

Functional renormalization of N scalars with $O(N)$ invarianceR. Percacci^{1,2,*} and M. Safari^{1,†}¹*International School for Advanced Studies, via Bonomea 265, I-34136 Trieste, Italy*²*INFN, Sezione di Trieste, via Valerio 2, I-34127 Trieste, Italy*

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We discuss general theories of N scalar fields with $O(N)$ symmetry. In addition to the standard case of linearly realized symmetry, there are also examples that carry nonlinear realizations, with the topology of a cylinder $R \times S^{N-1}$ or a sphere S^N . We write flow equations for the theory in the second order of the derivative expansion in the background field and discuss the properties of scaling solutions with vanishing potential.

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I. GENERAL SCALAR THEORIES WITH $O(N)$ SYMMETRY

The most economical nontrivial realization of $O(N)$ symmetry in a scalar field theory makes use of $N - 1$ fields. It corresponds to a nonlinear sigma model with values in the $(N - 1)$ -sphere $S^{N-1} = O(N)/O(N - 1)$, where the radius of the sphere is treated as a fixed parameter. In many applications it is preferable to turn the radius into a dynamical field. Then, one can reparametrize the theory in terms of N scalar fields ϕ^a , $a = 1, \dots, N$, transforming linearly under $O(N)$. Aside from the advantage of working with simpler, linear transformations, such models are also better behaved as quantum field theories.

In this paper we will consider the renormalization of $O(N)$ -invariant models with N fields but without relying on the linear structure. There are two aspects to this. As a first step, the metric in the target space could differ from the flat one, but maintain the topology R^N . This happens, for example, if one allows interactions of the form $Z(\phi^2)\partial_\mu\phi^a\partial^\mu\phi^a$ (in four dimensions, this requires operators of dimension at least six). As is always the case in mechanics, the term quadratic in time derivatives defines a metric in the configuration space. Here one has a conformally flat metric $Z(\phi^2)\delta_{ab}$, so the target space becomes effectively a Riemannian manifold and one could treat these theories as general nonlinear sigma models. However, there still exist global coordinates such that the fields transform in the familiar linear way under $O(N)$, and for this reason we will still say that the fields carry a linear realization of the global symmetry. The second aspect is the topology. By removing the origin, one can change the topology into that of a cylinder $S^{N-1} \times R$, and by adding a single point at infinity one can change the topology into that of a sphere S^N . In both cases there are now *local* coordinate systems such that the fields transform in the usual linear way, but such coordinates cannot be extended to cover the whole space. In fact, neither the cylinder nor

the sphere is a linear space, so in these cases we will say that the fields carry nonlinear realizations. Such models can be viewed as limits of deformations of linear models.

The motivation for studying these models comes at least in part from Higgs physics. Consider a linear model in the spontaneously broken phase where the field has a non-vanishing vacuum expectation value (VEV). Perturbative analysis of the fluctuations around the vacuum reveals the presence of $N - 1$ massless modes, the Goldstone bosons, and one massive mode, corresponding to the radius. Suppose we observe such a spectrum, and furthermore suppose that the scattering amplitudes exhibit $O(N)$ invariance. This is still not enough to validate the simple linear scalar theory: The target space could deviate significantly from flatness for fields that are either much larger or much smaller than the VEV or both.

Various nonlinear models of this type have been used in phenomenology. One class of models is based on the observation that the VEV of the Higgs is the only parameter in the standard model that breaks scale invariance. Instead of an explicit breaking, as in the SM, it is tempting to think that scale invariance is spontaneously broken. Then, the radius, parametrized as $\rho = e^\sigma$, where $-\infty < \sigma < \infty$, can be interpreted as a dilaton. In this model the topology of the target space is a cylinder, and the symmetry of the action is enhanced by scale invariance. Various applications of similar ideas to the standard model have been discussed in [1].

Another class of models that have enjoyed some popularity recently, the so-called composite Higgs models, is based on the assumption that the 4 real degrees of freedom of the Higgs doublet are the Goldstone bosons resulting from the breaking of some global symmetry group G to a subgroup H [2]. The minimal model of this type, giving rise to the Higgs doublet as the only Goldstone bosons, while preserving custodial symmetry, corresponds to $SO(5)$ spontaneously breaking to $SO(4)$ [3] (see also [4] and references therein). In this case the topology of the target space would be a four-sphere. Note that in both classes of models the radial mode can be interpreted as a Goldstone boson, on the same footing with the angular degrees of freedom. In both cases one loses the simple

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linear realization of the symmetry, but this may not be easily seen in perturbation theory around the VEV.

It is expected that these nonlinear models break down at some energy and that they are to be treated as low energy effective field theories. In this paper we shall discuss the renormalization of such theories using functional renormalization group (RG) methods. One powerful general method of studying quantum field theories is to integrate out quantum fluctuations gradually, one momentum shell at a time, beginning from some ultraviolet scale Λ_{UV} down to some scale k . The original implementation of this idea by Wilson gives rise to a k -dependent Wilsonian action S_k [5]. An exact functional equation was derived in [6,7] and applied to the study of scalar theories in [8], and later also to general $O(N)$ models [9]. For many purposes it has proven more convenient to study instead the k dependence of a one-particle irreducible functional Γ_k , called the “effective average action” (EAA), which is defined exactly like the effective action, but with a smooth cutoff in the functional integral (see [10–12] for introductory reviews and [13] for applications to gravity). More precisely, in the functional integral one adds (by hand) to the action a term quadratic in the fields that in momentum space looks like $\Delta S_k = \int dp \phi^a(-p) R_k(p^2) \phi^a(p)$, where $R_k(p^2)$ is a monotonically decreasing function with $R_k(0) = k^2$, $R_0(p^2) = 0$ and tending rapidly to zero for $p^2 > k^2$. The role of this term is to suppress the contribution to the functional integral of the modes with $p^2 < k^2$, so it is referred to as the infrared cutoff. The nice property of this functional is that it satisfies a simple functional equation [14,15]

$$k \frac{\partial \Gamma_k}{\partial k} = \frac{1}{2} \text{Tr} \left(\frac{\delta^2 \Gamma_k}{\delta \phi^2} + R_k \right)^{-1} k \frac{\partial R_k}{\partial k}. \quad (1)$$

This functional equation specifies the k dependence of the EAA, so the right-hand side (r.h.s.) can be considered as a “beta functional” of the theory. It contains the beta functions of all the couplings that are present in the EAA. Due to the falloff properties of R_k , the trace on the r.h.s. is finite. One can therefore take the equation as a basis for defining a quantum field theory: Given the field content and the symmetries, one can calculate the r.h.s. of the equation. It defines a flow on the “theory space” parametrized by the functionals Γ_k . Choosing some initial condition, one can then, at least in principle, calculate the effective action by solving the flow in the limit $k \rightarrow 0$.

It is of course impossible to do this in practice. A useful approximation method of this equation is the derivative expansion, where one retains terms up to some fixed number of derivatives. This is well motivated in statistical mechanics in applications to the theory of phase transitions and in particle physics for the study of low energy effective field theories. For a simple scalar theory, the lowest order of this expansion is the “local potential approximation” (LPA) where one retains a fixed quadratic kinetic term and the only interactions are given by a potential $V(\phi^2)$.

Inserting this ansatz in (1) one obtains a differential equation for the flow of the potential, which can be studied to analyze the scaling solutions [16,17]. A slightly improved version of the LPA, sometimes called LPA', takes into account the running of a wave function renormalization constant Z . This yields improved values for the critical exponents and also gives the anomalous dimension [18,19].

In the second order of the derivative expansion one retains the most general terms that are $O(N)$ invariant and contain at most two derivatives:

$$\Gamma_k(\phi) = \int d^d x \left[\frac{1}{2} Z(\phi^2) \partial_\mu \phi^a \partial^\mu \phi^a + Y(\phi^2) \phi^a \partial_\mu \phi^a \phi^b \partial^\mu \phi^b + V(\phi^2) \right], \quad (2)$$

where Z and Y are arbitrary, k -dependent functions of ϕ^2 . (We do not indicate the k dependence explicitly for typographical simplicity.) The scaling solutions of these models in $d = 3$ have been studied in [20], using standard methods of linear realizations. Since the functions $Z(\phi^2)$ and $Y(\phi^2)$ define a curved target space geometry, it makes sense to study such models also with methods that had been previously applied to the RG of nonlinear sigma models [21–24]. Here, however, the situation is more complicated. In typical nonlinear sigma models the symmetry group acts transitively on the target space, and therefore the only invariant potential is a constant. In the case we want to discuss, the potential need not be a constant (though it is constant on the orbits S^{N-1}) so we have to discuss its flow too.

In this paper we will lay the foundations for a study of these theories in the second order of the derivative expansion (the case $N = 1$ has already been discussed to next order of the derivative expansion [25]). For the quantization of the nonlinear sigma models we shall use the background field method. After choosing a background field $\bar{\varphi}^i$ any other field that is not too distant from it can be parametrized in terms of normal coordinates $\bar{\xi}^i(x)$. The action can then be expanded in powers of $\bar{\xi}$ and the cutoff is quadratic in $\bar{\xi}$, as we shall describe later. This method has the advantage of preserving invariance under simultaneous coordinate transformations of $\bar{\varphi}$ and $\bar{\xi}$, but it has the drawback that the EAA becomes a functional of two fields $\Gamma(\bar{\varphi}, \bar{\xi})$. In principle we would therefore have to allow many more invariants than in a functional of ϕ alone. In this paper we will not consider such complications, and we study the flow of background functionals only, which are the most important ones in the analysis of the phase structure of the theory. Regarding the wave function renormalization of $\bar{\xi}$, we shall make two separate approximation: One is to assume that it is k independent (i.e. that the anomalous dimension of $\bar{\xi}$ is zero); the other is to assume that the wave function renormalization of $\bar{\xi}$ can be approximated by the wave function renormalization of the background. We will see that even though we go to second order in derivatives of the background field, in both cases the results are close to those of the LPA in the linear formalism.

After deriving the flow equations for the theory, we will establish the existence of scaling solutions with three different target space topologies: the linear topology R^N , the cylinder $S^{N-1} \times R$ and the sphere S^N . We will only discuss scaling solutions with constant potential and determine in each case the set of relevant deformations. At least in the linear case and in dimensions $2 < d < 4$, there are also scaling solutions with nontrivial potential. We will not repeat here the analysis of these solutions, but it will be useful at some later stage to check that the geometrical methods used here agree with the standard ones used in [20].

II. FLOW EQUATIONS

We will study the RG flow of theories with action of the form (2), but we will not assume that the topology of the target space is R^N . In other words, the action (2) may only be valid locally in the target space. For this reason we shall use a formalism that is manifestly invariant under coordinate transformations in the target space. This formalism has been developed in the context of nonlinear sigma models [26] and has been applied previously to the RG flow of these theories [21–24]. Although one could actually work with completely arbitrary coordinates, it is convenient to assume that the coordinate system is adapted to the action of the group $O(N)$, in the sense that one coordinate ρ parametrizes the different orbits of the group and the remaining $N - 1$ coordinates χ^α are coordinates within the orbits S^{N-1} . Let us consider first what happens in the linear case. The coordinate transformation from the adapted coordinates to the linear coordinates ϕ^a is of the form $\phi^a = \rho \hat{\phi}^a(\chi^\alpha)$, with $\rho = \sqrt{\sum_i \phi_i^2}$. Then the action (2) may be rewritten in the form

$$\Gamma_k = \int d^d x \left[\frac{1}{2} J(\rho) \partial_\mu \rho \partial^\mu \rho + \frac{1}{2} K(\rho) g_{\alpha\beta} \partial_\mu \chi^\alpha \partial^\mu \chi^\beta + V(\rho) \right], \quad (3)$$

where $g_{\alpha\beta}$ is the metric on the unit $(N - 1)$ -dimensional sphere, $J = Z + 2\rho^2 Y$, $K = \rho^2 Z$. In the following we will assume that the EAA has the form (3), without assuming that it has been derived from (2). Thus the coordinate ρ need not have the meaning of ‘‘radius.’’

The terms in (3) with two derivatives can be rewritten as

$$\frac{1}{2} G_{ij} \partial_\mu \varphi^i \partial^\mu \varphi^j \quad (4)$$

where $\varphi^i = (h, \chi^\alpha)$ are adapted coordinates on the target space and

$$G_{ij} = \left(\begin{array}{c|c} J(\rho) & \\ \hline & K(\rho) g_{\alpha\beta} \end{array} \right) \quad (5)$$

is the metric. Note that this is a Euclidean Robertson-Walker metric with spherical ‘‘spatial’’ sections. The coordinate transformation

$$h(\rho) = \int_0^\rho \sqrt{J(x)} dx \quad (6)$$

brings the metric to standard form with $J = 1$. Notice that the field h is canonically normalized and therefore has dimension $\frac{d-2}{2}$. We will use this field reparametrization later on, but first we have to derive the beta functions of J , K and V .

It was found in [21] that in the nonlinear sigma models the flow of the metric G_{ij} is governed by the Ricci tensor of G_{ij} . Here we have to take into account the additional complications due to the presence of a nontrivial potential. We shall use the background field method. After choosing a background field $\bar{\varphi}^i(x)$, any other field $\varphi^i(x)$ that is not too distant from $\bar{\varphi}^i(x)$ can be parametrized in terms of the normal coordinates $\xi^i(x)$ as $\varphi^i(x) = \text{Exp}_{\bar{\varphi}(x)} \xi^i(x)$. Quantization produces an EAA that is a functional of two fields $\Gamma_k(\bar{\varphi}; \xi)$. We can expand the EAA in powers of ξ :

$$\Gamma_k(\bar{\varphi}; \xi) = \bar{\Gamma}_k(\bar{\varphi}) + \Gamma_k^{(1)}(\bar{\varphi}, \xi) + \Gamma_k^{(2)}(\bar{\varphi}, \xi) + \dots \quad (7)$$

where $\Gamma_k^{(n)}(\bar{\varphi}; \xi)$ contains n powers of ξ . In particular, $\bar{\Gamma}_k(\bar{\varphi}) \equiv \Gamma_k^{(0)}(\bar{\varphi}) = \Gamma_k(\bar{\varphi}; 0)$ depends only on the background. Throughout this paper we will work in a ‘‘single field truncation,’’ which means that the r.h.s. is assumed to be the Taylor expansion of some functional of the full field $\varphi = \text{Exp}_{\bar{\varphi}} \xi$ [and one can call that functional $\bar{\Gamma}_k(\varphi)$]. Furthermore, we assume that this functional has the form (3). By inserting this ansatz in the flow equation (1) we will derive ‘‘beta functionals’’ for J , K and V , which we will call

$$\zeta_J = \frac{d}{dt} \log J, \quad \zeta_K = \frac{d}{dt} \log K, \quad \zeta_V = \frac{d}{dt} \log V, \quad (8)$$

where $t = \log k$. To evaluate the r.h.s. of (1) we start by Taylor expanding (3) to second order in ξ . From the two-derivative terms we get

$$\frac{1}{2} \int d^d x \xi^i (-G_{ij} \nabla^2 - M_{ij}) \xi^j, \quad (9)$$

where $\nabla_\mu \xi^i = \partial_\mu \xi^i + \partial_\mu \varphi^k \Gamma_k^i{}^j \xi^j$ and $M_{ij} = \partial_\mu \varphi^m \partial^\mu \varphi^n R_{imjn}$. Here $\Gamma_k^i{}^j$ are the Christoffel symbols of the metric G_{ij} and R_{imjn} is its Riemann tensor, whose nonzero components are

$$R_{\alpha 0 \beta 0} = \frac{KK'J' + K'^2 J - 2KK''J}{4K^2 J} G_{\alpha\beta}, \quad (10)$$

$$R_{\alpha\gamma\beta\delta} = \frac{4KJ - K'^2}{4K^2 J} (G_{\alpha\beta} G_{\gamma\delta} - G_{\alpha\delta} G_{\gamma\beta}). \quad (11)$$

The expansion of the potential is

$$V(h) = V(\bar{h}) + V'(\bar{h}) \left(\xi^0 - \frac{1}{2} \Gamma_{ij}^0 \xi^i \xi^j + \dots \right) + \frac{1}{2} V''(\bar{h}) \left(\xi^0 - \frac{1}{2} \Gamma_{ij}^0 \xi^i \xi^j + \dots \right)^2 + \dots \quad (12)$$

where the nonzero components of Γ_{ij}^0 are $\Gamma_{\alpha\beta}^0 = -\frac{K'}{2J}g_{\alpha\beta}$, $\Gamma_{00}^0 = \frac{J'}{2J}$. This gives a contribution to $\Gamma_k^{(2)}$ of the form $\frac{1}{2} \int dx \xi^i S_{ij} \xi^j$, with

$$S_{ij} = V'' \delta_i^0 \delta_j^0 - V' \Gamma_{ij}^0 \\ = (V'' - V' J' / 2J) \delta_i^0 \delta_j^0 + (V' K' / 2J) g_{ij}. \quad (13)$$

(Here, we think of g_{ij} as an $N \times N$ matrix with $g_{00} = g_{0i} = 0$.) Altogether the second order expansion of (3) yields

$$\Gamma_k^{(2)}(\bar{\varphi}, \xi) = \frac{1}{2} \int d^d x \xi^i (-G_{ij} \nabla^2 - M_{ij} + S_{ij}) \xi^j. \quad (14)$$

The cutoff function is then conveniently chosen as

$$(\mathcal{R}_k)_{ij} = G_{ij} R_k(z) = \left(\frac{J R_k(z)}{K g_{\alpha\beta} R_k(z)} \right) \quad (15)$$

where z stands for the covariant Laplacian $-\nabla^2$. We now have all the pieces that enter in the r.h.s. of the functional renormalization group equation (FRGE). Adding the cutoff, the quadratic action can be written as

$$\Gamma^{(2)} + \Delta S_k = \frac{1}{2} \int d^d x \xi^i (\mathcal{P}_{ij} (-\nabla^2) - M_{ij} + S_{ij}) \xi^j \quad (16)$$

where $\mathcal{P}_{ij} = G_{ij} P_k$ and $P_k(z) = z + R_k(z)$. The derivative of the cutoff function with respect to t is

$$\dot{\mathcal{R}}_{ij} \equiv \frac{d\mathcal{R}_{ij}}{dt} \\ = \left(\frac{J(\dot{R}_k + \zeta_J R_k + R'_k \dot{z})}{K g_{\alpha\beta} (\dot{R}_k + \zeta_K R_k + R'_k \dot{z})} \right) \quad (17)$$

where $\dot{R}_k(z) = \partial_t R_k(z)$, $R'_k(z) = \partial_z R_k(z)$. The terms involving \dot{z} will give no contribution.

In order to calculate the beta functions for V, J, K we have to extract from the r.h.s. of the FRGE the terms that contain either no derivatives or two derivatives of the background field. Recalling that M contains two derivatives of the background field, the r.h.s. of the FRGE can be expanded in M as

$$\frac{1}{2} \text{Tr} \frac{\dot{\mathcal{R}}_k}{\mathcal{P}_k - M + S} \\ = \frac{1}{2} \text{Tr} (1 + (\mathcal{P}_k + S)^{-1} M + \dots) (\mathcal{P}_k + S)^{-1} \dot{\mathcal{R}}_k. \quad (18)$$

Only the first two terms are needed for our calculation, which is described in the Appendix.

Note that, in $\dot{\mathcal{R}}_k$, $-\zeta_J$ and $-\zeta_K$ play the role of anomalous dimensions of the fields. A more complete treatment, which goes beyond the single-field truncation, would consist in replacing the factors of J and K contained in the metric G_{ij} in (14) with independent wave function

renormalization constants Z_J and Z_K . Then in (17) ζ_J and ζ_K would be replaced by $-\eta_J = \frac{d \log Z_J}{d \log k}$ and $-\eta_K = \frac{d \log Z_K}{d \log k}$. These ‘‘genuine’’ anomalous dimensions would then be obtained from the t derivative of the two-point functions $\langle \xi^i \xi^j \rangle$. See [27] for such a calculation in a nonlinear sigma model. We will not attempt this calculation here and defer it to a future work. Instead, we shall evaluate the functional traces in two different approximations. In the first we will neglect all the derivatives of couplings in the r.h.s. of the FRGE and keep only the explicit dependence of the cutoff on k . In particular, we set $\zeta_J = 0$, $\zeta_K = 0$ in (17). We will refer to this as the one-loop approximation, since this is the result one would obtain by inserting a cutoff in the one-loop determinants and then deriving with respect to k . We will see that even though we are tracking the flow of terms with two derivatives of the background field, this approximation is very similar to the LPA in the linear formalism. The second approximation is to replace the genuine anomalous dimensions η_J and η_K by $-\zeta_J$ and $-\zeta_K$, as already indicated in (17). We will call this the single-field approximation. In the vicinity of the fixed points that we shall study here the anomalous dimensions are small anyway and both approximations should be good. We shall see *a posteriori* that in spite of the nonvanishing anomalous dimensions this approximation is also still quite close to the LPA.

In the Appendix we derive explicit formulas (A11)–(A13) for $\zeta_J, \zeta_K, \zeta_V$, which are essentially the beta functions for J, K and V . At this point we switch to dimensionless variables and simultaneously implement the coordinate transformation (6).¹ We define $\tilde{K} = k^{2-d} K$, $\tilde{V} = k^{-d} V$, both regarded as functions of the dimensionless field $\tilde{h} = k^{\frac{2-d}{2}} h$. The functions $\zeta_J, \zeta_K, \zeta_V$, being dimensionless, can be written in terms of the dimensionless variables simply by setting $k = 1$, $J = 1$ and putting a tilde on K and V .

The beta functions can be presented most compactly as follows. Define the quantities

$$\zeta_1 = \frac{c_d}{d+2} \frac{N-1}{\tilde{V}(1+\tilde{V}'\tilde{K}'/2\tilde{K})}, \\ \zeta_2 = \frac{c_d}{d+2} \frac{1}{\tilde{V}(1+\tilde{V}'')}, \\ \zeta_3 = \frac{c_d}{d+2} \frac{(N-2)(4-\tilde{K}''/\tilde{K})}{2\tilde{K}(1+\tilde{V}'\tilde{K}'/2\tilde{K})^2} \quad (19) \\ \zeta_4 = \frac{c_d}{d+2} \frac{\tilde{K}''/\tilde{K} - 2\tilde{K}'''}{2\tilde{K}(1+\tilde{V}'')^2}, \\ \zeta_5 = \frac{c_d}{d+2} \frac{(N-1)(\tilde{K}''/\tilde{K} - 2\tilde{K}''')}{2\tilde{K}(1+\tilde{V}'\tilde{K}'/2\tilde{K})^2},$$

¹This is equivalent to performing, after each functional integration over an infinitesimal momentum shell, a rescaling of momenta to restore the condition $k = 1$ and a redefinition of the field to restore its canonical normalization.

where $c_d = \frac{1}{(4\pi)^{d/2}\Gamma(d/2+1)}$. The ζ 's defined in (8) are given, in the one-loop approximation, by

$$\zeta_J = (d+2)\zeta_5, \quad (20)$$

$$\zeta_K = (d+2)(\zeta_3 + \zeta_4), \quad (21)$$

$$\zeta_V = (d+2)(\zeta_1 + \zeta_2), \quad (22)$$

and, in the single-field approximation, by

$$\zeta_J = (d+2) \frac{(1 + \zeta_4)\zeta_5}{1 - \zeta_3 - \zeta_4\zeta_5}, \quad (23)$$

$$\zeta_K = (d+2) \frac{\zeta_3 + \zeta_4(1 + \zeta_5)}{1 - \zeta_3 - \zeta_4\zeta_5}, \quad (24)$$

$$\zeta_V = (d+2) \frac{\zeta_1(1 + \zeta_4) + \zeta_2(1 - \zeta_3 + \zeta_5)}{1 - \zeta_3 - \zeta_4\zeta_5}. \quad (25)$$

From the definition of \tilde{h} and (6) one finds

$$\frac{d\tilde{h}}{dt} = \frac{2-d}{2}\tilde{h} + \frac{1}{2}\int_0^{\tilde{h}} dx \zeta_J(x). \quad (26)$$

Using the definitions of \tilde{K} and \tilde{V} in the relations (8), we arrive at the flow equations

$$\frac{d\tilde{K}}{dt} = (\zeta_K - d + 2)\tilde{K} + \frac{d-2}{2}\tilde{h}\tilde{K}' - \frac{1}{2}\tilde{K}'\int_0^{\tilde{h}} dx \zeta_J(x), \quad (27)$$

$$\frac{d\tilde{V}}{dt} = (\zeta_V - d)\tilde{V} + \frac{d-2}{2}\tilde{h}\tilde{V}' - \frac{1}{2}\tilde{V}'\int_0^{\tilde{h}} dx \zeta_J(x). \quad (28)$$

These t derivatives take into account, besides the integration over fluctuations, also the t -dependent field redefinition that is necessary to keep the field h canonically normalized. As expected, the redundant variable J has disappeared from the equations: ζ_J , ζ_K and ζ_V are functions of \tilde{K} and \tilde{V} only. Note that if we think of expanding J in Taylor series, we have not only normalized the kinetic term but also eliminated infinitely many redundant interaction terms.

It is interesting to consider first the case $N = 1$. This corresponds to $g_{\alpha\beta} = 0$, so the term involving the function \tilde{K} is not present in either side of the FRGE. There are only two equations which govern the running of V and J . One has $\zeta_J = 0$, and after the appropriate field redefinitions Eq. (28) becomes

$$\frac{d\tilde{V}}{dt} = \frac{c_d}{1 + \tilde{V}''(\tilde{h})} - d\tilde{V}(\tilde{h}) + \left(\frac{d}{2} - 1\right)\tilde{h}\tilde{V}'(\tilde{h}). \quad (29)$$

This reproduces the well-known flow equation for the potential in the LPA.

If we restrict J to be a constant, it can be viewed as the wave function renormalization of h . Then, one would be tempted to identify ζ_J with (minus) the anomalous dimension of the field h . However, we must stress that the formulas (20) or (23) for ζ_J are quite different from the formula for the anomalous dimension for linear scalar field theories in the LPA':

$$\begin{aligned} \eta &= c_d \frac{(\tilde{V}''')^2}{(1 + \tilde{V}''')^4} \quad \text{for } N = 1; \\ \eta &= c_d \frac{2(\tilde{h}\tilde{V}'' - \tilde{V}')^2}{\tilde{h}^2(\tilde{h} + \tilde{h}\tilde{V}'' - \tilde{V}')^2} \quad \text{for } N > 1. \end{aligned} \quad (30)$$

This is actually to be expected because they are different quantities. In order to compute the anomalous dimension η , to be compared with the preceding formula, one should compute the two-point function of ξ^0 . We will not discuss this in the present paper. We note however that for the Gaussian fixed point η is expected to be zero, and so is ζ_J . For the fixed points that we shall discuss in this paper we expect our approximations to be acceptable, but for a quantitatively accurate discussion of the Wilson-Fisher fixed point one should calculate the anomalous dimension separately.

In the rest of the paper we discuss some scaling solutions and their infinitesimal deformations.

III. THE FLAT (GAUSSIAN) FIXED POINT

The choice $K = h^2$ corresponds to the flat metric. This gives $\zeta_3 = \zeta_4 = \zeta_5 = 0$. Plugging this into Eqs. (20)–(22) or into Eqs. (23)–(25) gives, both in one-loop approximation and in the single-field approximation,

$$\begin{aligned} \zeta_J &= 0; & \zeta_K &= 0; \\ \zeta_V &= \frac{c_d}{\tilde{V}} \left[\frac{N-1}{1 + \tilde{V}'(\tilde{h})/\tilde{h}} + \frac{1}{1 + \tilde{V}''(\tilde{h})} \right]. \end{aligned} \quad (31)$$

Then Eqs. (27) and (28) become

$$\begin{aligned} \frac{d\tilde{K}}{dt} &= 0; \\ \frac{d\tilde{V}}{dt} &= c_d \left[\frac{N-1}{1 + \tilde{V}'(\tilde{h})/\tilde{h}} + \frac{1}{1 + \tilde{V}''(\tilde{h})} \right] \\ &\quad - d\tilde{V}(\tilde{h}) + \left(\frac{d}{2} - 1\right)\tilde{h}\tilde{V}'(\tilde{h}). \end{aligned} \quad (32)$$

This flow equation for the potential agrees with that of the standard linear theory in the LPA. From (32) we see that the fixed point condition for $\tilde{K}(\tilde{h})$ is already satisfied and the fixed point condition for $\tilde{V}(\tilde{h})$ is satisfied by a constant potential $\tilde{V}(\tilde{h}) = c_d N/d$. We thus have a fixed point solution

$$\tilde{K}_* = \tilde{h}^2, \quad \tilde{V}_* = c_d N/d. \quad (33)$$

This is a Gaussian fixed point, corresponding to a free theory at which, in addition to $O(N)$, the theory is also translation

invariant. It exists in any dimension. In $d = 3$ there is also a solution with nontrivial \tilde{V}_* , corresponding the Wilson-Fisher fixed point, but we shall not discuss it in this paper. One expects that the infinitesimal deformations around the Gaussian fixed point are characterized by canonical critical exponents. Let us check this explicitly and obtain the corresponding eigenvectors.

A. Linearized equations at one loop

Linearizing the flow equations (27) and (28), around the fixed point (33) one finds

$$\lambda \delta \tilde{K} = \tilde{K}_* \delta \zeta_K - (d-2) \delta \tilde{K} + \frac{d-2}{2} \tilde{h} \delta \tilde{K}' - \frac{1}{2} \tilde{K}'_* \int_0^{\tilde{h}} dx \delta \zeta_J(x), \quad (34)$$

$$\lambda \delta \tilde{V} = \tilde{V}_* \delta \zeta_V + \frac{d-2}{2} \tilde{h} \delta \tilde{V}', \quad (35)$$

where λ are scaling exponents to be determined and $\delta \zeta_V$, $\delta \zeta_J$ and $\delta \zeta_K$ are given by

$$\delta \zeta_J = -\frac{2(N-1)c_d}{\tilde{h}^4} \delta \tilde{K} + \frac{2(N-1)c_d}{\tilde{h}^3} \delta \tilde{K}' - \frac{(N-1)c_d}{\tilde{h}^2} \delta \tilde{K}'', \quad (36)$$

$$\delta \zeta_K = \frac{2(N-3)c_d}{\tilde{h}^4} \delta \tilde{K} - \frac{2(N-3)c_d}{\tilde{h}^3} \delta \tilde{K}' - \frac{c_d}{\tilde{h}^2} \delta \tilde{K}'', \quad (37)$$

$$\delta \zeta_V = -\frac{d^2}{Nc_d} \delta \tilde{V} - \frac{d(N-1)}{N\tilde{h}} \delta \tilde{V}' - \frac{d}{N} \delta \tilde{V}''. \quad (38)$$

Equation (34) contains the second derivative $\delta \tilde{K}''$ and an integral. So this is in principle a third order equation. To get rid of one derivative we divide the equation by $\tilde{K}'_* = 2\tilde{h}$, and take its derivative with respect to \tilde{h} . This gives

$$\lambda \left(\frac{\delta \tilde{K}}{\tilde{K}'_*} \right)' = \left(\frac{K_*}{\tilde{K}'_*} \delta \zeta_K \right)' - (d-2) \left(\frac{\delta \tilde{K}}{\tilde{K}'_*} \right)' + \frac{d-2}{4} \delta \tilde{K}'' - \frac{1}{2} \delta \zeta_J. \quad (39)$$

If we define $\Delta K \equiv \delta \tilde{K} / \tilde{K}'_*$ we have

$$\delta \zeta_J = -\frac{2(N-1)c_d}{\tilde{h}} \Delta \tilde{K}'', \quad (40)$$

$$\delta \zeta_K = -\frac{4(N-2)c_d}{\tilde{h}^2} \Delta \tilde{K}' - \frac{2c_d}{\tilde{h}} \Delta \tilde{K}'', \quad (41)$$

$$\delta \zeta_V = -\frac{d^2}{Nc_d} \delta \tilde{V} - \frac{d(N-1)}{\tilde{h}N} \delta \tilde{V}' - \frac{d}{N} \delta \tilde{V}''. \quad (42)$$

Then we observe that only derivatives of $\Delta \tilde{K}$ appear in these expressions and not $\Delta \tilde{K}$ itself.

Therefore, defining the variable $\kappa \equiv (\Delta \tilde{K})'$, the first derivative of Eq. (34) becomes the following second order equation:

$$0 = \kappa'' + \left(\frac{N-3}{\tilde{h}} - \frac{d-2}{2c_d} \tilde{h} \right) \kappa' - \left(\frac{2(N-2)}{\tilde{h}^2} - \frac{\lambda}{c_d} \right) \kappa, \quad (43)$$

whereas Eq. (35) becomes

$$0 = \delta \tilde{V}'' + \frac{2c_d(N-1) - (d-2)\tilde{h}^2}{2c_d\tilde{h}} \delta \tilde{V}' + \frac{d+\lambda}{c_d} \delta \tilde{V}. \quad (44)$$

Equation (44) is well known in the literature [see e.g. Eq. (A.3) of [16]]. Imposing regularity in the origin and boundedness by polynomials for large fields leads to

$$\delta \tilde{V} = {}_1F_1(-i, N/2, \tilde{h}^2), \quad \lambda = (d-2)i - d, \quad (45)$$

$$i = 0, 1, 2, \dots$$

where

$$\tilde{h} = \left(\frac{d-2}{4c_d} \right)^{1/2} \tilde{h}. \quad (46)$$

In [16], the eigenfunctions were represented in terms of Laguerre polynomials. In fact our result (45) agrees with [16], noticing that

$$L_n^{N/2-1}(z) = \frac{(N/2)_n}{n!} {}_1F_1(-n, N/2, z), \quad (47)$$

$$(\alpha)_n \equiv \alpha(\alpha+1)\dots(\alpha+n-1), \quad (\alpha)_0 \equiv 1.$$

Similarly, the regular and polynomial-bounded solutions to Eq. (43) are

$$\kappa = \tilde{h}^2 {}_1F_1(-i+1, 1+N/2, \tilde{h}^2), \quad (48)$$

$$\lambda = (d-2)i, \quad i = 1, 2, 3, \dots$$

Again, these hypergeometric functions are polynomials. Note that there is degeneracy with the solutions (45) when $d/(d-2)$ is an integer. The only dimensions where this happens are $d = 0, 1, 3, 4$. We can convert this to solutions for $\delta \tilde{K}$ by using the definition $\kappa = (\delta \tilde{K} / \tilde{K}'_*)'$,

$$\delta \tilde{K} = 2\tilde{h} \int_0^{\tilde{h}} dx x^2 {}_1F_1(-i+1, 1+N/2, \bar{x}^2) \quad (49)$$

$$\bar{x} = \left(\frac{d-2}{4c_d} \right)^{1/2} x.$$

The integration constant has been put to zero using Eq. (34). These solutions have $\kappa \neq 0$. There is also a solution $\delta \tilde{K} \propto \tilde{h}$ (and hence $\kappa = 0$) with eigenvalue $\lambda = 1 - d/2$. The only dimension for which the equation for δV also has a solution with $\lambda = 1 - d/2$ is $d = 6$.

B. Linearized equations in the single-field approximation

In the single-field approximation the linearized equations have the same form except for the replacement of (42) by

$$\begin{aligned} \delta\zeta_V = & -\frac{d^2}{Nc_d}\delta\tilde{V} - \frac{d(N-1)}{N\tilde{h}}\delta\tilde{V}' - \frac{d}{N}\delta\tilde{V}'' \\ & + \frac{2d(N-1)(N-4)c_d}{N(d+2)\tilde{h}^4}\delta\tilde{K} \\ & - \frac{2d(N-1)(N-4)c_d}{N(d+2)\tilde{h}^3}\delta\tilde{K}' \\ & - \frac{2d(N-1)c_d}{N(d+2)\tilde{h}^2}\delta\tilde{K}'' . \end{aligned} \quad (50)$$

The last three terms introduce a coupling between the equations for δK and δV . Proceeding as in the previous section, one arrives at a system of equations for κ and $\delta\tilde{V}$, where (43) is unchanged and (44) is replaced by

$$\begin{aligned} 0 = & \delta\tilde{V}'' + \frac{2c_d(N-1) - (d-2)\tilde{h}^2}{2c_d\tilde{h}}\delta\tilde{V}' \\ & + \frac{d+\lambda}{c_d}\delta\tilde{V} + \frac{4(N-1)c_d}{(d+2)\tilde{h}}\left[\kappa' + \frac{N-2}{\tilde{h}}\kappa\right] . \end{aligned} \quad (51)$$

We note that (43) is automatically satisfied when $\delta\tilde{K} = 0$. Therefore, the system of equations has an infinite set of solutions where $\delta K = 0$ and δV is as in the preceding section. Next we look for solutions to Eqs. (51) and (43) with $\kappa \neq 0$. To this end we plug the eigenfunction (48) into Eq. (51) and solve for $\delta\tilde{V}$. The most general solution to this equation consists of any of its solutions plus the solution to the homogeneous equation (44). Since (48) is a polynomial of order $2i$, the two terms in Eq. (51) involving κ are of order $2(i-1)$. Therefore a solution of the inhomogeneous equations (51) with κ given by (48) can be found by making a general order $2(i-1)$ polynomial ansatz for $\delta\tilde{V}$ and solving for the unknown coefficients. The solution to the homogeneous equation which is well behaved at $\tilde{h} = 0$ is given by (45), but now the eigenvalues are found from Eq. (48). Thus, these solutions can be written as

$$\begin{aligned} \delta\tilde{V} = & {}_1F_1(-i - d/(d-2), N/2, \tilde{h}^2), \quad \lambda = (d-2)i, \\ i = & 1, 2, \dots, \quad \tilde{h} = \left(\frac{d-2}{4c_d}\right)^{1/2} h . \end{aligned} \quad (52)$$

IV. THE CYLINDRICAL FIXED POINT

A. The cylindrical fixed point in the one-loop approximation

Let us now see if there is a fixed point with $\tilde{K} = \text{const}$. From Eqs. (20)–(22) one has

$$\zeta_J = 0, \quad (53)$$

$$\zeta_K = 2c_d \frac{N-2}{\tilde{K}}, \quad (54)$$

$$\zeta_V = c_d \left[\frac{N-1}{\tilde{V}} + \frac{1}{\tilde{V}(1+\tilde{V}'')} \right]. \quad (55)$$

Using $\zeta_J = 0$ and Eq. (27) the fixed point condition implies $\zeta_K = d-2$. Plugging this back into Eq. (54) one obtains the fixed point value of $\tilde{K} = c_d \frac{N-2}{d-2}$. Also combining Eq. (55) with Eq. (28) gives the fixed point condition

$$0 = c_d \left[N-1 + \frac{1}{1+\tilde{V}''} \right] - d\tilde{V} + \left(\frac{d}{2} - 1 \right) h\tilde{V}' \quad (56)$$

which is a nonlinear second order differential equation. This equation has a solution for a constant \tilde{V} . Then the fixed point is

$$\tilde{K}_* = c_d \frac{N-2}{d-2}; \quad \tilde{V}_* = \frac{c_d N}{d}. \quad (57)$$

We will not consider more general solutions in this work. However it is worth mentioning that we have verified numerically, following the same method used, for example, in [17], that in $d=4$ this is the only scaling solution, while in $d=3$ there also exists a fixed point with nontrivial potential. The fixed point (57) has the geometry of a cylinder $R \times S^{N-1}$. So at the fixed point the symmetry is enhanced: In addition to $O(N)$ it is also invariant under translations in the \tilde{h} direction.

From the flow equations (27) and (28), the linearized equations around the fixed point with constant potential described above have the same form as (34) and (35), except that now $\tilde{K}'_* = 0$, so

$$\lambda\delta\tilde{K} = \tilde{K}_*\delta\zeta_K - (d-2)\delta\tilde{K} + \frac{d-2}{2}\tilde{h}\delta\tilde{K}', \quad (58)$$

$$\lambda\delta\tilde{V} = \tilde{V}_*\delta\zeta_V + \frac{d-2}{2}\tilde{h}\delta\tilde{V}'. \quad (59)$$

In the above equations, $\delta\zeta_J$, $\delta\zeta_K$ and $\delta\zeta_V$ are given by

$$\delta\zeta_J = -\frac{(d-2)(N-1)}{2(N-2)}\delta\tilde{K}'', \quad (60)$$

$$\delta\zeta_K = -\frac{(d-2)^2}{2(N-2)c_d}\delta\tilde{K} - \frac{d-2}{2(N-2)}\delta\tilde{K}'', \quad (61)$$

$$\delta\zeta_V = -\frac{d^2}{Nc_d}\delta\tilde{V} - \frac{d}{N}\delta\tilde{V}'' . \quad (62)$$

The second order equations (58) and (59) can now be written as the following decoupled system:

$$0 = c_d \delta \tilde{K}'' - \frac{d-2}{2} \tilde{h} \delta \tilde{K}' + (d-2+\lambda) \delta \tilde{K}, \quad (63)$$

$$0 = c_d \delta \tilde{V}'' - \frac{d-2}{2} \tilde{h} \delta \tilde{V}' + (d+\lambda) \delta \tilde{V}. \quad (64)$$

Except for the coefficients of $\delta \tilde{K}$ and $\delta \tilde{V}$, these are essentially the same equation. As in the flat case, regularity constrains the eigenvalue λ to discrete values. The solutions are

$$\delta \tilde{V} = {}_1F_1(-i, 1/2, \tilde{h}^2), \quad \lambda = -d + (d-2)i, \quad (65)$$

$$i = 0, 1, 2, \dots,$$

$$\delta \tilde{K} = {}_1F_1(-i, 1/2, \tilde{h}^2), \quad \lambda = -d + 2 + (d-2)i, \quad (66)$$

$$i = 0, 1, 2, \dots$$

Common eigenvalues exist only when $(d-2)i = 2$ for integer values of i , which means when $2/(d-2)$ is an integer, or $d = 0, 1, 3, 4$.

B. The cylindrical fixed point in the single-field approximation

Now we look for a fixed point with $\tilde{K} = \text{const}$ in the single-field approximation. From Eqs. (23)–(25) one has

$$\zeta_J = 0, \quad (67)$$

$$\zeta_K = c_d \left(1 + \frac{\zeta_K}{d+2}\right) 2 \frac{N-2}{\tilde{K}}, \quad (68)$$

$$\zeta_V = c_d \left[\left(1 + \frac{\zeta_K}{d+2}\right) \frac{N-1}{\tilde{V}} + \frac{1}{\tilde{V}(1+\tilde{V}'')} \right]. \quad (69)$$

Using $\zeta_J = 0$, Eq. (27) also gives $\zeta_K = d-2$. Plugging this back into Eqs. (69) and (68), gives

$$\tilde{K} = c_d \frac{4d(N-2)}{d^2-4}; \quad (70)$$

$$\zeta_V \tilde{V} = c_d \left[\frac{2d(N-1)}{d+2} + \frac{1}{1+\tilde{V}''} \right].$$

The first equation gives the fixed point value of the constant function \tilde{K} . Combining the second equation with Eq. (28) gives the fixed point condition

$$0 = c_d \left[\frac{2d(N-1)}{d+2} + \frac{1}{1+\tilde{V}''} \right] - d\tilde{V} + \left(\frac{d}{2} - 1\right) h \tilde{V}'. \quad (71)$$

This equation has a solution for \tilde{V} constant. The fixed point is

$$\tilde{K}_* = c_d \frac{4d(N-2)}{d^2-4}; \quad \tilde{V}_* = c_d \frac{2d(N-1) + d + 2}{d(d+2)}. \quad (72)$$

It is shifted relative to the one-loop solution (57), but has the same general properties.

The linearized equations around the fixed point have the same form as in (58) and (59), but now

$$\delta \zeta_J = -\frac{(d-2)(N-1)}{2(N-2)} \delta \tilde{K}'', \quad (73)$$

$$\delta \zeta_K = -\frac{(d-2)^2}{2(N-2)c_d} \delta \tilde{K} - \frac{d-2}{2(N-2)} \delta \tilde{K}'', \quad (74)$$

$$\delta \zeta_V = -\frac{d^2(d+2)}{(d(2N-1)+2)c_d} \delta \tilde{V} - \frac{d(d+2)}{d(2N-1)+2} \delta \tilde{V}''$$

$$- \frac{(d-2)^2 d(N-1)}{2(N-2)(d(2N-1)+2)c_d} \delta \tilde{K}$$

$$- \frac{(d-2)d(N-1)}{(N-2)(d(2N-1)+2)} \delta \tilde{K}''. \quad (75)$$

The two second order equations (58) and (59) can now be written as

$$0 = 4dc_d \delta \tilde{K}'' - (d^2-4)\tilde{h} \delta \tilde{K}' + 2(2d^2 + d(\lambda-4) + 2\lambda) \delta \tilde{K}, \quad (76)$$

$$0 = c_d \delta \tilde{V}'' - \frac{d-2}{2} \tilde{h} \delta \tilde{V}' + (d+\lambda) \delta \tilde{V}$$

$$+ \frac{(d-2)(N-1)c_d}{(d+2)(N-2)} \delta \tilde{K}'' + \frac{(d-2)^2(N-1)}{2(d+2)(N-2)} \delta \tilde{K}. \quad (77)$$

To solve this eigenvalue problem, we start again by restricting ourselves to the case $\delta \tilde{K} = 0$. In this case the first equation above is automatically satisfied and the second becomes

$$c_d \delta \tilde{V}'' - (d/2 - 1)h \delta \tilde{V}' + (d+\lambda) \delta \tilde{V} = 0. \quad (78)$$

This was solved in the preceding section where we found the solution (65).

Now we look for solutions to Eqs. (76) and (77) with $\delta \tilde{K} \neq 0$. The general regular solution to (76) is

$$\delta \tilde{K} = {}_1F_1(-i, 1/2, \tilde{h}^2), \quad \lambda = (d-2)i - \frac{2d(d-2)}{d+2},$$

$$i = 0, 1, 2, 3, \dots \quad (79)$$

We can now plug this solution into Eq. (77) and solve for $\delta \tilde{V}$. The most general solution to this equation consists of any of its solutions plus a solution to the homogeneous equation (78). A solution of (77) can be found by making a polynomial ansatz of order $2i$ and solving for its coefficients. The solutions to the homogeneous equation which are well behaved at $\tilde{h} = 0$ are

$$\delta\tilde{V} = {}_1F_1\left(-i + \frac{d(3d-2)}{d^2-4}, 1/2, \tilde{h}^2\right), \quad (80)$$

$$\lambda = (d-2)i - \frac{2d(d-2)}{d+2}, \quad i = 0, 1, 2, \dots$$

This is a polynomial with a finite number of terms only when $i - \frac{d(3d-2)}{d^2-4}$ is a non-negative integer. This happens only for $d = 6$.

V. THE SPHERICAL FIXED POINT

For $J = 1$, $K = f^2 \sin^2(h/f)$ and $V = \text{const}$, the symmetry of the Lagrangian is enhanced to $SO(N+1)$. The quantization procedure used here preserves global symmetries [28], and since $SO(N+1)$ symmetry fixes completely the form of the metric, up to the overall factor f^2 , and the constant value of the potential, only these two parameters can flow. A constant potential automatically satisfies $\frac{d\tilde{V}}{dt} = 0$, and it is known from [21] that the flow equation for \tilde{f}^2 has a fixed point. Therefore it is already clear that there is a fixed point corresponding to the geometry of S^N . We will nevertheless check this with our equations. In terms of the rescaled dimensionless variables we have $\tilde{K} = \tilde{f}^2 \sin^2(\tilde{h}/\tilde{f})$ and \tilde{V} a real constant. Keeping \tilde{h} fixed, the derivative of \tilde{K} with respect to the scale will be

$$\partial_t |_{\tilde{h}} \tilde{K} = 2\partial_t \tilde{f} / \tilde{f} [\tilde{K} - (\tilde{h}/2)\tilde{K}']. \quad (81)$$

We discuss first the one-loop approximation.

A. The spherical fixed point in the one-loop approximation

For $\tilde{V}(\tilde{h}) = \text{const}$ and $\tilde{K}(\tilde{h}) = \tilde{f}^2 \sin^2 \tilde{h}/\tilde{f}$, Eqs. (A11)–(A13) are

$$\zeta_J = \zeta_K = 2c_d \frac{N-1}{\tilde{f}^2} \equiv \eta; \quad \zeta_V = c_d \frac{N}{\tilde{V}}. \quad (82)$$

Equations (27) and (28) then give

$$\frac{d\tilde{K}}{dt} = (\tilde{K} - (\tilde{h}/2)\tilde{K}')(\eta - d + 2), \quad (83)$$

$$\frac{d\tilde{V}}{dt} = \tilde{V}(\zeta_V - d) = c_d N - d\tilde{V}. \quad (84)$$

The first equation leads to

$$2\frac{d\tilde{f}}{dt} = (\eta - d + 2)\tilde{f}, \quad (85)$$

and using (82) this reproduces the one-loop result of [21]. The fixed point values are found to be $\tilde{f}_*^2 = \frac{2c_d(N-1)}{d-2}$, $\tilde{V}_* = \frac{c_d N}{d}$. Summarizing, the spherical fixed point is given by

$$\tilde{V}_* = \frac{c_d N}{d}, \quad \tilde{K}_* = \frac{2c_d(N-1)}{d-2} \sin^2 \tilde{h}, \quad (86)$$

$$\tilde{h} = \sqrt{\frac{d-2}{2c_d(N-1)}} \tilde{h}.$$

In the argument of \sin an arbitrary additive constant has been set to zero. We have $0 \leq \tilde{h} \leq \pi$, which ensures regularity of the metric at $\tilde{h} = 0$, $\tilde{h} = \pi$.

The linearized flow equations are

$$\lambda \delta\tilde{K} = \tilde{K}_* \delta\zeta_K - \frac{1}{2} \tilde{K}'_* \int_0^{\tilde{h}} dx \delta\zeta_J(x), \quad (87)$$

$$\lambda \delta\tilde{V} = \tilde{V}_* \delta\zeta_V. \quad (88)$$

It is convenient at this point to use \tilde{h} instead of \tilde{h} as the argument of all functions and define $\tilde{f}_*^2 \tilde{K}(\tilde{h}) = \tilde{K}(\tilde{h})$, $\delta\tilde{K}(\tilde{h}) = \delta\tilde{K}(\tilde{h})$ and $\delta\tilde{V}(\tilde{h}) = \delta\tilde{V}(\tilde{h})$. Note that a prime on a bar function differs by a factor \tilde{f}_* from a prime on a tilde function. In terms of these bar functions, $\delta\zeta_J$, $\delta\zeta_K$ and $\delta\zeta_V$ are given by

$$\tilde{f}_*^2 \delta\zeta_J = -(d-2) \left[2 \cot(\tilde{h}) \delta\tilde{V}' + \csc^4(\tilde{h}) \delta\tilde{K} - \cot(\tilde{h}) \csc^2(\tilde{h}) \delta\tilde{K}' + \frac{1}{2} \csc^2(\tilde{h}) \delta\tilde{K}'' \right], \quad (89)$$

$$\tilde{f}_*^2 \delta\zeta_K = -\frac{d-2}{N-1} \left[2(N-2) \cot(\tilde{h}) \delta\tilde{V}' + 2\delta\tilde{V}'' - \csc^4(\tilde{h}) ((N-2) \cos(2\tilde{h}) - 1) \delta\tilde{K} + (N-3) \cot(\tilde{h}) \csc^2(\tilde{h}) \delta\tilde{K}' + \frac{\csc^2(\tilde{h})}{2} \delta\tilde{K}'' \right], \quad (90)$$

$$\tilde{V}_* \delta\zeta_V = -d\delta\tilde{V} - \frac{d-2}{2} \cot(\tilde{h}) \delta\tilde{V}' - \frac{d-2}{2(N-1)} \delta\tilde{V}'' \quad (91)$$

Furthermore, in order to reduce the linearized equations to second order, we proceed as in the flat case. Reexpressing $\delta\tilde{K}$ in terms of $\Delta\tilde{K} = \delta\tilde{K}/\tilde{K}'_*$, we have

$$\tilde{f}_*^2 \delta\zeta_J = -(d-2) [2 \cot(\tilde{h}) \delta\tilde{V}' - 2\Delta\tilde{K}' + \cot(\tilde{h}) \Delta\tilde{K}''], \quad (92)$$

$$\tilde{f}_*^2 \delta\zeta_K = -\frac{d-2}{N-1} [2(N-2) \cot(\tilde{h}) \delta\tilde{V}' + 2\delta\tilde{V}'' + 2((N-2) \cot^2(\tilde{h}) - 1) \Delta\tilde{K}' + \cot(\tilde{h}) \Delta\tilde{K}''], \quad (93)$$

$$\tilde{V}_* \delta \zeta_V = -d\delta\tilde{V} - \frac{d-2}{2} \cot(\bar{h})\delta\tilde{V}' - \frac{d-2}{2(N-1)} \delta\tilde{V}''.$$
(94)

Note that there are no undifferentiated $\Delta\bar{K}$ in these expressions, and the coefficient of the undifferentiated $\delta\tilde{V}$ is constant. Therefore, choosing to work with the variables $v \equiv \delta\tilde{V}'$, $\kappa \equiv (\Delta\bar{K})'$, the first derivatives of Eqs. (87) and (88) become the following second order equations:

$$0 = v'' + (N-1) \cot(\bar{h})v' + (N-1) \left(\frac{2(d+\lambda)}{d-2} - \csc^2(\bar{h}) \right) v,$$
(95)

$$0 = 2 \tan(\bar{h})v'' + 2(\sec^2(\bar{h}) + N-2)v' - 2(N-1) \cot(\bar{h})v + \kappa'' + ((N-3) \cot(\bar{h}) - 2 \tan(\bar{h}))\kappa' + 2 \left(\frac{(N-1)(d+\lambda-2)}{d-2} - (N-2) \csc^2(\bar{h}) - \sec^2(\bar{h}) \right) \kappa.$$
(96)

As in the preceding two cases, the one-loop linearized equation for v contains only v , but the equation for κ involves also v . Notice that we need not have differentiated Eq. (88), in which case the equation would have been

$$0 = \delta\tilde{V}'' + (N-1) \cot(\bar{h})\delta\tilde{V}' + \frac{2(N-1)(d+\lambda)}{d-2} \delta\tilde{V}.$$
(97)

To define an eigenvalue problem one has to impose some homogeneous boundary conditions. Assuming regularity of the solutions on the boundaries, Eqs. (95) and (96), force the conditions $\kappa(0) = \kappa(\pi) = 0$ and $v(0) = v(\pi) = 0$. This can be seen by Taylor expanding the equations around these points.

Consider the ‘‘equatorial reflection’’ $\bar{h} \mapsto \pi - \bar{h}$. Every solution of Eqs. (95) and (96) has the property that out of the two functions v , κ , always one is even and the other is odd. To see this note that the coefficients of the equations are either even or odd. Then, defining $\bar{v}(h) = v(\pi - h)$ and $\bar{\kappa}(h) = -\kappa(\pi - h)$ the functions \bar{v} and $\bar{\kappa}$ satisfy the same equations as v and κ . Since the solutions are unique up to a (common) factor, we must have that $v = \pm\bar{v}$ and $\kappa = \pm\bar{\kappa}$, or

$$v(h) = \pm v(\pi - h), \quad \kappa(h) = \mp \kappa(\pi - h).$$
(98)

In other words, out of the pair of functions v , κ which satisfy Eqs. (117) and (118), one is even and one is odd. In the case where κ is even, $\delta K'$ does not vanish either at $\bar{h} = 0$ or at $\bar{h} = \pi$, which means that the metric is singular at least at one end point. For this reason we restrict ourselves to the case when κ is odd and v is even.

We will now describe a method to construct the solutions to Eqs. (95) and (96) analytically. To begin with, we rewrite these equations in the compact form

$$\mathfrak{L}_{\kappa\kappa}\kappa + \mathfrak{L}_{\kappa v}v = \lambda\kappa,$$
(99)

$$\mathfrak{L}_{vv}v = \lambda v,$$
(100)

where $\mathfrak{L}_{\kappa\kappa}$, $\mathfrak{L}_{\kappa v}$, \mathfrak{L}_{vv} are differential operators of second order. One can check that

$$\mathfrak{L}_{\kappa\kappa}\mathfrak{L}_{\kappa v} - \mathfrak{L}_{\kappa v}\mathfrak{L}_{vv} = 2\mathfrak{L}_{\kappa v}.$$
(101)

This tells us that if v is an eigenfunction of \mathfrak{L}_{vv} with eigenvalue λ , then $\mathfrak{L}_{\kappa v}v$ is an eigenfunction of $\mathfrak{L}_{\kappa\kappa}$ with eigenvalue $\lambda + 2$. Using this we can easily find the solution to Eq. (99). Take g to be a solution to Eq. (100). Since $\mathfrak{L}_{\kappa v}g$ is an eigenfunction of $\mathfrak{L}_{\kappa\kappa}$, we have $\mathfrak{L}_{\kappa v}g = \eta f$ for some η which has to be computed, where f is an eigenfunction of $\mathfrak{L}_{\kappa\kappa}$ with eigenvalue $\lambda + 2$. Of course η depends on the choice of normalization for f and g . We then plug the ansatz $\kappa = Cf$ into Eq. (99) to get

$$(\lambda + 2)Cf + \eta f = C\lambda f \Rightarrow 2Cf + \eta f = 0 \Rightarrow C = -\eta/2.$$
(102)

So with g given, the function $-\eta/2f$ found in this way will be a solution to Eq. (99). Thus, the general solution to Eqs. (99) and (100) will be $v = g$ and $\kappa = -\eta/2f$. Also, if there is a solution to $\mathfrak{L}_{\kappa\kappa}\kappa = \lambda\kappa$ with the same eigenvalue λ as that corresponding to g , this must also be added to $\kappa = -\eta/2f$. Of course, in practice we can solve Eq. (97) instead of Eq. (95).

For $\delta\tilde{V} = 0$ our equations reduce to $\mathfrak{L}_{\kappa\kappa}\kappa = \lambda\kappa$. A nonzero constant $\delta\tilde{V}$ instead satisfies Eq. (97) only if $\lambda = -d$.

Having found the solutions $\delta\tilde{V}$ and κ , the function $\delta\tilde{K}(\bar{h})$ is of the form

$$\delta\tilde{K}(\bar{h}) = \sin(2\bar{h}) \int_0^{\bar{h}} dx \kappa(x).$$
(103)

Here an integration constant has been put to zero using Eq. (87).

B. The spherical fixed point in the single-field approximation

In the single-field approximation we find that for $\tilde{V}(\tilde{h}) = \text{const}$ and $\tilde{K}(\tilde{h}) = \tilde{f}^2 \sin^2 \tilde{h} / \tilde{f}$ Eqs. (A11)–(A13) are

$$\zeta_J = c_d \left(1 + \frac{\zeta_K}{d+2} \right) 2 \frac{N-1}{\tilde{f}^2},$$
(104)

$$\zeta_K = c_d \left[\left(1 + \frac{\zeta_K}{d+2} \right) (N-2) + \left(1 + \frac{\zeta_J}{d+2} \right) \right] \frac{2}{\tilde{f}^2},$$
(105)

$$\zeta_V = c_d \left[\left(1 + \frac{\zeta_K}{d+2} \right) (N-1) + \left(1 + \frac{\zeta_J}{d+2} \right) \right] \frac{1}{\tilde{V}}, \quad (106)$$

so also in this case ζ_J , ζ_K and ζ_V are constant, and from the first two equations above it is clear that $\zeta_K = \zeta_J \equiv \eta$. Equations (27) and (28) then give

$$\frac{d\tilde{K}}{dt} = (\tilde{K} - (\tilde{h}/2)\tilde{K}')(\eta - d + 2), \quad (107)$$

$$\frac{d\tilde{V}}{dt} = \tilde{V}(\zeta_V - d) = c_d \left(1 + \frac{\eta}{d+2} \right) N - d\tilde{V}. \quad (108)$$

The first equation has the same structure as Eq. (81), and comparing them we have

$$2 \frac{d\tilde{f}}{dt} = (\eta - d + 2)\tilde{f}. \quad (109)$$

The value of η can be found easily from Eq. (104):

$$\eta = - \frac{2c_d(d+2)(N-1)}{2c_d(N-1) - (d+2)\tilde{f}^2}. \quad (110)$$

This reproduces the result of [21]. Using (110), Eq. (108) also becomes

$$\frac{d\tilde{V}}{dt} = - \frac{c_d(d+2)\tilde{f}^2 N}{2c_d(N-1) - (d+2)\tilde{f}^2} - d\tilde{V}. \quad (111)$$

From (109) and (111) the fixed point values of \tilde{f}^2 and \tilde{V} are found to be

$$\tilde{f}_*^2 = \frac{4c_d d(N-1)}{d^2 - 4}, \quad \tilde{V}_* = \frac{2c_d N}{d+2}. \quad (112)$$

The value of \tilde{V}_* could also be found more easily from (108) using the fact that at the fixed point $\eta = d - 2$. So, the spherical fixed point is given by

$$\tilde{V}_* = \frac{2c_d N}{d+2}, \quad \tilde{K}_* = \frac{4c_d d(N-1)}{d^2 - 4} \sin^2 \tilde{h}, \quad (113)$$

$$\tilde{h} = \sqrt{\frac{d^2 - 4}{4c_d d(N-1)}} h.$$

In the argument of sin an arbitrary additive constant has been set to zero. We have $0 \leq \tilde{h} \leq \pi$, which ensures regularity of the metric at $h = 0$, $h = \pi$.

The linearized flow equations have the same structure as Eqs. (87) and (88). We also apply the same definitions explained after Eqs. (87) and (88). In this case we have

$$\begin{aligned} f_*^2 \delta \zeta_J = & - \frac{8d^2(d-2)(N-1) \cot(\tilde{h})}{(d+2)(d(2N-1)-2)} \delta \tilde{V}' - \frac{4d(d-2)^2}{(d+2)(d(2N-1)-2)} \delta \tilde{V}'' \\ & - \frac{2d(d-2)((d(2N-1)-2) \cos(2\tilde{h}) - 3d - 4N + 10) \csc^2(\tilde{h})}{(d+2)(d(2N-1)-2)} \Delta \tilde{K}' \\ & - \frac{2d(d-2)(dN + d + 2N - 6) \cot(\tilde{h})}{(d+2)(d(2N-1)-2)} \Delta \tilde{K}'', \end{aligned} \quad (114)$$

$$\begin{aligned} f_*^2 \delta \zeta_K = & - \frac{4d(d-2)(d(2N-3)-2) \cot(\tilde{h})}{(d+2)(d(2N-1)-2)} \delta \tilde{V}' - \frac{8d^2(d-2)}{(d+2)(d(2N-1)-2)} \delta \tilde{V}'' \\ & - \frac{2d(d-2)((d(2N-1)-2) \cos(2\tilde{h}) + d(2N-7) + 2) \csc^2(\tilde{h})}{(d+2)(d(2N-1)-2)} \Delta \tilde{K}' - \frac{2d(3d-2)(d-2) \cot(\tilde{h})}{(d+2)(d(2N-1)-2)} \Delta \tilde{K}'', \end{aligned} \quad (115)$$

$$\begin{aligned} \tilde{V}_* \delta \zeta_V = & - d \delta \tilde{V} - \frac{(3d-2)(d-2) \cot(\tilde{h})}{2(d+2)} \delta \tilde{V}' - \frac{(3d-2)(d-2)}{2(d+2)(N-1)} \delta \tilde{V}'' \\ & - \frac{(d-2)^2(N \cos(2\tilde{h}) + N - 4) \csc^2(\tilde{h})}{2(d+2)(N-1)} \Delta \tilde{K}' - \frac{(d-2)^2 \cot(\tilde{h})}{(d+2)(N-1)} \Delta \tilde{K}''. \end{aligned} \quad (116)$$

Note that there are no undifferentiated $\Delta \tilde{K}$ in these expressions, and the coefficient of the undifferentiated $\delta \tilde{V}$ is constant. Therefore, choosing to work with the variables $v \equiv \delta \tilde{V}'$, $\kappa \equiv (\Delta \tilde{K})'$, the first derivatives of Eqs. (87) and (88) become the following second order equations:

$$\begin{aligned} 0 = v'' + (N-1) \cot(\tilde{h}) v' + \frac{(N-1)(2(d+2)(d+\lambda) - (3d-2)(d-2) \csc^2(\tilde{h}))}{(3d-2)(d-2)} v \\ + \frac{2(d-2) \cot(\tilde{h})}{3d-2} [\kappa'' + (N \cos(2\tilde{h}) + N - 6) \csc(2\tilde{h}) \kappa' - 2(N-2) \csc^2(\tilde{h}) \kappa], \end{aligned} \quad (117)$$

$$0 = \frac{4d \tan(\bar{h})}{3d-2} [v'' + (\sec^2(\bar{h}) + N - 2) \cot(\bar{h})v' - (N - 1) \cot(\bar{h})^2 v] + \kappa'' + ((N - 1) \cos(2\bar{h}) + N - 5) \csc(2\bar{h})\kappa' + \frac{(d(2N - 1) - 2)((d + 2)\lambda + 2d(d - 2)) - d(d - 2)(3d - 2)(2(N - 2)\csc^2(\bar{h}) + 2\sec^2(\bar{h}))}{(d - 2)d(3d - 2)} \kappa. \quad (118)$$

Again, assuming regularity of the solutions on the boundaries, Eqs. (117) and (118) force the conditions $\kappa(0) = \kappa(\pi) = 0$ and $v(0) = v(\pi) = 0$. As in the one-loop approximation, we are interested in the solutions where κ is odd and v is even under the equatorial reflection $\bar{h} \mapsto \pi - \bar{h}$.

Having found v , κ , the solutions to Eqs. (87) and (88) are of the form

$$\delta\bar{V}(\bar{h}) = \int_0^{\bar{h}} dx v(x), \quad \delta\bar{K}(\bar{h}) = \sin(2\bar{h}) \int_0^{\bar{h}} dx \kappa(x). \quad (119)$$

Here two integration constants have been put to zero, as can be deduced from Eqs. (87) and (88) and parity considerations. In addition to the above solutions with nonvanishing κ or v , there are two solutions, $\delta\bar{V} = C$, $\delta\bar{K} = 0$ with eigenvalue $\lambda = -d$ and $\delta\bar{V} = 0$, $\delta\bar{K} = C \sin(2\bar{h})$ with eigenvalue $\lambda = 0$.

In order to solve Eqs. (117) and (118) analytically, we rewrite them in the compact form

$$\mathfrak{L}_{\kappa\kappa}\kappa + \mathfrak{L}_{\kappa v}v = \lambda\kappa, \quad (120)$$

$$\mathfrak{L}_{v\kappa}\kappa + \mathfrak{L}_{vv}v = \lambda v, \quad (121)$$

where $\mathfrak{L}_{\kappa\kappa}$, $\mathfrak{L}_{\kappa v}$, $\mathfrak{L}_{v\kappa}$, \mathfrak{L}_{vv} are second order differential operators satisfying

$$\mathfrak{L}_{\kappa\kappa}\mathfrak{L}_{\kappa v} - \alpha\mathfrak{L}_{\kappa v}\mathfrak{L}_{vv} = \beta\mathfrak{L}_{\kappa v}, \quad (122)$$

$$\mathfrak{L}_{v\kappa}\mathfrak{L}_{\kappa\kappa} - \alpha\mathfrak{L}_{vv}\mathfrak{L}_{v\kappa} = \beta\mathfrak{L}_{v\kappa}, \quad (123)$$

where

$$\alpha = \frac{2d(N - 1)}{2dN - d - 2}, \quad \beta = -\frac{2d(d(d - 6)N + 2d + 4)}{(d + 2)(d(2N - 1) - 2)} \quad (124)$$

and also

$$\mathfrak{L}_{\kappa\kappa}^2 + \gamma_1\mathfrak{L}_{\kappa v}\mathfrak{L}_{v\kappa} + \gamma_2\mathfrak{L}_{\kappa\kappa} + \gamma_3 1 = 0, \quad (125)$$

$$\mathfrak{L}_{vv}^2 + \delta_1\mathfrak{L}_{v\kappa}\mathfrak{L}_{\kappa v} + \delta_2\mathfrak{L}_{vv} + \delta_3 1 = 0, \quad (126)$$

where

$$\begin{aligned} \gamma_1 &= \frac{(2 - 3d)^2(N - 1)}{4(d - 2)(-2dN + d + 2)} \\ \gamma_2 &= \frac{(d - 2)d(5dN - 4d + 2N - 8)}{(d + 2)(d(2N - 1) - 2)} \\ \gamma_3 &= \frac{2(d - 2)^2d^2(N - 2)}{(d + 2)(d(2N - 1) - 2)}, \\ \delta_1 &= -\frac{(2 - 3d)^2(d(2N - 1) - 2)}{16(d - 2)d^2(N - 1)} \\ \delta_2 &= -\frac{d(8 - 16N) - d^2(N - 4) + 4N}{2(d + 2)(N - 1)} \\ \delta_3 &= -\frac{d(N(d(d - 12) + 4) + 2d(d + 2))}{2(d + 2)(N - 1)}. \end{aligned} \quad (127)$$

Now suppose that f , g are eigenfunctions of $\mathfrak{L}_{\kappa\kappa}$, \mathfrak{L}_{vv} with eigenvalues σ , σ' :

$$\mathfrak{L}_{\kappa\kappa}f = \sigma f, \quad \mathfrak{L}_{vv}g = \sigma'g. \quad (128)$$

Then, using the identities (122) we find that $\mathfrak{L}_{v\kappa}f$ is an eigenvector of \mathfrak{L}_{vv} with eigenvalue $(\sigma - \beta)/\alpha$ and $\mathfrak{L}_{\kappa v}g$ is an eigenvector of $\mathfrak{L}_{\kappa\kappa}$ with eigenvalue $\alpha\sigma' + \beta$,

$$\mathfrak{L}_{vv}(\mathfrak{L}_{v\kappa}f) = \frac{\sigma - \beta}{\alpha}\mathfrak{L}_{v\kappa}f, \quad (129)$$

$$\mathfrak{L}_{\kappa\kappa}(\mathfrak{L}_{\kappa v}g) = (\alpha\sigma' + \beta)\mathfrak{L}_{\kappa v}g. \quad (130)$$

Assuming nondegeneracy of the eigenvalues of $\mathfrak{L}_{\kappa\kappa}$, \mathfrak{L}_{vv} , this means that given f , there is a g such that

$$\mathfrak{L}_{v\kappa}f = \eta g, \quad \mathfrak{L}_{\kappa v}g = \eta'f \quad (131)$$

for some η , η' . Notice that the transformations

$$\sigma \rightarrow \frac{\sigma - \beta}{\alpha}, \quad \sigma' \rightarrow \alpha\sigma' + \beta \quad (132)$$

are the inverse of each other; this means that there is a one-to-one correspondence between the eigenvalues of $\mathfrak{L}_{\kappa\kappa}$ and \mathfrak{L}_{vv} given by $\sigma \leftrightarrow \frac{\sigma - \beta}{\alpha}$. From now on we take f , g to be the eigenvectors which correspond to each other, i.e. the eigenvectors corresponding to the eigenvalues σ and $\frac{\sigma - \beta}{\alpha}$, respectively.

Since we have not fixed the normalization of f , g we should not expect to be able to find the values of both η and η' because, for example, making the redefinition $f \rightarrow \tau f$ leads to $\eta \rightarrow \tau^{-1}\eta$ and $\eta' \rightarrow \tau\eta'$. The product $\eta\eta'$, however, is fixed, and in fact it is an eigenvalue of $\mathfrak{L}_{v\kappa}\mathfrak{L}_{\kappa v}$ and $\mathfrak{L}_{\kappa v}\mathfrak{L}_{v\kappa}$. To find it we exploit one of the identities (125) and (126). For example, acting on f by the first equation leads to

$$\sigma^2 + \gamma_1\eta\eta' + \gamma_2\sigma + \gamma_3 = 0. \quad (133)$$

As anticipated, η and η' appear only as $\eta\eta'$. Solving this equation we find

$$\eta\eta' = \frac{4(d-2)((d+2)\sigma + 2(d-2)d)(\sigma(d(2N-1)-2) + (d-2)d(N-2))}{(2-3d)^2(d+2)(N-1)}. \quad (134)$$

Acting on g by Eq. (126) will lead to the same result. Let us now plug our two eigenfunctions accompanied by unknown factors $c_1 f$ and $c_2 g$ into Eqs. (120) and (121). Doing this we get the linear equation

$$\begin{pmatrix} \sigma & \eta' \\ \eta & \frac{\sigma-\beta}{\alpha} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \lambda \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad (135)$$

whose solution gives us two eigenvalues λ^\pm and the corresponding eigenvectors $c^\pm = (c_1^\pm, c_2^\pm)^T$. In this way we find two solutions to Eqs. (120) and (121): $\kappa = c_1^\pm f$, $\nu = c_2^\pm g$, corresponding to $\lambda = \lambda^\pm$.

It is possible to find explicit expressions in terms of d , N and σ for the eigenvalues λ^\pm by solving the characteristic equation of the matrix in Eq. (135). This is because it is only the product $\eta\eta'$ which appears in the characteristic equation. The result in terms of α , β and $\eta\eta'$ (which all depend on d , N) and σ is

$$\lambda^\pm = \frac{\sigma(\alpha+1) - \beta \pm \sqrt{4\alpha^2\eta\eta' + ((\alpha-1)\sigma + \beta)^2}}{2\alpha}. \quad (136)$$

The eigenvectors are

$$\begin{aligned} c_1^\pm &= \frac{\sigma(\alpha-1) + \beta \pm \sqrt{4\alpha^2\eta\eta' + ((\alpha-1)\sigma + \beta)^2}}{2\alpha} \\ c_2^\pm &= \eta. \end{aligned} \quad (137)$$

We see in this case that there is a dependence on η . After specifying d , N , one can compute η from the formula $\mathcal{L}_{\kappa\nu}g = \eta f$. Having found η one can follow the method

described above to find analytic solutions for the eigenfunctions. Analytic formulas for the eigenperturbations in $d = 4$ and $N = 4$ are reported in Sec. VI.

VI. THE CASE $d = 4$, $N = 4$

Up to this point all formulas hold in arbitrary dimension d and for arbitrary N . As an example we now give the scaling dimensions (eigenvalues of the stability matrix) and operators (eigenvectors of the stability matrix) in the special case $d = 4$ and $N = 4$, which is directly relevant for standard model physics. With all three topologies there is a relevant perturbation with eigenvalue $\lambda = -4$ and eigenfunctions $\delta\tilde{K} = 0$, $\delta\tilde{V} = 1$. This is just a change of the vacuum energy and is only important in a gravitational context. In fact, we have seen that it is convenient and customary to study the derivative of the potential, and constants can be neglected. We will not discuss this mode further in the following.

We begin by considering the Gaussian fixed point. There is a sequence of perturbations with $\delta\tilde{K} = 0$ with eigenvalues $-2, 0, 2, 4, \dots$; these are just the canonical dimensions of the couplings that multiply the operators ϕ^{2n} , $n = 1, 2, 3, 4, \dots$. Another series of perturbations with nonzero $\delta\tilde{K}$ have eigenvalues $0, 2, 4, \dots$, corresponding to the canonical dimensions of the couplings that multiply the operators $(\partial\phi)^2\phi^{2n}$, $n = 0, 1, 2, \dots$. For positive eigenvalues there is therefore a double degeneracy. The eigenfunctions are not simply given by these field monomials: They correspond to certain linear combinations. In the single-field approximation the first few eigenfunctions are

$$\begin{aligned} \lambda = -2 & \quad \delta\tilde{K} = 0 \quad \delta\tilde{V} = 1 - 8\pi^2\tilde{h}^2 \\ \lambda = 0 & \quad \delta\tilde{K} = 0 \quad \delta\tilde{V} = 1 - 16\pi^2\tilde{h}^2 + \frac{128\pi^4}{3}\tilde{h}^4 \\ \lambda = 2 & \quad \delta\tilde{K} = \frac{2}{3}\tilde{h}^4 C_2 \quad \delta\tilde{V} = -\frac{1}{768\pi^4}C_2 + C_1\left(1 - 24\pi^2\tilde{h}^2 + 128\pi^4\tilde{h}^4 - \frac{512\pi^6}{3}\tilde{h}^6\right) \\ \lambda = 4 & \quad \delta\tilde{K} = C_2\left(\frac{2}{3}\tilde{h}^4 - \frac{32\pi^2}{15}\tilde{h}^6\right) \\ & \quad \delta\tilde{V} = C_2\left(-\frac{1}{768\pi^4} + \frac{1}{96\pi^2}\tilde{h}^2\right) + C_1\left(1 - 32\pi^2\tilde{h}^2 + 256\pi^4\tilde{h}^4 - \frac{2048\pi^6}{3}\tilde{h}^6 + \frac{8192\pi^8}{15}\tilde{h}^8\right) \\ \lambda = 6 & \quad \delta\tilde{K} = C_2\left(\frac{2}{3}\tilde{h}^4 - \frac{64\pi^2}{15}\tilde{h}^6 + \frac{128\pi^4}{21}\tilde{h}^8\right) \\ & \quad \delta\tilde{V} = C_2\left(-\frac{1}{768\pi^4} + \frac{1}{48\pi^2}\tilde{h}^2 - \frac{1}{18}\tilde{h}^4\right) \\ & \quad + C_1\left(1 - 40\pi^2\tilde{h}^2 + \frac{1280\pi^4}{3}\tilde{h}^4 - \frac{5120\pi^6}{3}\tilde{h}^6 + \frac{8192\pi^8}{3}\tilde{h}^8 - \frac{65536\pi^{10}}{45}\tilde{h}^{10}\right). \end{aligned}$$

Here C_1 and C_2 are arbitrary coefficients; there are degenerate eigenfunctions corresponding to taking $C_1 = 0, C_2 \neq 0$ or $C_1 \neq 0, C_2 = 0$. In the one-loop approximation the equations for $\delta\tilde{K}$ and $\delta\tilde{V}$ are independent and the eigenfunctions of type $\delta\tilde{V}$ can be obtained from the preceding formulas by setting $\tilde{C}_2 = 0$ (in other words, they are the coefficients of C_1).

In the case of the cylindrical fixed point in the one-loop approximation, the equations for $\delta\tilde{K}$ and $\delta\tilde{V}$ are decoupled, so we have independent eigenfunctions for $\delta\tilde{K}$ and $\delta\tilde{V}$. The first few eigenvalues and eigenfunctions are

$$\begin{aligned} \lambda = -2 & \quad \delta\tilde{K} = 1 & \quad \delta\tilde{V} = 1 - 32\pi^2\tilde{h}^2 \\ \lambda = 0 & \quad \delta\tilde{K} = 1 - 32\pi^2\tilde{h}^2 & \quad \delta\tilde{V} = 1 - 64\pi^2\tilde{h}^2 + \frac{1024\pi^4}{3}\tilde{h}^4 \\ \lambda = 2 & \quad \delta\tilde{K} = 1 - 64\pi^2\tilde{h}^2 + \frac{1024\pi^4}{3}\tilde{h}^4 & \quad \delta\tilde{V} = 1 - 96\pi^2\tilde{h}^2 + 1024\pi^4\tilde{h}^4 - \frac{32768\pi^6}{15}\tilde{h}^6 \\ \lambda = 4 & \quad \delta\tilde{K} = 1 - 96\pi^2\tilde{h}^2 + 1024\pi^4\tilde{h}^4 - \frac{32768\pi^6}{15}\tilde{h}^6 \\ & \quad \delta\tilde{V} = 1 - 128\pi^2\tilde{h}^2 + 2048\pi^4\tilde{h}^4 - \frac{131072\pi^6}{15}\tilde{h}^6 + \frac{1048576\pi^6}{105}\tilde{h}^8. \end{aligned}$$

In this case we have not written the arbitrary coefficients C_1 and C_2 : One can take arbitrary linear combinations of the degenerate $\delta\tilde{K}$ and $\delta\tilde{V}$. Note that also in this case the eigenvalues for the potential are just the canonical dimensions of the operators with the highest power of h minus four, while those for K are the dimension of the highest power of h minus two (accounting for the extra two derivatives of the Goldstone bosons).

In the case of the cylindrical fixed point in the single-field approximation there are again two sequences of eigenperturbations, but they are no longer degenerate. Those with $\delta\tilde{K} = 0$ and $\delta\tilde{V} \neq 0$ are the same as in the one-loop approximation, and also have the same eigenvalues. This is because the field h is just a free field decoupled from the Goldstone bosons. The eigenfunctions with $\delta\tilde{K} \neq 0$ now must also have $\delta\tilde{V} \neq 0$ and non-integer eigenvalues that differ by multiples of two. The eigenfunctions listed below have been obtained by solving the inhomogeneous equation with the method described in Sec. IV B:

$$\begin{aligned} \lambda = -\frac{8}{3} & \quad \delta\tilde{K} = 1 & \quad \delta\tilde{V} = -\frac{3}{8} \\ \lambda = -\frac{2}{3} & \quad \delta\tilde{K} = 1 - 24\pi^2\tilde{h}^2 & \quad \delta\tilde{V} = -\frac{3}{32} + 9\pi^2\tilde{h}^2 \\ \lambda = \frac{4}{3} & \quad \delta\tilde{K} = 1 - 48\pi^2\tilde{h}^2 + 192\pi^4\tilde{h}^4 & \quad \delta\tilde{V} = \frac{69}{512} + \frac{9\pi^2}{2}\tilde{h}^2 - 72\pi^4\tilde{h}^4 \\ \lambda = \frac{10}{3} & \quad \delta\tilde{K} = 1 - 72\pi^2\tilde{h}^2 + 576\pi^4\tilde{h}^4 - \frac{4608\pi^6}{5}\tilde{h}^6 & \quad \delta\tilde{V} = \frac{7239}{22528} - \frac{621\pi^2}{64}\tilde{h}^2 - 54\pi^4\tilde{h}^4 + \frac{1728\pi^6}{5}\tilde{h}^6. \end{aligned}$$

The eigenfunction with the most negative eigenvalue (-2 at one loop and $-8/3$ in the single-field approximation) have already been mentioned in [21], as the slope of the beta function of the sigma model coupling.

Finally, let us come to the spherical fixed point. In the one-loop approximation we proceed as explained in Sec. VA. There will be a series of solutions with $\delta\tilde{V} = 0$ and κ satisfying $\mathcal{L}_{\kappa\kappa}\kappa = \lambda\kappa$. The (odd) eigenfunctions f_i and corresponding eigenvalues σ_i which satisfy this equation are

$$f_i = \cot(\bar{h}) \csc(\bar{h}) {}_2F_1(i + 1/2, -i, 5/2, \cos^2(\bar{h})) \quad \sigma_i = \frac{2}{3}(2i^2 + i - 4), \quad i = 2, 3, \dots \quad (138)$$

where ${}_2F_1(a, b, c, z)$ is the hypergeometric function. For $\delta\tilde{V} \neq 0$ the solution of (97) is

$$g_i = \csc(h) P_{2i}^1(\cos(h)) \quad \sigma'_i = \frac{2}{3}(2i^2 + i - 7), \quad i = 1, 2, \dots \quad (139)$$

where $P_n^m(x)$ is the associated Legendre polynomial. Now, given g_i , the function $-\frac{1}{2}\mathcal{L}_{\kappa\nu}g'_i = -\frac{\eta}{2}f_i$ will satisfy Eq. (96). Furthermore, there is no solution to Eq. (96) with $\nu = 0$ which has the same eigenvalue as that of g_i . So $-\frac{1}{2}\mathcal{L}_{\kappa\nu}g'_i = -\frac{\eta}{2}f_i$ will be the unique solution to Eq. (96). The first few eigenvectors and their corresponding eigenvalues are

$$\lambda_1 = -\frac{8}{3} \approx -2.667, \quad \delta\bar{K} = 0; \quad \delta\bar{V} = -3 \cos(\bar{h})$$

$$\lambda_2 = 2, \quad \delta\bar{K} = -140 \sin^4(\bar{h}) \cos(\bar{h}); \quad \delta\bar{V} = -\frac{5}{4} \cos(\bar{h})(1 + 7 \cos(2\bar{h}))$$

$$\lambda_3 = 4, \quad \delta\bar{K} = \frac{2}{3} \sin^4(\bar{h}) \cos(\bar{h}); \quad \delta\bar{V} = 0$$

$$\lambda_4 = \frac{28}{3} \approx 9.33, \quad \delta\bar{K} = -84 \sin^4(\bar{h}) \cos(\bar{h})(9 + 11 \cos(2\bar{h})) \quad \delta\bar{V} = -\frac{21}{128} (50 \cos(\bar{h}) + 45 \cos(3\bar{h}) + 33 \cos(5\bar{h}))$$

$$\lambda_5 = \frac{34}{3} \approx 11.33, \quad \delta\bar{K} = -\frac{1}{25} \sin^4(\bar{h}) \cos(\bar{h})(9 + 11 \cos(2\bar{h})); \quad \delta\bar{V} = 0$$

$$\lambda_6 = \frac{58}{3} \approx 19.33, \quad \delta\bar{K} = -\frac{33}{16} \sin^4(\bar{h}) \cos(\bar{h})(1581 + 1924 \cos(2\bar{h}) + 975 \cos(4\bar{h}))$$

$$\delta\bar{V} = -\frac{9}{512} \cos(\bar{h})(178 + 869 \cos(2\bar{h}) + 286 \cos(4\bar{h}) + 715 \cos(6\bar{h})).$$

In the single-field approximation, the solutions of Eqs. (128) are

$$f_i = \cot(\bar{h}) \csc(\bar{h}) {}_2F_1(i + 1/2, -i, 5/2, \cos^2(\bar{h})) \quad \sigma_i = \frac{8}{39} (10i^2 + 5i - 18), \quad (140)$$

$$g_i = \csc(\bar{h}) P_{2i}^2(\cos(\bar{h})) \quad \sigma'_i = \frac{2}{9} (10i^2 + 5i - 23), \quad (141)$$

With these solutions one constructs two series of eigenvectors ($\delta\bar{V}_i^\pm$, $\delta\bar{K}_i^\pm$) and eigenvalues λ_i^\pm , whose first few members are listed below:

$$\lambda_1^- = -\frac{16}{9} \approx -1.778;$$

$$\delta\bar{K}_1^- = 0, \quad \delta\bar{V}_1^- = \frac{2}{3} \cos \bar{h}$$

$$\lambda_2^\pm = \frac{1}{39} (245 \pm \sqrt{70009}) \approx (-0.502, 13.07)$$

$$\delta\bar{K}_2^\pm = \frac{2(11 \pm \sqrt{70009})}{117} \sin^4(\bar{h}) \cos(\bar{h})$$

$$\delta\bar{V}_2^\pm = -\frac{1}{18} (9 \cos(\bar{h}) + 7 \cos(3\bar{h}))$$

$$\lambda_3^\pm = \frac{2}{117} (1055 \pm \sqrt{898681}) \approx (1.829, 34.24)$$

$$\delta\bar{K}_3^\pm = \frac{2(11 \mp \sqrt{898681})}{2925} \sin^4(\bar{h}) \cos(\bar{h})(9 + 11 \cos(2\bar{h}))$$

$$\delta\bar{V}_3^\pm = \frac{1}{120} (50 \cos(\bar{h}) + 45 \cos(3\bar{h}) + 33 \cos(5\bar{h}))$$

$$\lambda_4^\pm = \frac{1}{117} (3985 \pm \sqrt{11540929}) \approx (5.024, 69.10)$$

$$\delta\bar{K}_4^\pm = -\frac{(97 \mp \sqrt{11540929})}{573300} \sin^4(\bar{h}) \cos(\bar{h})(1581 + 1924 \cos(2\bar{h}) + 975 \cos(4\bar{h}))$$

$$\delta\bar{V}_4^\pm = -\frac{1}{3360} (1225 \cos(\bar{h}) + 11(105 \cos(3\bar{h}) + 91 \cos(5\bar{h}) + 65 \cos(7\bar{h})))$$

$$\lambda_5^\pm = \frac{8}{39} (265 \pm 4\sqrt{3046}) \approx (9.074, 99.64)$$

$$\delta\bar{K}_5^\pm = \frac{2(2 \mp \sqrt{3046})}{51597} \sin^4(\bar{h}) \cos(\bar{h})(8859 \cos(2\bar{h}) + 4794 \cos(4\bar{h}) + 2261 \cos(6\bar{h}) + 5590)$$

$$\delta\bar{V}_5^\pm = \frac{1}{24192} (7938 \cos(\bar{h}) + 13(588 \cos(3\bar{h}) + 540 \cos(5\bar{h}) + 459 \cos(7\bar{h}) + 323 \cos(9\bar{h})))$$

$$\lambda_6^\pm = \frac{1}{117} (9235 \pm \sqrt{57751849}) \approx (13.98, 143.88)$$

$$\delta\bar{K}_6^\pm = -\frac{307 \mp \sqrt{57751849}}{1997835840} \sin^4 \bar{h} \cos \bar{h} (4175357 + 6595320 \cos(2\bar{h}) + 4655076 \cos(4\bar{h}) + 2405704 \cos(6\bar{h}) + 1092063 \cos(8\bar{h}))$$

$$\delta\bar{V}_6^\pm = -\frac{1}{354816} (106772 \cos \bar{h} + 103950 \cos(3\bar{h}) + 17(5775 \cos(5\bar{h}) + 5225 \cos(7\bar{h}) + 4389 \cos(9\bar{h}) + 3059 \cos(11\bar{h}))).$$

The eigenvalues of the (−) series are systematically smaller than those of the (+) series. Thus, the lowest eigenvalues are as follows:² $\lambda_1^- = -1.78$, $\lambda_2^- = -0.50$, $\lambda_3^- = 1.83$, $\lambda_4^- = 5.03$, $\lambda_5^- = 9.07$, $\lambda_2^+ = 13.07$, $\lambda_6^- = 13.98$

Finally, let us ask what these results imply for the coupling of the Higgs field h to the Goldstone bosons χ^α . Following the notation of [29] for the general parametrization of Higgs couplings at low energies, we Taylor expand the function K which appears in (3) in powers of the shifted field $H = h - \langle h \rangle$, and the Goldstone boson Lagrangian reads

$$\frac{v^2}{2} \left(1 + 2a \frac{H}{v} + b \frac{H^2}{v^2} + \dots \right) g_{\alpha\beta} \partial_\mu \chi^\alpha \partial^\mu \chi^\beta. \quad (142)$$

Here $v = 246$ GeV is the weak scale and $\langle h \rangle$ is defined as the position of the minimum of V .

The Gaussian fixed point is the basis of the perturbative treatment of linear scalar theory. It describes N free massless scalar fields. In particular, there is no Higgs VEV at the fixed point. The eigenvalues of the linearized flow are just the canonical dimensions of the operators appearing in the eigenperturbation. There are only two nonirrelevant perturbations: the mass and the quartic self-interaction. A combination of these perturbations with suitable coefficients generates a Higgs VEV, and the coefficients a and b in (142) are both equal to 1, while all higher order couplings vanish.

At the cylindrical fixed point the Higgs and the Goldstone bosons are decoupled. At one loop there are then nonirrelevant perturbations, with $\delta\tilde{K}$ a polynomial up to second order in h and $\delta\tilde{V}$ a polynomial up to fourth order. By a suitable choice of coefficients one can generate a VEV for h . At the same time, there are also perturbations describing quartic Higgs-Goldstone interactions which have a similar form as in the standard model, but their strength is unrelated to the VEV. Specifically, we parametrize the perturbations as

$$K(\rho) = k^2(\tilde{K}_* + \epsilon_i \delta\tilde{K}_i(\tilde{h})), \quad (143)$$

$$V(\rho) = k^4(\tilde{V}_* + \epsilon'_i \delta\tilde{V}_i(\tilde{h})), \quad (144)$$

where $\epsilon_i = c_i e^{i\lambda_i}$ and $\epsilon'_i = c'_i e^{i\lambda_i}$, with c_i, c'_i being scale-independent quantities and $i = 1, 2$. Here $\delta\tilde{K}_1(\tilde{h}), \delta\tilde{V}_1(\tilde{h})$ are the two relevant and $\delta\tilde{K}_2(\tilde{h}), \delta\tilde{V}_2(\tilde{h})$ the two marginal eigenperturbations. Doing this the Higgs VEV will be given by

$$\langle \tilde{h} \rangle^2 = \frac{3(\epsilon'_1 + 2\epsilon'_2)}{64\pi^2 \epsilon'_2} \quad (145)$$

and the couplings by

²For the eigenvalue $\lambda_1^+ = -8/13 = -0.615385$ the eigenfunction κ is singular at $h = 0, \pi$ and $\delta\tilde{K}$ tends to a nonzero constant at the end points, so in this case the metric will be singular at both ends.

$$a = -\frac{4\sqrt{6}\pi\sqrt{\epsilon'_1 + 2\epsilon'_2}\epsilon_2}{\sqrt{2\epsilon'_2(\epsilon_1 - 2\epsilon_2 + \tilde{K}_*) - 3\epsilon_2\epsilon'_1}}; \quad (146)$$

$$b = -32\pi^2\epsilon_2.$$

One can also compute the mass and the weak scale to get

$$\frac{m^2}{k^2} = 128\pi^2(\epsilon'_1 + 2\epsilon'_2);$$

$$\frac{v^2}{k^2} = \epsilon_1 - \epsilon_2 \left(2 + \frac{3\epsilon'_1}{2\epsilon'_2} \right) + \tilde{K}_*. \quad (147)$$

Solving for ϵ_1 and ϵ_2 to set $a = b = 1$, we find the following expression for the weak scale:

$$\frac{v^2}{k^2} = \frac{3(\epsilon'_1 + 2\epsilon'_2)}{64\pi^2 \epsilon'_2}, \quad (148)$$

so that the ratio of the mass squared to the weak scale squared is

$$\frac{m^2}{v^2} = \frac{8192\pi^4 \epsilon'_2}{3}. \quad (149)$$

In the single-field approximation instead, we parametrize the perturbations as

$$K(\rho) = k^2(\tilde{K}_* + \epsilon_i \delta\tilde{K}_i(\tilde{h})), \quad (150)$$

$$V(\rho) = k^4(\tilde{V}_* + \epsilon_i \delta\tilde{V}_i(\tilde{h})), \quad (151)$$

with ϵ_i defined as before and $i = 1, 2, 3, 4$. Here $(\delta\tilde{K}_i(\tilde{h}), \delta\tilde{V}_i(\tilde{h}))$ are the relevant or marginal eigenperturbations in increasing order of eigenvalues. In terms of ϵ_i the Higgs VEV is given by

$$\langle \tilde{h} \rangle^2 = \frac{3(32\epsilon_2 - 9\epsilon_3 + 64\epsilon_4)}{2048\pi^2 \epsilon_4} \quad (152)$$

and the couplings are

$$a = -\frac{6\sqrt{6}\pi\sqrt{32\epsilon_2 - 9\epsilon_3 + 64\epsilon_4}\epsilon_3}{81\epsilon_3^2 + 256\epsilon_4(\epsilon_1 + \tilde{K}_*) - 288\epsilon_2\epsilon_3 - 320\epsilon_3\epsilon_4};$$

$$b = -24\pi^2\epsilon_3. \quad (153)$$

The mass and the weak scale in this case will be

$$\frac{m^2}{k^2} = 4\pi^2(32\epsilon_2 - 9\epsilon_3 + 64\epsilon_4);$$

$$\frac{v^2}{k^2} = \epsilon_1 + \frac{\epsilon_3(81\epsilon_3 - 320\epsilon_4 - 288\epsilon_2)}{256\epsilon_4} + \tilde{K}_*. \quad (154)$$

As in the previous case we solve for ϵ_1 and ϵ_3 to set $a = b = 1$. Doing this we find the following expressions for the mass and weak scale:

$$\frac{m^2}{k^2} = \frac{3 + 256(\epsilon_2 + 2\epsilon_4)\pi^2}{2};$$

$$\frac{v^2}{k^2} = \frac{9 + 768(\epsilon_2 + 2\epsilon_4)\pi^2}{16384\pi^4 \epsilon_4}. \quad (155)$$

Their ratio reads

$$\frac{m^2}{v^2} = \frac{8192\pi^4\epsilon_4}{3}. \quad (156)$$

We see that by adjusting the free parameters one can mimic the results of the standard model.

For the spherical fixed point in the single-field approximation there are two relevant eigenperturbations, one of which has $\delta\tilde{K} \neq 0$. This will lead to modifications of Higgs couplings to Goldstone bosons. A linear combination of the two eigenperturbations in the potential $r\delta\tilde{V}_1^-(\tilde{h}) + \delta\tilde{V}_2^-(\tilde{h})$, where $r = r_0 \exp(t(\lambda_1^- - \lambda_2^-))$, will give rise to a nonvanishing VEV, $\langle\tilde{h}\rangle \equiv \theta$, given by

$$\sin^2\theta = \frac{6-r}{7}. \quad (157)$$

Using this notation we find the couplings to be

$$a = \cos\theta - \epsilon\sin^2\theta(3 - 4\sin^2\theta), \quad (158)$$

$$b = \cos(2\theta) - \epsilon\sin^2\theta\cos\theta(12 - 25\sin^2\theta), \quad (159)$$

where $\epsilon = \epsilon_0 \exp(t\lambda_2^-)$ is some small parameter. The parameter r can always be tuned to give a small enough θ . On the other hand, the mass and the weak scale will be given by

$$m^2 = -k^2\tilde{f}_*^2\frac{28}{3}\sin^2\theta\cos\theta\epsilon, \quad (160)$$

$$v^2 = k^2\tilde{f}_*^2(\sin^2\theta - 2\epsilon\sin^4\theta\cos\theta), \quad (161)$$

with the ratio being

$$\frac{m^2}{v^2} = -\frac{28}{3}\cos\theta\epsilon. \quad (162)$$

Very roughly, the values $r \approx 6$ and $\epsilon \approx -0.028$ will reproduce the correct ratio $m^2/v^2 \approx 0.26$. So also in this case we can get arbitrarily close to the result of the standard model.

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APPENDIX: CALCULATION OF BETA FUNCTIONS

We give some details of the calculation of the two terms on the r.h.s. of (18), in the single-field approximation (for the adiabatic approximation one has to neglect the ζ 's in the r.h.s.). The (infinite) matrices whose trace we must evaluate are

$$(\mathcal{P}^{-1}\dot{\mathcal{R}})_j^i = \frac{\dot{R}_k + \zeta_K R_k + R_k' \dot{z}}{P + V'K'/2KJ} (\delta_j^i - \delta_0^i \delta_j^0) + \frac{\dot{R}_k + \zeta_J R_k + R_k' \dot{z}}{P + V''/J - V'J'/2J^2} \delta_0^i \delta_j^0 \quad (A1)$$

and

$$(M\mathcal{P}^{-1}\dot{\mathcal{R}}\mathcal{P}^{-1})_i^j = \frac{\dot{R}_k + \zeta_K R_k + R_k' \dot{z}}{K(P + V'K'/2KJ)^2} M_{im} g^{mj} + \frac{\dot{R}_k + \zeta_J R_k + R_k' \dot{z}}{J(P + V''/J - V'J'/2J^2)^2} M_{im} \delta_0^m \delta_0^j. \quad (A2)$$

These can be evaluated using the following general formulas for the trace of a function of the Laplacian:

$$\text{Tr}[W(\Delta)] = \sum_{\lambda} W(\lambda) = \frac{1}{(4\pi)^{\frac{d}{2}}} \sum_{n=0}^{\infty} B_{2n}(\Delta) Q_{\frac{d}{2}-n}(W), \quad (A3)$$

$$\text{Tr}[W(\Delta)\partial_t\Delta] = -\frac{1}{(4\pi)^{\frac{d}{2}}} \sum_{n=0}^{\infty} \partial_t B_{2n}(\Delta) Q_{\frac{d}{2}-n+1}(W), \quad (A4)$$

where B_{2n} are the coefficients appearing in the heat kernel expansion,

$$\text{Tr}e^{-s\Delta} = \frac{1}{(4\pi)^{\frac{d}{2}}} \sum_{n=0}^{\infty} B_{2n}(\Delta) s^{-\frac{d}{2}+n}, \quad (A5)$$

and the Q functionals are given (for m a non-negative integer) by $Q_m(W) = \frac{1}{\Gamma(m)} \int_0^{\infty} dz z^{m-1} W(z)$. For convenience we choose the optimized cutoff $R_k(z) = (k^2 - z)\theta(k^2 - z)$ [16], for which one can easily calculate

$$Q_n\left(\frac{\dot{R}_k + \eta R_k}{(P_k + q)^l}\right) = \frac{k^{2(n-l+1)}}{\Gamma(n+1)} \frac{2 + \frac{\eta}{n+1}}{(1 + \tilde{q})^l}. \quad (A6)$$

With the aid of these formulas one gets

$$\text{Tr}_0[\mathcal{P}^{-1}\dot{\mathcal{R}}] = 2c_d k^d \int d^d x \left[\left(1 + \frac{\zeta_K}{d+2}\right) \frac{N-1}{1 + \frac{V'K'}{2KJk^2}} + \left(1 + \frac{\zeta_J}{d+2}\right) \frac{1}{1 + \frac{V''}{Jk^2} - \frac{V'J'}{2J^2k^2}} \right] \quad (\text{A7})$$

and

$$\begin{aligned} \text{Tr}_0[\mathcal{P}^{-1}M\mathcal{P}^{-1}\dot{\mathcal{R}}] &= 2c_d k^{d-2} \int d^d x \left[\left(1 + \frac{\zeta_K}{d+2}\right) \frac{N-1}{4} \frac{KK'J' + K'^2J - 2KK''J}{K^2J(1 + \frac{V'K'}{2KJk^2})^2} \right] \partial^\mu \varphi^0 \partial_\mu \varphi^0 \\ &+ 2c_d k^{d-2} \int d^d x \left[\left(1 + \frac{\zeta_K}{d+2}\right) \frac{N-2}{4} \frac{4KJ - K'^2}{KJ(1 + \frac{V'K'}{2KJk^2})^2} \right] \partial^\mu \varphi^\alpha \partial_\mu \varphi^\beta g_{\alpha\beta} \\ &+ 2c_d k^{d-2} \int d^d x \left[\left(1 + \frac{\zeta_J}{d+2}\right) \frac{KK'J' + K'^2J - 2KK''J}{4KJ^2(1 + \frac{V''}{Jk^2} - \frac{V'J'}{2J^2k^2})^2} \right] \partial^\mu \varphi^\alpha \partial_\mu \varphi^\beta g_{\alpha\beta} \end{aligned} \quad (\text{A8})$$

where $c_d = \frac{1}{(4\pi)^{d/2}\Gamma(d/2+1)}$ and by Tr_0 we mean the B_0 term in the heat kernel expansion. In the final equation we have used

$$J^{-1}M_{00} = \frac{KK'J' + K'^2J - 2KK''J}{4KJ^2} \partial^\mu \varphi^\alpha \partial_\mu \varphi^\beta g_{\alpha\beta} \quad (\text{A9})$$

and

$$K^{-1}M_{\alpha\beta}g^{\alpha\beta} = \frac{N-1}{4} \frac{KK'J' + K'^2J - 2KK''J}{K^2J} \partial^\mu \varphi^0 \partial_\mu \varphi^0 + \frac{N-2}{4} \frac{4KJ - K'^2}{KJ} \partial^\mu \varphi^\alpha \partial_\mu \varphi^\beta g_{\alpha\beta}. \quad (\text{A10})$$

Collecting these results, one obtains

$$\zeta_J = \frac{2c_d k^{d-2}}{J} \left(1 + \frac{\zeta_K}{d+2}\right) \frac{N-1}{4} \frac{KK'J' + K'^2J - 2KK''J}{K^2J(1 + V'K'/2KJk^2)^2}, \quad (\text{A11})$$

$$\zeta_K = \frac{2c_d k^{d-2}}{K} \left[\left(1 + \frac{\zeta_K}{d+2}\right) \frac{N-2}{4} \frac{4KJ - K'^2}{KJ(1 + V'K'/2KJk^2)^2} + \left(1 + \frac{\zeta_J}{d+2}\right) \frac{KK'J' + K'^2J - 2KK''J}{4KJ^2(1 + V''/Jk^2 - V'J'/2J^2k^2)^2} \right], \quad (\text{A12})$$

$$\zeta_V = \frac{c_d k^d}{V} \left[\left(1 + \frac{\zeta_K}{d+2}\right) \frac{N-1}{1 + V'K'/2KJk^2} + \left(1 + \frac{\zeta_J}{d+2}\right) \frac{1}{1 + V''/Jk^2 - V'J'/2J^2k^2} \right]. \quad (\text{A13})$$

This can be rewritten in terms of dimensionless variables as in Sec. II.

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- [1] W. Buchmüller and N. Dragon, *Phys. Lett. B* **195**, 417 (1987); *Nucl. Phys.* **B321**, 207 (1989); Hung Cheng, *Phys. Rev. Lett.* **61**, 2182 (1988); M. Pawłowski and R. Raczka, *Found. Phys.* **24**, 1305 (1994); K. A. Meissner and H. Nicolai, *Phys. Lett. B* **648**, 312 (2007); **660**, 260 (2008); W. D. Goldberger, B. Grinstein, and W. Skiba, *Phys. Rev. Lett.* **100**, 111802 (2008).
- [2] D. B. Kaplan and H. Georgi, *Phys. Lett.* **136B**, 183 (1984).
- [3] K. Agashe, R. Contino, and A. Pomarol, *Nucl. Phys.* **B719**, 165 (2005).
- [4] G. Panico, M. Safari, and M. Serone, *J. High Energy Phys.* **02** (2011) 103; D. Marzocca, M. Serone, and J. Shu, *J. High Energy Phys.* **08** (2012) 013.
- [5] K. Wilson and J. Kogut, *Phys. Rep.* **12**, 75 (1974); K. Wilson, *Rev. Mod. Phys.* **47**, 773 (1975).
- [6] F. J. Wegner and A. Houghton, *Phys. Rev. A* **8**, 401 (1973).
- [7] J. Polchinski, *Nucl. Phys.* **B231**, 269 (1984).
- [8] A. Hasenfratz and P. Hasenfratz, *Nucl. Phys.* **B270**, 687 (1986).
- [9] S.-B. Liao and J. Polonyi, *Phys. Rev. D* **51**, 4474 (1995).
- [10] T. R. Morris, *Prog. Theor. Phys. Suppl.* **131**, 395 (1998); C. Bagnuls and C. Bervillier, *Phys. Rep.* **348**, 91 (2001).
- [11] J. Berges, N. Tetradis, and C. Wetterich, *Phys. Rep.* **363**, 223 (2002).
- [12] B. Delamotte, *Lect. Notes Phys.* **852**, 49 (2012).
- [13] M. Niedermaier and M. Reuter, *Living Rev. Relativity* **9**, 5 (2006); M. Niedermaier, *Classical Quantum Gravity* **24**, R171 (2007); R. Percacci, in *Approaches to Quantum Gravity: Towards a New Understanding of Space, Time and Matter*, edited by D. Oriti (Cambridge University Press, Cambridge, England, 2009); [arXiv:0709.3851](https://arxiv.org/abs/0709.3851); D. F. Litim, *Proc. Sci., QG-Ph* (2008) 024 [[arXiv:0810.3675](https://arxiv.org/abs/0810.3675)]; A. Codello, R. Percacci, and

- C. Rahmede, *Int. J. Mod. Phys. A* **23**, 143 (2008); *Ann. Phys. (Amsterdam)* **324**, 414 (2009).
- [14] C. Wetterich, *Phys. Lett. B* **301**, 90 (1993).
- [15] T.R. Morris, *Phys. Lett. B* **329**, 241 (1994); **334**, 355 (1994).
- [16] D.F. Litim, *Nucl. Phys.* **B631**, 128 (2002).
- [17] A. Codello, *J. Phys. A* **45**, 465006 (2012); A. Codello and G. D'Odorico, *Phys. Rev. Lett.* **110**, 141601 (2013).
- [18] N. Tetradis and D. F. Litim, *Nucl. Phys.* **B464**, 492 (1996).
- [19] L. Canet, B. Delamotte, D. Mouhanna, and J. Vidal, *Phys. Rev. D* **67**, 065004 (2003).
- [20] T.R. Morris and M.D. Turner, *Nucl. Phys.* **B509**, 637 (1998).
- [21] A. Codello and R. Percacci, *Phys. Lett. B* **672**, 280 (2009).
- [22] M. Fabbrichesi, R. Percacci, A. Tonero, and O. Zanusso, *Phys. Rev. D* **83**, 025016 (2011).
- [23] M. Fabbrichesi, R. Percacci, A. Tonero, and L. Vecchi, *Phys. Rev. Lett.* **107**, 021803 (2011).
- [24] F. Bazzocchi, M. Fabbrichesi, R. Percacci, A. Tonero, and L. Vecchi, *Phys. Lett. B* **705**, 388 (2011).
- [25] L. Canet, B. Delamotte, D. Mouhanna, and J. Vidal, *Phys. Rev. B* **68**, 064421 (2003).
- [26] J. Honerkamp, *Nucl. Phys.* **B36**, 130 (1972); L. Alvarez-Gaume, D. Z. Freedman, and S. Mukhi, *Ann. Phys. (N.Y.)* **134**, 85 (1981); D. G. Boulware and L. S. Brown, *Ann. Phys. (N.Y.)* **138**, 392 (1982); P. S. Howe, G. Papadopoulos, and K. S. Stelle, *Nucl. Phys.* **B296**, 26 (1988).
- [27] R. Flore, A. Wipf, and O. Zanusso, *Phys. Rev. D* **87**, 065019 (2013).
- [28] R. Percacci and O. Zanusso, *Phys. Rev. D* **81**, 065012 (2010).
- [29] R. Contino, C. Grojean, M. Moretti, F. Piccinini, and R. Rattazzi, *J. High Energy Phys.* **05** (2010) 089.