

Light-front analysis of the Casimir effect

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The Casimir force between conducting plates at rest in an inertial frame is usually computed in equal-time quantization, the natural choice for the given boundary conditions. We show that the well-known result obtained in this way can also be obtained in Dirac's light-front coordinates. This differs from a light-front analysis where the plates are at "rest" in an infinite momentum frame, rather than an inertial frame; in that case, as shown by Lenz and Steinbacher, the result does not agree with the standard result. As is usually done, the analysis is simplified by working with a scalar field and periodic boundary conditions, in place of the complexity of quantum electrodynamics. The two key ingredients are a careful implementation of the boundary conditions, following the work of Almeida *et al.* on oblique light-front coordinates, and computation of the ordinary energy density, rather than the light-front energy density. The analysis demonstrates that the physics of the effect is independent of the coordinate choice, as it must be. This is meant not as a new derivation of the Casimir effect but as a demonstration that light-front quantization is not somehow flawed in its treatment of such vacuum effects.

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I. INTRODUCTION

The Casimir effect [1,2] is the existence of a force between conducting plates due to the exclusion of vacuum modes by boundary conditions at the plates. The vacuum energy density between the plates differs from the free density and defines an effective potential energy for the plates that varies with the plate separation. The energy density is computed by summing over the allowed modes, and the variation of the effective potential yields the force. To simplify these calculations, the physical boundary conditions at the conducting surfaces can be replaced by periodic boundary conditions, and the photon field replaced by a massless scalar field.

Because the plates are at rest in an inertial frame, the natural choice of coordinates for the analysis is the standard set of equal-time coordinates, rather than Dirac's light-front coordinates [3,4]. Nevertheless, attempts at analysis in light-front quantization have been made [5,6]. In Ref. [5], the analysis considered the light-front analog of spatial periodicity, with boundary conditions periodic in $x^- \equiv t - z$ rather than z . However, this corresponds to plates moving with the speed of light, which cannot be realized experimentally, and, in any case, the calculation did not lead to a well-defined Casimir force. In Ref. [6], the coordinate choice was modified away from Dirac's light-front coordinates in such a way as to avoid the difficulties encountered by Lenz and Steinbacher. In their analysis, Almeida *et al.* arrived at suitable boundary conditions, conditions periodic in z that mix their time and space coordinates. Such a choice is also not "natural" for light-front coordinates because it mixes x^- with light-front time $x^+ \equiv t + z$.

In both of these attempts, light-front quantization appeared somehow deficient and Dirac's light-front coordinates

seemed unusable for the computation of this vacuum effect. We wish to rectify these impressions and exonerate light-front coordinates. In this paper we show that the Casimir effect can be calculated in Dirac's light-front coordinates, without any modifications, given the proper choice of boundary conditions.

A careful choice of boundary conditions is not the entire story. To complete a calculation of the Casimir effect in light-front coordinates, we must calculate the true vacuum energy,¹ not the light-front energy $p^- \equiv E - p_z$. The ordinary energy E is what determines the effective potential, which in turn determines the Casimir force. That the physics of a system is determined by E was seen in a light-front variational analysis of ϕ^4 theory by Harindranath and Vary [7] and in finite-temperature calculations by Elser and Kalloniotis [8]. In the latter case, the important point is that a partition function should be computed for contact with a heat bath at rest in an inertial frame; if the light-front energy p^- is used instead of E , the heat bath must be interpreted as moving with the speed of light, an unphysical situation.²

Our definition of light-front coordinates is illustrated in Fig. 1. In addition, we define transverse spatial coordinates $\vec{x}_\perp = (x, y)$ and the light-front momentum $\underline{p} = (p^+ \equiv E + p_z, \vec{p}_\perp)$. The scalar product of four-momentum and coordinates is then given by $p \cdot x = \frac{1}{2}(p^+x^- + p^-x^+) - \vec{p}_\perp \cdot \vec{x}_\perp$, and the mass-shell condition $p^2 = m^2$ yields $p^- = (p_\perp^2 + m^2)/p^+$. We will make some use of these relations in the following sections.

The remainder of the paper contains our analysis of the Casimir effect for plates perpendicular to the z axis, in

¹This is not an issue for oblique light-front coordinates, where the light-front energy may be used directly [6].

²For further discussion of this point, see Ref. [9].

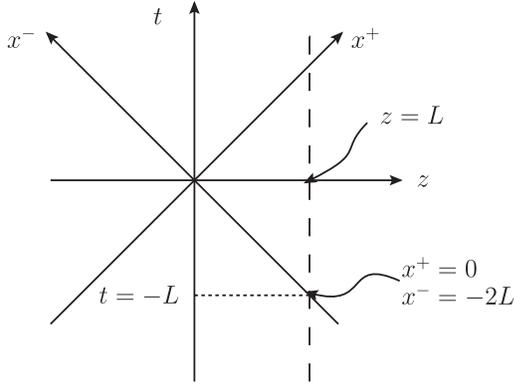


FIG. 1. Light-front coordinates for the longitudinal case. The periodicity is in z , from 0 to L . The line for $z = L$ intersects the x^- axis at $x^- \equiv t - z = -2L$.

Sec. II, and for plates perpendicular to the x axis, in Sec. III. The latter case, where periodicity in the spatial coordinate is the same for both equal-time and light-front coordinates, is considered in order to check that our approach has not destroyed the agreement between coordinate systems already obtained by Lenz and Steinbacher [5]. A summary is given in Sec. IV.

II. LONGITUDINAL CASE

We first consider plates separated in the z direction, as shown in Fig. 1. One plate is at $z = 0$, and the other at $z = L$. The standard result for the expectation value of the energy density is a sum over zero-point energies [2],

$$\langle \mathcal{H} \rangle = \frac{1}{2L} \sum_{n=-\infty}^{\infty} \int \frac{d^2 p_{\perp}}{(2\pi)^2} E_n, \quad (2.1)$$

with

$$E_n = \sqrt{p_{\perp}^2 + \left(\frac{2\pi n}{L}\right)^2}. \quad (2.2)$$

The sum is typically regulated by a heat-bath factor³ $e^{-\Lambda E_n}$. The sum then yields

$$\langle \mathcal{H} \rangle = \frac{3}{2\pi^2 \Lambda^4} - \frac{\pi^2}{90L^4}. \quad (2.3)$$

The second term provides the effective potential, independent of the regulator, and determines the Casimir force.

To simulate the Casimir effect, we impose periodic boundary conditions on a neutral massless scalar field and compute the vacuum energy density. The mode expansion for the scalar field is

³In light-front quantization, the same factor should be used, since the system should be in contact with a heat bath at rest in an inertial frame.

$$\phi = \int \frac{d\underline{p}}{\sqrt{16\pi^3 p^+}} \{a(\underline{p})e^{-ip \cdot x} + a^\dagger(\underline{p})e^{ip \cdot x}\}, \quad (2.4)$$

with the modes quantized such that

$$[a(\underline{p}), a^\dagger(\underline{p}')] = \delta(\underline{p} - \underline{p}'). \quad (2.5)$$

The periodicity imposed is $\phi(z + L) = \phi(z)$; in light-front coordinates, this is

$$\phi(x^+ + L, x^- - L, \vec{x}_{\perp}) = \phi(x^+, x^-, \vec{x}_{\perp}). \quad (2.6)$$

This implies $-p^+ L/2 + p^- L/2 = 2\pi n$ or

$$\frac{p_{\perp}^2}{p^+} - p^+ = \frac{4\pi}{L} n, \quad (2.7)$$

with n any integer between $-\infty$ and ∞ . The positive solution of this constraint is

$$p_n^+ \equiv \frac{2\pi}{L} n + \sqrt{\left(\frac{2\pi}{L} n\right)^2 + p_{\perp}^2}. \quad (2.8)$$

Then $n = -\infty$ corresponds to $p^+ = 0$, and $n = \infty$ to $p^+ = \infty$.

The mode expansion of the field is restricted to a discrete sum for the longitudinal modes. We define discrete annihilation operators

$$a_n(\vec{p}_{\perp}) = \sqrt{\left|\frac{dp^+}{dn}\right|} a(p_n^+, \vec{p}_{\perp}), \quad (2.9)$$

for which the commutation relation becomes

$$[a_n(\vec{p}_{\perp}), a_{n'}^\dagger(\vec{p}'_{\perp})] = \delta_{nn'} \delta(\vec{p}_{\perp} - \vec{p}'_{\perp}), \quad (2.10)$$

and change the integration over p^+ to a sum over n ,

$$\int dp^+ = \int \frac{dp^+}{dn} dn \rightarrow \sum_n \frac{dp^+}{dn}, \quad (2.11)$$

where $\frac{dp^+}{dn} = \frac{2\pi}{L} \frac{p_n^+}{E_n}$. Substitution then gives

$$\begin{aligned} \phi(x^+ = 0) &= \frac{1}{\sqrt{2L}} \sum_n \int \frac{d^2 p_{\perp}}{2\pi\sqrt{E_n}} \\ &\times \{a_n(\vec{p}_{\perp})e^{-ip_n^+ x^-/2 + i\vec{p}_{\perp} \cdot \vec{x}_{\perp}} \\ &+ a_n^\dagger(\vec{p}_{\perp})e^{ip_n^+ x^-/2 - i\vec{p}_{\perp} \cdot \vec{x}_{\perp}}\}, \end{aligned} \quad (2.12)$$

where the leading $\frac{1}{\sqrt{2L}}$ factor is consistent with the normalization of the discrete basis functions $e^{-ip_n^+ x^-/2 + i\vec{p}_{\perp} \cdot \vec{x}_{\perp}}$ on the interval $-2L < x^- < 0$.

For the free scalar, the light-front (LF) energy and longitudinal momentum densities are $\mathcal{H}^- = \frac{1}{2} |\vec{\partial}_{\perp} \phi|^2$ and $\mathcal{H}^+ = 2 |\partial_{\perp} \phi|^2$. Their vacuum expectation values are

$$\begin{aligned}
 \langle 0|\mathcal{H}^-|0\rangle &= \frac{1}{4L} \sum_{n,n'} \int \frac{d^2 p_\perp d^2 p'_\perp}{(2\pi)^2 \sqrt{E_n E_{n'}}} \vec{p}_\perp \cdot \vec{p}'_\perp \\
 &\quad \times \langle 0|a_n(\vec{p}_\perp) a_{n'}^\dagger(\vec{p}'_\perp)|0\rangle \\
 &= \frac{1}{4L} \sum_n \int \frac{d^2 p_\perp}{(2\pi)^2 E_n} p_\perp^2
 \end{aligned} \quad (2.13)$$

and

$$\begin{aligned}
 \langle 0|\mathcal{H}^+|0\rangle &= \frac{2}{2L} \sum_{n,n'} \int \frac{d^2 p_\perp d^2 p'_\perp}{(2\pi)^2 \sqrt{E_n E_{n'}}} \frac{p_n^+ p_{n'}^+}{4} \\
 &\quad \times \langle 0|a_n(\vec{p}_\perp) a_{n'}^\dagger(\vec{p}'_\perp)|0\rangle \\
 &= \frac{1}{4L} \sum_n \int \frac{d^2 p_\perp}{(2\pi)^2 E_n} (p_n^+)^2.
 \end{aligned} \quad (2.14)$$

These yield an energy density

$$\mathcal{E}_{\text{LF}} \equiv \frac{1}{2} (\langle 0|\mathcal{H}^-|0\rangle + \langle 0|\mathcal{H}^+|0\rangle) \quad (2.15)$$

$$= \frac{1}{8L} \sum_n \int \frac{d^2 p_\perp}{(2\pi)^2 E_n} \left(2E_n^2 + 2\frac{2\pi}{L} n E_n \right) \quad (2.16)$$

relative to light-front coordinates. The second term is zero, because it is proportional to $\sum_{n=-\infty}^{\infty} n = 0$. We then obtain

$$\mathcal{E}_{\text{LF}} = \frac{1}{4L} \sum_n \int \frac{d^2 p_\perp}{(2\pi)^2} E_n. \quad (2.17)$$

However, we still need to relate this to the energy density relative to equal-time coordinates, which we denote simply by \mathcal{E} .

Integration over a finite volume between the plates yields

$$\mathcal{E} = \frac{1}{LL_\perp^2} \int_{-2L}^0 dx^- \int_0^{L_\perp} d^2 x_\perp \mathcal{E}_{\text{LF}}. \quad (2.18)$$

A change of variable from x^- to $z = (x^+ + x^-)/2$ at fixed x^+ simplifies this to

$$\mathcal{E} = \frac{1}{LL_\perp^2} \int_0^L 2dx^- \int_0^{L_\perp} d^2 x_\perp \mathcal{E}_{\text{LF}} = 2\mathcal{E}_{\text{LF}}. \quad (2.19)$$

Thus, the energy density is

$$\mathcal{E} = \frac{1}{2L} \sum_n \int \frac{d^2 p_\perp}{(2\pi)^2} E_n, \quad (2.20)$$

which matches exactly the standard result (2.1). When properly regulated, the sum can be performed to extract the regulator-independent piece and the force calculated from the derivative with respect to the separation.

III. TRANSVERSE CASE

The transverse case is less problematic. In fact, a direct implementation of light-front coordinates, without any of

the considerations made here, does yield the correct result [5]. Therefore, there could be concern that the additional steps that we have introduced will somehow destroy this agreement. However, this does not happen, as we show in this section.

Without loss of generality, let the periodicity be in the x direction, so that we require $\phi(x^+, x^-, x + L_\perp, y) = \phi(x^+, x^-, x, y)$. This is satisfied if p_x is restricted to the discrete values $p_n \equiv 2\pi n/L_\perp$. We define discrete annihilation operators

$$a_n(p^+, p_y) = \sqrt{\frac{2\pi}{L}} a(p^+, p_n, p_y), \quad (3.1)$$

with the commutation relation

$$[a_n(p^+, p_y), a_{n'}^\dagger(p'^+, p'_y)] = \delta_{nn'} \delta(p^+ - p'^+) \delta(p_y - p'_y). \quad (3.2)$$

The scalar field is then

$$\begin{aligned}
 \phi(x^+ = 0) &= \frac{1}{\sqrt{L_\perp}} \sum_n \int \frac{dp^+ dp_y}{\sqrt{8\pi^2 p^+}} \\
 &\quad \times \{ a_n(p^+, p_y) e^{-ip^+ x^-/2 + ip_n x + ip_y y} \\
 &\quad + a_n^\dagger(p^+, p_y) e^{ip^+ x^-/2 - ip_n x - ip_y y} \}.
 \end{aligned} \quad (3.3)$$

The leading factor is consistent with the normalization of the wave functions $e^{-ip^+ x^-/2 + ip_n x + ip_y y}$ on the interval $0 < x < L_\perp$.

The energy and longitudinal momentum densities are

$$\begin{aligned}
 \langle 0|\mathcal{H}^-|0\rangle &= \frac{1}{2L_\perp} \sum_{nn'} \int \frac{dp^+ dp_y dp'^+ dp'_y}{8\pi^2 \sqrt{p^+ p'^+}} (p_n p_{n'} + p_y p'_y) \\
 &\quad \times \langle 0|a_n(p^+, p_y) a_{n'}^\dagger(p'^+, p'_y)|0\rangle \\
 &= \frac{1}{2L_\perp} \sum_n \int \frac{dp^+ dp_y}{8\pi^2} \frac{p_n^2 + p_y^2}{p^+}
 \end{aligned} \quad (3.4)$$

and

$$\begin{aligned}
 \langle 0|\mathcal{H}^+|0\rangle &= \frac{2}{L_\perp} \sum_{nn'} \int \frac{dp^+ dp_y dp'^+ dp'_y}{8\pi^2 \sqrt{p^+ p'^+}} \frac{p^+ p'^+}{4} \\
 &\quad \times \langle 0|a_n(p^+, p_y) a_{n'}^\dagger(p'^+, p'_y)|0\rangle \\
 &= \frac{1}{2L_\perp} \sum_n \int \frac{dp^+ dp_y}{8\pi^2} p^+.
 \end{aligned} \quad (3.5)$$

Averaged together, these determine \mathcal{E}_{LF} to be

$$\mathcal{E}_{\text{LF}} = \frac{1}{2L_\perp} \sum_n \int \frac{dp^- dp^+ dp_y p^- + p^+}{8\pi^2} \delta\left(p^- - \frac{p_n^2 + p_y^2}{p^+}\right). \quad (3.6)$$

The delta function is equivalent to the mass-shell condition:

$$\delta\left(p^- - \frac{p_n^2 + p_y^2}{p^+}\right) = p^+ \delta(p^2) = p^+ \delta(E^2 - E_n^2), \quad (3.7)$$

with $E_n = \sqrt{(\frac{2\pi}{L_\perp} n)^2 + p_z^2 + p_y^2}$, and facilitates a conversion to integration over the equal-time variables $E = (p^+ + p^-)/2$ and $p_z = (p^+ - p^-)/2$. The conversion yields

$$\mathcal{E}_{\text{LF}} = \frac{1}{2L_\perp} \sum_n \int \frac{2dE dp_z dp_y}{8\pi^2} E(E + p_z) \frac{1}{2E_n} \delta(E - E_n). \quad (3.8)$$

The E integral can be done immediately, with use of the delta function. The contribution from the p_z term is zero, because that part of the p_z integral is trivially odd.⁴ This leaves an energy density relative to light-front coordinates of

$$\mathcal{E}_{\text{LF}} = \frac{1}{4L_\perp} \sum_n \int \frac{dp_z dp_y}{(2\pi)^2} E_n. \quad (3.9)$$

The transformation to the energy density relative to equal-time coordinates is, as before, just multiplication by two. Therefore, we obtain in the transverse case

$$\mathcal{E} = \frac{1}{2L_\perp} \sum_n \int \frac{dp_z dp_y}{(2\pi)^2} E_n, \quad (3.10)$$

which matches the usual equal-time result and is of the same form as in the longitudinal case.

⁴This explains why our approach yields the same result as a calculation of the light-front energy density alone in the transverse case; the difference is just contributions proportional to p_z , which integrate to zero.

IV. SUMMARY

By a physical choice of boundary conditions and vacuum energy, we have computed in Dirac's light-front coordinates the vacuum energy density appropriate for the Casimir effect [1] and obtained the standard result (2.1). This is meant not as a new derivation of the Casimir effect, although it is, but as a demonstration that light-front quantization is not somehow flawed in its treatment of such vacuum effects. Unlike previous attempts [5,6], we have invoked the physics of plates at rest in an inertial frame and have not resorted to alteration away from Dirac's light-front coordinates. Keeping the plates at rest is not natural in light-front coordinates, but is physically correct. Similarly, the vacuum energy is computed as the standard equal-time energy, which is the correct input to the calculation of a Casimir force. Again, this is not the natural choice in light-front coordinates, where one would usually calculate the light-front energy. In other words, by being careful to calculate the same physical quantity, we have obtained the standard result, though with use of a different coordinate system.

Clearly, light-front coordinates are not the preferred system for this calculation. However, there are many situations in nonperturbative field-theoretic calculations where light-front coordinates are much superior [4]. In particular, Fock-state expansions are well defined and the associated wave functions are boost invariant, making the calculation of observables relatively straightforward. That something such as the Casimir effect, which is much less natural for light-front coordinates, can also be calculated correctly provides additional confidence in the usefulness of the approach.

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