factor $|\varphi_{n_\perp}(r_{\mathrm{int}})|^2$ does not reduce to $\frac{eB}{2\pi}$, when $eB > \frac{1}{r^2_{\mathrm{int}}}$.

IV. FORMALISM FOR THREE-BODY FINAL STATES

In this section we consider an example of the neutron β decay, $n \to p + e^- + \bar{\nu}_e$, in the presence of the homogeneous magnetic field $\mathbf{B}$ along the z axis

Writing the matrix element as

$$\mathcal{M}_{if} = G\cos\theta(\bar{\psi}_p(x)O^{(1)}_\mu\psi_n(x))(\bar{\psi}_e(x)O^{(2)}_\mu\psi_\nu), \qquad (20)$$

where $O^{(i)}_\mu = \gamma_\mu(1 - \alpha_i\gamma_5)$, and $G\cos\theta \equiv \bar{G} = \frac{1.0\times10^{-5}}{m_p^2}$, $\alpha_1 = 1, 262$, $\alpha_2 = 0$ and for $B = 0$ $\psi_k(x) = \frac{ue^{i\mathbf{p}_k\mathbf{x}}}{\sqrt{V_3}}$, one obtains for the decay probability,

$$dw = \bar{G}^2|(\bar{u}0_iu)(\bar{u}0_iu)|^2\left(\prod_{k=2,3,4} d^3k\right) \times \frac{\delta^{(4)}(p_1 - p_2 - p_3 - p_4)}{(2\pi)^5} = \bar{G}^2|(\bar{u}0_iu)(\bar{u}0_iu)|^2 \frac{(M_n - \varepsilon_2 - \varepsilon_3)\varepsilon_2 d\varepsilon_2 \varepsilon_3 d\varepsilon_3}{(4\pi)^3}, \quad (21)$$

and $\varepsilon_i = \sqrt{\mathbf{p}_i^2 + m_i^2}$, $i = 2, 3$ refers to the proton and electron, respectively.

The integration in (21) proceeds inside the phase space of the ep system.

We now want to apply the m.f. to our system, and to this end we realize that it will act only on the relative coordinate $\boldsymbol{\eta}_\perp$ of the (ep) system, perpendicular to the $\mathbf{B}$ direction, and we must separate out the $\boldsymbol{\eta}_\perp$ dependence both in the (ep) wave function and in the phase space integration. One can consider (20) as a matrix element of the operator $\hat{T}$ between initial and final states (see Appendix) $\mathcal{M}_{if} = \langle \Psi_f^+|\hat{T}|\Psi_i\rangle$, and both Ψ_f and Ψ_i enter at one space-time point in the limit of large W mass.

Therefore $\Psi_f(x) = \psi_p(x)\psi_e(x) \to \Psi_{ep}(x)$, and the latter wave function is defined by the m.f. Hamiltonian (4).

According to [5], the Hamiltonian for the neutral (ep) system can be written as in (4) with $\omega_1 = \omega_p \approx m_p$, $\omega_2 = \omega_e \ll m_p$.

Noting that $\omega_1 \equiv \omega_p \approx m_p \gg \omega_2 \equiv \omega_e$, one obtains

$$E_{ep}^{(0)} \equiv E_{ep}(\omega_p^{(0)}, \omega_e^{(0)}) \cong \frac{\mathbf{P}^2}{2m_p} + \sqrt{m_p^2 + \pi_z^2 + eB(2n_\perp + 1 - \sigma_{pz})} + \sqrt{m_e^2 + \pi_z^2 + eB(2n_\perp + 1 - \sigma_{ez})}. \quad (22)$$

Now the phase space can be rewritten as follows,

$$\frac{d^3p_p d^3p_e}{(2\pi)^6} = \frac{d^3P d^3\pi}{(2\pi)^6} = \frac{d^3P d\pi_z}{(2\pi)^4}\frac{d^2\pi_\perp}{(2\pi)^2}, \quad (23)$$

and

$$d\Phi \equiv \frac{d^3p_\nu d^3p_p d^3p_e}{(2\pi)^5}\delta^{(4)}(p_n - p_p - p_e - p_\nu) = \frac{d^3P d\pi_z d^2\pi_\perp}{(2\pi)^5}\delta(m_n - |\mathbf{P}| - E_{ep}^{(0)}) = \frac{P^2 d\pi_z}{2\pi^2(1 + \frac{P}{m_p})}\frac{d^2\pi_\perp}{(2\pi)^2}. \quad (24)$$

Writing $E_{ep}^{(0)} \cong m_p + \varepsilon_e$, $\varepsilon_e \equiv \varepsilon$, one can express (24) in the form

$$d\Phi = \frac{(\Delta m - \varepsilon)^2}{4\pi^3} d\pi_z \pi_\perp d\pi_\perp = (\Delta m - \varepsilon)^2 \frac{\sqrt{\varepsilon^2 - m_e^2}}{2\pi^3}\varepsilon d\varepsilon; \qquad \Phi = \int d\Phi \cong \frac{1.63}{2\pi^2} m_e^5 \quad (25)$$

$\Delta m = 1.29$ MeV, and finally

$$w = G^2(1 + 3\alpha^2)\Phi; \qquad \alpha \equiv g_A/g_V \approx 1.262. \quad (26)$$

In the presence of m.f., the values of $\pi_\perp$ are quantized in the Hamiltonian (4), so that one should substitute as in (15), and finally the decay probability becomes

$$dw = g^2|(\bar{u}0_iu)(\bar{u}0_iu)|^2 \frac{P^2 d\pi_z}{4\pi^2(1 + \frac{P}{m_p})}\sum_{n_\perp=0}^{n_\perp(\max)} \varphi_{n_\perp}^2(0), \quad (27)$$

and $n_\perp(\max)$ is defined by the condition $m_n = P + E_{ep}^{(0)}(n_\perp)$.

As was shown in (19) for $eB \to 0$, one has the answer (21).

In the opposite limit, when eB is large, $eB \gtrsim m_e^2$ and $n_\perp(\max) = 0$, one can read in (22) that $E_{ep}^{(0)}$ is

$$E_{ep}^{(0)}(eB \to \infty) \cong m_p + \sqrt{m_e^2 + \pi_z^2}, \qquad \sigma_{ez} = -1 \quad (28)$$

and

$$\Delta m \equiv m_n - m_p = P + \sqrt{m_e^2 + \pi_z^2}, \quad (29)$$

which defines allowable phase space for (P, π_z). In this case for $n_\perp = 0$, one has the amplifying factor as in (7), $\varphi_0^2(0) = \frac{|eB|}{2\pi}$, which can be much larger than $\int^{\pi_\perp(\max)} \frac{d^2\pi_\perp}{(2\pi)^2} = \frac{\pi_\perp^2(\max)}{4\pi} \le \frac{(\Delta m)^2 - m_e^2}{4\pi}$ for $eB \gg (\Delta m)^2 - m_e^2$ (see [23,24]).

Inserting the values of $\Delta m = 2.53 m_e$, one obtains the amplifying factor for $eB \gg m_e^2$, when only one term $n_\perp = 0$ should be kept in (26), $\frac{w_n(eB)}{w_n(0)} = \frac{eB}{\kappa^2}$, and $\kappa^2 \approx m_e^2$. Exact calculation in [23] yields $\frac{w_n(eB)}{w_n(0)} = 0.77\frac{eB}{m_e^2}$.

Note, however, that for small $\pi_\perp(\max)$ and hence small κ^2, this ratio can be arbitrarily large.

For more discussion of this subject and additional references, see [25].

V. OTHER INTERACTIONS AND OTHER SYSTEMS

One immediate application of the magnetic focusing is the hyperfine interaction in all systems. In hydrogene, this effect was studied in [15] and it was shown that in the standard form of the hyperfine shift,

$$\Delta E_{\rm hf} = \frac{32\pi}{3} g_p \mu_B \mu_n |\Psi(0)|^2, \quad (30)$$

where $g_p = 2.79$, $\mu_B = \frac{|e|}{2m_e}$, and $\mu_n = \frac{|e|}{2m_p}$, the wave function of the ground state can be chosen as

$$\Psi_0(\eta_\perp, z) = N \exp\left(-\frac{\eta_\perp^2 a^2}{2} - \frac{z^2 b^2}{2}\right) \qquad (31)$$

where $|\Psi(0)|^2 = \frac{a^2 b}{\pi^{3/2}}$, and a, b are fitting parameters; for the free electron $a = \frac{eB}{2m_e}$, and for large $eB > (m_e\alpha)^2$, $a^2 b$ grows with eB almost linearly, which gives the possibility to study this effect experimentally.

The same effect was found in relativistic hadronic systems in [16], where again the hf shift has the same form as in (25), namely,

$$\langle V_{\rm hf}\rangle = \frac{8\pi\alpha_s(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)}{9\omega_1\omega_2}|\Psi_{q\bar q}(0)|^2, \qquad (32)$$

and one can show that (32), when ω_1, ω_2 are decreasing and $\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 = -3$ leads to the absurd result of the negative pseudoscalar masses at large eB, contradicting the stability theorem, proved in [16], which tells us that energy eigenvalues in magnetic and vacuum Euclidean fields cannot be negative and, hence, $V_{\rm hf}$ cannot be treated perturbatively, when $\omega_i \to 0$ and $\langle V_{\rm hf}\rangle \to \infty$. Consequently, the relativistic smearing must be introduced, replacing $\delta^{(3)}(\boldsymbol{\eta})$ in $V_{\rm hf}$ by $(\frac{1}{\mu\sqrt{\pi}})^3 \exp(-\mu^2\eta^2)$, $\mu \approx (1 \div 2)$ GeV.

In any case, the behavior (32) signifies the strong increase of the hyperfine splitting for hadrons in magnetic field, which is stabilized by the smearing [16].

The calculations of hadron masses, subject to strong hf interaction have been done in [10–13]. In the case of π^0, ρ^0 mesons, the large eB asymptotics correspond to the $\langle\sigma_1\sigma_2| = \langle + - |$ state for π^0 and the $\langle - + |$ state for ρ^0, and the strong hf interaction splits the masses, creating a deep minimum for the π^0 mass. However, this calculation in [10] refers to the purely $q\bar q$ components of π^0, whereas the chiral dynamics in m.f. needs a special treatment, which was performed in [13] and again showed a decreasing with eB π^0 mass, with a minimum.

The calculation in [13] was done based on the unified theory of chiral dynamics with $q\bar q$ degrees of freedom, developed earlier in [26]. In its turn in [17] it was shown that the chiral condensate grows with m.f., first quadratically for $eB < \sigma$ and then for $eB \gg \sigma$ linearly with eB. This behavior found in [17] agrees quantitatively with the lattice data of [7], the first reference that contradicts chiral perturbative theory. In this way one can conclude that the standard chiral theory can be applied only for $eB \lesssim m_\pi^2$, as was noticed before in [27].

The subsequent analysis of Nambu-Goldstone mesons in m.f. was done in [13], where it was shown that Gell-Mann-Oakes-Renner relations are valid for neutral Nambu-Goldstone mesons but are violated for the charged ones, and the π^+ mass was calculated in agreement with lattice data from [28]. One should stress that the phenomenon of "magnetic catalysis," discussed in [6,8] and implying the growth of chiral condensate $\langle\bar q q\rangle$ in m.f. in our treatment, occurs due to magnetic focusing, as was shown in [17], since $\langle\bar q q\rangle \sim \psi^2(0) \sim eB$.

We now turn to other processes, where the oppositely charged particles appear in the initial state. One example is the $\mu^- + p \to n + \nu$. In the case when the process occurs from the ground state of the $(\mu^- p)$ atom, one can use the form (26) with $r_\perp$, r_z obtained in [15], which are reduced by the ratio $\frac{m_e}{\tilde m_\mu}$, $\hat m_\mu = \frac{m_\mu m_p}{m_\mu = m_p}$.

Since the probability $w_\mu \equiv w(\mu^- + p \to n + \nu)$ is proportional to $|\Psi_{\mu p}(0)|^2$, one can obtain the amplification coefficient

$$\rho(B) \equiv \frac{w_\mu(B)}{w_\mu(0)} = \frac{r_\perp^2(0) r_z(0)}{r_\perp^2(B) r_z(B)}, \qquad \rho(0) = 1. \qquad (33)$$

Using the data from Fig. 2 of [15], one easily obtains for $H = \frac{eB}{\tilde m_\mu^2 \alpha^2}$, $0 \le H \le 2$,

$$\rho(H = 1) \cong 1.05; \qquad \rho(H = 2) = 3.35.$$

One can see that in this case one needs much larger fields $[(\frac{\tilde m_\mu}{m_e})^2$ times larger] to produce the same kind of effect as in the electron case.

Let us turn now to the case of the positronium annihilation in m.f. Since the latter does not conserve the spin, but only total spin projection S_z, it is convenient to discuss separately case (a) of $S_z = 0$ parapositronium and orthopositronium and (b) $S_z \pm 1$ orthopositronium. In the first case, as in the $q\bar q$ case, both states are a mixture of $\alpha = (\sigma_z = +1, \sigma_z = -1) \equiv \langle + - |$ and $\beta = \langle - + |$, and parapositronium in strong m.f. tends to be a pure α state, whereas orthopositronium is a pure β state (see [10,12] for a similar discussion in the $q\bar q$ case). For case (b) $S_z = \pm 1$, in orthopositronium the corresponding ground state masses in strong m.f. grow as $\sim\sqrt{eB}$ with additional amplification due to the strong hf contribution. As a result the decay phase space increases with eB and the decay probability in case (b) is growing both due to phase space and as in (33), where in m.f. $\tilde m_\mu \to \frac{1}{2} m_e$. The same happens in the β state of orthopositronium, since the total energy in the β state also grows as $\sim\sqrt{eB}$.

However, in the α state of parapositronium the mass is slightly decreasing, while the $|\psi(0)|^2$ factor is growing as eB, and we expect the same situation, as in the example of (e^+e^-) pair creation in m.f., discussed in Sec. II. At this point it is necessary also to stress the difference between the two examples: for positronium annihilation the factor $\rho(B)$ is proportional to $|\psi(r_{\rm an})|^2$, where $r_{\rm an}$ is of the order of $\sqrt{r_\perp^2 + r_z^2}$, while the (e^+e^-) pair creation occurs at one point and brings about factor $|\psi(0)|^2$. The resulting difference is expected to be of the order of unity for $eBr_z^2 \lesssim 1$.

However, for large $eB > (m_e\alpha)^2$, one expects the behavior to be $w_{+-}(B) \approx \int |\psi_{+-}(r_{\rm an})|^2 w^{(0)} d\tau$, where $w^{(0)} d\tau$ is the phase space integration (depending on B)

with the two-photon annihilation amplitude $A^{(2)}$, $w^{(0)} = |A^{(2)}|^2$. Note that both $\langle + - |$ and $\langle - + |$ states can be superpositions of $S = 1$ and $S = 0$ states. As a result one obtains the distributions $w_{+-}(B)$ and $w_{-+}(B)$ as functions of eB and notice that the linear growth for $eB \gg (m_e\alpha)^2$ is saturated for larger m.f., as was discussed previously.

In [29] the authors have used the Gaussian form of positronium wave function and found the almost linear growth of two-photon annihilation with eB for $eB \lesssim 10^{13}$ G: $w(H = 1) = 3.35 \times 10^{12}\ \mathrm{s}^{-1}$, $w(H = 3) = 12.3 \times 10^{12}\ \mathrm{s}^{-1}$, $w(H = 10) = 5.09 \times 10^{13}\ \mathrm{s}^{-1}$, $w(H = 44) = 11.8 \times 10^{13}\ \mathrm{s}^{-1}$, where $H = \frac{B}{10^{12}}$ G. One can see that for $B = 10^{13}$ G, the growth of w is weakening.

VI. SUMMARY AND DISCUSSION

We have demonstrated that the acting of m.f. on the system of two opposite charges in the initial or final state produces an amplifying factor, which can grow linearly with eB for eB in the range $(eB)_{\min} \lesssim (eB) \lesssim (eB)_{\max}$.

In particular, for the $(e, -e)$ continuum production the relative probability $w(eB)/w(0) \simeq |\psi_{\rm cont}(r_{\rm (int)}|^2/\kappa^2$ can grow as $\frac{eB}{\kappa^2}$, where κ^2 is the effective phase space for the perpendicular motion in the $B = 0$ case. In this case $(eB)_{\min}$ is also of the order of κ^2, while $(eB)_{\max}$ is defined by the $(eB)_{\max} \sim 1/r^2_{\rm int}$, where $r_{\rm int}$ refers to the range of the internal production process (e.g., $r_{\rm int} \sim 1/m_W$ for neutron β decay).

We have illustrated this behavior by the processes of the e^+e^- pair creation and the neutron β decay; in the last case this amplification was known and calculated by many authors (see, e.g., [23–25]) using QPS in m.f. as in [1,2].

We have shown that the FSI method of our paper gives the same result as QPS for $r_{\rm int} = 0$ or for $eB \ll 1/r^2_{\rm int}$; however, for $eB \sim 1/r^2_{\rm int}$ the QPS prediction overestimates the probability.

For the $(e, -e)$ bound state production, the amplification factor is proportional to $|\psi_{\rm b.s.}(r_{\rm int})|^2$, and for $r_{\rm int} = 0$ it grows linearly with eB only for $eB \gg r^2_{\rm b.s.}$, where $r_{\rm b.s.}$ is the bound state radius at zero m.f., e.g., for positronium $r_{\rm b.s.} = \frac{2}{m_e\alpha}$.

The same arguments can be applied, in principle, to the effect of constant m.f. in the initial state, as it is illustrated above in the paper by the example of the two-photon positron annihilation (see, e.g., [29]). However, in this case one should distinguish different physical situations; in astrophysics these processes are part of a set of reactions in m.f. at finite temperature and chemical potential, while in experiment one should take into account boundary conditions of the experimental device.

Incidentally, one should stress that all above derivations disregarded the Coulomb potential role in initial state interaction and FSI.

As it is known, the latter gives the factor for the $(e, -e)$ system $|\psi_{\rm coul}(0)|^2 = \frac{2\pi\eta}{0\exp(-2\pi\eta)+1}$, $\eta = \frac{e^2}{v}$, which may bring an additional amplification of the magnetic focusing effect, discussed above. As it is, strictly speaking, we can discuss the Coulomb amplification for $2\pi\eta \gg 1$ and $eB \ll \kappa^2$. However, the explicit amplification factor in the case when both Coulomb interaction and m.f. are acting is still not available and should be derived using solutions when both interactions are present.

Finally, probably the most important conclusion of our paper is that for $(e, -e)$ systems in continuum, there occurs an amplification factor $\frac{eB}{\kappa^2}$, which does not depend on the masses of the charges and their sizes R_i (provided $eB < 1/R^2$) and, therefore, can be important for stimulating different reactions in atomic, molecular, nuclear, or hadronic reactions.

APPENDIX: INITIAL AND FINAL INTERACTION FACTORS IN PROCESS PROBABILITY

One can use the standard formalism [18] to define the matrix element of a transition from the state β to α, generated by the interaction V in the total Hamiltonian,

$$H = K + U + V, \tag{A1}$$

where K is the kinetic term and U and V have different ranges, r_U, r_V, correspondingly. To the first order in V one can write

$$f_{\beta\alpha} = (\varphi^-_\beta V \varphi^+_\alpha) + O(V^2), \tag{A2}$$

where φ^-_β, φ^+_α are ingoing and outgoing solutions of the operator $K + U$,

$$\varphi^-_\beta = \chi_\beta + \frac{1}{E - K - i\varepsilon} U \varphi^-_\beta, \tag{A3}$$

and for φ^+_α one should replace $\beta \to \alpha$, $\varepsilon \to -\varepsilon$.

Introducing for the orbital momentum $l = 0$ the internal amplitude due to V only as $f_{\rm in}(E)$, one can write (see [18] for a detailed derivation)

$$f_{\alpha\alpha} = f_{\rm in}(E)(\eta_{\rm Jost})^{-2}, \tag{A4}$$

where $\eta_{\rm Jost} = \eta(0)$, and $\eta(r)$ is the Jost solution of the external problem, satisfying the condition

$$\lim_{r\to\infty} \exp(-ikr)\eta(r) = 1, \tag{A5}$$

while for the Coulomb type interaction one should replace $-ikr \to -ikr + \frac{i\alpha}{v}\ln 2kr$.

$\eta_{\rm Jost}$ can be expressed via the regular solution $\chi(r)$ of the external potential with the asymptotics

$$\chi_{\rm ex}(r) \sim \frac{\sin(kr + \delta_{\rm ext})e^{i\delta_{\rm ext}}}{kr}, \qquad r \to \infty, \tag{A6}$$

and the connection is

$$\lim_{r\to 0} \chi(r) = \eta^{-1}_{\rm Jost}. \tag{A7}$$

As a result one has (see [18])

$$f_{\alpha\alpha}(E) = \frac{f_{in}(E)(\chi_{\rm ex}(0))^2}{1 + f_{in}(E)g(E,0)}, \quad \text{(A8)}$$

where $g(E, r_{\rm in})$ is expressed via Green's functions,

$$g(E, r) = G_0(r, r_{\rm in}) - G_{\rm ex}(r, r_{\rm in}). \quad \text{(A9)}$$

In particular, for the Coulomb external problem, one finds [19]

$$\eta_{\rm Jost} = \exp\left(\frac{\pi\gamma}{2}\right)/\Gamma(1 + i\gamma) = |\eta_{\rm Jost}| \exp(-i\sigma_0), \quad \text{(A10)}$$

$$\exp(2i\sigma_0) = \frac{\Gamma(1 + i\gamma)}{\Gamma(1 - i\gamma)}, \quad \text{(A11)}$$

$$|\eta_{\rm Jost}|^{-2} \equiv C_0^2 = \frac{2\pi\gamma}{\exp(2\pi\gamma) - 1}, \quad \text{(A12)}$$

where $\gamma = \frac{ze^2\mu}{k}$, and $\gamma = -|\gamma|$ for the oppositely charged system. Note, however, that (A4) and (A12) refer to the combination of the short-range (nuclear) and Coulomb (external) interactions, and the latter acts both in the initial and final states [φ_β^- and φ_α^+ in (A2)]. In the more general case, when initial and final states are different (i.e., short-range reaction with different particles and interactions in the initial and final states), one should keep track of indices β and α, which are different, and φ_β^- and φ_α^+ satisfy equations of the form (A3) for φ_β^- with $U \equiv U_\beta$, $K = K_\beta$, and φ_α^+ satisfies

$$\varphi_\alpha^+ = \chi_\alpha + \frac{1}{E - K_\alpha + i\varepsilon} U_\alpha \varphi_\alpha^+. \quad \text{(A13)}$$

To clarify this situation we consider, as in [18], the second reference, a simple model for the internal interaction

$$V(\mathbf{r}, \mathbf{r}') = \lambda g_\beta(r) g_\alpha(r'), \quad \text{(A14)}$$

and as a result the reaction amplitude becomes

$$f_{\beta\alpha} = \lambda I_\beta I_\alpha, \qquad I_\beta = \int \varphi_\beta^{-*} g_\beta d^3r. \quad \text{(A15)}$$

Taking into account different radii of internal and external motion, one can approximate

$$I_\beta = \varphi_\beta^{-*}(0) \int g_\beta d^3r, \qquad I_\alpha = \varphi_\alpha^+(0) \int g_\alpha d^3r. \quad \text{(A16)}$$

A generalization of $f_{\alpha\alpha}$ to the case of $\alpha \neq \beta$ is (to the first order in $f_{\rm in}$)

$$f_{\alpha\beta} = (\eta_{\rm Jost}^{(\beta)^*})^{-1} f_{\rm in}(E) (\eta_{\rm Jost}^{(\alpha)})^{-1}. \quad \text{(A17)}$$

We are especially interested in the case when both Coulomb interaction and external magnetic field are present simultaneously. The first interaction is effective when $\gamma = |\frac{Z\alpha}{v}| \sim 1$, and for $\gamma \ll 1$ one can retain only m.f. effects. In this case both φ_β^- and φ_α^+ are products of a plane wave in the z direction and a bound state eigenfunction in the (x, y) plane,

$$\varphi_\beta^- = \varphi_\alpha^+ = \frac{e^{ik_z z}}{\sqrt{L}} \varphi_{n_\perp}(\mathbf{x}_\perp). \quad \text{(A18)}$$

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