Isorotating baby Skyrmions

A. Halavanau[1] and Ya. Shnir[1,2,3]

[1]Department of Theoretical Physics and Astrophysics, Belarusian State University, 220030 Minsk, Belarus
[2]Bogoliubov Laboratory of Theoretical Physics, JINR, 141980 Dubna, Moscow region, Russia
[3]Department of Physics, Carl von Ossietzky University Oldenburg, D-26111 Oldenburg, Germany

(Received 18 September 2013; published 22 October 2013)

We discuss how internal rotation with fixed angular frequency can affect the solitons in the baby Skyrme model in which the global $O(3)$ symmetry is broken to the $SO(2)$. Two particular choices of the potential term are considered, the ''old'' potential and the ''new'' double vacuum potential, We do not impose any assumptions about the symmetry on the fields. Our results confirm existence of two types of instabilities determined by the relation between the mass parameter of the potential μ and the angular frequency ω. It is shown that multi-Skyrmions in the model with old potential at some critical value of the angular frequency become unstable with respect to decay into single Skyrmion constituents.

DOI: 10.1103/PhysRevD.88.085028 PACS numbers: 11.27.+d, 11.10.Lm

I. INTRODUCTION

The so-called baby Skyrme model is a modified version of the nonlinear $O(3)$ σ model in $2+1$ dimensions [1], a low-dimensional simplified theory which resembles the conventional Skyrme model in many important respects. This model has a number of applications, e.g. in condensed matter physics where Skyrmion configurations were observed experimentally [2], or in the topological quantum Hall effect [3].

Together with the original Skyrme model in $d = 3 + 1$ [4] and the Faddeev-Skyrme model [5], the baby Skyrme model can be considered as a member of the Skyrme family. Indeed, the Lagrangian of all these models has similar structure; it includes the usual $O(3)$ sigma model kinetic term, the Skyrme term, which is quartic in derivatives, and the potential term which does not contain the derivatives.

According to the Derrick's theorem [6], the Skyrme term is always required to support existence of soliton configurations. In $d = 2 + 1$ the potential term is also necessary to obtain the localized field configurations with finite energy. In $d = 3 + 1$ the last term is optional; it gives a mass to the excitations of the scalar field, so in the context of the usual Skyrme model it is referred to as ''pion mass term.'' It is known that its appearance might seriously affect the structure of the solutions of the Skyrme model [7]. On the other hand, the form of the potential term in the baby Skyrme model is largely arbitrary; there are different choices related with various ways of symmetry breaking [8–10].

Note that the solitons of the models of the Skyrme family possess both rotational and internal rotational (or isorotational) degrees of freedom. The traditional approach to study the spinning solitons is related with rigid body approximation, both in the context of the Skyrme model [11,12] and in the baby Skyrme model [13]. The assumption is that the solitons could rotate without changing its shape. This restriction can be weakly relaxed by consideration of the radial deformations which would not violate the rotational symmetry of the hedgehog configuration [13,14]. Evidently, this approximation is not very satisfactory; a consistent approach is to solve the full system of field equations without imposing any spatial symmetries on the isospinning solitons. Furthermore, almost all previous studies of spinning solitons (see e.g. [15,16]) were concerned with minimization of the total energy functional $E_J[\boldsymbol{\phi}]$ for fixed value of the isospin J. However, if we do not assume the spinning soliton will have precisely the same shape as the static soliton, this approach becomes rather involved; it is related to the numerical solution of the complicated differential-integral equation.

Very recently the isospinning soliton solutions were considered in the Faddeev-Skyrme model beyond rigid body approximation [15,17]. The approach of the paper [17] is to consider the static pseudoenergy minimization problem, where the pseudoenergy functional $F_\omega[\boldsymbol{\phi}]$ depends parametrically on the angular frequency ω. The important conclusion, which is general for all models of the Skyrme family, is that there is a new type of instability of the solitons due to the extra nonlinear velocity dependence generated by the Skyrme term [17].

In this paper, we aim to perform the analysis of the critical behavior of the isospinning solitons of the baby Skyrme model without imposing any spacial symmetries. We confirm existence of two types of instabilities determined by the relation between the mass parameter of the potential μ and the frequency ω similar to the pattern observed in the Faddeev-Skyrme model [15,17]. Interestingly, we observe that the critical behavior of the isospinning baby Skyrmions depends also on the structure of the potential of the model, for example in the case of the ''old'' model [1] the isospinning configurations of higher degree may become unstable with respect to decay into constituents.

Note that we do not consider the very interesting case of the family of the potentials discussed by Karliner and Hen [9]. Here we also restrict our consideration to configurations of degree $1 \leq B \leq 5$. A more systematic investigation of the spinning baby Skyrmions for various potentials and for larger values of B will be presented elsewhere in our work in collaboration with Battye and Haberichter [18].

In the next section we briefly review our approach to the minimization problem of finding isospinning solitons of the baby Skyrme model. The numerical results are presented in Sec. III where we considered behavior of the isorotating solitons in two different cases, the old baby Skyrme model [1] and the "new" double vacuum model [19]. We give our conclusions and remarks in the final section.

Last but not least, we consider our results to be complementary to the independent findings of Battye and Haberichter presented in the recent paper [20].

II. BABY SKYRME MODEL

As a starting point we consider the rescaled Lagrangian[1] of the $O(3)$ σ model with the Skyrme term in $2+1$ dimensions [1]:

$$L = \partial_\mu \boldsymbol{\phi} \cdot \partial^\mu \boldsymbol{\phi} - \frac{1}{4}(\partial_\mu \boldsymbol{\phi} \times \partial_\nu \boldsymbol{\phi})^2 - U[\boldsymbol{\phi}], \quad (1)$$

where $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)$ denotes a triplet of scalar fields which satisfy the constraint $|\boldsymbol{\phi}|^2 = 1$. Topologically the field is the map ϕ: $\mathbb{R}^2 \to S^2$ characterized by the topological charge $B = \pi_2(S^2) = \mathbb{Z}$. Explicitly,

$$B = \frac{1}{4\pi}\int_{\mathbb{R}_2} \boldsymbol{\phi} \cdot \partial_1 \boldsymbol{\phi} \times \partial_2 \boldsymbol{\phi}. \quad (2)$$

Note that the first two terms in the functional (1) are invariant under the global $O(3)$ transformations; this symmetry becomes broken via the potential term. The standard choice of the potential of the baby Skyrme model is [1]

$$U[\boldsymbol{\phi}] = \mu^2[1 - \phi_3], \quad (3)$$

thus the symmetry is broken to $SO(2)$ and there is a unique vacuum $\boldsymbol{\phi}_\infty = (0, 0, 1)$. The corresponding solitons of degree $B = 1, 2$ are axially symmetric [1]; however, the rotational symmetry of the configurations of higher degree becomes broken [13].

The residual symmetry of the configurations with respect to the rotations around the third axis in the internal space allows us to consider the stationary isospinning (i.e. internally rotating) solitons,

$$(\phi_1 + i\phi_2) \mapsto (\phi_1 + i\phi_2)e^{i\omega t}, \quad (4)$$

where ω is the angular frequency. The corresponding conserved quantity is the angular momentum $J = \omega\Lambda[\boldsymbol{\phi}]$, where $\Lambda[\boldsymbol{\phi}]$ is the moment of inertia; thus the total energy of the spinning field configuration is

$$E_J[\boldsymbol{\phi}] = V[\boldsymbol{\phi}] + \frac{J^2}{2\Lambda[\boldsymbol{\phi}]}.$$

Evidently, the isorotations (4) of the energy functional of the baby Skyrme model yield the pseudoenergy functional

$$\begin{aligned} F_\omega[\boldsymbol{\phi}] = \int_{\mathbb{R}_2} \Big\{ & \frac{1}{2}([2 - \omega^2(\boldsymbol{\phi}_\infty \times \boldsymbol{\phi})^2](\partial_i \boldsymbol{\phi} \cdot \partial_i \boldsymbol{\phi}) \\ & + \omega^2[\boldsymbol{\phi}_\infty \cdot (\boldsymbol{\phi} \times \partial_i \boldsymbol{\phi})]^2) \\ & + \frac{1}{4}(\partial_i \boldsymbol{\phi} \times \partial_j \boldsymbol{\phi})^2 + (U[\boldsymbol{\phi}] - \omega^2(\boldsymbol{\phi}_\infty \times \boldsymbol{\phi})^2)\Big\} \\ = V - & \frac{1}{2}\omega^2 \Lambda(\boldsymbol{\phi}), \end{aligned} \quad (5)$$

where the V is the potential energy of the nonrotated configuration and the moment of inertia is

$$\Lambda(\boldsymbol{\phi}) = \int_{\mathbb{R}_2} \{(\boldsymbol{\phi}_\infty \times \boldsymbol{\phi})^2[1 + (\partial_i \boldsymbol{\phi} \cdot \partial_i \boldsymbol{\phi})] - [\boldsymbol{\phi}_\infty \cdot (\boldsymbol{\phi} \times \partial_i \boldsymbol{\phi})]^2\}. \quad (6)$$

The isospinning solitons correspond to the stationary points of the functional (5), which has the same dimensional structure as the total energy functional; thus, the standard scaling arguments in two spatial dimensions [6] yield the virial relation

$$\mathbb{G} = \mathbb{V}, \quad (7)$$

where $\mathbb{G} = \frac{1}{4}\int_{\mathbb{R}_2}(\partial_i \boldsymbol{\phi} \times \partial_j \boldsymbol{\phi})^2$ and $\mathbb{V} = \int_{\mathbb{R}_2}(U[\boldsymbol{\phi}] - \omega^2(\boldsymbol{\phi}_\infty \times \boldsymbol{\phi})^2)$ are two integrals which must be positively defined.

However, the pseudoenergy (5) is not bounded from below for $\omega > \omega_1 = \sqrt{2}$ independently from the particular choice of the potential $U[\boldsymbol{\phi}]$ [17]. Indeed, the first term in (5) effectively defines the geometry of the deformed sphere S^2 squashed along the direction $\boldsymbol{\phi}_\infty$, the metric on this space becomes singular at $\omega = \omega_1 = \sqrt{2}$.

The second critical frequency is related to the condition of positiveness of the effective potential,

$$U_\omega[\boldsymbol{\phi}] = U[\boldsymbol{\phi}] - \omega^2(1 - \phi_3^2),$$

it approaches zero at some critical value $\omega = \omega_2$. In this limit the isospinning solitons of the baby Skyrme model cease to exist because the vanishing of the potential makes the configuration unstable; the virial relation (7) becomes

[1]Our conventions are slightly different from usual choice [1]; the kinetic term differs from the standard one by a factor of 2. Evidently, corresponding rescaling of the mass parameter μ allows us to recover the latter conventions.

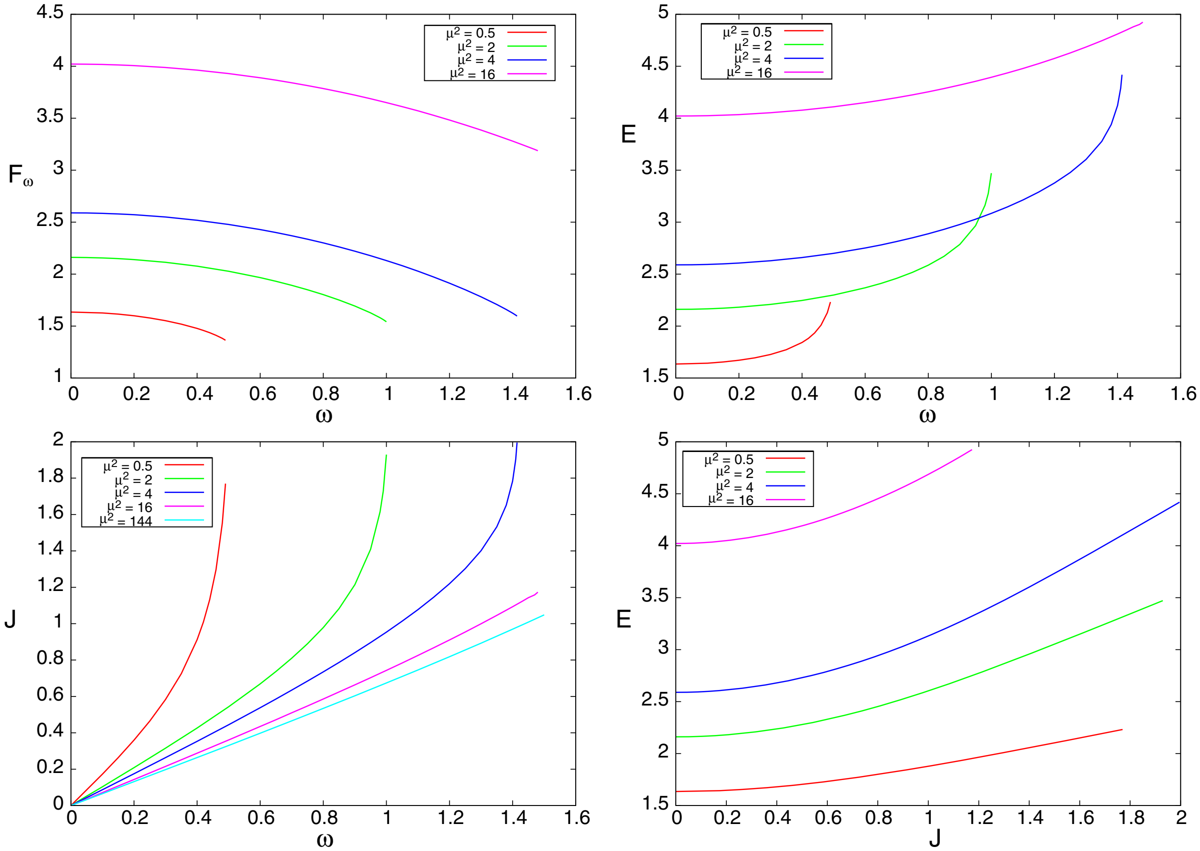

FIG. 1 (color online). Pseudoenergy, energy and isospin of the $B = 1$ baby Skyrmion in the model with potential (3) are plotted as functions of angular frequency ω, and the energy is plotted as a function of isospin J at $\mu^2 = 0.5, 2, 4, 16$.

violated. It is particularly convenient to investigate the critical behavior of that type not in the case of the potential (3) but for the double vacuum model [19] with another choice of the rotationally invariant potential,

$$U[\boldsymbol{\phi}] = \mu^2[1 - \phi_3^2]. \tag{8}$$

Evidently in that case the critical value $\omega_2 = \mu$. Below we consider the pattern of critical behavior in both models.

The traditional approach to study the solitons of the model (1) is related to separation of the radial and angular variables [1,13]; thus, the consideration becomes restricted to the case of rotationally invariant configurations and the corresponding Euler-Lagrange equations are reduced to a single ordinary differential equation on radial function $f(\rho)$. However, more detailed analyses reveal that the higher charge $B \geq 3$ baby Skyrmions may not possess rotational symmetry [1,21]; starting from some critical value of the mass parameter μ the global minimum of the energy functional corresponds to the configurations with discrete symmetries.

The violation of the rotation invariance in the baby Skyrme model attracted a lot of attention recently; it was demonstrated that the effect strongly depends on the particular choice of the potential of the model [8–10]. Thus, considering the isorotating baby Skyrmions we will consider the complete system of coupled partial differential equations on the triplet of functions $\boldsymbol{\phi}(\rho, \theta)$ which follows from the Lagrangian (1) in two cases of the rotationally invariant potentials, the standard potential (3) and the double vacuum model (8). Note that our numerical results indicate that another possible choice of the rotationally invariant holomorphic potential $U[\boldsymbol{\phi}] = \mu^2[1 - \phi_3]^4$ [22] does not admit isorotations; the configuration becomes unstable for any nonzero value of the parameter ω.

III. NUMERICAL RESULTS

In this section the results of numerical simulations of the isospinning baby Skyrme model will be presented. The numerical calculations are mainly performed on an equidistant grid in polar coordinates ρ and θ, employing the compact radial coordinate $x = \rho/(1 + \rho) \in [0:1]$ and $\theta \in [0, 2\pi]$. To find stationary points of the functional (5), which depends parametrically on ω, we implement a simple forward differencing scheme on a square lattice with lattice spacing $\Delta x = 0.01$. Typical grids used have sizes 120×120. The relative errors of the solutions are of order of 10^{-4} or smaller. To check our results for correctness we checked that the corresponding virial relation (7) holds; as

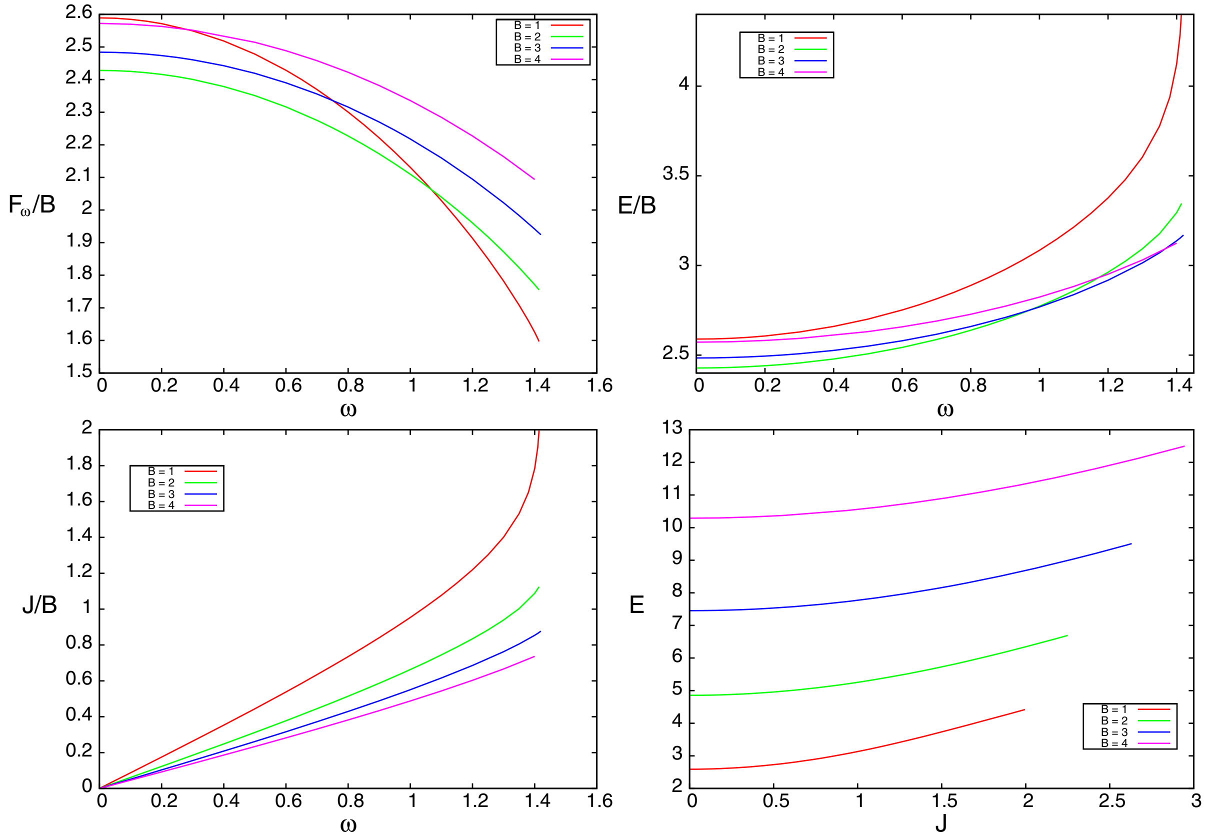

FIG. 2 (color online). Pseudoenergy, energy and isospin of the rotationally invariant baby Skyrmion solutions in the model with potential (3) are plotted as functions of angular frequency ω. The energy is presented as a function of isospin J at $\mu^2 = 4$.

a further verification we evaluated the value of the topological charge by direct integration of (2).

Each of our simulations began at $\omega = 0$ at fixed value of μ, then we proceed by making small increments in ω. Initial configurations we created using ansatz

$$\begin{aligned}\phi_1 &= \sin f(\rho)\cos(B\theta);\\ \phi_2 &= \sin f(\rho)\sin(B\theta);\\ \phi_1 &= \cos f(\rho),\end{aligned} \tag{9}$$

where the input profile function is defined as $f(\rho) = 4\arctan e^{-\rho}$. Evidently this corresponds to the configuration of degree B with standard boundary conditions on the profile function $f(\rho)$. Generally, in our calculations we do not impose any assumptions about the spatial symmetries of the components of the field $\boldsymbol{\phi}$ although in the next section we briefly considered the axially symmetric isospinning multi-Skyrmions.

A. Old baby Skyrme model

First we consider evolution of the baby Skyrmions in the model with potential (3). Note that the rotationally invariant ansatz (9) corresponds to the ring baby Skyrmions which in the standard conventions, for $B > 2$ and $\mu^2 = 0.1$, are not absolute minima but rather the saddle points of the energy functional [1,21]; they are unstable with respect to perturbations which break this symmetry.[2] However, the situation may change as the mass parameter increases, for larger values of μ the solutions may preserve the axial symmetry. Indeed the larger parameter μ is the smaller the soliton size is; in the limit of very large mass the potential term can be considered as a sort of constraint on the field component ϕ_3 with μ^2 acting like a Lagrange multiple. Note that in that limit the baby Skyrmions are, in fact, compactons, the fields reach the vacuum values at finite distance from the center of the soliton and they do not have asymptotic tails [23]. In our calculations we mainly considered relatively large values of $\mu > 1$.

In Fig. 1 we present typical graphs of the soliton energies, both as functions of ω and as functions of isospin J for a range of values of μ. When the mass parameter $\mu^2 < 2$ we observe critical behavior of the first type, the effective potential vanishes and both the energy and the angular momentum diverge. When μ^2 increases further, a second type of critical behavior is observed, our algorithm ceases to find any critical points when ω is taking the values $\omega > \sqrt{2}$ though the energy and the angular

[2]We remind that the kinetic term in standard conventions differs from our term by a factor of $1/2$.

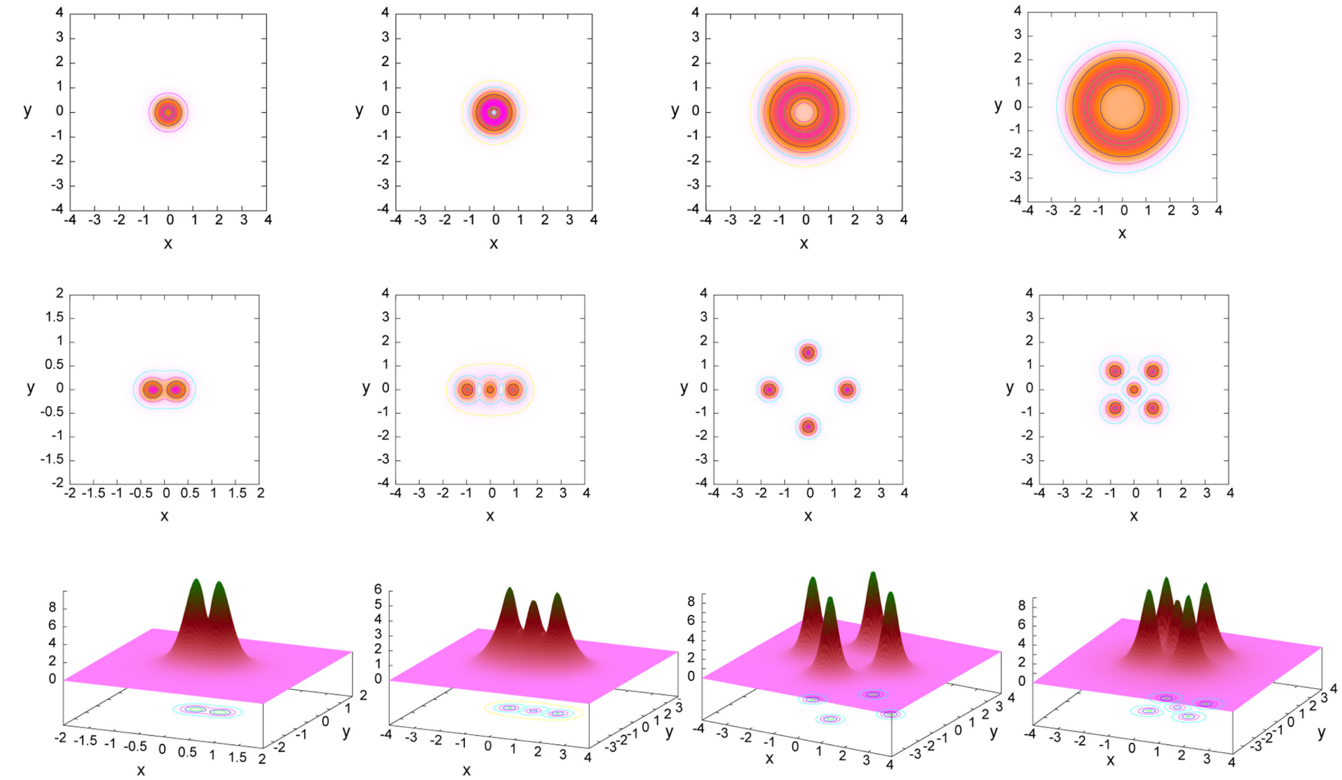

FIG. 3 (color online). Critical behavior of the rotationally invariant soliton solutions of the model with potential (3). The contour plots of the energy density of the rotationally invariant (upper row) baby Skyrmions with charges $B = 2, 3, 4, 5$ and $\mu^2 = 8$ at $\omega = 0.8$ and their decay into B charge one solitons (2nd and 3rd rows).

momentum remain finite. Note that the plots of the energy of the baby Skyrmions as a function of isospin look similar with the dependencies $E(J)$ in the Faddeev-Skyrme model [17], up to some value of J the energy remains almost constant; i.e. the configuration spins as a rigid rotator, then the curve $E(J)$ becomes linear up to a critical value at which the solution breaks up.

Interestingly, for the rotationally invariant configurations which we can construct using the hedgehog ansatz [1] and considering relatively large values of the mass parameter μ, we observe crossing in both $F_\omega(\omega)$ and $E(\omega)$ curves as displayed in Fig. 2. Indeed, our numerical simulations confirm that for some (third) critical value of frequency ω_3 the pseudoenergy of the axially symmetric $B \geq 2$ multi-Skyrmion becomes higher than the pseudoenergy of the system of B charge one baby Skyrmions, so the configurations are unstable with respect to decay into constituents as shown in Fig. 3. Typically, increasing the value of the mass parameter μ will increase the stability of the rotationally invariant multisolitons, the critical values of the frequencies which correspond to the crossing between the $F_\omega(\omega)$ curves then increase.

B. Double vacuum baby Skyrme model

Let us now briefly consider the new baby Skyrme model with double vacuum potential (8). First, we observed some similarity with the case of the model with old potential considered above. As the angular frequency ω increases the radius of the configuration is getting larger whereas its qualitative shape does not change. In Fig. 4 we present typical plots of the $B = 1$ Skyrmion energies, both as functions of ω and as functions of isospin J for a range of values of μ. Again, we observe critical behavior of two different types, if the mass parameter satisfies the condition $\mu^2 \leq \omega_1^2 = 2$ the effective potential vanishes at $\mu = \omega_2$ and both the energy and the angular momentum diverge (cf. Fig. 1). In the second case $\mu^2 > 2$ we observe another type of the critical behavior; as ω approaches the critical value ω_1 the rotational invariance of the configuration becomes slightly broken as shown in Fig. 6 and then the configuration ceases to exist. However, both the energy and the angular momentum of the Skyrmion remain finite up to the critical value ω_1.

Note that the charge B solutions of the model with double vacuum potential are rotationally invariant. Both the pseudoenergy and the total energy of these multi-Skyrmions are always lower than the pseudoenergy and the energy of the system of B individual charge one solitons, so the system remains stable with respect to decay into constituents. Indeed, we do not observe crossings in $F_\omega(\omega)$ curves displayed in Fig. 5 which one has to compare with similar curves for the rotationally invariant soliton

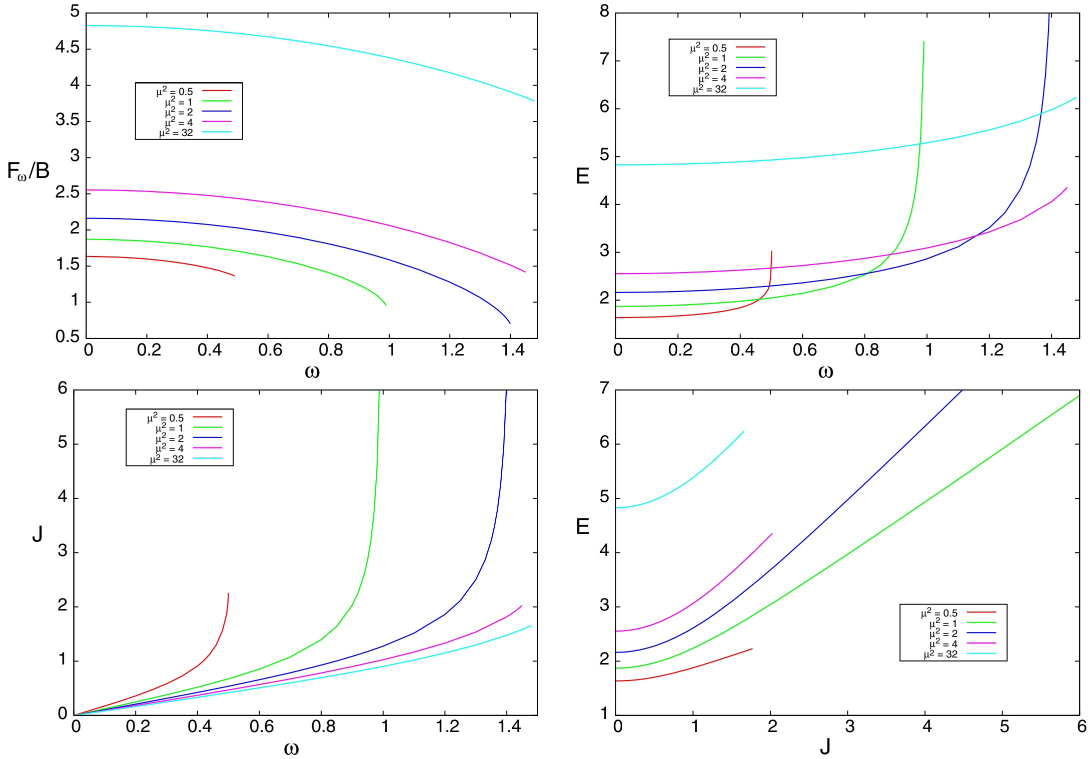

FIG. 4 (color online). Pseudoenergy, energy and isospin of the $B = 1$ baby Skyrmion in the model with potential (8) are plotted as functions of the angular frequency ω. The energy E is plotted as a function of isospin J at $\mu^2 = 0.5$, 1, 2, 4, 32.

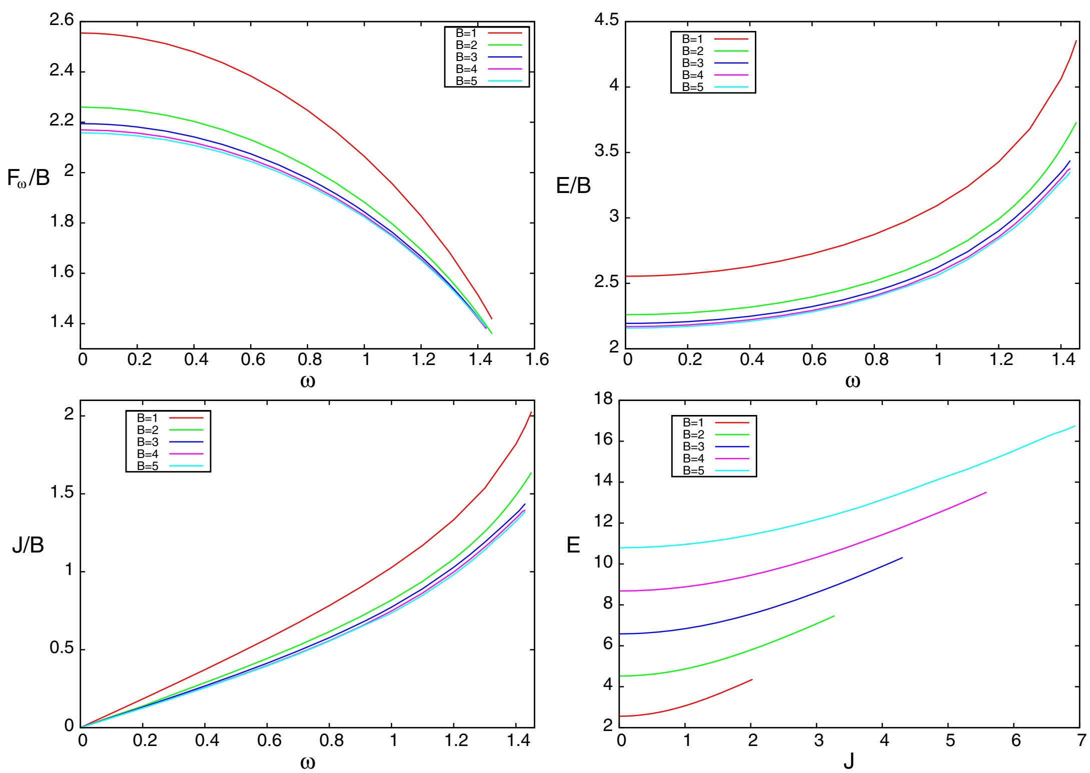

FIG. 5 (color online). Pseudoenergy, energy and isospin of the baby Skyrmion solutions in the model with potential (8) are plotted as functions of angular frequency ω, and the energy as a function of isospin J at $\mu^2 = 4$.

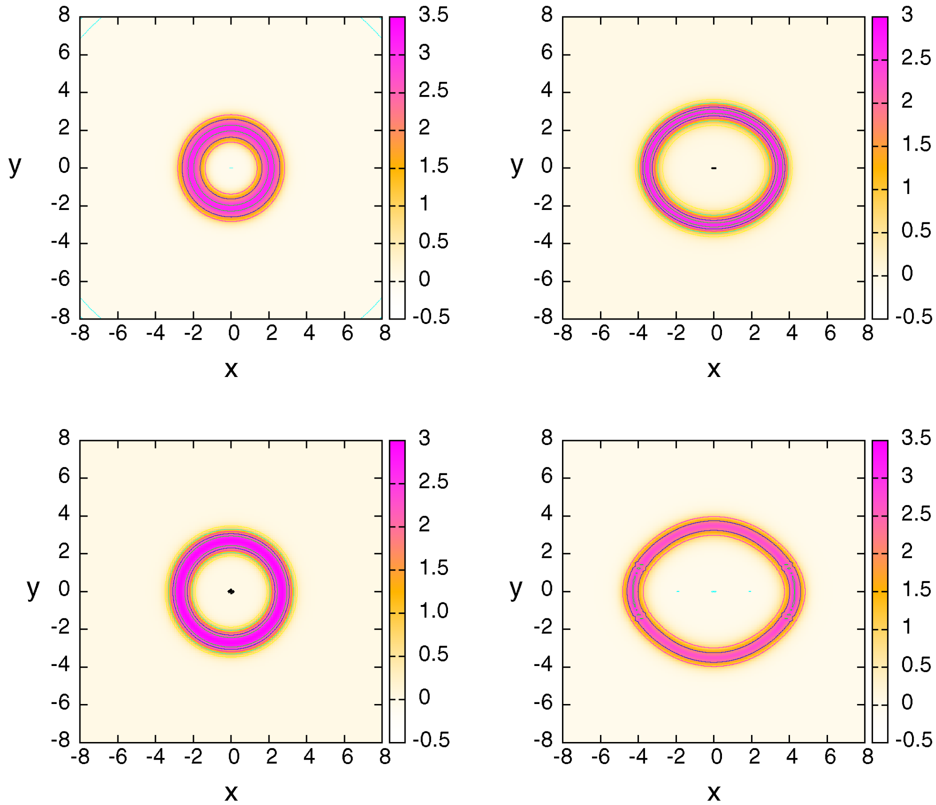

FIG. 6 (color online). The contour plots of the energy density of the soliton solutions of the model with double vacuum potential (8) with charges $B = 4$ and $B = 5$ for $\mu^2 = 4$ at $\omega = 0$ (left panels) and $\omega = \sqrt{2}$ (right panels).

solutions of the old baby Skyrme model we presented in Fig. 2.

IV. CONCLUSIONS

We have studied isospinning soliton solutions of the low-dimensional baby Skyrme model of degree $1 \leq B \leq 5$ with two types of potential, the old model and the new double vacuum model. Similar to the case of the isospinning solitons of the Faddeev-Skyrme model [15,17,24], we used reformulation of the minimization problem considering the stationary points of the pseudoenergy functional $F_\omega(\omega)$ which we found numerically without imposing any assumptions about the spatial symmetries. Our results confirm that the solitons persist for all range of values of $\omega \leq \min\{\sqrt{2}, \mu\}$, where μ is the mass of the scalar excitations, and their qualitative shape is independent of the frequency ω. Thus, there are two types of instabilities of the solitons, one is due to radiation of the scalar field and another one is related to destabilization of the rotating solitons by nonlinear velocity terms [17].

Apart from the above feature, we have found that the critical behavior of isospinning configurations strongly depends on the structure of the potential term of the model. The multisolitons of the new baby Skyrme model remain stable with respect to decay into constituents of smaller charge up to the critical value of the angular parameter ω at which the configuration ceases to exist; however, the charge $B > 3$ solitons of the old model decay into constituents at some value of ω at which the pseudoenergy of the isospinning configuration per unit charge becomes higher than the pseudoenergy of the system of isospinning baby Skyrmions of lower charge. Also note that our results indicate that the solitons of the holomorphic model do not admit isorotations.

Similar results about the critical behavior of the isospinning solitons in the old baby Skyrme model were reported in a very recent paper [20]. More systematic investigation of the spinning baby Skyrmions for various potentials and for larger values of B will be presented elsewhere in our work in collaboration with Battye and Haberichter [18].

ACKNOWLEDGMENTS

We thank Richard Battye, Mareike Haberichter, Itay Hen, Marek Karliner, Olga Kichakova, Jutta Kunz, Paul Sutcliffe, and Wojtek Zakrzewski for useful discussions and valuable comments. This work was financially supported by Alexander von Humboldt Foundation in the framework of the Institutes linkage Programm. Some of the work of Ya. S. was carried out at University of Vilnius supported by the EU BMU-MID visiting grant. Ya. S. is very grateful to the Institute of Physics, Carl von Ossietzky University for hospitality in Oldenburg.

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