

General relativistic polytropes for anisotropic matter: The general formalism and applications

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We set up in detail the general formalism to model polytropic general relativistic stars with anisotropic pressure. We shall consider two different possible polytropic equations, all of which yield the same Lane-Emden equation in the Newtonian limit. A heuristic model based on an ansatz to obtain anisotropic matter solutions from known solutions for isotropic matter is adopted to illustrate the effects of the pressure anisotropy on the structure of the star. In this context, the Tolman mass, which is a measure of the active gravitational mass, is invoked to explain some features of the models. Prospective extensions of the proposed approach are pointed out.

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I. INTRODUCTION

The polytropic equations of state have a long and a venerable history, they were introduced in the context of Newtonian gravity, in order to deal with a variety of astrophysical problems (see Refs. [1–6] and references therein), their great success stemming mainly from the simplicity of the equation of state and the ensuing main equation (Lane-Emden).

The theory of polytropes is based on the polytropic equation of state, which in the Newtonian case reads

$$P = K\rho_0^\gamma = K\rho_0^{1+1/n}, \quad (1)$$

where P and ρ_0 denote the isotropic pressure and the mass (baryonic) density, respectively. Constants K , γ , and n are usually called the polytropic constant, polytropic exponent, and polytropic index, respectively.

The polytropic equation of state may be used to model two very different types of situations, namely:

- (i) When the polytropic constant K is fixed and can be calculated from natural constants. This is the case of a completely degenerate gas in the nonrelativistic ($\gamma = 5/3$; $n = 3/2$) and relativistic limit ($\gamma = 4/3$; $n = 3$). Polytropes of this kind are particularly useful to model compact objects such as white dwarfs, and they lead in a rather simple way to the Chandrasekhar mass limit.
- (ii) When K is a free parameter, as, for example, in the case of isothermal ideal gas, or in a completely convective star. Models related to isothermal ideal gas are relevant in the so-called Schönberg-Chandrasekhar limit (see Ref. [3] for details).

However, in some cases the object under study may be compact enough as to require the use of general relativity (e.g., neutron stars). In the context of this theory, polytropic equations of state have also been widely used (see Refs. [7–15] and references therein).

Both in the Newtonian and in the general-relativistic regime, the fluid under consideration is generally assumed to be endowed with isotropic pressure (Pascal principle). However, in a recent work [16] we have extended the framework used to deal with Newtonian polytropic equations of state to the case of anisotropic fluids (principal stresses unequal).

The motivation to undertake such an endeavor was based on the fact that the local anisotropy of pressure may be caused by a large variety of physical phenomena of the kind we expect to find in compact objects (see Refs. [17–25] and references therein for an extensive discussion on this point).

Among all possible sources of anisotropy, there are two particularly related to our primary interest. The first one is the intense magnetic field observed in compact objects such as white dwarfs, neutron stars, or magnetized strange quark stars (see, for example, Refs. [26–30] and references therein). Indeed, it is a well-established fact that a magnetic field acting on a Fermi gas produces pressure anisotropy (see Refs. [31–34] and references therein). In some way, the magnetic field can be addressed as a fluid anisotropy.

Another source of anisotropy expected to be present in neutron stars and, in general, in highly dense matter, is the viscosity (see [35–42] and references therein).

An alternative approach to anisotropy comes from kinetic theory using the spherically symmetric Einstein-Vlasov equations, which admits a very rich class of static solutions, none of them isotropic (Refs. [43–45] and references therein). The advantages or disadvantages of either approach are related to the specific problem under consideration.

Based in all the arguments above, it is our main purpose in this paper to develop the general formalism to describe

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polytropes in the presence of pressure anisotropy, within the context of general relativity.

For the sake of completeness, we shall first review very briefly the theory of polytropes for a perfect (isotropic) general relativistic fluid. In this latter case there exist two possible versions of the polytropic equation of state leading to the same Newtonian limit.

Next, we shall display the general formalism for anisotropic fluids. In this case, we shall consider the same two possible polytropic equations of state of the isotropic case, but now applied to the radial pressure alone. Both cases share the same Newtonian limit.

In order to integrate the obtained system of equations we need to provide further information about the anisotropy inherent to the problem under consideration. For doing that, we shall assume an ansatz, allowing us to specific modeling. For these models we shall also calculate the Tolman mass, whose behavior allows one to understand some of their peculiar features. As we shall see below, our method links our models continually with the isotropic case (see Sec. IV), thereby allowing us to bring out the influence of anisotropy on the structure of the object. However, it should be stressed that the above-mentioned models are presented with the sole purpose to illustrate the method, the natural way to obtain models, which consists in providing the specific information about the kind of anisotropy present in each specific problem. Finally, we shall conclude with a summary of results and some possible extensions of our formalism.

II. THE GENERAL RELATIVISTIC POLYTROPE FOR A PERFECT FLUID

A. The field equations

We consider spherically symmetric static distributions of fluid, bounded by a spherical surface Σ . The line element is given in Schwarzschild-like coordinates by

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (2)$$

where ν and λ are functions of r . We number the coordinates: $x^0 = t$; $x^1 = r$; $x^2 = \theta$; $x^3 = \phi$. We use geometric units and therefore we have $c = G = 1$. The metric (2) has to satisfy Einstein field equations which in our case read [18]

$$\rho = -\frac{1}{8\pi} \left[-\frac{1}{r^2} + e^{-\lambda} \left(\frac{1}{r^2} - \frac{\lambda'}{r} \right) \right], \quad (3)$$

$$P = -\frac{1}{8\pi} \left[\frac{1}{r^2} - e^{-\lambda} \left(\frac{1}{r^2} + \frac{\nu'}{r} \right) \right], \quad (4)$$

$$P = \frac{1}{8\pi} \left[\frac{e^{-\lambda}}{4} \left(2\nu'' + \nu'^2 - \lambda'\nu' + 2\frac{\nu' - \lambda'}{r} \right) \right], \quad (5)$$

where prime denotes derivative with respect to r , P is the isotropic pressure, and ρ is the energy density.

At the outside of the fluid distribution, the spacetime is that of Schwarzschild, given by

$$ds^2 = \left(1 - \frac{2M}{r} \right) dt^2 - \left(1 - \frac{2M}{r} \right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (6)$$

In order to match smoothly the two metrics above on the boundary surface $r = r_\Sigma$, we must require the continuity of the first and the second fundamental form across that surface (Darmois conditions). Then it follows

$$e^{\nu_\Sigma} = 1 - \frac{2M}{r_\Sigma}, \quad (7)$$

$$e^{-\lambda_\Sigma} = 1 - \frac{2M}{r_\Sigma}, \quad (8)$$

$$P_\Sigma = 0, \quad (9)$$

where the subscript Σ indicates that the quantity is evaluated at the boundary surface Σ .

Next, it will be useful to calculate the radial component of the conservation law,

$$T^\mu_{\nu;\mu} = 0, \quad (10)$$

where

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu - P g_{\mu\nu}, \quad (11)$$

with

$$u^\mu = (e^{-\nu/2}, 0, 0, 0), \quad (12)$$

where u^μ denotes the four velocity of the fluid.

After simple calculations we get

$$P' = -\frac{\nu'}{2}(\rho + P), \quad (13)$$

or using

$$\nu' = \frac{2(m + 4\pi P r^3)}{r(r - 2m)}, \quad (14)$$

we may also write

$$P' = -\frac{(m + 4\pi P r^3)}{r(r - 2m)}(\rho + P), \quad (15)$$

which is the well-known Tolman-Oppenheimer-Volkoff (TOV) equation, and where the mass function $m(r)$, as usual, is defined by

$$e^{-\lambda} = 1 - 2m/r. \quad (16)$$

B. The relativistic polytrope

As mentioned in the Introduction, the Newtonian polytrope is characterized by (1). However, when considering the polytropic equation of state within the context of general relativity, two different possibilities arise, leading to the same equation (1) in the Newtonian limit, namely.

1. Case I

In this case the original polytropic equation of state, (1), is conserved, see for example [8,9,14].

Now, the first and the second law of thermodynamics may be written as

$$d\left(\frac{\rho + P}{\mathcal{N}}\right) - \frac{dP}{\mathcal{N}} = Td\left(\frac{\sigma}{\mathcal{N}}\right), \quad (17)$$

where T denotes temperature, σ is entropy per unit of proper volume, and \mathcal{N} is the particle density, such that

$$\rho_0 = \mathcal{N}m_0. \quad (18)$$

Then for an adiabatic process it follows

$$d\left(\frac{\rho}{\mathcal{N}}\right) + Pd\left(\frac{1}{\mathcal{N}}\right) = 0, \quad (19)$$

which together with (1) leads to

$$K\rho_0^{\gamma-2} = \frac{d(\rho/\rho_0)}{d\rho_0}, \quad (20)$$

with

$$\gamma = 1 + 1/n. \quad (21)$$

If $\gamma \neq 1$, (20) may be easily integrated to give

$$\rho = C\rho_0 + P/(\gamma - 1), \quad (22)$$

where C is a constant of integration. In the nonrelativistic limit we should have $\rho \rightarrow \rho_0$, and therefore $C = 1$.

Thus, the polytropic equation of state amounts to

$$\rho = \rho_0 + P/(\gamma - 1). \quad (23)$$

It is worth noticing that the familiar ‘‘barotropic’’ equation of state,

$$\rho = P/(\gamma - 1), \quad (24)$$

is a particular case of (22) with $C = 0$.

In the very special case $\gamma = 1$, one obtains

$$K\rho_0^{-1} = \frac{d(\rho/\rho_0)}{d\rho_0}, \quad (25)$$

whose solution is

$$\rho = P \log \rho_0 + \rho_0 C, \quad (26)$$

or, if putting $C = 1$ from the nonrelativistic limit

$$\rho = P \log \rho_0 + \rho_0. \quad (27)$$

From now on we shall only consider the $\gamma \neq 1$ case.

Next, let us introduce the following variables:

$$\alpha = P_c/\rho_c, \quad r = \xi/A, \quad A^2 = 4\pi\rho_c/\alpha(n+1), \quad (28)$$

$$\psi_0^n = \rho_0/\rho_{0c}, \quad v(\xi) = m(r)A^3/(4\pi\rho_c), \quad (29)$$

where subscript c indicates that the quantity is evaluated at the center. It is worth noticing that some of the variables defined above differ from those used in [8].

Then the Tolman-Oppenheimer-Volkoff equation (15) becomes

$$\xi^2 \frac{d\psi_0}{d\xi} \left[\frac{1 - 2(n+1)\alpha v/\xi}{(1-n\alpha) + (n+1)\alpha\psi_0} \right] + v + \alpha\xi^3 \psi_0^{n+1} = 0, \quad (30)$$

and from the definition of mass function (16) and Eq. (3), we have

$$m' = 4\pi r^2 \rho \quad (31)$$

or

$$\frac{dv}{d\xi} = \xi^2 \psi_0^n (1 - n\alpha + n\alpha\psi_0). \quad (32)$$

Once again, recall that our definitions (28) and (29) differ from Eqs. (15)–(18) in [8], therefore (30) and (32) also differ in form from Eqs. (19) and (20) in [8], although, of course they are absolutely equivalent.

The boundary of the surface of the sphere is defined by $\xi = \xi_n$ such that $\psi_0(\xi_n) = 0$ and the following boundary conditions apply:

$$\psi_0(\xi = 0) = 1, \quad v(\xi = 0) = 0. \quad (33)$$

Combining (30) and (32) we obtain the generalized Lane-Emden equation for this case, which reads

$$\begin{aligned} & a \frac{d^2 \psi_0}{d\xi^2} + \frac{2}{\xi} \frac{d\psi_0}{d\xi} \left[a - \alpha\xi(n+1) \right. \\ & \quad \times \left. \left[\xi \psi_0^n (b - \alpha\psi_0) - \frac{v}{\xi^2} - \frac{a}{2b} \frac{d\psi_0}{d\xi} - \frac{b\xi \psi_0^n}{2} \right] \right] \\ & \quad + b\psi_0^n (3\alpha\psi_0 + b - \alpha\psi_0) = 0, \end{aligned} \quad (34)$$

where

$$a \equiv 1 - 2(n+1)\alpha v/\xi$$

and

$$b \equiv (1 - n\alpha) + (n+1)\alpha\psi_0.$$

In the Newtonian limit ($\alpha \rightarrow 0$), (34) becomes

$$\frac{d^2 \psi_0}{d\xi^2} + \frac{2}{\xi} \frac{d\psi_0}{d\xi} + \psi_0^n = 0, \quad (35)$$

which is the classical Lane-Emden equation.

2. Case II

Another possibility consists in assuming that the relativistic polytrope is defined by

$$P = K\rho^{1+1/n}, \quad (36)$$

instead of (1), see for example [7,11,15].

In this case one obtains from (19) and (36)

$$\rho = \frac{\rho_0}{(1 - K\rho_0^{1/n})^n}. \quad (37)$$

Using (36) and (28) it can be easily demonstrated that if α is small enough, (37) becomes

$$\rho \approx \rho_0(1 + nK\rho_0^{1/n}), \quad (38)$$

where only terms linear in α have been kept. It should be observed that (38) and (23) are identical, implying that both cases, I and II, differ in terms of order α^2 and higher (obviously they coincide in the Newtonian limit).

Then introducing

$$\psi^n = \rho/\rho_c, \quad (39)$$

we obtain for the TOV equation

$$\xi^2 \frac{d\psi}{d\xi} \left[\frac{1 - 2(n+1)\alpha v/\xi}{1 + \alpha\psi} \right] + v + \alpha\xi^3 \psi^{n+1} = 0, \quad (40)$$

and from (31)

$$\frac{dv}{d\xi} = \xi^2 \psi^n. \quad (41)$$

From the two equations above, we can obtain the generalized Lane-Emden equation for this case, which reads

$$\begin{aligned} a \frac{d^2\psi}{d\xi^2} + \frac{2}{\xi} \frac{d\psi}{d\xi} \left\{ a - \alpha\xi^2(n+1)\psi^n + \frac{v(n+1)\alpha}{\xi} \right. \\ \left. - \frac{d\psi}{d\xi} \frac{\xi a \alpha}{2c} + \frac{\xi^2 \alpha c(n+1)\psi^n}{2} \right\} \\ + c\psi^n + 3\alpha c\psi^{(n+1)} = 0, \end{aligned} \quad (42)$$

where $c \equiv 1 + \alpha\psi$.

Once again, in the Newtonian limit ($\alpha \rightarrow 0$), the Lane-Emden equation (35) is recovered in this case too, as it should be. Obviously equations of state in both cases differ from each other, especially in the highly relativistic regime, and therefore resulting models will also differ.

We shall next generalize the scheme above for the case when the pressure is no longer isotropic.

III. THE POLYTROPE FOR ANISOTROPIC FLUIDS

If we allow the principal stresses to be unequal, then the energy-momentum tensor in the canonical form reads

$$T_\nu^\mu = \rho u^\mu u_\nu - Ph_\nu^\mu + \Pi_\nu^\mu, \quad (43)$$

where P is the isotropic pressure, and $\Pi_{\mu\nu}$ the anisotropic pressure tensor, with

$$h_\nu^\mu = \delta_\nu^\mu - u^\mu u_\nu, \quad \Pi_\nu^\mu = \Pi \left(s^\mu s_\nu + \frac{1}{3} h_\nu^\mu \right), \quad (44)$$

and s^μ is defined as

$$s^\mu = (0, e^{-\frac{\lambda}{2}}, 0, 0), \quad (45)$$

with the properties $s^\mu u_\mu = 0$, $s^\mu s_\mu = -1$.

It is immediate to see that

$$\rho = T_{\alpha\beta} u^\alpha u^\beta, \quad (46)$$

$$P = -\frac{1}{3} h^{\alpha\beta} T_{\alpha\beta}, \quad \Pi_{\alpha\beta} = h_\alpha^\mu h_\beta^\nu (T_{\mu\nu} + Ph_{\mu\nu}). \quad (47)$$

For our purposes in this work, it would be more convenient to introduce the following two auxiliary variables (P_r and P_\perp) by

$$P_r = s^\alpha s^\beta T_{\alpha\beta}, \quad P_\perp = K^\alpha K^\beta T_{\alpha\beta}, \quad (48)$$

where K^α is a unit spacelike vector (orthogonal to u^α and s^α).

In terms of the above variables, we have

$$\Pi = P_r - P_\perp; \quad P = \frac{P_r + 2P_\perp}{3}, \quad (49)$$

from where the physical meaning of P_r and P_\perp becomes evident, and the energy momentum can be written under the form

$$T_{\mu\nu} = (\rho + P_\perp) u_\mu u_\nu - P_\perp g_{\mu\nu} + (P_r - P_\perp) s_\mu s_\nu. \quad (50)$$

From these last expressions it is a simple matter to prove that the hydrostatic equilibrium equation now reads

$$P_r' = -\frac{v'}{2}(\rho + P_r) + \frac{2(P_\perp - P_r)}{r}, \quad (51)$$

where P_r and P_\perp will be hereafter called the radial and tangential pressures, respectively. This is the generalized Tolman-Oppenheimer-Volkoff equation for anisotropic matter. Alternatively, using

$$v' = 2 \frac{m + 4\pi P_r r^3}{r(r - 2m)}, \quad (52)$$

we may write

$$P_r' = -\frac{(m + 4\pi P_r r^3)}{r(r - 2m)}(\rho + P_r) + \frac{2(P_\perp - P_r)}{r}. \quad (53)$$

Based on the considerations of the previous section, we shall consider the following two cases to extend the polytropic equation of state to anisotropic matter.

(1)

$$P_r = K\rho_0^\gamma = K\rho_0^{1+1/n}. \quad (54)$$

(2)

$$P_r = K\rho^\gamma = K\rho^{1+1/n}. \quad (55)$$

We shall next proceed to analyze each case in detail.

A. Case I

Assuming (54), then with the help of (17) we obtain

$$\rho = \rho_0 + P_r/(\gamma - 1). \quad (56)$$

Then repeating the same procedure as in the isotropic case we get

$$\xi^2 \frac{d\psi_0}{d\xi} \left[\frac{1 - 2(n+1)\alpha v/\xi}{(1-n\alpha) + (n+1)\alpha\psi_0} \right] + v + \alpha\xi^3 \psi_0^{n+1} + \frac{2\Delta\psi_0^{-n}\xi}{P_{rc}(n+1)} \left[\frac{1 - 2\alpha(n+1)v/\xi}{(1-n\alpha) + (n+1)\alpha\psi_0} \right] = 0, \quad (57)$$

where now $\alpha = P_{cr}/\rho_c$ and $\Delta = -\Pi = P_\perp - P_r$.

On the other hand, we obtain from the mass function definition (16) the same equation (32) as in the isotropic case.

Combining (57) and (32) we are led to the generalized Lane-Emden equation for this case:

$$\begin{aligned} & a \frac{d^2\psi_0}{d\xi^2} + \frac{2}{\xi} \frac{d\psi_0}{d\xi} \left\{ a - \alpha\xi(n+1) \left[\xi\psi_0^n(b - \alpha\psi_0) - \frac{v}{\xi^2} - \frac{a}{2b} \frac{d\psi_0}{d\xi} - \frac{b\xi\psi_0^n}{2} \right] \right. \\ & \quad \left. - \frac{\xi^2}{(n+1)P_{rc}} \left[\psi_0^{-n}\Delta \frac{a}{b} \right] \left[\frac{(n+1)\alpha}{b} + \frac{n}{\psi_0} \right] \right\} + b\psi_0^n(3\alpha\psi_0 + b - \alpha\psi_0) \\ & \quad + \frac{2}{(n+1)P_{rc}} \left[\psi_0^{-n}\Delta \xi \frac{a}{b} \right] \left[\frac{1}{\Delta} \frac{d\Delta}{d\xi} + \frac{1}{\xi} + \frac{2(n+1)\alpha}{a} \left[-\xi\psi_0^n[1 + n\alpha(\psi_0 - 1)] + \frac{v}{\xi^2} \right] \right\} = 0. \end{aligned} \quad (58)$$

It is a simple matter to check that the equation above reduces to Eq. (20) in [16] in the Newtonian limit (taking care of the changes in notation).

B. Case II

In this case the assumed equation of state is (55), then the TOV equation becomes

$$\xi^2 \frac{d\psi}{d\xi} \left[\frac{1 - 2(n+1)\alpha v/\xi}{1 + \alpha\psi} \right] + v + \alpha\xi^3 \psi^{n+1} + \frac{2\Delta\psi^{-n}\xi}{P_{rc}(n+1)} \left[\frac{1 - 2\alpha(n+1)v/\xi}{1 + \alpha\psi} \right] = 0, \quad (59)$$

and from the definition of the mass function (16), we obtain (41).

Once again, the combination of (59) and (41) leads to the generalized Lane-Emden equation for this case which reads

$$\begin{aligned} & a \frac{d^2\psi}{d\xi^2} + \frac{2}{\xi} \frac{d\psi}{d\xi} \left\{ a - \alpha\xi^2(n+1)\psi^n + \frac{v(n+1)\alpha}{\xi} - \frac{d\psi}{d\xi} \frac{\xi a \alpha}{2c} + \frac{\xi^2 \alpha c(n+1)\psi^n}{2} - \frac{\xi^2}{(1+n)P_{rc}} \left[\Psi^{-n}\Delta \frac{a}{c} \right] \left[\frac{n}{\psi} + \frac{\alpha}{c} \right] \right\} \\ & \quad + c\psi^n + 3\alpha c\psi^{(n+1)} + \frac{2}{(1+n)P_{rc}} \left[\psi^{-n}\Delta \xi \frac{a}{c} \right] \left[\frac{1}{\Delta} \frac{d\Delta}{d\xi} + \frac{1}{\xi} + \frac{2(n+1)\alpha}{a} \left[-\xi\psi^n + \frac{v}{\xi^2} \right] \right\} = 0. \end{aligned} \quad (60)$$

In the Newtonian limit we recover Eq. (20) in [16].

As mentioned before, it is obvious that in order to proceed further with the modeling of a compact object, we need to prescribe the specific anisotropy of the problem (Δ). Such information, of course, depends on the specific physical problem under consideration. Here we shall not follow that direction; instead, we shall assume an ansatz already used in the modeling of relativistic anisotropic stars [46,47], whose main virtue (besides its simplicity) is the fact that the obtained models are continuously connected with the isotropic case, thus allowing one to compare both cases, and thereby to illustrate the influence of the anisotropy on the structure of the object.

IV. MODELING ANISOTROPIC POLYTROPES

In order to obtain specific models to illustrate our method, we shall here adopt the heuristic procedure used in Ref. [46], which allows one to obtain solutions for anisotropic matter from known solutions for isotropic matter. The procedure mentioned above may be summarized as follows (see [46]) for details:

(i) Let

$$\Delta = Cf(P_r, r)(\rho + P_r)r^N, \quad (61)$$

where C is a parameter which measures the anisotropy, and the function f and the number N are to be specific for each model.

(ii) Assume

$$f(P_r, r)r^{N-1} = \frac{\nu'}{2}. \quad (62)$$

(iii) With the assumption above, (51) becomes

$$\frac{dP_r}{dr} = -h(\rho + P_r)\frac{\nu'}{2}, \quad (63)$$

where $h = 1 - 2C$. For simplicity, we assume h to be constant throughout the sphere, which of course does not imply the constancy of either pressure.

Then we obtain for case I the following two equations:

$$\xi^2 \frac{d\psi_0}{d\xi} \left[\frac{1 - 2(n+1)\alpha\nu/\xi}{(1-n\alpha) + (n+1)\alpha\psi_0} \right] + h(\nu + \alpha\xi^3\psi_0^{n+1}) = 0, \quad (64)$$

and

$$\frac{d\nu}{d\xi} = \xi^2\psi_0^n(1 - n\alpha + n\alpha\psi_0), \quad (65)$$

whereas for case II, the corresponding equations read

$$\xi^2 \frac{d\psi}{d\xi} \left[\frac{1 - 2(n+1)\alpha\nu/\xi}{1 + \alpha\psi} \right] + h(\nu + \alpha\xi^3\psi^{n+1}) = 0, \quad (66)$$

and

$$\frac{d\nu}{d\xi} = \xi^2\psi^n. \quad (67)$$

The above equations shall be integrated numerically, with the corresponding boundary conditions.

It will be useful to calculate the Tolman mass, which is a measure of the active gravitational mass (see [48] and references therein), defined by

$$m_T = \frac{1}{2}r^2e^{(\nu-\lambda)/2}\nu'. \quad (68)$$

Alternatively we can calculate the Tolman mass from the expression [17]

$$m_T = e^{(\nu+\lambda)/2}(m + 4\pi P_r r^3). \quad (69)$$

The functions λ , m , and P_r in the above expression are obtained directly by integration of (64) and (65) for case I, and (66) and (67) for case II. Thus, we only need an expression for ν , which can be obtained as follows, for either case. Let us first consider case I. Then from (54), (29), and (56), we may write

$$P_r = K\rho_{0c}^{1+1/n}\psi_0^{(n+1)} \quad (70)$$

and

$$\rho = \rho_{0c}\psi_0^n(1 + nk\rho_{0c}^{1/n}\psi_0), \quad (71)$$

with the help of these two expressions we may write Eq. (63) in the form

$$2(n+1)\beta d\psi_0 + h\nu[1 + \beta\psi_0(n+1)] = 0, \quad (72)$$

where the parameter $\beta \equiv K\rho_{0c}^{1/n}$ is related to α by

$$\beta = \frac{\alpha}{1 - n\alpha}. \quad (73)$$

Equation (72) can be easily integrated to obtain

$$e^\nu = \frac{C}{[1 + (n+1)\beta\psi_0]^{2/h}}, \quad (74)$$

where C is a constant of integration. Next, using the fact that at the center ($r = 0$) $\psi_0 = 1$, we may write (74) as

$$e^\nu = e^{\nu_c} \frac{[1 + (n+1)\beta]^{2/h}}{[1 + (n+1)\beta\psi_0]^{2/h}}. \quad (75)$$

Finally, using (7) and the fact that the radial pressure vanishes at the boundary surface we obtain

$$e^\nu = \left(1 - \frac{2M}{r_\Sigma}\right)[1 + (n+1)\beta\psi_0]^{-2/h}. \quad (76)$$

Thus, the (dimensionless) Tolman mass can be written as

$$v_T = (\nu + \alpha\xi^3\psi_0^{1+n})(a_\Sigma/a)^{1/2} \times \left[\frac{1 - n\alpha}{1 - n\alpha + \alpha(1+n)\psi_0} \right]^{1/h}, \quad (77)$$

where

$$v_T = \frac{m_T A^3}{4\pi\rho_c}.$$

For case II, we follow a similar procedure. Thus, using (55) and (39), we transform (63) into

$$2(n+1)\alpha d\psi + h(1 + \alpha\psi)d\nu = 0, \quad (78)$$

which after integration produces

$$e^\nu = \left(1 - \frac{2M}{r_\Sigma}\right)(1 + \alpha\psi)^{-2(n+1)/h}, \quad (79)$$

where boundary conditions have been used. In this case the Tolman mass reduces to

$$v_T = (\nu + \alpha\xi^3\psi^{1+n})(a_\Sigma/a)^{1/2}(1 + \alpha\psi)^{-(1+n)/h}. \quad (80)$$

In order to see how the Tolman mass distributes through the sphere in the process of contraction (slow and adiabatic), it would be convenient to introduce the following dimensionless variables:

$$x = \frac{r}{r_\Sigma} = \frac{\xi}{\tilde{A}}, \quad y = \frac{M}{r_\Sigma}, \quad \tilde{m} = \frac{m}{M}, \quad \tilde{A} = r_\Sigma A. \quad (81)$$

In terms of the above variables the Tolman mass for the two cases I and II reads

$$v_T = (v + \alpha x^3 \tilde{A}^3 \psi_0^{1+n}) \left[\frac{x(1-2y)}{x - 2\alpha(n+1)v/\tilde{A}} \right]^{1/2} \times \left[\frac{1-n\alpha}{1-n\alpha + \alpha(1+n)\psi_0} \right]^{1/h} \quad (82)$$

and

$$v_T = (v + \alpha x^3 \tilde{A}^3 \psi_0^{1+n}) \times \left[\frac{x(1-2y)}{x - 2\alpha(n+1)v/\tilde{A}} \right]^{1/2} [1 + \alpha\psi]^{-(1+n)/h}, \quad (83)$$

respectively. Observe that $\tilde{A} = \xi_\Sigma$, therefore

$$y = \alpha(n+1) \frac{v_\Sigma}{\xi_\Sigma}. \quad (84)$$

This means that y is not independent of the anisotropy parameter h . In other words, y is constant for a given pair (n, α) if $v_\Sigma/\xi_\Sigma = \text{constant}$, and this is possible only for each h . One potential at the surface y corresponds univocally to each anisotropic and relativistic polytrope.

The full set of equations deployed above has been integrated for both cases (I and II), and a wide range of values of different parameters. In what follows we analyze the most relevant results obtained from a selection of the whole set of obtained models.

In case I, Fig. 1 shows the integration of (64) and (65) for the values of the parameters indicated in the figure legend. However as mentioned there, the ensuing qualitative behavior, namely, larger values of ψ_0 for smaller values of h (everywhere throughout the sphere), is maintained for a wide range of values of the triplet (α, n, h) . There exists though, some range of values of the parameters, for which such a behavior is disrupted, as indicated in Fig. 2, where ψ_0 close to the boundary surface is larger for $h = 1$ than for $h = 0.5$. We shall comment further on this ‘‘anomaly’’ later on.

In the Newtonian polytropes there exist bounded configurations for a specific range of the parameter n , whereas for the general relativistic polytropes of an isotropic fluid, bounded configurations are also restricted by the value of the parameter α . In our case, as it should be expected, the existence of bounded configurations is restricted by the triplet (α, n, h) , as indicated in Figs. 3 and 4.

Finally, always for case I, in Fig. 5 we plot the total mass function (v_Σ) as a function of the relativistic parameter α for different values of h , and $n = 2$. The exhibited behavior, larger values of v_Σ for smaller values of h , is representative for a wide range of (n, h) .

Case II is described by the integration of (66) and (67). Results are exhibited in Figs. 6–10, and besides quantitative differences, the analysis of these figures is essentially the same as in case I.

Next, we have plotted the ‘‘surface potential’’ (y) for both cases, in Figs. 11 and 14. This is a relevant variable since it measures the compactness of the configuration. For case I (Fig. 11) we observe that the degree of compactness

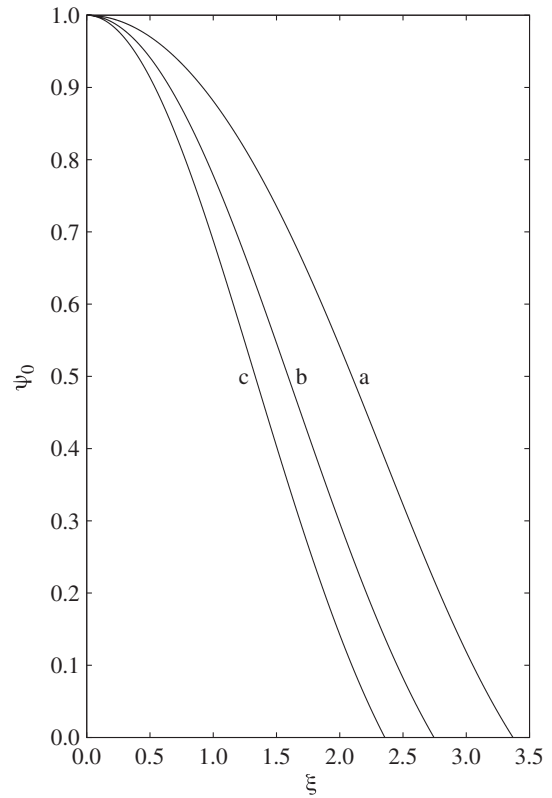


FIG. 1. ψ_0 as a function of ξ for $\alpha = 0.1$, $n = 1$ and $h = 0.5$ (curve a); $h = 1.0$ (curve b); $h = 1.5$ (curve c). For a wide range of (α, n, h) this figure is qualitatively representative.

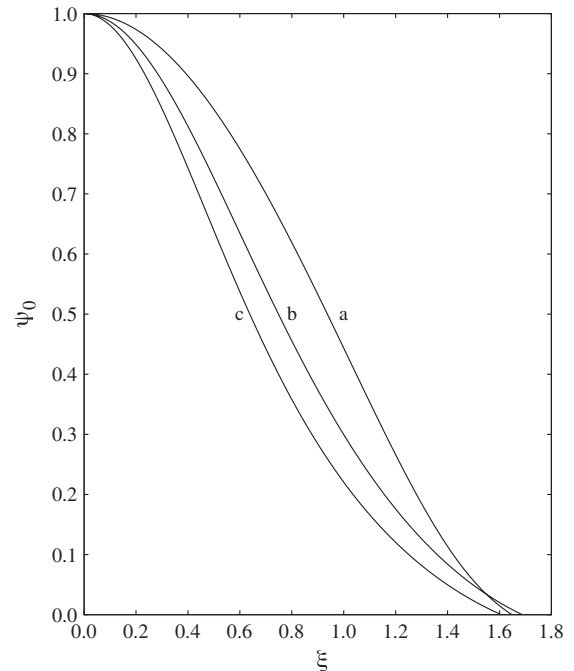


FIG. 2. ψ_0 as a function of ξ for $\alpha = 1$, $n = 0.5$ and $h = 0.5$ (curve a); $h = 1.0$ (curve b); $h = 1.5$ (curve c). Note the anomaly for $h = 0.5$, that is, $\psi_0(\xi; h = 1) > \psi_0(\xi; h = 0.5)$ in some sector close to the surface ξ_Σ .

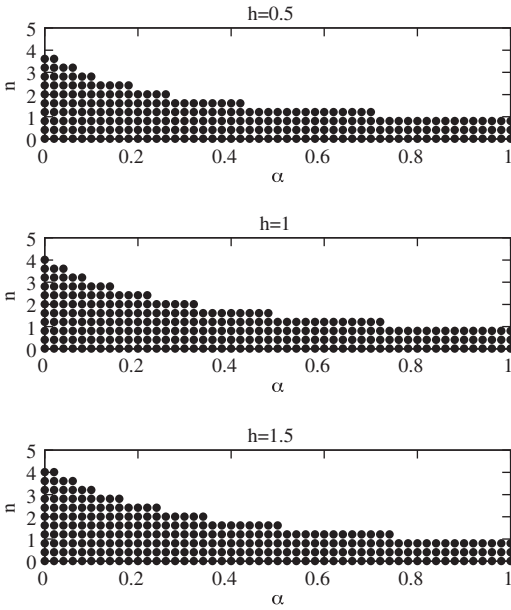


FIG. 3. Case I: Duplets (α, n) for bounded sources with h : 0.5 1.0; 1.5. Clearly $h > 1$ favors the bounded configurations whereas $h < 1$ does not.

monotonically decreases with h . For case II (Fig. 14) the situation is essentially the same (decreasing of y with increasing h), except for some range of values of h in the ultrarelativistic case and $n = 2$ (curve d). At any rate, maximal values of y correspond to minimal values of h in the four curves.

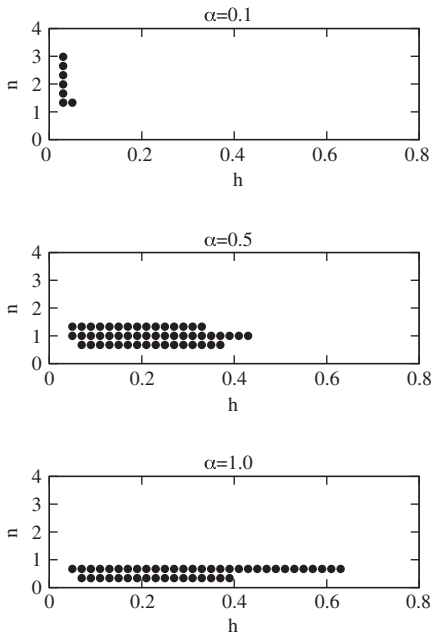


FIG. 4. Case I: Duplets (h, n) for different α of bounded sources contained in Fig. 3, which show anomalous behavior like in Fig. 2. To find this subset of anomalous models, we do a search of parameters for which the behavior appears at least in some interior point of the distribution.

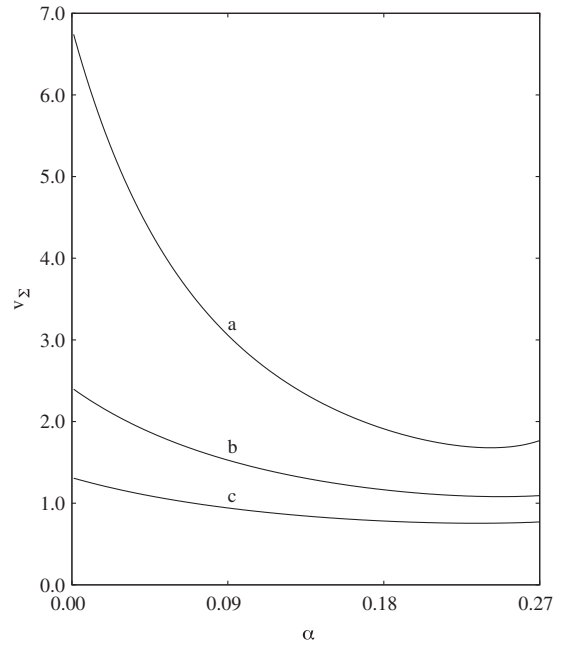


FIG. 5. Case I: ν_Σ as a function of α for $n = 2$ and $h = 0.5$ (curve a); $h = 1.0$ (curve b); $h = 1.5$ (curve c). In this case the limit value for α is 0.27 to get a bounded source. For a wide range of (n, h) this figure is qualitatively representative.

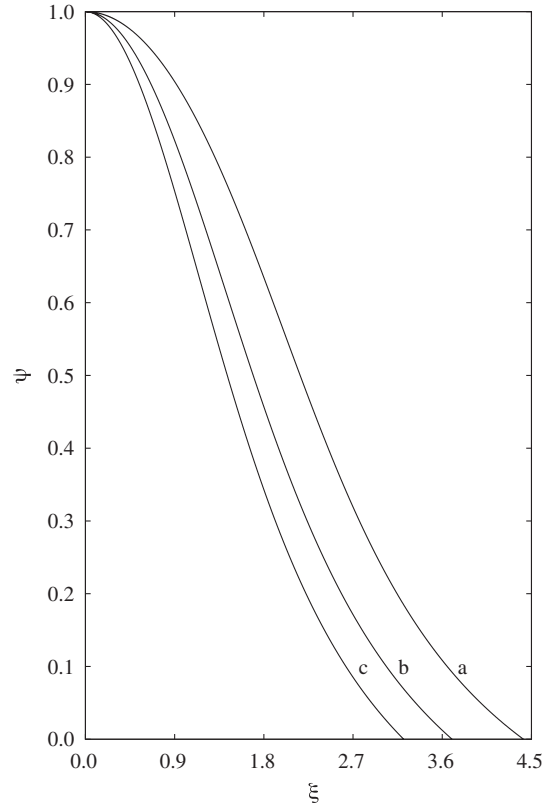


FIG. 6. ψ as a function of ξ for $\alpha = 0.1$, $n = 2$ and $h = 0.5$ (curve a); $h = 1.0$ (curve b); $h = 1.5$ (curve c). For a wide range of (α, n, h) this figure is qualitatively representative.

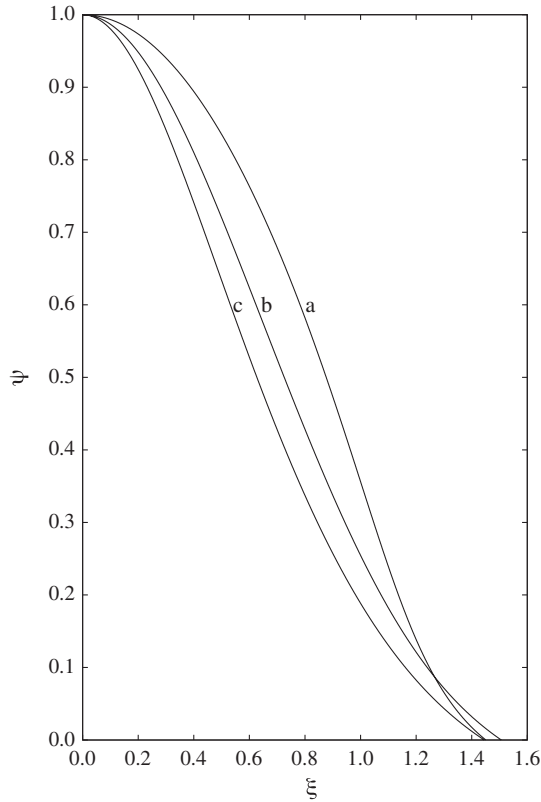


FIG. 7. ψ as a function of ξ for $\alpha = 1$, $n = 1$ and $h = 0.5$ (curve a); $h = 1.0$ (curve b); $h = 1.5$ (curve c). Note the anomaly for $h = 0.5$, that is, $\psi_0(\xi; h = 1) > \psi_0(\xi; h = 0.5)$ in some sector close to the surface ξ_Σ .

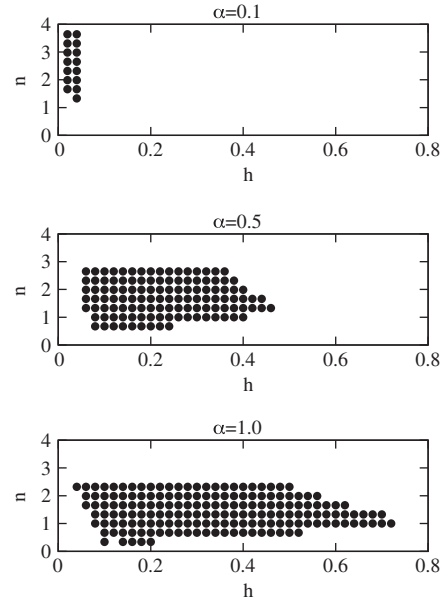


FIG. 9. Case II: Duplets (h, n) for different α of bounded sources contained in Fig. 8, which show anomalous behavior like in Fig. 7. To find this subset of anomalous models, we do a search of parameters for which the behavior appears at least in some interior point of the distribution.

It should be emphasized that the relationship between the maximal values of y (maximal surface redshift) and the local anisotropy of pressure has been discussed in great detail in the past (see [49–55] and references therein). The explanation for such interest is easily understood, if we

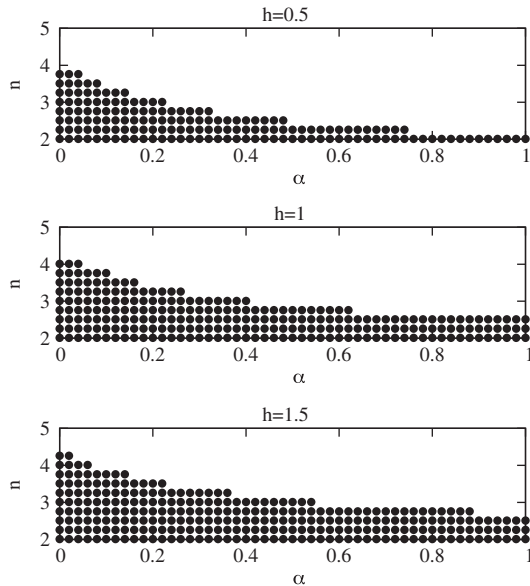


FIG. 8. Case II: Duplets (α, n) for bounded sources with h : 0.5; 1.0; 1.5. Clearly $h > 1$ favors the bounded configurations whereas $h < 1$ does not. Below $n = 2$ (not shown) all the sources are bounded too.

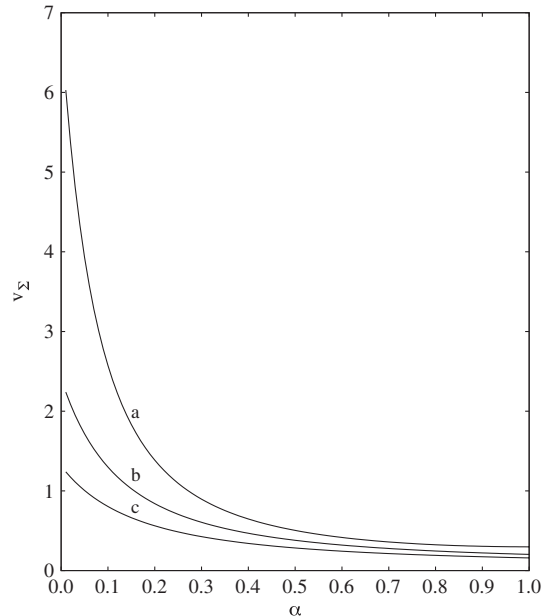


FIG. 10. Case II: v_Σ as a function of α for $n = 2$ and $h = 0.5$ (curve a); $h = 1.0$ (curve b); $h = 1.5$ (curve c). For a wide range of (n, h) this figure is qualitatively representative.

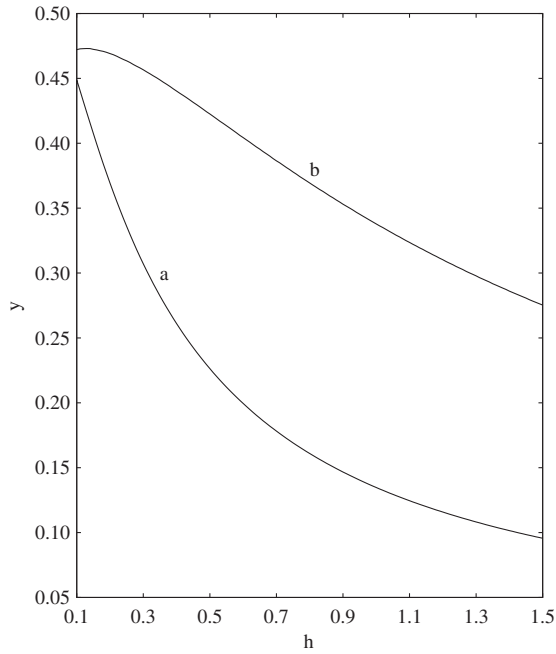


FIG. 11. Case I: Potential at the surface y as a function of the anisotropy parameter, h for pairs (n, α) : (1.0,0.1) (curve a); (0.5,1) (curve b).

recall that the surface redshift is an observable variable, which thereby might provide information about the structure of the source, in particular about its degree of anisotropy. As mentioned before each anisotropic polytropic

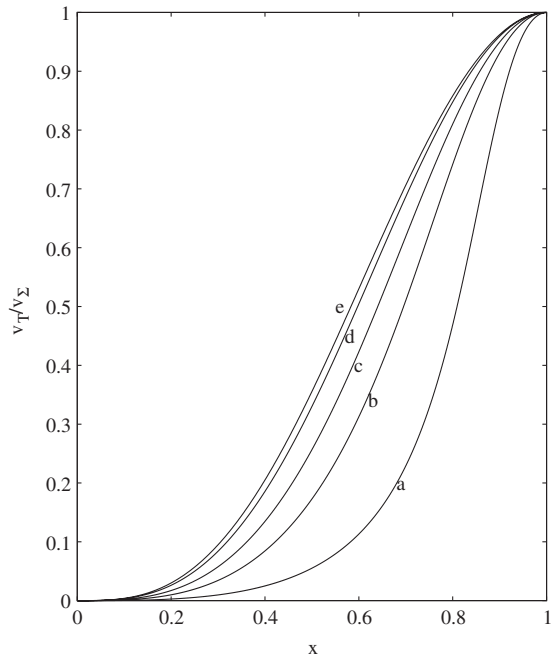


FIG. 12. Case I: v_T/v_Σ as a function of x for $n = 1$, $\alpha = 0.1$, and different values of $h(y)$: 0.15(≈ 0.41) (curve a); 0.3(≈ 0.31) (curve b); 0.5(≈ 0.23) (curve c), 1.0(≈ 0.13) (curve d), 1.5(≈ 0.1) (curve e). Values of y are read off Fig. 11. The value of $h = 0.1$ corresponds to $y = 0.448(>4/9)$.

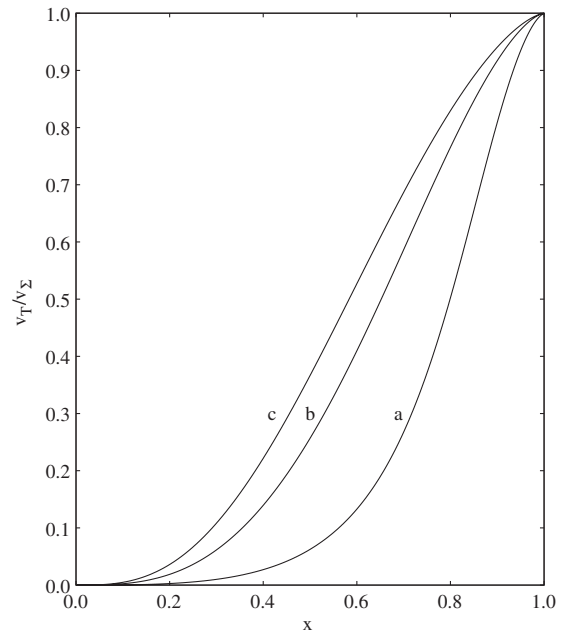


FIG. 13. Case I: v_T/v_Σ as a function of x for $n = 0.5$, $\alpha = 1.0$, and different values of $h(y)$: 0.5(≈ 0.42) (curve a), 1.0(≈ 0.34) (curve b), 1.5(≈ 0.28) (curve c). Values of y are read off Fig. 11.

model is characterized by a unique y [see (84)]. The polytropic models favoring higher redshifts are clearly exhibited in Figs. 11 and 14.

The correspondence mentioned above, between y and h , suggests an increasing of the stability of the models with a

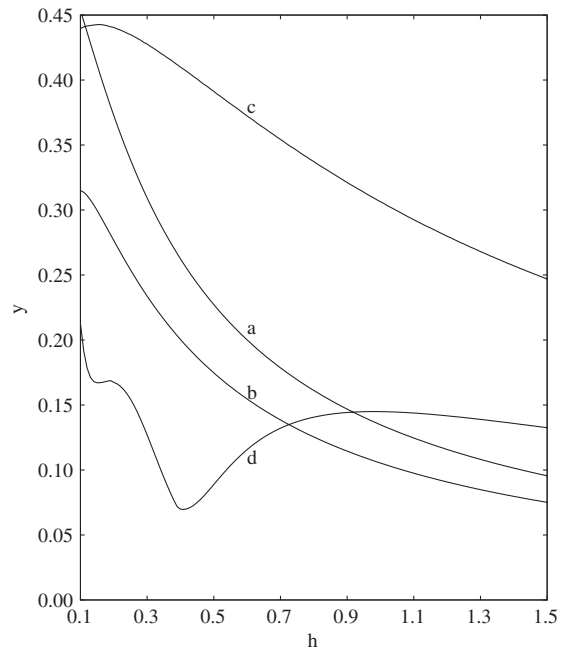


FIG. 14. Case II: Potential at the surface y as a function of the anisotropy parameter, h for pairs (n, α) : (1.0,0.1) (curve a); (2.0,0.1) (curve b), (1.0,1.0) (curve c), (2.0,1.0) (curve d).

decreasing h , except for the cases where such a correspondence is broken. In order to delve deeper into this interesting feature of the models, we have investigated the behavior of the Tolman mass within the sphere.

Figure 12 displays the Tolman mass (normalized by the total mass), for case I, as function of x for the selection of values of the parameters indicated in the legend. As we move from the less compact configuration (curve e) to the more compact one (curve a), the Tolman mass tends to concentrate on the outer regions of the sphere. This behavior persists for a wide range of values of the parameters (see Fig. 13). Now in order to better understand this effect, we recall that we can regard the slow, adiabatic evolution of a sphere, as a sequence of spheres in equilibrium. Thus in the process of contraction we move from e to a, i.e., the system shifts to more compact configurations with smaller values of h and with the Tolman mass decreasing in the inner regions. In other words as the sphere gets more compact, equilibrium configurations correspond to smaller values of h , which in turn produce a sharper effect of migration of Tolman mass towards the boundary surface. In terms of stability, and keeping in mind the physical meaning of the Tolman mass, it may be said that more stable configurations correspond to smaller values of h , since these are associated to a sharper reduction of the active gravitational mass in the inner regions, thereby providing a clear physical picture of the described scenario. It is worth mentioning that this effect was observed some years ago [56] in the case of the Florides solution [57].

For case II the situation is essentially the same, as shown in Fig. 15, although with an important difference. Indeed, for the values of the parameters considered in Fig. 15 and for a wide range of values, the behavior of the Tolman mass is basically the same as in case I. However for the values of Fig. 16 we observe a deviation from that behavior, in a small region close to the boundary surface, where the Tolman mass is larger for smaller values of h (curve a), than for larger ones (curves b and c). Again, speaking in terms of stability, it appears that for the case of Fig. 16, the stability is enhanced by decreasing of h , in the inner regions of the sphere, while the opposite seems to happen in the outer layers. An extreme example of the effect mentioned above is shown in Fig. 17. As we can see, curve a corresponding to the minimal value of h , describes a configuration with the maximal value of the Tolman mass inside the sphere. As the adiabatic collapse proceeds, we move into curve c corresponding to the maximal value of h , and from thereof, to curve b corresponding to the isotropic case. Thus, the less stable configuration in this example is that corresponding to a. On the other hand case b is more stable than c in some inner region, while the opposite happens in the outer layers.

So, it appears that the stability of the “core” and the “envelope” may respond differently to different degrees of

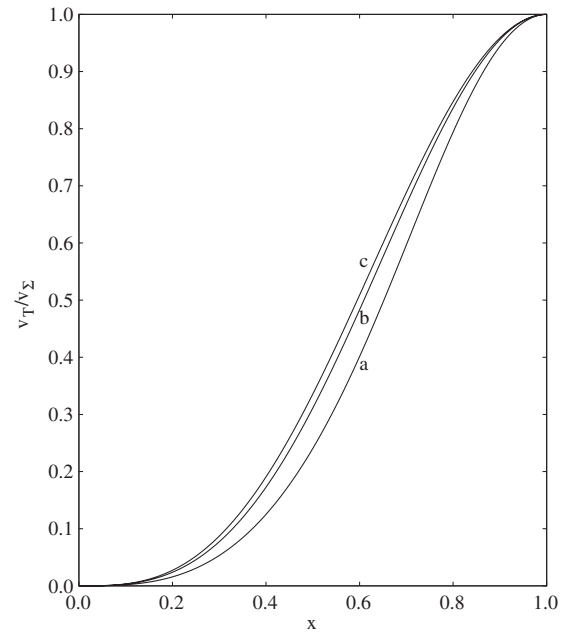


FIG. 15. Case II: v_T/v_Σ as a function of x for $n = 1$, $\alpha = 0.1$, and different values of $h(y)$: $0.5(\approx 0.23)$ (curve a), $1.0(\approx 0.13)$ (curve b), $1.5(\approx 0.1)$ (curve c). Values of y are read off Fig. 14.

anisotropy (different h). This fact was already pointed out, and discussed in some detail for the Bowers-Liang solution [58], in [59]. Here we have arrived at the same result by analyzing the behavior of the Tolman mass, whereas in [59] such a peculiar effect was obtained by studying the behavior of the adiabatic index.

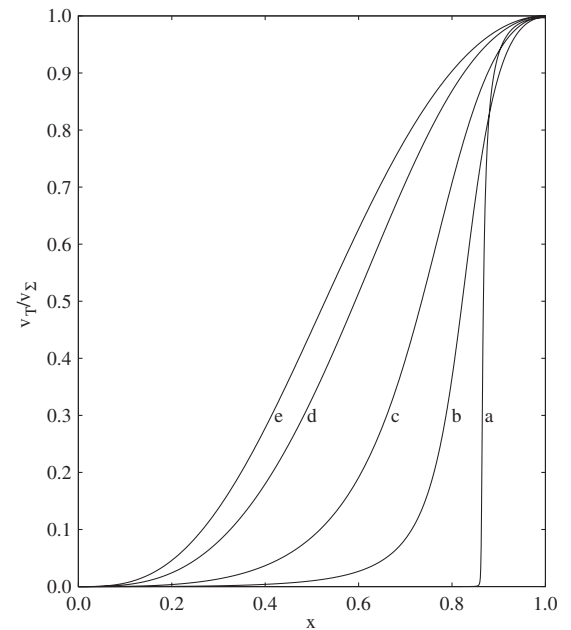


FIG. 16. Case II: v_T/v_Σ as a function of x for $n = 1$, $\alpha = 1.0$, and different values of $h(y)$: $0.1(\approx 0.44)$ (curve a), $0.3(\approx 0.43)$ (curve b), $0.5(\approx 0.39)$ (curve c), $1.0(\approx 0.31)$ (curve d), $1.5(\approx 0.25)$ (curve e). Values of y are read off Fig. 14.

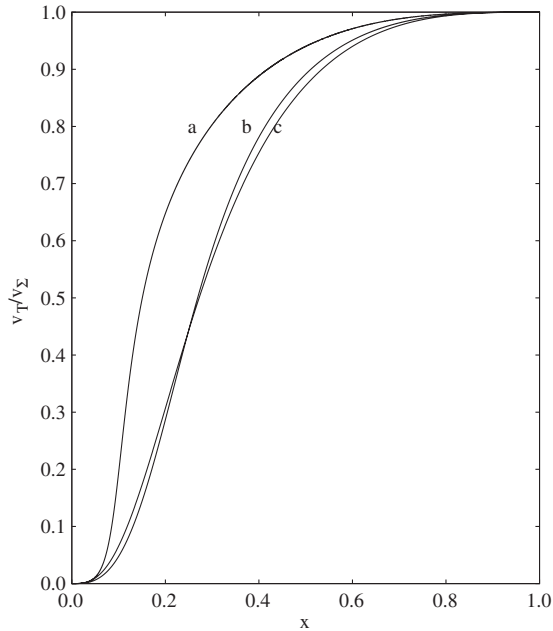


FIG. 17. Case II: v_T/v_Σ as a function of x for $n = 2$, $\alpha = 1.0$, and different values of $h(y)$: $0.5 (\approx 0.09)$ (curve a), $1.0 (\approx 0.14)$ (curve b), $1.5 (\approx 0.13)$ (curve c). Values of y are read off Fig. 14.

Finally, the following two comments are in order:

- (i) For both cases I and II we have found that for some selection of values of the parameters, the behavior of functions ψ_0 and ψ , close to the boundary surface, deviates from the “standard” behavior depicted in Figs. 1 and 6. This “anomalous” behavior is shown in Figs. 2 and 7. On the other hand, another anomalous behavior in the stability (Tolman mass) was found for case II, close to the boundary (in some range of values of the parameters). Both behaviors occur for similar values of the parameters. However, while the former appears in both cases, the latter was only found for case II. Therefore the possible link between both kinds of anomalous behaviors, if any, has not been established.
- (ii) The fact that some features of our models (e.g., the different stability response to different degrees of anisotropy) are shared by a completely different kind of anisotropic models, suggests that such features might be common to all, or at least to a wide family of, anisotropic compact stars.

V. CONCLUSIONS

We have presented the general framework for the modeling of general relativistic polytropes in the presence of anisotropic pressure. As mentioned in the Introduction, we undertook this task, motivated on the one hand by the conspicuous presence of such an anisotropy in compact objects and its influence on their structure, and on the other by the fact that polytropes represent fluid systems with a wide range of applications in astrophysics (e.g., Fermi fluids).

Thus, we have identified two possible polytropic equations of state. For each case we have found the full set of equations, which become the generalized Lane-Emden equation for the anisotropic matter in the Newtonian limit.

As it should be expected (since an additional variable is introduced), the above-mentioned set of equations requires additional information to be integrated. This information should be provided through the specific description of the anisotropy present in each problem. However, in order to illustrate our formalism we have proceeded to adopt a specific ansatz in order to integrate the system for both cases. The motivation for such a choice was already explained, however it should be emphasized that the main reason to present such models was not to describe any specific astrophysical scenario, but to convince the reader that the formalism works.

Nevertheless the obtained models exhibit some interesting features which deserve to be commented. We did so in the previous section with the help of the Tolman mass concept. Particularly interesting is the fact that some of the more relevant features of the models are present in models not related in any way to the ansatz used here.

At this point, we envisage two possible lines of research to apply the formalism presented in Sec. III, namely:

- (i) to identify the Δ associated to a specific astrophysical problem from purely physics considerations and then proceed to integrate the corresponding set of equations;
- (ii) to introduce a heuristic ansatz (different from the one assumed here) allowing to integrate the system (64) and (65) or (66) and (67). One example of which, could be the method proposed by Lake to obtain anisotropic solutions to Einstein equations from isotropic Newtonian solutions [60].

Also, among the many unanswered questions, related to the issue considered here, two are called particularly to our attention, namely:

- (i) is there a link between the two anomalous behaviors described in the previous section?
- (ii) is the different response of the stability, to different degrees of anisotropy, somehow related to the possible “cracking” of the configurations as described in [25,61] (and references therein)?

Finally, we want to stress that all we have done here requires spherical symmetry, at least as an approximation. It is possible that this symmetry can be broken by a strong magnetic field, rendering the distribution anisotropic and nonspherical. In such a case, of course, the method presented here does not apply, or eventually applies only approximatively.

ACKNOWLEDGMENTS

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