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Twisted-mass potential on the non-Abelian string world sheet induced by bulk masses

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We derive the twisted-mass potential in $\mathcal{N} = (2,2)$ CP^{N-1} theory on the world sheet of the non-Abelian string from the bulk $\mathcal{N} = 2$ theory with massive (s)quarks by determining the profile functions of the adjoint fields. Although this potential was indirectly found some time ago, this is the first direct derivation from the bulk. As an application of the adjoint field profiles, we compute and confirm the $|\mu\sigma|$ potential (where σ is a scalar field in the gauge supermultiplet), which arises in the effective two-dimensional theory on the string due to the supersymmetry breaking bulk mass term $\mu\mathcal{A}^2$ for the adjoint matter.

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I. INTRODUCTION

Phenomena on the non-Abelian flux tubes (strings) in supersymmetric QCD, such as $2D$-$4D$ correspondence (see, e.g., the review publications [1,2]) attract exceeding attention now [3]. A wide variety of nonperturbative effects was addressed in theories which support such flux tubes [4]. Supersymmetry plays a special role in a number of aspects. Typically, the flux tubes require the existence of scalar fields. $\mathcal{N} = 2$ supersymmetric QCD supplies both scalar quarks and adjoint scalars. In addition, the power of supersymmetry manifests itself in providing a setting for obtaining exact results (see, e.g., Refs. [5–8]).

The string becomes *non-Abelian* if it gives rise to the so-called orientational moduli living on its world sheet [9–12]. In the context of gauge theories, this typically requires $\mathrm{U}(N)_C \times \mathrm{SU(N)}_F$ spontaneously broken down to color-flavor locked diagonal $\mathrm{SU}(N)_{\mathrm{C+F}}$. Then the orientational moduli span a CP^{N-1} space, and the latter becomes the target space of the two-dimensional theory on the world sheet [2].

A soft breaking of $\mathcal{N} = 2$ supersymmetry down to $\mathcal{N} = 1$ in the bulk gives rise to a richer set of theories on the string world sheet. For the most part in this paper, however, we will deal with the $\mathcal{N} = 2$ gauge theory.

When nonvanishing (s)quark mass parameters are introduced in the bulk theory, the global $\mathrm{SU}(N)_{\mathrm{C+F}}$ group is explicitly broken, and, strictly speaking, the non-Abelian strings are no longer. The moduli parameters are lifted, and a shallow potential is generated. The only true minima are the so-called Z_N strings. In terms of the two-dimensional world sheet theory, these strings are described by the vacua of the two-dimensional potential.

The fact of its existence and the form of this potential has been known for a long time [13,14]. Indeed, the only form compatible with $\mathcal{N} = (2,2)$ supersymmetry in two dimensions is

$$V_{1+1}^{\text{twisted-mass}} = \sum |m_j|^2 |n_j|^2 - \Big| \sum m_j |n_j|^2 \Big|^2. \tag{1.1}$$

Here m_j are the mass parameters and n_j the orientational (*quasi*)moduli. On geometrical grounds, this potential was found in Ref. [12] by Hanany and Tong. Derivation of this potential from the bulk theory was only carried out in the SU(2) case [11]. As we will discuss below, the quark mass parameters induce a nonvanishing expectation value for the adjoint fields. An *ansatz* was proposed for the adjoint field $a^{\mathrm{SU}(2)}$ in Ref. [11], the substitution of which into the bulk action produced the expected result (1.1). This paper extends the $\mathrm{SU}(2)_C \times \mathrm{SU}(2)_F$ bulk theory to the general case of $\mathrm{SU}(N)_C \times \mathrm{SU(N)}_F$. We propose an ansatz for the adjoint fields in the general case for the first time. We confirm our expressions by substituting the adjoints into the bulk action. This procedure produces a consistent expression both for the two-dimensional action and for its normalization integral and in this way provides us with a direct derivation of the world sheet potential (1.1).

As another application of our ansatz for the adjoint fields, we are able to confirm the potential

$$V_{1+1} = 4\pi |\mu_{\mathrm{U}} m - \mu_N(\sqrt{2}\sigma + m)|, \tag{1.2}$$

arising on the world sheet [15] once the $\mathcal{N} = 2$ supersymmetry in the bulk is broken by a quadratic superpotential for the adjoint superfields $\mu\mathcal{A}^2$ down to $\mathcal{N} = 1$. Here σ is a scalar field of the gauge multiplet in the world sheet CP^{N-1} model, m is the average quark mass, while μ_{U} and μ_N are the mass terms of the bulk U(1) and $\mathrm{SU}(N)$ adjoint matter, respectively. This potential becomes nontrivial once the quark masses are nondegenerate and breaks $\mathcal{N} = (2,2)$ world sheet supersymmetry down to $\mathcal{N} = (0,2)$.

For the case of a single-trace bulk deformation operator (i.e., $\mu_{\mathrm{U}} = \mu_N$) this potential acquires a particularly simple form:

$$V_{1+1} = 4\pi|\sqrt{2}\mu_N \sigma|. \tag{1.3}$$

Although our derivation is valid only to the linear order in μ, it is carried out starting directly from the bulk theory.

II. ADJOINT FIELDS

We start with the $\mathcal{N} = 2$ Supersymmetric Quantum Chromodynamics with $N_f = N_c = N$ flavors transforming according to the fundamental representation of the gauge group $\mathrm{U}(1) \times \mathrm{SU}(N)$. In order for the theory to support non-Abelian strings, we introduce the Fayet–Illiopolous (FI) terms into the theory. The bosonic part of the Lagrangian is as follows:

$$\begin{aligned}\mathcal{L} = {}& \frac{1}{2g_2^2}\,\mathrm{Tr}(F_{\mu\nu}^{\mathrm{SU}(N)})^2 + \frac{1}{g_1^2}(F_{\mu\nu}^{\mathrm{U}(1)})^2 + \frac{2}{g_2^2}\,\mathrm{Tr}|\mathcal{D}_\mu a^{\mathrm{SU}(N)}|^2 + \frac{4}{g_1^2}|\partial_\mu a^{\mathrm{U}(1)}|^2 \\ & + \mathrm{Tr}|\mathcal{D}_\mu q|^2 + \mathrm{Tr}|\mathcal{D}_\mu \bar{\tilde{q}}|^2 + V(q, \tilde{q}, a^{\mathrm{U}(1)}, a^{\mathrm{SU}(N)}).\end{aligned} \tag{2.1}$$

Here $F_{\mu\nu}^{\mathrm{SU}(N)}$ and $F_{\mu\nu}^{\mathrm{U}(1)}$ are the field strengths of the non-Abelian and Abelian gauge fields, correspondingly, and $a^{\mathrm{SU}(N)}$ and $a^{\mathrm{U}(1)}$ are the scalar adjoint fields (scalar superpartners of the gauge fields). The quark fields q and $\tilde{q}$ which comprise the quark hypermultiplet are written in the color–flavor matrix notation (the first index of such a matrix refers to color and the second to flavor). The potential in the theory with $\mathcal{N} = 2$ supersymmetry is

$$\begin{aligned}V(q, \tilde{q}, a^{\mathrm{U}(1)}, a^{\mathrm{SU}(N)}) = {}& g_2^2\,\mathrm{Tr}\Big(\frac{1}{g_2^2}[a^{\mathrm{SU}(N)}\bar{a}^{\mathrm{SU}(N)}] + \frac{1}{2}\,\mathrm{Ts}(q\bar{q} - \bar{\tilde{q}}\,\tilde{q})\Big)^2 + \frac{g_1^2}{8}(\mathrm{Tr}(q\bar{q} - \bar{\tilde{q}}\,\tilde{q}) - N\xi_3)^2 + g_2^2\,\mathrm{Tr}|\mathrm{Ts}\,q\tilde{q}|^2 \\ & + \frac{g_1^2}{2}\left|\mathrm{Tr} q\tilde{q} - \frac{N}{2}\xi\right|^2 + 2\,\mathrm{Tr}\left|(a^{\mathrm{U}(1)} + a^{\mathrm{SU}(N)})q + q\cdot\frac{\hat{m}}{\sqrt{2}}\right|^2 + 2\,\mathrm{Tr}\left|(a^{\mathrm{U}(1)} + a^{\mathrm{SU}(N)})\bar{\tilde{q}} + \bar{\tilde{q}}\cdot\frac{\hat{m}}{\sqrt{2}}\right|^2.\end{aligned} \tag{2.2}$$

Here Ts takes a traceless part of an expression. Parameter ξ_3 denotes the (real) D-term FI parameter, while ξ is the (complex) F-term FI parameter. When the $\mathcal{N} = 2$ supersymmetry is not broken, these parameters are equivalent, and only one is necessary. We will therefore only use ξ_3 but will still call it ξ for brevity. Matrix $\hat{m}$ here denotes the diagonal matrix of the quark mass parameters:

$$\hat{m} = \begin{pmatrix} m_1 & & & \\ & m_2 & & \\ & & \ddots & \\ & & & m_N \end{pmatrix}. \tag{2.3}$$

Because this is a matrix in the flavor space, it multiplies matrix q on the right. For the theory to be accessible semiclassically, we canonically assume the FI parameter to be large,

$$\sqrt{\xi} \gg \Lambda_{\mathrm{SU}(N)}, \quad m.$$

A. Zero masses

We start from the case in which the (s)quark masses vanish. Again, in this section we assume the FI F-term equal to zero, with the D-term denoted as

$$\xi_3 \equiv \xi \neq 0.$$

When the bare quark mass matrix vanishes,

$$\hat{m} = 0,$$

the theory supports non-Abelian string solutions. We will not review the perturbative spectrum of this model, referring the reader to Ref. [2]. We will just point out that the $r = N$ vacuum of the potential (2.2) can always be chosen in the color-flavor locked form:

$$\langle q^{kA}\rangle = \sqrt{\xi}\begin{pmatrix} 1 & 0 & \dots \\ \dots & \dots & \dots \\ \dots & 0 & 1 \end{pmatrix}, \qquad \langle \tilde{q}_{Ak}\rangle = 0. \tag{2.4}$$

As currently we hold $\hat{m} = 0$, the adjoint fields vanish in this vacuum,

$$\langle a^{\mathrm{SU}(N)}\rangle = \langle a^{\mathrm{U}(1)}\rangle = 0. \tag{2.5}$$

The string solutions are found as profile functions of the quark and gauge fields, which tend to the vacuum values at the infinity, but with a winding of one of their components in the plane perpendicular to the string—that is what keeps the string stable. The string ansatz for the scalar fields is

$$q = \bar{q} = \phi, \qquad \tilde{q} = \bar{\tilde{q}} = 0, \qquad a^{\mathrm{U}(1)} = a^{\mathrm{SU}(N)} = 0. \tag{2.6}$$

The quark matrix ϕ is described in terms of the profile functions $\phi_1(r)$ and $\phi_2(r)$,

$$\phi(r) = \phi_2 + n\bar{n}\cdot(\phi_1 - \phi_2). \tag{2.7}$$

We chose here a singular gauge in which the quarks do not wind at all, but the gauge fields do, for which purpose they have to be singular at the core of the string $r = 0$. The ansatz for the gauge fields is

$$A_j^{\mathrm{SU}(N)} = \epsilon_{jk}\frac{x^k}{r^2} f_N(r)(n\bar{n} - 1/N),$$
$$A_j^{\mathrm{U}(1)} = \frac{1}{N}\epsilon_{jk}\frac{x^k}{r^2} f(r). \tag{2.8}$$

These string profiles obey the first-order Bogomol'nyi-Prasad-Sommerfield (BPS) equations

$$\partial_r\phi_1(r) = \frac{1}{Nr}(f(r) + (N-1)f(r))\phi_1(r),$$
$$\partial_r\phi_2(r) = \frac{1}{Nr}(f(r) - f_N(r))\phi_2(r),$$
$$\partial_r f(r) = \frac{Ng_1^2}{4} r(\phi_1(r)^2 + (N-1)\phi_2(r)^2 - N\xi),$$
$$\partial_r f_N(r) = \frac{g_2^2}{2} r(\phi_1(r)^2 - \phi_2(r)^2), \tag{2.9}$$

supplemented with the appropriate boundary conditions

$$\phi_1(0)=0, \quad \phi_2(0)\neq 0, \quad \phi_1(\infty)=\sqrt{\xi}, \quad \phi_2(\infty)=\sqrt{\xi},$$
$$f_N(0)=1, \quad f(0)=1, \quad f_N(\infty)=0, \quad f(\infty)=0. \tag{2.10}$$

The latter conditions at infinity ensure that the fields tend to their vacuum values, while the conditions in the string core are needed for the finiteness of the string tension [and do not restrict the value of $\phi_2(0)$ other than that it cannot vanish].

The above ansatz describes a family of solutions, labeled by the CP^{N-1} moduli variables n^l,

$$\vec{n} \in \mathcal{C}^N, \qquad |\vec{n}|^2 = 1. \tag{2.11}$$

These so-called *orientational* moduli "rotate" the solution in the $\mathrm{SU}(N)\times\mathrm{U}(1)$ space. Each solution actually breaks the color-flavor group $\mathrm{SU}(N)_{\mathrm{C+F}}$ down to $\mathrm{SU}(N-1)\times\mathrm{U}(1)$. Thus, there are as many as

$$\frac{\mathrm{SU}(N)}{\mathrm{SU}(N-1)\times\mathrm{U}(1)} \sim CP^{N-1} \tag{2.12}$$

solutions, which are labeled by the vector $\vec{n}$. Note that in the ansatz (2.6), (2.7), and (2.8), in our notation, $n\bar{n}$ is a matrix.

It is these moduli that give the string the name non-Abelian. They live on this string. In order to see this, one allows them to be weekly dependent on t and z (longitudinal) coordinates. Then it can be shown [2] that the bulk theory induces a "live" action for $\vec{n}$ on the world sheet of the strings. The way this happens is that when t, z dependence is introduced, the ansatz (2.8) has to be extended—the longitudinal components of the gauge field now get excited,

$$A_\mu^{\mathrm{SU}(N)} = i[n\bar{n}, \partial_\mu(n\bar{n})]\rho(r), \qquad \mu = 0, 3. \tag{2.13}$$

Here $\rho(r)$ is a new profile function with a boundary condition,

$$\rho(0) = 1, \tag{2.14}$$

which again is needed for finiteness of the string tension. When now all the profiles (2.6), (2.7), (2.8), and (2.13) are substituted into the bulk action (2.1), and integrated over the transverse coordinates, the following theory emerges on the world sheet of the string:

$$S = 2\beta\int d^2x(|\partial_\mu n|^2 + (\bar{n}\partial_\mu n)^2), \tag{2.15}$$

with the summation index μ running over the longitudinal coordinates (0 and 3). Here β is a normalization constant, arising due to the transverse integration of the profile functions,

$$\beta = \frac{2\pi}{g_2^2}\times\int rdr\Big((\partial_r\rho)^2 + \frac{1}{r^2}f_N^2(1-\rho)^2$$
$$+ g_2^2\Big((1-\rho)(\phi_1-\phi_2)^2 + \frac{1}{2}\rho^2(\phi_1^2+\phi_2^2)\Big)\Big), \tag{2.16}$$

and effectively becoming the coupling constant of the two-dimensional theory. Minimization of Eq. (2.16) with respect to $\rho(r)$ gives

$$\rho(r) = 1 - \frac{\phi_1}{\phi_2}. \tag{2.17}$$

If one now takes into account the BPS equations (2.9) for the profiles, then the integral in Eq. (2.16) reduces to unity, and

$$\beta = \frac{2\pi}{g_2^2}. \tag{2.18}$$

Note that the action (2.15) could and would actually have higher-order derivative corrections, running in powers of

$$\frac{\partial_\mu}{g_2\sqrt{\xi}}. \tag{2.19}$$

Below the scale of the inverse thickness of the string, $g_2\sqrt{\xi}$, where the world sheet description (2.15) is valid, such corrections are negligible.

B. Nonvanishing masses

When nonzero masses are introduced in the theory (2.1), the situation changes significantly. The non-Abelian strings cease to be solutions of equations of motion, and the orientational moduli $\vec{n}$ are lifted.[1] They become

[1]These moduli are lifted at the quantum level even if all mass terms vanish. But this is a quantum effect. The above statement can be reformulated more accurately as follows: the orientational moduli $\vec{n}$ are lifted at the classical level if $m_i \neq m_j \neq 0$.

quasimoduli, as a shallow potential is generated on the world sheet. Only when $\vec{n}$ equals one of

$$\vec{n}_{\text{vac}} = (0, \ldots, 1, \ldots, 0) \tag{2.20}$$

does the string become a BPS solution again, in the sense of the low-energy Abelian theory. As there are N such strings, they are called the $\mathcal{Z}_N$ strings.

The ansatz for the squarks and gauge fields remains the same:

$$\begin{aligned} q &= \bar{q} = \phi, \\ \tilde{q} &= \bar{\tilde{q}} = 0, \\ A_j^{\text{SU}(N)} &= \epsilon_{jk}\frac{x^k}{r^2} f_N(r)(n\bar{n} - 1/N), \\ A_j^{\text{U}(1)} &= \frac{1}{N}\epsilon_{jk}\frac{x^k}{r^2} f(r). \end{aligned} \tag{2.21}$$

The first obvious change, revealed by inspecting the last two lines of Eq. (2.2),

$$2\,\text{Tr}\left|(a^{\text{U}(1)} + a^{\text{SU}(N)})q + q\cdot\frac{\hat{m}}{\sqrt{2}}\right|^2 + 2\,\text{Tr}\left|(a^{\text{U}(1)} + a^{\text{SU}(N)})\bar{\tilde{q}} + \bar{\tilde{q}}\cdot\frac{\hat{m}}{\sqrt{2}}\right|^2, \tag{2.22}$$

is that the vacuum values of the adjoint scalars are no longer zero,

$$a^{\text{U}(1)} = \langle a^{\text{U}(1)}\rangle = -\frac{m}{\sqrt{2}}, \qquad \langle a^{\text{SU}(N)}\rangle = -\frac{\hat{\Delta m}}{\sqrt{2}}. \tag{2.23}$$

Here m is the average mass parameter, and $\hat{\Delta m}$ is the diagonal matrix of the mass differences,

$$\hat{\Delta m}_j = \hat{m}_j - m, \qquad m = \frac{1}{N}\sum \hat{m}_j. \tag{2.24}$$

The second "massive" F term in Eq. (2.22) is responsible for making the non-Abelian string a quasisolution, except when $\vec{n}$ takes one of its vacuum values (2.20).

As is shown in the first line of Eq. (2.23), the U(1) scalar $a^{\text{U}(1)}$ does not develop any profile and always sits in its vacuum. Its sole purpose is to cancel the average mass m in the above F terms (since the average mass is essentially a unit matrix, it commutes with q, and the cancellation happens everywhere). In fact, the average quark mass can be eliminated by the shift of $a^{\text{U}(1)}$.

A very different thing happens to the SU(N) field $a^{\text{SU}(N)}$. As the average mass has been canceled everywhere, it is only $\hat{\Delta m}$ that is left to cancel. However, the latter is not generically proportional to the unit matrix, and so the complete cancellation can only happen at infinity (or whenever $\vec{n} = \vec{n}_{\text{vac}}$, in which case q commutes with everything). Therefore, field $a^{\text{SU}(N)}$ does have a profile, which asymptotically tends to the vacuum value given by the mass differences in Eq. (2.23).

The ansatz for the non-Abelian adjoint field $a^{\text{SU}(N)}$ has been known for the case of the SU(2) gauge group [11]. In this case the CP^1 moduli variables n^l can be traded for O(3) variables S^a,

$$S^a = (\bar{n}\tau^a n). \tag{2.25}$$

In terms of these, the known ansatz looks as

$$a^{\text{SU}(2)} = a^a\frac{\tau^a}{2} = -\frac{\Delta m}{\sqrt{2}}(\tau^3\omega(r) + S^3 S^a\tau^a(1 - \omega(r))). \tag{2.26}$$

Here Δm is the only mass difference $(m_1 - m_2)/2$, and the reason that the third direction enters explicitly is because $\hat{\Delta m} \propto \tau^3$ in this case. The profile function $\omega(r)$ satisfies the boundary conditions

$$\omega(0) = 0, \qquad \omega(\infty) = 1 \tag{2.27}$$

and is found by a minimization procedure, giving

$$\omega(r) = \frac{\phi_1(r)}{\phi_2(r)}. \tag{2.28}$$

The role of this profile function is to give $a^{\text{SU}(2)}$ an interpolation between the vacuum value (when $\omega = 1$),

$$a^{\text{SU}(2)}(\infty) = -\frac{\hat{\Delta m}}{\sqrt{2}} = -\frac{\Delta m\cdot\tau^3}{\sqrt{2}}, \tag{2.29}$$

and its value at the core of the string (when $\omega = 0$),

$$a^{\text{SU}(2)}(0) = -\frac{\Delta m}{\sqrt{2}} S^3(S^a\tau^a). \tag{2.30}$$

The latter expression is proportional to $S^a\tau^a$ and commutes with the gauge field (which is proportional to the same matrix structure). This is needed so that the kinetic term of $a^{\text{SU}(2)}$ containing the commutator $[A_\mu^{\text{SU}(2)}, a^{\text{SU}(2)}]$ does not produce a divergent contribution to the string tension, due to the singularity of the gauge field at the core. At the same time, if $\vec{S}$ happens to be parallel to the third axis (i.e., the string is in the vacuum), then $\omega(r)$ in Eq. (2.26) cancels away, and the adjoint field takes its vacuum value everywhere in the space.

We now give the generalization of the ansatz (2.26) to the case of the SU(N) gauge group. The expression appears to be more involved than its SU(2) counterpart, namely,

$$a^{\text{SU}(N)} = -\frac{1}{\sqrt{2}}(\hat{\Delta m} - (1 - \omega(r))[n\bar{n}[n\bar{n}, \hat{\Delta m}]]). \tag{2.31}$$

We will show that $\omega(r)$ is the same profile function as in Eq. (2.26).

Before discussing the properties of this ansatz, we first bring a few useful relations involving matrix $n\bar{n}$. These relations owe to the fact that

$$(n\bar{n})^2 = n\bar{n}. \tag{2.32}$$

We notice that expression (2.31) involves the second commutator of $n\bar{n}$ and the mass difference matrix $\Delta\hat{m}$. It appears that the *third* commutator of $n\bar{n}$ and any matrix actually equals the first commutator of these,

$$[n\bar{n}[n\bar{n}[n\bar{n},\hat{M}]]] = [n\bar{n},\hat{M}]. \tag{2.33}$$

Expression (2.31) takes the vacuum value $\Delta\hat{m}$ at infinity and rotates it as r goes to zero. The only available "color" parameter for such a rotation is $n\bar{n}$. Let us show that indeed such a rotation takes place. Note that, because of the property (2.32), an exponent involving $n\bar{n}$ will always reduce to trigonometric functions. Then a "rotation" of any matrix $\hat{M}$ will look as follows:

$$e^{i\alpha n\bar{n}}\cdot\hat{M}\cdot e^{-i\alpha n\bar{n}} = \hat{M} + i\sin\alpha\,[n\bar{n},\hat{M}] - (1-\cos\alpha)[n\bar{n}[n\bar{n},\hat{M}]]. \tag{2.34}$$

Getting rid of the imaginary part, expression (2.31) can then be written as

$$-\sqrt{2}\cdot a^{\mathrm{SU}(N)} = \frac{1}{2}e^{i\alpha n\bar{n}}\cdot\Delta\hat{m}\cdot e^{-i\alpha n\bar{n}} + \frac{1}{2}e^{-i\alpha n\bar{n}}\cdot\Delta\hat{m}\cdot e^{i\alpha n\bar{n}}, \tag{2.35}$$

where

$$\cos\alpha(r) = \omega(r). \tag{2.36}$$

Another way of writing this is to notice that an exponent of commutators of $n\bar{n}$ with any matrix (i.e., a commutator exponent analogous to that in the kinetic term of the adjoint scalar) will similarly be reducible to trigonometric functions owing to Eq. (2.33). Then our ansatz can be written as a "cosine":

$$-\sqrt{2}\cdot a^{\mathrm{SU}(N)} = \frac{e^{i\alpha[n\bar{n}\cdot]} + e^{-i\alpha[n\bar{n}\cdot]}}{2}\Delta\hat{m}. \tag{2.37}$$

Now let us discuss the properties of this ansatz. First of all, it is easy to see that it is a traceless matrix. Next, we repeat, as r goes to infinity [$\omega(r)\to 1$, and $\alpha\to 0$], the adjoint field approaches the vacuum value:

$$a^{\mathrm{SU}(N)} \xrightarrow{r\to\infty} \langle a^{\mathrm{SU}(N)}\rangle = -\frac{\Delta\hat{m}}{\sqrt{2}}. \tag{2.38}$$

On the other hand, at the core of the string, the solution turns into a matrix,

$$-\sqrt{2}\cdot a^{\mathrm{SU}(N)}(0) = \Delta\hat{m} - [n\bar{n}[n\bar{n},\Delta\hat{m}]], \tag{2.39}$$

which, because of property (2.33), commutes with $n\bar{n}$. This way, at the string core, the adjoint field commutes with the gauge field [proportional to $n\bar{n} - 1/N$; see Eq. (2.21)], and the gauge field singularity is avoided. Note that, unlike in the case of SU(2), the adjoint field does not become proportional solely to $n\bar{n} - 1/N$ at the core.

It is also easy to check the BPS condition on the solution (2.31). Indeed, when $\vec{n} = \vec{n}_{\mathrm{vac}}$, matrix $n\bar{n}$ commutes with anything, and the right-hand side in Eq. (2.31) reduces to the vacuum value

$$a^{\mathrm{SU}(N)}(\vec{n}_{\mathrm{vac}}) = \langle a^{\mathrm{SU}(N)}\rangle = -\frac{\Delta\hat{m}}{\sqrt{2}} \tag{2.40}$$

everywhere in the space.

Finally, it is slightly more technical, but straightforward, to check that Eq. (2.31) reduces to Eq. (2.26) for the gauge group SU(2), i.e., is a correct generalization.

The ansatz (2.31) is not the only generalization of the SU(2) formula (2.26). In fact, if one took the "direct" correspondence rules [see the definition (2.25)],

$$\Delta m\tau^3 \to \Delta\hat{m}, \qquad \frac{S^a\tau^a}{2}\to n\bar{n} - 1/N, \qquad S^3\to(\bar{n}\tau^3 n), \tag{2.41}$$

and applied them to Eq. (2.26), the following expression would emerge:

$$-\frac{1}{\sqrt{2}}(\Delta\hat{m}\cdot\omega(r) + 2(1-\omega(r))\cdot(\bar{n}\,\Delta\hat{m}\,n)(n\bar{n} - 1/N)).$$

The latter expression certainly does reduce to Eq. (2.26) if one again assumes $N = 2$. However, this expression does not work for generic N. Most obvious is the fact that it does not satisfy the BPS condition—it does not reduce to the constant vacuum value when $\vec{n} = \vec{n}_{\mathrm{vac}}$.

At the same time, when one takes Eq. (2.31) and substitutes it into the bulk action (2.1), the following potential emerges on the world sheet of the string:

$$\frac{4\pi}{g_2^2}\int rdr\Big((\partial_r\omega)^2 + \frac{1}{r^2}f_N^2\omega^2 + g_2^2\Big(\omega(\phi_1-\phi_2)^2 + \frac{1}{2}(1-\omega)^2(\phi_1^2+\phi_2^2)\Big)\Big) \times\int d^2x((\bar{n}|\Delta\hat{m}|^2 n) - |(\bar{n}\,\Delta\hat{m}\,n)|^2) + O(\Delta\hat{m}^4). \tag{2.42}$$

We notice that the normalization integral here appears to be the same as in Eq. (2.16), which, therefore, gives us via minimization

$$\omega(r) = 1-\rho(r) = \frac{\phi_1(r)}{\phi_2(r)}, \tag{2.43}$$

and the whole integral in the first two lines of expression (2.42) reduces to unity. As for the corrections $O(\Delta\hat{m}^4)$, they look as [here the representation (2.35) is helpful in finding their form]

$$O(\Delta\hat{m}^4) = 2\pi\int rdr\frac{1}{2}(1-\omega^2)^2\cdot(\Delta\hat{m})^4, \tag{2.44}$$

where $(\Delta\hat{m})^4$ is an expression involving $\vec{n}$ and the fourth power of $\Delta\hat{m}$. The profile integral in the above expression is saturated at the thickness of the string. Therefore, by dimensional counting, these corrections are suppressed by a power of ξ,

$$O(\Delta\hat{m}^4) \sim |\Delta\hat{m}|^2 \cdot \frac{|\Delta\hat{m}|^2}{g_2^2 \xi}, \tag{2.45}$$

and can be ignored on the same grounds as the higher-order derivatives (2.19).

Taking Eq. (2.43) into account, we write the result for $a^{\mathrm{SU}(N)}$ as

$$a^{\mathrm{SU}(N)} = -\frac{1}{\sqrt{2}}(\Delta\hat{m} - \rho(r)[n\bar{n}[n\bar{n}, \Delta\hat{m}]]). \tag{2.46}$$

We observe that this expression provides us with the expected form of the twisted-mass potential on the world sheet of the string,

$$\begin{aligned} 2\beta \int d^2x((\bar{n}|\Delta\hat{m}|^2 n) - |(\bar{n}\,\Delta\hat{m}\,n)|^2) \\ = 2\beta \int d^2x\Big(\sum |m_k|^2 |n_k|^2 - \Big|\sum m_k |n_k|^2\Big|^2\Big). \end{aligned} \tag{2.47}$$

Here we use the well-known shift invariance of this potential in order to replace $\Delta\hat{m}_k$ by m_k.

To conclude this section, we note that the twisted-mass-deformed CP^{N-1} model can be nicely rewritten as a strong coupling limit of a U(1) gauge theory [16]. In this description the meaning of the twisted-mass potential becomes transparent. Namely, the potential reduces to the mass terms for $\vec{n}$ fields. The bosonic part of the action reads

$$\begin{aligned} S = \int d^2x\Big\{2\beta|\nabla_\mu n_k|^2 + \frac{1}{4e^2}F_{\mu\nu}^2 + \frac{1}{e^2}|\partial_\mu \sigma|^2 \\ + 2\beta|\sqrt{2}\sigma + m_k|^2|n_k|^2 + 2e^2\beta^2(|n_k|^2 - 1)^2\Big\}. \end{aligned} \tag{2.48}$$

Here σ is a scalar superpartner of the U(1) gauge field. In the limit $e^2 \to \infty$, fields A_μ and σ can be excluded by virtue of the algebraic equations of motion, namely

$$A_\mu = -\frac{i}{2}(\bar{n}\partial_\mu n - \partial_\mu \bar{n} n), \qquad \sigma = -\sum \frac{m_j}{\sqrt{2}}|n_j|^2. \tag{2.49}$$

Substitution of this into Eq. (2.48) brings us back to the CP^{N-1} model with the potential (2.47).

III. POTENTIAL ON THE HETEROTIC VORTEX STRING

One interesting kind of deformation of the $\mathcal{N} = 2$ theory supporting vortex strings is achieved by introducing quadratic terms for the adjoint fields in the superpotential,

$$\mathcal{W}_{\mathcal{A}} \supset \mathrm{Tr}(\mu_{\mathrm{U}}(\mathcal{A}^{\mathrm{U}(1)})^2 + \mu_N(\mathcal{A}^{\mathrm{SU}(N)})^2). \tag{3.1}$$

Here we have introduced parameters μ_{U}, μ_N, which are related to μ_1, μ_2 of Ref. [17] via[2]

$$\mu_{\mathrm{U}} = \sqrt{\frac{2}{N}}\mu_1, \qquad \mu_N \equiv \mu_2. \tag{3.2}$$

Such a superpotential breaks supersymmetry to $\mathcal{N} = 1$. The world sheet theory on the heterotic vortex string was studied in detail in Refs. [17–19] for the bulk theory with massless quarks and nonzero FI D term ξ_3 and in Ref. [15] for the theory with massive quarks and zero ξ_3.

We have a chance now to directly confirm the moduli potential arising on string to the linear order in the supersymmetry-breaking parameters μ_{U} and μ_N [15]. In such a theory, the FI F terms are induced implicitly, due to the superpotential (3.1),

$$\begin{aligned} \frac{1}{2}g_1^2|\mathrm{Tr}q\tilde{q} + \sqrt{2}N\mu_{\mathrm{U}} \cdot a^{\mathrm{U}(1)}|^2 \\ + g_2^2\,\mathrm{Tr}|\mathrm{T}s q\tilde{q} + \sqrt{2}\mu_2 \cdot a^{\mathrm{SU}(N)}|^2. \end{aligned} \tag{3.3}$$

From now on we assume that $\xi_3 = 0$, while the effective FI F components ξ are generated due to nonzero vacuum values (2.23) of the adjoint fields. In particular, the average quark mass cannot be excluded any longer. It becomes a new parameter which determines the average quark condensate. More precisely, classically the quark vacuum expectation values (VEVs) are determined by

$$\xi_j \approx 2(\mu_{\mathrm{U}} m + \mu_N \Delta\hat{m}_j). \tag{3.4}$$

If the quark mass differences vanish, these parameters reduce to a single FI term which does not break $\mathcal{N} = 2$ supersymmetry in the bulk and $\mathcal{N} = (2, 2)$ supersymmetry on the world sheet in the linear order in μ [20,21]. However, once the quark mass differences are small but nonvanishing, the color-flavor group $\mathrm{SU}_{C+F}(N)$ is broken because both the adjoint and quark VEVs are no longer equal (i.e., flavour-universal). In this case a shallow potential is generated in the world sheet CP^{N-1} model breaking $\mathcal{N} = (2, 2)$ supersymmetry down to $\mathcal{N} = (0, 2)$ [15].[3] The non-Abelian string becomes a heterotic string [17,19].

To derive the world-sheet potential, we substitute the expression (2.46) into the F terms (3.3) and expand the latter to the linear order in $\Delta\hat{m}$. The first term in Eq. (3.3) does not contain $\Delta\hat{m}$ and is just part of the average (i.e., zero-order) string tension

$$2\pi|\hat{\xi}| = 2\pi \cdot |2\mu_{\mathrm{U}} m|. \tag{3.5}$$

[2] One of the advantages of the new notation is that the so-called "single-trace" operator corresponds to the case $\mu_{\mathrm{U}}/\mu_N = 1$. We, however, will duplicate the key results in both notations.

[3] Note that this does not happen in the theory with the FI D term. Namely, the twisted-mass potential of the previous section does not break $\mathcal{N} = (2, 2)$ supersymmetry on the world sheet.

As for the second term, we notice that when plugging in the adjoint field

$$a^{\mathrm{SU}(N)} = -\frac{1}{\sqrt{2}}(\Delta\hat{m} - \rho(r)[n\bar{n}[n\bar{n}, \Delta\hat{m}]]),$$

its commutator part does not contribute at the linear order—the traceless part of $q\tilde{q}$ is proportional to $n\bar{n} - 1/N$, and

$$\mathrm{Tr}\, n\bar{n}[n\bar{n}, *] = 0.$$

Therefore, only the vacuum value $\langle a^{\mathrm{SU}(N)}\rangle$ plays a role here. The profile integral involving $\phi_1(r)$ and $\phi_2(r)$ in $q\tilde{q}$ reduces to an integral of a total derivative due to the BPS equations (2.9),

$$2\pi \int r dr g_2^2(\phi_1 - \phi_2)^2 = 4\pi \int dr \partial_r f_N(r) = -4\pi,$$

and the resulting linear terms are

$$2\pi \cdot \left(\mu_N(\bar{n}\,\Delta\hat{m}\,n) \cdot \frac{\overline{\mu_{\mathrm{U}} m}}{|\mu_{\mathrm{U}} m|} + \bar{\mu}_N(\bar{n}\Delta\hat{m}^\dagger n) \cdot \frac{\mu_{\mathrm{U}} m}{|\mu_{\mathrm{U}} m|}\right). \tag{3.6}$$

Now it is obvious that this expression comprises the linear terms in the expansion of the absolute value in a series in $\Delta\hat{m}$,

$$\begin{aligned} V_{1+1} &= 4\pi \cdot |\mu_{\mathrm{U}} m + \mu_N(\bar{n}\,\Delta\hat{m}\,n)| \\ &= 4\pi \cdot \left(\mu_{\mathrm{U}} m + \mu_N(\bar{n}\,\Delta\hat{m}\,n) \cdot \frac{\overline{\mu_{\mathrm{U}} m}}{|\mu_{\mathrm{U}} m|}\right. \\ &\quad \left. + \bar{\mu}_N(\bar{n}\Delta\hat{m}^\dagger n) \cdot \frac{\mu_{\mathrm{U}} m}{|\mu_{\mathrm{U}} m|} + \cdots\right). \end{aligned} \tag{3.7}$$

In terms of parameters μ_1 and μ_2, this formula reads

$$\begin{aligned} V_{1+1} &= 4\pi \cdot \left|\sqrt{\frac{2}{N}}\mu_1 m + \mu_2(\bar{n}\,\Delta\hat{m}\,n)\right| \\ &= 4\pi \cdot \left(\sqrt{\frac{2}{N}}\mu_1 m + \mu_2(\bar{n}\,\Delta\hat{m}\,n) \cdot \frac{\overline{\mu_1 m}}{|\mu_1 m|}\right. \\ &\quad \left. + \bar{\mu}_2(\bar{n}\Delta\hat{m}^\dagger n) \cdot \frac{\mu_1 m}{|\mu_1 m|} + \cdots\right). \end{aligned} \tag{3.8}$$

The above formulas perfectly agree with the two-dimensional potential found in Ref. [15]. Adding and subtracting $\mu_N m = \mu_N m(\bar{n}n)$ inside the absolute value, and trading variables $\vec{n}$ for an auxiliary variable σ via (2.49),[4] we have

$$\begin{aligned} V_{1+1}(\sigma) &= 4\pi \cdot |\mu_{\mathrm{U}} m - \mu_N(\sqrt{2}\sigma + m)| \\ &= 4\pi \cdot \left|\sqrt{\frac{2}{N}}\mu_1 m - \mu_2(\sqrt{2}\sigma + m)\right|. \end{aligned} \tag{3.9}$$

[4]Here we still can use Eq. (2.49) assuming that the μ-induced potential V_{1+1} is a small correction to the action (2.48).

Note that now (in contrast to the case of the FI D term) the vacuum energies of this world sheet potential give the string tensions,

$$T_j = V_{1+1}(\sigma_j), \tag{3.10}$$

where σ_j are VEVs of the field σ in the N vacua of the CP^{N-1} model. Classically $\sqrt{2}\sigma_j = -m_j$. This is the way the potential (3.9) was conjectured in Ref. [15]. Indeed, using Eq. (3.4) we find correct string tensions

$$T_j = 2\pi|\xi_j|. \tag{3.11}$$

We can see that in the CP^{N-1} model with potential (3.9) for generic quark masses the $\mathcal{N} = (0, 2)$ supersymmetry of the action is broken by the choice of the vacuum already at the classical level. The vacuum energies in the N vacua of the CP^{N-1} model are generically all different.

To conclude this section, let us note that the potential (3.9) gives quantum corrections to the string tensions [15]. In the quantum theory, the VEV of the σ field in each of the N vacua of the CP^{N-1} model with a weak deformation (3.9) is given by solutions of the equation [16,22–24]

$$\prod_{i=1}^{N}(\sqrt{2}\sigma + m_i) = \Lambda_{CP}^N, \tag{3.12}$$

where Λ_{CP} is the scale of the CP^{N-1} model. Solutions σ_i to this equation give exact string tensions via Eq. (3.10) with all corrections in powers of Λ_{CP}/m_i included.

IV. CONCLUSIONS

We found an expression [Eq. (2.46)] for the adjoint field profiles for the non-Abelian vortex configuration in $\mathcal{N} = 2$ supersymmetric QCD with the gauge group U(N) and N flavors. This expression enabled us to derive the twisted-mass potential (2.47) on the vortex world sheet starting from the bulk theory.

In the case in which $\mathcal{N} = 2$ supersymmetry is softly broken by an operator $\mu\mathcal{A}^2$, which at the same time stabilizes the string acting as an effective FI F term, we managed to use expression (2.46) to derive and confirm to the linear order the potential (3.9) generated on the world sheet. Our result is in agreement with the potential found in Ref. [15] and removes the ambiguity of adding a potential vanishing in the critical points of Eq. (3.9).

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[1] D. Tong, Ann. Phys. (Amsterdam) **324**, 30 (2009); M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, J. Phys. A **39**, R315 (2006); K. Konishi, Lect. Notes Phys. **737**, 471 (2008); M. Eto, Y. Hirono, M. Nitta, and S. Yasui, arXiv:1308.1535.
[2] M. Shifman and A. Yung, *Supersymmetric Solitons* (Cambridge University Press, Cambridge, England, 2009).
[3] D. Gaiotto, S. Gukov, and N. Seiberg, arXiv:1307.2578.
[4] M. Shifman and A. Yung, Phys. Rev. D **81**, 085009 (2010); P. Koroteev, M. Shifman, W. Vinci, and A. Yung, Phys. Rev. D **84**, 065018 (2011); M. Eto, T. Fujimori, S. B. Gudnason, Y. Jiang, K. Konishi, M. Nitta, and K. Ohashi, J. High Energy Phys. 12 (2011) 017; S. Chen, R. Zhang, and M. Zhu, arXiv:1201.1602; K. Konishi, M. Nitta, and W. Vinci, J. High Energy Phys. 09 (2012) 014; K. Konishi, arXiv:1209.1376; M. Eto, T. Fujimori, M. Nitta, and K. Ohashi, J. High Energy Phys. 07 (2013) 034; Q.-H. Huo, Y. Jiang, R.-Z. Wang, and H. Yan, Europhys. Lett. **101**, 27001 (2013); S. Yasui, Y. Hirono, K. Itakura, and M. Nitta, Phys. Rev. E **87**, 052142 (2013).
[5] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. **B229**, 381 (1983).
[6] N. Seiberg and E. Witten, Nucl. Phys. **B426**, 19 (1994); **B430**, 485(E) (1994).
[7] N. Seiberg and E. Witten, Nucl. Phys. **B431**, 484 (1994).
[8] M. Shifman and A. Yung, Phys. Rev. D **87**, 106009 (2013); **83**, 105021 (2011); **86**, 065003 (2012); **86**, 025001 (2012); **87**, 085044 (2013).
[9] A. Hanany and D. Tong, J. High Energy Phys. 07 (2003) 037.
[10] R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi, and A. Yung, Nucl. Phys. **B673**, 187 (2003).
[11] M. Shifman and A. Yung, Phys. Rev. D **70**, 045004 (2004).
[12] A. Hanany and D. Tong, J. High Energy Phys. 04 (2004) 066.
[13] N. Dorey, J. High Energy Phys. 11 (1998) 005.
[14] M. Shifman, A. Vainshtein, and R. Zwicky, J. Phys. A **39**, 13005 (2006).
[15] M. Shifman and A. Yung, Phys. Rev. D **82**, 066006 (2010).
[16] A. Hanany and K. Hori, Nucl. Phys. **B513**, 119 (1998).
[17] M. Shifman and A. Yung, Phys. Rev. D **77**, 125016 (2008).
[18] P. A. Bolokhov, M. Shifman, and A. Yung, Phys. Rev. D **81**, 065025 (2010).
[19] M. Edalati and D. Tong, J. High Energy Phys. 05 (2007) 005.
[20] A. Hanany, M. J. Strassler, and A. Zaffaroni, Nucl. Phys. **B513**, 87 (1998).
[21] A. I. Vainshtein and A. Yung, Nucl. Phys. **B614**, 3 (2001).
[22] A. D'Adda, A. C. Davis, P. DiVeccia, and P. Salamonson, Nucl. Phys. **B222**, 45 (1983).
[23] S. Cecotti and C. Vafa, Commun. Math. Phys. **158**, 569 (1993).
[24] E. Witten, Nucl. Phys. **B403**, 159 (1993).