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Disformal invariance of second order scalar-tensor theories: Framing the Horndeski action

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The Horndeski action is the most general one involving a metric and a scalar field that leads to second-order field equations in four dimensions. Being the natural extension of the well-known scalar-tensor theories, its structure and properties are worth analyzing along the experience accumulated in the latter context. Here, we argue that disformal transformations play, for the Horndeski theory, a similar role to that of conformal transformations for scalar-tensor theories *a là* Brans–Dicke. We identify the most general transformation preserving second-order field equations and discuss the issue of viable frames for this kind of theory, in particular, the possibility to cast the action in the so-called Einstein frame. Interestingly, we find that only for a subset of the Horndeski Lagrangian such a frame exists. Finally, we investigate the transformation properties of such frames under field redefinitions and frame transformations and their reciprocal relationship.

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I. INTRODUCTION

There are very few theories that can rival general relativity (GR) for elegance, simplicity, and longevity. This pillar of modern physics, thanks to its remarkable agreement with experiments, is nowadays unanimously considered as the standard model of classical gravitational interactions [1,2]. It might seem then that the growing attention that in the last years has been devoted to generalized theories of gravitation is a preposterous attitude. There are, however, very good reasons to not be satisfied with the present state of affairs.

On the theoretical side, GR remains poorly understood in its foundations: we can construct very many alternative theories of gravitation, but we do lack an axiomatic derivation of such theories and hence an authentic understanding of their reciprocal relation. Moreover, generalized theories of gravitation can be as well considered as different effective actions induced by physics beyond the Planck energy, and, as such, their study as alternative models of gravitation could provide some insight on the long-standing problem of building a quantum gravity theory.

On the experimental side, we do lack severe experimental constraints on GR from galactic scales upward. Of course, we do know from both cosmology and astrophysics that GR plus a cosmological constant so far provides a very good description of the observed Universe [3]. However, this comes at the price of accepting that 95% of the energy/matter content of the Universe is of unknown nature. Indeed, dark matter and dark energy have been among the most pressing motivations for the recent outburst of attention toward alternative theories, to name a few, $f(R)$ theories of gravity [4], generalized scalar-tensor theories [5], Tensor-Vector-Scalar theories [6], and modified newtonian dynamics [7–9] (see also Ref. [10] and references therein for a thorough presentation).

In particular, generalized Brans–Dicke scalar-tensor theories have acquired, since their initial proposal more than half a century ago [11], a most relevant role as the standard alternative theories of gravitation. The investigation of formal aspects of these theories has played a fundamental role for several theoretical and observational issues in gravitation. In particular, scalar-tensor theories have represented an ideal setting for understanding the thorny issue of the different representations of a given gravitational theory. For example, it has been realized that a whole class of higher-curvature theories, $f(R)$ theories, can be recast as special cases of scalar-tensor theories (with the number of scalars related to the order of the initial field equations). Even more interestingly, the invariance of the action of generalized scalar-tensor theories under metric conformal transformations and redefinitions of the scalar field can be used to relate several equivalent frames, for example, trading off a space-time varying gravitational constant (i.e., a nonminimal coupling) for a GR-like gravitational sector (i.e., minimally coupled) associated to a matter action with field-dependent mass and coupling constants.

It is worth stressing that such features are not only theoretically interesting but are also relevant for the actual observational tests of the theory, so much so that the question of whether conformally related frames are physically distinguishable is still an open issue in the literature (see, e.g., Ref. [12]). Furthermore, these kinds of investigations become even more important as one moves further away from GR into more general theories.

Further generalizations of the scalar-tensor theories have been extensively investigated in the contexts of cosmology [13], dark energy (DE) [14–16] , and Inflationary models

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[17,18] and have indeed provided very efficient frameworks for explaining (in a alternative ways with respect to GR) the observed properties of the Universe.

An extension of the scalar-tensor theories framework that has attracted a lot of interest is represented by the Horndeski action [19], recently rediscovered in the context of the covariant Galileon theory [20]. This action provides the most general Lagrangian for a metric and a scalar field that gives second-order field equations and as such is a well-motivated effective field theory. This class of scalar-tensor theories has been extensively investigated since it includes, as subcases, basically all known models of DE and single scalar field inflation. However, this generality comes at a dear price. In fact, the physics derived from the full action is rather obscure, and the theory has been investigated only in a few regimes or for particular models, like the Friedman-Lamaitre-Robertson-Walker universe, so that a systematic investigation is still missing (see, however, Refs. [21,22] for a first attempt in this direction and Refs. [23,24] for a method to derive constraints in the context of DE models).

Given the above-mentioned fruitful interplay between scalar-tensor theories and conformal transformations, one may wonder whether a generalization along this line might help shed some light on the properties and structure of the Horndeski theory. This is the main motivation of the present work. As we shall see in what follows, simple conformal transformations are not enough for this task, due to the more complicated structure of the Horndeski actions, and the use of generalized metric transformation will be required.

An example of such a generalized metric transformation is given by disformally related metrics. These were proposed in Ref. [25] and applied first in the context of relativistic extensions of modified newtonian dynamics-like theories [26] in order to account for measured light deflection by galaxies. Later, they found applications in varying speed-of-light models [27], dark energy models [21,28–30], inflation [31], and modified dark matter models [32,33]. More recently, empirical tests of these ideas have been proposed in laboratory experiments [34] as well as in cosmological observations [35–37], signaling the important role that disformal transformation is playing in contemporary cosmology and gravitation theory.

The paper is organized as follows. In Sec. II we briefly introduce the Horndeski action and the disformal transformations, discussing the most general set of transformations that leave the action invariant. In Sec. III we discuss some specific cases of disformal transformations and identify the subset of Horndeski actions admitting an Einstein frame. In Sec. IV we discuss in more detail the issue of disformal frames and their properties under field redefinitions and metric transformations, providing some explicit examples and discussing their reciprocal relationship. Finally, in Sec. V we draw our conclusions.

II. HORNDESKI ACTION AND DISFORMAL TRANSFORMATIONS

The Horndeski Lagrangian [19] is the most general Lagrangian that involves a metric and a scalar field that gives second-order field equations in both fields in four dimensions. Recently generalized to arbitrary dimensions by Deffayet *et al.* in Ref. [20], it is the natural extension of scalar-tensor theories *a là* Brans–Dicke.

Horndeski theory remained a sort of theoretical curiosity for more than thirty years, but it was recently rediscovered as a powerful tool in cosmology. In fact, its generality (within the bound of second-order field equations) made of it an ideal metatheory for scalar-tensor models of dark energy and dark matter. However, up to now, no structural analysis analogous to the one carried out for standard scalar-tensor theories was performed. In particular, there is no obvious extension of the concept of equivalent frames and no principle to fix the shape of the free parameter functions. To address these points, after briefly reviewing the Horndeski action and disformal transformations, we shall discuss here the behavior of this theory under such an extended class of metric transformations.

A. Horndeski Lagrangian

The Horndeski action, rephrased in the modern language of Galileons [38],[1] can be written as

$$\mathcal{L} = \sum_i \mathcal{L}_i, \tag{1}$$

where

$$\mathcal{L}_2 = K(\phi, X), \tag{2}$$

$$\mathcal{L}_3 = G_3(\phi, X)\Box\phi, \tag{3}$$

$$\mathcal{L}_4 = G_4(\phi, X)R + -G_{4,X}(\phi, X)[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2], \tag{4}$$

$$\mathcal{L}_5 = G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi + \frac{G_{5,X}}{6}[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\nu\nabla_\mu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3], \tag{5}$$

where $X = \nabla_\mu\phi\nabla^\mu\phi/2$, $(\nabla_\mu\nabla_\nu\phi)^2 = \nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi$ and $(\nabla_\mu\nabla_\nu\phi)^3 = \nabla_\nu\nabla_\mu\phi\nabla^\nu\nabla^\lambda\phi\nabla_\lambda\nabla_\mu\phi$, while $G_{i,X} = \partial G_i/\partial X$, R is the Ricci scalar, and $G_{\mu\nu}$ is the Einstein tensor. The coefficient function G_4 has the dimensions of a mass square, and it plays the role of a varying gravitation constant, while G_5 has those of a mass to the fourth power. The field ϕ is taken to have mass dimension 1. As

[1]Notice that we have a different sign convention with respect to Ref. [38] due to the different definition of the function $X \equiv \nabla_\mu\phi\nabla^\mu\phi/2$.

said, this gravitational action is the most general one that can be built with a metric and a scalar field, providing second-order field equations in four dimensions. Notice that, despite the presence of second-order derivatives in the action, no new degree of freedom is introduced, thus evading Ostrogradski's theorem, which states that such extra degrees of freedoms lead to classical instabilities [39]. We will not discuss here the equations of motion, referring the interested reader to Refs. [20,40] for a general analysis. Let us instead focus our attention on some important properties of this Lagrangian.

First of all notice that, beyond the usual conformal nonminimal coupling, there is another source that couples the Einstein tensor to second-order derivatives of the field. This represents a novelty as, contrary to what happens for the coupling to the Ricci scalar, in this case, we have a direction-dependent coupling. Second, all the sub-Lagrangians give second-order field equations independently so that one could, in principle, neglect some of them without spoiling the second-order nature of the field equations. However, as is shown in Appendix C, eventually, neglected terms can always be generated through redefinitions of the field variables. Finally, we notice that, compared with the standard scalar-tensor theories action, the nonminimal coupling (NMC) coefficients now depend also on the kinetic term.

Given that this model is a generalization of standard, one may wonder whether suitable metric transformations can be introduced also in this case, leaving the action invariant and linking alternative frames. It not hard to realize that simple conformal transformations have limited power in this sense. In standard scalar-tensor theories , these transformations allow us to replace by constants some of the field-dependent coefficients. However, the various terms appearing in the Horndeski action $[K(\phi, X), G_i(\phi, X)]$ are also dependent on the kinetic term X, and hence more general transformations are clearly needed.

The most natural extension of the conformal transformation in this sense would be $A(\phi) \rightarrow A(\phi, X)$. However, even if this can remove the nonminimal coupling in the $\mathcal{L}_4$, it is basically ineffective on the the nonminimal coupling provided by $\mathcal{L}_5$. Moreover, this generalized conformal transformation contains derivatives of the field, and hence one must be careful that those do not end up introducing higher derivatives in the equations of motion. In this sense, the next natural candidate for a suitable set of metric transformations is then represented by the disformal ones.

B. Disformal transformations

Disformal transformations are defined by the relation[2]

$$\bar{g}_{\mu\nu} = A(\phi, X) g_{\mu\nu} + B(\phi, X) \phi_\mu \phi_\nu, \tag{6}$$

where the disformal functions A and B now depend on both the scalar field ϕ and its kinetic term X and where we have defined for convenience $\phi_\mu = \nabla_\mu \phi$. We can classify the properties of this generalization in two main categories: first, the new functions do not simply depend on the local value of the field but also on the metric itself, hidden inside the definition of the kinetic term; second, we have a translation along the lines of variation of the field, which means that the new metric will also depend on the way the field is changing through space-time.

When dealing with metric transformations, one has to ensure that the new metric is still a good one. We can formally define the goodness of a metric transformation with a set of properties: it must preserve a Lorentzian signature, it must be causal, and it has to be invertible, with a nonzero volume element. All these properties directly translate into constraints on the two free functions A and B, which we are going to discuss one by one.

Lorentzian signature Consider a frame in which $\phi_\mu = (\phi_0, \vec{0})$. Then, the Lorentzian requirement can be translated into

$$\bar{g}_{00} = A(\phi, X) g_{00} + B(\phi, X) \phi_0 \phi_0 < 0. \tag{7}$$

This constraint must hold true for all values of the field and its derivative. Given that we cannot exclude that for some values of the field variables the function B can be zero, a first requirement is that $A > 0$. This is the usual requirement made also for standard scalar-tensor theories. Then, by multiplying Eq. (7) with g^{00}, we found that the condition to be fulfilled for preventing $\bar{g}_{00}$ from sign inversion is

$$A(\phi, X) + 2B(\phi, X) X > 0. \tag{8}$$

As a consequence, to have this relation to hold true for all values of X, it is necessary to have some kinetic dependence at least in one of the two disformal functions. This result was first derived in Ref. [41] (see also the original paper by Bekenstein [25]). However, in Ref. [21] it was argued that the dynamics of the scalar field can be such that it is possible to keep the metric Lorentzian even with no X dependences in the disformal functions A and B. For example, this can happen when the scalar fields enters a slow-roll phase, e.g., when thought to be the field responsible for dark energy. However, this subject is not yet fully understood, and, being not mandatory for our purposes in this investigation, in the following, we will assume that both metrics are Lorentzian for all the values of the scalar field and its kinetic term.

Causal behavior The disformal metric can have, depending on the sign of the B function, light cones wider or narrower than those of the metric g. This may lead one to think that particles moving along one metric may show superluminal or acausal behavior. However, the requirement of the invariance of the squared line element and recalling that physical particles satisfies $ds^2 < 0$ is enough

[2] More general formulations may be possible, for example, including higher derivatives of the scalar field or by adding vector fields [25].

to ensure causal behavior. This objection has been discussed in some details in Ref. [42].

Invertible We also must be sure that an inverse of the metric and the volume element are never singular. The inverse disformal metric is given by

$$\bar{g}^{\mu\nu} = \frac{1}{A(\phi,X)} g^{\mu\nu} - \frac{B(\phi,X)/A(\phi,X)}{A(\phi,X)+2B(\phi,X)X} \nabla^\mu \phi \nabla^\nu \phi, \quad (9)$$

while the volume element is given by $\sqrt{-\bar{g}} = A(\phi)^2 \times (1 + 2XB/A)^{1/2}\sqrt{-g}$. The constraints derived from these requirements are weaker than those already obtained; hence, there are no new potential issues.

From this analysis we learn that the extension of conformal transformations to disformal ones is well posed, even if all previous points deserve a deeper analysis, which, in any case, is beyond the scope of the present paper and is left for further studies.

Disformal metrics seem to be good candidates for our purposes as they possess, beyond a purely conformal term, another one that is a deformation of the metric along the direction of variation of the field, and indeed disformal transformations have for the Horndeski action a role very similar to that of conformal transformations for standard scalar-tensor theories.

C. Invariance of the Horndeski Lagrangian under disformal transformations

The ability of the Horndeski action to give second-order field equations resides in a fine cancelation between higher derivatives coming from NMC terms and those produced from derivative counterterms, which requires the coefficient functions of the second field derivatives in $\mathcal{L}_4$ and $\mathcal{L}_5$ to be proportional to the derivative of G_i with respect to X [43].

It is easy to see that this requirement already reduces the freedom in the disformal functions A and B. In fact, any kinetic dependence of these two terms would lead unavoidably to the breaking of the Horndeski structure, i.e., to higher-order equations of motion. We prove this through some examples in Appendix A, while, here, we give a first principle argument as to why one should expect this to happen. The ability of the Horndeski action to give second -order field equations lies on the antisymmetric structure of second derivative terms, as was made clear in Ref. [20]. Consider the $\mathcal{L}_4$ part of the Lagrangian. This can be rewritten in the form

$$\mathcal{L}_4 = (g^{\mu\beta}g^{\nu\alpha} - g^{\mu\nu}g^{\alpha\beta})[G_4(\phi,X)R_{\mu\nu\alpha\beta} - G_{4,X}(\phi,X)\nabla_\mu\nabla_\nu\phi\nabla_\alpha\nabla_\beta\phi], \quad (10)$$

where the antisymmetric structure is made clear. Given that we have to preserve this structure in order to keep the equations of motion second order, any transformation operated on the fundamental variables φ and $g_{\mu\nu}$ has to be necessarily reabsorbed into the coefficient function G4 and its derivative modulo a surface term. However, any kinetic dependence in the disformal functions will spoil this structure. In fact, consider the transformation property of the second derivatives of the scalar field under the conformal transformation $\hat{g}_{\mu\nu} = A(X)g_{\mu\nu}$:

$$\nabla_\mu\nabla_\nu\phi \rightarrow \nabla_\mu\nabla_\nu\phi + \frac{A_{,X}}{A}[g_{\mu\nu}\phi^\alpha\phi^\beta\nabla_\alpha\nabla_\beta\phi + -\phi_\mu\phi^\alpha\nabla_\alpha\nabla_\nu\phi - \phi_\nu\phi^\alpha\nabla_\alpha\nabla_\mu\phi]. \quad (11)$$

When inserted in Eq. (10), among other terms, the following one is generated:

$$\sim 4G_{4,X}\left(\frac{A_{,X}}{A}\right)^2 \phi^\mu\phi^\nu\phi^\alpha\phi^\beta\nabla_\mu\nabla_\nu\phi\nabla_\alpha\nabla_\beta\phi, \quad (12)$$

which is clearly symmetric in the four indices and hence will produce higher than second derivatives in the equations of motion. One may wonder whether there may be counterterms coming from curvature that eliminate this, but, as shown in the Appendix, this is not the case. This result may seem an artifact of the transformation used. In this sense, it has been recently shown how, in some specific cases, such higher-than-second derivatives can be eliminated in the equations of motion using a hidden dynamical constraint [44]. However, further analysis is required to see whether this result can be generalized to the full Horndeski action.

We hence conclude that, in order to be sure to preserve second-order field equations, we have to restrict our analysis to the following class of disformal transformations[3]:

$$\bar{g}_{\mu\nu} = A(\phi)g_{\mu\nu} + B(\phi)\phi_\mu\phi_\nu. \quad (13)$$

In Appendix C we show that this transformation preserves the antisymmetric structure of the Horndeski action as its effects happen to simply renormalize the coefficient functions K and G_is. We refer the reader to this Appendix for the detailed transformation properties, while here we discuss the meaning of this and analyze some relevant subcases. As a concluding remark, let us add that the Horndeski action is clearly also invariant under the field rescaling $\phi \rightarrow s(\phi)\phi$ (this is explicitly discussed Appendix D, in which we consider the effects of this transformation on the Horndeski coefficient functions). This property will play an important role later on in our discussion when we shall deal with the equivalence of disformal frames.

III. SPECIAL CASES

The structural invariance of the action under disformal transformations translates into the statement that such

[3]Even though we do not provide a formal proof that this relation is the most general that leaves the Horndeski action invariant, we notice that more general transformations, despite being possible, have to introduce higher derivatives of the scalar field [44].

transformations represent a symmetry of the Horndeski action so that all the functions are defined modulo a conformal and a disformal transformation. This reminds us very closely of the case of standard scalar-tensor theories in which invariance under conformal transformations is used to reduce the number of free functions that defines the theory. However, the generalization of this reasoning to the case of the Horndeski theory is not straightforward. In fact, the subset of disformal transformations (13) does not allow for kinetic term dependent coefficients; consequently, one cannot generically rescale the functions $(K(\phi, X), G_i(\phi, X)$ characterizing the Horndeski action. The next two subsections are devoted to analyze this issue further. We will study the transformation properties of the Horndeski action under pure conformal and disformal transformations separately, and, in particular, we will provide the subclass of Horndeski theories that admits a representation in which all NMC terms are eliminated via a disformal transformation.

A. Purely conformal transformations

Let us first consider the effects of conformal transformations $\bar{g}_{\mu\nu} = A(\phi) g_{\mu\nu}$ on the Horndeski action (1), extending the well-known results for these transformations in scalar-tensor theories to this more general class of actions. The transformed Lagrangian coefficient functions read

$$\bar{K}(\phi, X) = A^2 K(\phi, X_C) + 2XG_3 AA' + 3X\frac{G_4 A'[1-2A]}{A} + \frac{6G_5 X^2 A'}{A}\left[\frac{A''}{A} - \frac{A'^2}{A^2}\right] - 2XH_{5,\phi} + \frac{2G_{5,X}X^3}{A^3}A'^2, \tag{14}$$

$$\bar{G}_3(\phi, X) = AG_3(\phi, X_C) - 2G_{4,X}A' + X\left(-2H_{\Box,\phi} - \frac{G_5 A'^2}{2A^2} + \frac{2G_5 A''}{A} + \frac{G_{5,X} X A'^2}{A^2}\right) - H_5 \tag{15}$$

$$\begin{aligned}\bar{G}_4(\phi, X) &= A(\phi) G_4(\phi, X_C),\\ \bar{G}_5(\phi, X) &= G_5(\phi, X_C),\end{aligned} \tag{16}$$

where

$$X_C = X/A(\phi), \qquad H_\Box = G_5 \frac{A'}{A}, \tag{17}$$

$$H_5 = \int dX\left[H_{\Box,\phi} + \frac{G_5 A''}{A} + \frac{5G_5}{2}\frac{A'^2}{A^2} + 2G_{5,X}\frac{A'}{A}\right]. \tag{18}$$

The effects of the transformation are manifolds. First, notice that there is a hierarchical propagation of terms from higher derivatives Lagrangians toward lower ones; that is to say $\mathcal{L}_i$ generates terms that contribute to all $\mathcal{L}_{j<i}$ so that, even if in the original Lagrangian some terms were neglected, they will inevitably appear after a conformal transformation. A special case is represented by the $\mathcal{L}_5$ Lagrangian that cannot be generated this way. Second, the conformal NMC $G_4(\phi, X)$, is modified by a multiplicative factor while the NMC with the Einstein tensor is unaffected apart from a redefinition of the kinetic term inside $G_5(\phi, X)$.

Given that, in general, all the coefficient functions depend on both the scalar field and its kinetic term, it is clear that using only a conformal transformation we shall not be able to eliminate nonminimal couplings for any choice of the conformal factor $A(\phi)$. Even in the special case in which the coefficient functions depend only on the field and not on its derivatives, we are able to set at most $G_4(\phi) = 1$ while retaining the generalized NMC between the Einstein tensor and the field derivatives (5), given that part of the Lagrangian is not affected by conformal transformations; see Eq. (16).

Notice that, even if we were to take $G_5(\phi, X)$ to be a function of the scalar field only, we would not be able to eliminate it. In fact, we then have the relation

$$\begin{aligned}G_5(\phi) G_{\mu\nu}\nabla^\mu\nabla^\nu\phi &= G_{5,\phi} XR - G_{5,\phi}[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2]\\ &\quad - G_{5,\phi\phi}[2X\Box\phi - \phi^\mu\phi^\nu\nabla_\mu\nabla_\nu\phi],\end{aligned} \tag{19}$$

that shows how, in this case, $\mathcal{L}_5$ is a contribution to the $\mathcal{L}_4$ (as well as to $\mathcal{L}_3$ and $\mathcal{L}_2$) that depends explicitly on the kinetic term and cannot be eliminated by a simple conformal transformation.

B. Purely disformal transformations

We turn our attention now to the case of a pure disformal transformation, i.e., when the conformal factor $A(\phi)$ is set to 1 while the disformal function $B(\phi)$ is left unspecified. Given that we are mainly interested in the effects of transformations on the NMC terms, we will report here only the relevant coefficient functions. The remaining ones can be easily derived from the equations in Appendix C, and the discussion of the effects of the disformal transformation on them is analogous to that for conformal transformation.

In the case under consideration, we have that the transformed NMC coefficient functions read

$$\bar{G}_4(\phi, X) = (1 + 2XB)^{1/2} G_4(\phi, X_D) + \frac{G_5(\phi, X_D) B'(\phi) X^2}{(1 + 2XB)^{3/2}} - H_{R,\phi}(\phi, X) X, \tag{20}$$

$$\bar{G}_5(\phi, X) = \frac{G_5(\phi, X_D)}{(1 + 2XB)^{1/2}} + H_R(\phi, X), \tag{21}$$

where

$$\begin{aligned}X_D &= X/(1 + 2BX),\\ H_R(\phi, X) &= B\int dX \frac{G_5(\phi, X_D)}{(1 + 2XB)^{3/2}}.\end{aligned} \tag{22}$$

Here, we notice that the effects of the disformal transformation are richer than those of the conformal one. In fact, besides a conformal modification of G_4, we have other contributions to $\mathcal{L}_4$, and, in this case, G_5 is modified as well. In particular, the modified coefficient functions receive corrections that depend on the kinetic term, but, as can be seen from Eqs. (20) and (21), even in this case, one cannot generically eliminate the NMC.

Let us focus on this last point and study which constraints can be imposed on the coefficient functions of the Hornedski action so as to be able to eliminate all the NMC, i.e., to use the disformal transformation to obtain $\bar{G}_4 = 1$ and $\bar{G}_5 = 0$. The latter condition is satisfied if

$$\frac{G_5(\phi, X_D)}{(1 + 2XB)^{1/2}} + B \int \frac{G_5(\phi, X_D)}{(1 + 2XB)^{3/2}} dX = 0$$
$$\Rightarrow \int dX \left[\frac{G_{5,X}(\phi, X_D)}{(1 + 2XB)^{1/2}} \right] = 0. \quad (23)$$

In general, if $G_5 = G_5(\phi)$, the above constraint is automatically satisfied. We cannot exclude the existence of other solutions in which an X dependence is also allowed, for example, if the integrand function is fast oscillating. However, these will depend on the specific model chosen and would need to be investigated case by case. Finally, notice that this constraint is not influenced by the freedom in rescaling the scalar field.

To have no conformal coupling to gravity, we have to impose

$$1 = (1 + 2XB)^{1/2} G_4(\phi, X_D) + \frac{G_{5,\phi}(\phi, X_D) X}{(1 + 2XB)^{1/2}} - \tilde{G}_5 X, \quad (24)$$

where

$$\tilde{G}_5(\phi, X) = \int dX \frac{G_{5,X}(\phi, X_D)}{(1 + 2BX)^{1/2}}. \quad (25)$$

This allows us to find the form of the untransformed function G_4 in terms of the transformed variable X_D. Inverting relation (22),[4] we can find the form of the function in terms of the untransformed variable X that is needed to satisfy the requirement, namely,

$$G_4(\phi, X) = (1 - 2B(\phi)X)^{1/2} - G_{5,\phi}(\phi)X + \tilde{G}_{5,\phi}(\phi, X)X, \quad (26)$$

with

$$\tilde{G}_5(\phi, X) = \int dX (1 - 2BX)^{1/2} G_{5,X}(\phi, X). \quad (27)$$

Given that we want both constraints to be satisfied at the same time, we have then

$$G_5 = G(\phi) \quad \text{and}$$
$$G_4(\phi, X) = (1 - 2B(\phi)X)^{1/2} - G_{5,\phi}(\phi)X, \quad (28)$$

which fixes once and for all the functional dependence of the $G_4(\phi, X)$ function on the kinetic term.

We conclude that the Lagrangian

$$S_{\rm NMC} = \int d^4x \sqrt{-g} [G_4(\phi, X) R - G_{4,X}[(\Box\phi)^2 - (\nabla_\mu \nabla_\mu \phi)^2] + G_5(\phi) G_{\mu\nu} \nabla^\mu \nabla^\nu \phi], \quad (29)$$

where G_4 is given by Eq. (28), is the only one that admits a disformal map able to eliminate all the NMC terms in the context of Horndeski theory.

However, it is worth noticing that, inserting Eqs. (19) and (28) in Eq. (29), all the terms depending on $G_5(\phi)$ end up cancelling. Hence, if the function G_5 depends only on the scalar field, we conclude that the existence of a disformal metric able to cancel all NMC *requires* the absence of $\mathcal{L}_5$.[5] We are hence left with the action

$$S_{\rm NMC} = \int d^4x \sqrt{-g} [G_E(\phi, X) R - G_{E,X}[(\Box\phi)^2 - (\nabla_\mu \nabla_\mu \phi)^2]], \quad (30)$$

where

$$G_E = (1 - 2B(\phi)X)^{1/2}. \quad (31)$$

While we have considered here a special subset of the ϕ-dependent disformal transformations (13), we can easily extend our conclusions to transformations including a conformal factor $A(\phi)$. Indeed, in this case, the most general action allowing for a full elimination of the NMC would be the same as Eq. (30) modulo a conformal rescaling of the $G_E(\phi, X)$ function.

As a final remark, it is perhaps worth is to stress that, as noted in Ref. [21], the nonrelativistic limit of the action (30) corresponds to the quartic covariant term of the Galileon action with the appropriate nonminimal coupling to yield second-order field equations [45].

IV. DISFORMAL FRAMES

The invariance of an action under metric transformations implies the possibility to fix some of the free functions characterizing the theory, similarly to what is done when choosing a gauge. Consequently, the number of the independent functions is reduced. In our specific case, the Horndeski action (5) is invariant under both purely conformal and disformal transformations. This freedom allows us

[4]This can always be done, as the Jacobian of the transformation $\frac{dX_D}{dX} = (1 + 2BX)^{-2}$ is never singular.

[5]It may seem that a constant G_5 could be included without spoiling our request of no NMC. However, in this case, $\mathcal{L}_5$ reduces to a surface term and hence does not contribute to the dynamics.

to define an infinite set of equivalent frames defined by different fixings of two of the free functions in the action (see Refs. [46,47] for a similar reasoning in standard scalar-tensor theories).

Among all these equivalent representations of the theory, two are the most relevant as they correspond to somewhat opposite situations: the Einstein and Jordan frames. For the sake of clarity, we provide here generalized definitions relevant for the Horndeski actions under consideration here.

Jordan Frame In the *Jordan frame*, the Lagrangian of the gravitational sector includes a nonminimally coupled scalar field; meanwhile, all the matter fields follow the geodesics of the gravitational metric (the stress-energy tensor of the matter fields is covariantly conserved with respect to the gravitational metric).

Einstein Frame In the *Einstein frame*, the gravitational dynamics is described by the standard Einstein–Hilbert Lagrangian (plus possibly a cosmological constant). However, matter fields are coupled to the gravitational metric via some function of the scalar field and its derivatives. They hence move on geodesics that can be different from the one determined by the metric defining the Ricci scalar. Moreover, the gravitational equations in the absence of matter do not reduce to $R = 0$, as in GR but, in general, will retain the scalar field as a possible source.

We now proceed to recall the issue of frames and their equivalence in standard scalar-tensor theories and then extend this to the case of the Horndeski action.

A. Scalar-tensor theories and conformal transformations

Scalar-tensor theories of gravity [5,10,13] represent one of the simplest and most studied extensions of GR, in which a scalar degree of freedom is added to the Lagrangian besides the metric and matter fields. A minimal prescription for generalizing GR is to promote the gravitational constant to a scalar field that must be provided with its own dynamics in order to preserve diffeomorphism invariance. Furthermore, the Einstein equivalence principle (EEP) allows such a scalar field to also mediate the coupling of the matter to the metric (albeit in a universal way). This reasoning then leads to the action

$$S = \int d^4x \left[G(\phi)R - \frac{f(\phi)}{2}\nabla_\mu\phi\nabla^\mu\phi - V(\phi)\right] + S_m[e^{2\alpha(\phi)}g, \psi], \tag{32}$$

where the four functions $G(\phi)$, $f(\phi)$, $V(\phi)$, and $\alpha(\phi)$ are general functions of their argument. We will not enter into the details of the applications of this theory, referring to the above-cited papers and to references therein for details, but we will focus on some more formal properties of this action.

First of all, the above-mentioned free functions in the action are actually redundant for fixing a particular action [46,47]. Indeed, the invariance of action (32) under the conformal transformations $\bar{g}_{\mu\nu} = \Omega^2(\phi)g_{\mu\nu}$ and the scalar field redefinitions $\bar{\phi} = F(\phi)$ imply the possibility to freely choose two out of the four functions. Hence, implementations of Eq. (32), differing only for the fixing of two of the four coefficient functions, are indeed just different representations of the same physical theory [46].

For this class of theories, the Einstein frame is defined by the choice $G(\phi) = 1$ and $f(\phi) = 1$ so that gravity is described by the standard Einstein–Hilbert action; the scalar field has a canonical kinetic term, while matter fields follow the geodesics of a physical metric conformally related to the gravitational one. The Jordan frame is instead obtained choosing $\alpha(\phi) = 0$ and $A(\phi) = \phi$. In this case, we have that all fields follow the same metric, but now the scalar field is nonminimally coupled to curvature, and it may possess a nonstandard kinetic term. The fact that the above two frames are picked up from Eq. (32) by just fixing two of the four coefficient functions implies their mathematical equivalence (i.e., a varying gravitational coupling in the Jordan frame is translated into field-dependent matter masses and couplings when the action is in the Einstein frame).

The lesson that we want to capture with this short introduction is that, when dealing with generalized actions like Eq. (32), one has to pay attention to their symmetries in order to correctly identify the set of equivalent frames (i.e., different representations of the same theory), which one can alternatively use for more conveniently dealing with different physical issues.

This considerations become even more important as further modifications of gravity are introduced and complicated terms are added. In what follows, we shall first investigate the issue for that class of Horndeski actions, admitting an Einstein frame (as this frame is often adopted for physical investigations). Later, we shall extend the discussion to more general actions.

B. Horndeski action and the Einstein frame

In Sec. III, we derived the most general action in the Jordan frame for which all NMC can be eliminated via the disformal transformation (13). However, the discussion of the possible equivalence of frames requires us to include also the action for matter fields with possible generalized coupling to the metric. This leads to the completion of Eq. (30),

$$S = \int d^4x\sqrt{-g}[G(\phi,X)R + -G_{,X}(\phi,X)[(\Box\phi)^2 - (\nabla_\mu\nabla_\mu\phi)^2] + K(\phi,X) + G_3(\phi,X)\Box\phi] + S_m[\bar{g},\psi], \tag{33}$$

where

$$G(\phi,X) = C(\phi)^2\left(1 - 2\frac{D(\phi)}{C(\phi)}X\right)^{1/2}, \tag{34}$$

S_m is the total matter action defined in terms of the physical metric

$$\bar{g}_{\mu\nu} = e^{\alpha(\phi)} g_{\mu\nu} + \beta(\phi)\phi_\mu\phi_\nu, \tag{35}$$

and ψ stands generically for matter fields.

In the action appear six free functions, four related to the field-metric couplings, $C(\phi)$, $D(\phi)$, $\alpha(\phi)$, and $\beta(\phi)$, and two defining the minimally coupled scalar field Lagrangian, $K(\phi, X)$ and $G_3(\phi, X)$. Thanks to the invariance under both conformal and disformal transformations, we can fix two out of the four metric functions $C(\phi)$, $D(\phi)$, $\alpha(\phi)$, and $\beta(\phi)$ with the appropriate choice of the functions $A(\phi)$ and $B(\phi)$ appearing in the disformal transformation (13). In principle, we could act on $K(\phi, X)$ and $G_3(\phi, X)$, but, given their generic dependence on the kinetic term, Eq. (13) is not effective for fixing them. Hence, with a general disformal transformation (13), we can define a Jordan and an Einstein frame in the same sense as it can be done for standard scalar-tensor theories.

However, we can also use the invariance of the Horndeski action under field rescaling to further constrain the number of independent functions [as in the case of action (32)]. In fact, as shown in Appendix D, we can always rescale the field ϕ by a function, provided that this does not lead to a constant. This amounts to saying that we can fix one more of the free functions $\alpha(\phi)$, $\beta(\phi)$, $C(\phi)$, and $D(\phi)$ to arbitrary values so that the Einstein and Jordan frames defined above represent a class of equivalent theories that can be further fixed with a field redefinition. We conclude that implementations of Eq. (33), which differ only by the fixing of three out of six functions, are nothing but equivalent representations of the same physical theory.

It is worth noticing that the invariance under two metric transformations allows the definition of more physically interesting equivalent frames with respect to standard scalar-tensor theories. In fact, we can actually define the following four equivalent frames, all obtained from the action (33) with different fixing of the free functions:

Jordan Frame The Jordan frame is defined by the action

$$S_J = \int d^4x\sqrt{-g}[G_J(\phi,X)R + -G_{J,X}[(\Box\phi)^2 - (\nabla_\mu\nabla_\mu\phi)^2] + +K(\phi,X) + G_3(\phi,X)\Box\phi] + S_m[g,\psi], \tag{36}$$

where we have fixed $\alpha = 1$ and $\beta = 1$ so that matter is minimally coupled to the metric that defines the curvature terms appearing in the action. As a consequence a conformal nonminimal coupling term, described by the presence of the function $G_J = C(\phi)^2(1 - 2D(\phi)X)^{1/2}$, is present and can be further constrained with a field redefinition.

Einstein Frame The Einstein frame is given by the action

$$S_E = \int d^4x\sqrt{-g}[R + K(\phi, X) + G_3(\phi, X)\Box\phi] + S_m[\bar{g}, \psi], \tag{37}$$

where the NMC has been eliminated by the fixing $C(\phi) = 1$ and $D(\phi = 0)$ in the action (33), but now matter feels a physical metric related via a disformal transformation to that defining curvature terms, i.e., $\bar{g}_{\mu\nu} = e^{\alpha(\phi)} g_{\mu\nu} + \beta(\phi)\phi_\mu\phi_\nu$. Again, we can fix one of the two functions α and β via a field rescaling.

Galileon Frame This frame is given by the action

$$S_G = \int d^4x\sqrt{-g}[G_G(\phi,X)R - G_{G,X}[(\Box\phi)^2 - (\nabla_\mu\nabla_\mu\phi)^2] + K(\phi,X) + G_3(\phi,X)\Box\phi] + S_m[\bar{g},\psi], \tag{38}$$

where

$$G_G = (1 - 2D(\phi)X)^{1/2}; \qquad \bar{g}_{\mu\nu} = e^{\alpha(\phi)} g_{\mu\nu}, \tag{39}$$

which amounts to the choice $C(\phi) = 1$ and $\beta(\phi) = 1$. In this case, we have both NMC and matter fields feeling a physical metric that is now conformally related to the gravitational one.

Disformal Frame This frame is given by the action

$$S_D = \int d^4x\sqrt{-g}[G_G(\phi, X)R + K(\phi, X) + G_3(\phi, X)\Box\phi] + S_m[\bar{g}, \psi], \tag{40}$$

where

$$G_G = C(\phi)^2; \qquad \bar{g}_{\mu\nu} = g_{\mu\nu} + \beta(\phi)\phi_\mu\phi_\nu, \tag{41}$$

which amounts to the choice $D = 0$ and $\alpha = 1$.

It is worth stressing here that the last two frames, which can be seen as some sorts of intermediate frames between the Jordan and Einstein ones, can actually reduce to the latter for suitable choices of the rescaling of the field, as can be seen from the last columns of Table I. This is a consequence of the fact that we have four free metric functions, $\alpha(\phi)$, $\beta(\phi)$, $C(\phi)$, and $D(\phi)$, three of which can be arbitrarily fixed, and hence some overlap is expected.

As a final remark, while all these equivalent frames are connected by disformal transformations and field rescaling, one has to also be careful about accordingly rescaling also the so-far neglected functions $K(\phi, X)$ and $G_i(\phi, X)$ in order to preserve the equivalence of frames.

The above-mentioned frames were first proposed in Ref. [21] and partially discussed in Ref. [45], in which it was pointed out how disformal transformations relate them, albeit no discussion about their actual equivalence was provided. Here, we have rederived the same results in a different way, and, in addition, we have proven the frames' equivalence. This has relevant consequences; for example, it implies that not only Dirac-Born-Infeld Galileon models with a nonminimally coupled scalar field can be cast via a disformal transformation into the simpler Einstein frame but also guarantees the equivalence of these representations. Furthermore, the equivalence of the frames allows us to claim the equivalence of many apparently unrelated models as those reported in Ref. [28], given that we can

TABLE I. Disformal frames obtained for different fixings of the Horndeski coefficient functions of Eq. (33). The first two columns show the results of the fixing after a disformal transformation, while the last two show the effects of the further freedom associated with the invariance under field rescaling (there are two possibilities in each slot in this case, as one can alternatively rescale the metric or the field ϕ derivative terms). Λ is a dimensional constant introduced to keep track of the dimensions of the coefficient functions, while $\varphi(\phi)$ is the rescaled conformal function.

	Disformal transformation		Field rescaling	
Frame	Matter metric	NMC function	Matter metric	NMC function
Jordan frame	$g_{\mu\nu}$	$C(\phi)^2(1+2D(\phi)X)^{1/2}$	$g_{\mu\nu}$ $g_{\mu\nu}$	$(1-2D(\phi)X)^{1/2}$ $C(\phi)^2(1-2\Lambda X)^{1/2}$
Einstein frame	$e^{\alpha(\phi)}g_{\mu\nu}+\beta(\phi)\phi_\mu\phi_\nu$	1	$\varphi(\phi)g_{\mu\nu}+\beta(\phi)\phi_\mu\phi_\nu$ $e^{\alpha(\phi)}g_{\mu\nu}+\Lambda\phi_\mu\phi_\nu$	1 1
Galileon frame	$e^{\alpha(\phi)}g_{\mu\nu}$	$(1-2D(\phi)X)^{1/2}$	$\varphi(\phi)g_{\mu\nu}$ $e^{\alpha(\phi)}g_{\mu\nu}$	$(1-2D(\phi)X)^{1/2}$ $(1-2\Lambda X)^{1/2}$
Disformal frame	$g_{\mu\nu}+\beta(\phi)\phi_\mu\phi_\nu$	$C(\phi)^2$	$g_{\mu\nu}+\Lambda\phi_\mu\phi_\nu$ $g_{\mu\nu}+\beta(\phi)\phi_\mu\phi_\nu$	$C(\phi)^2$ 1

move from one to the other through appropriately chosen disformal transformations and field redefinitions.

C. More general disformal frames

We have seen in the previous section that the requirement of an Einstein frame strongly constrains the shape of the Horndeski Lagrangian with a specific form for $G_4(\phi, X)$ and forcing $G_5(\phi, X) = 0$. However, there is no real physical need to have an Einstein frame so that one may wonder about the existence of more general Lagrangians that do not posses an Einstein frame but that show in any case interesting properties under disformal transformation. We list and analyze here some examples.

Disformal matter When we add the matter Lagrangian to the full Horndeski action, the EEP allows matter fields to be coupled to a metric that is disformally related to the one defining the Horndeski action,

$$S = S_H[g,\phi] + S_m[\bar{g},\psi], \tag{42}$$

where S_H is the full Horndeski action (1), ψ collectively defines matter fields, and where $\bar{g}_{\mu\nu} = e^{\alpha(\phi)}g_{\mu\nu} + \beta(\phi)\phi_\mu\phi_\nu$. Thanks to the invariance of the full Horndeski action under disformal transformations and field rescaling, we are free to fix both $\alpha(\phi)$ and $\beta(\phi)$ in such a way that, after the transformation, matter propagates along the geodesics defined by the metric $g_{\mu\nu}$ that appears in the Horndeski action. These transformations will, of course, affect the Horndeski Lagrangian, but only in the shape of its coefficient functions, not in its structure. Hence, a Horndeski theory in which matter propagates on the metric $\bar{g}_{\mu\nu} = e^{\alpha(\phi)}g_{\mu\nu} + \beta(\phi)\phi_\mu\phi_\nu$ is equivalent to another Horndeski theory, with redefined coefficient functions, in which matter propagates along the same metric $g_{\mu\nu}$ that enters the Horndeski action.

This fact is not particularly surprising, but it is nonetheless interesting as it shows how, without any assumption on the shape of the Horndeski action, we can see that apparently different matter behaviors are in fact different representations of the same theory.

Einstein coupling Another possible extension is to include the $\mathcal{L}_5$ Lagrangian while keeping the requirement of having a frame with no conformal coupling to gravity. Using the relations derived in Appendix C, we see that this requirements translate into a condition on the initial shape of the $G_4(\phi, X)$ function

$$G_4(\phi,X) = (1-2B(\phi)X)^{1/2} - G_{5,\phi}(\phi,X)X + \tilde{G}_{5,\phi}(\phi,X)X, \tag{43}$$

where

$$\tilde{G}_5(\phi,X) = \int dX(1-2BX)^{1/2}G_{5,X}(\phi,X). \tag{44}$$

With this requirement, we can consider the action

$$\begin{aligned} S = &\int d^4x\sqrt{-g}[G_4(\phi,X)R - G_{4,X}(\phi,X)[(\Box\phi)^2 \\ &- (\nabla_\mu\nabla_\mu\phi)^2] + K(\phi,X) + G_3(\phi,X)\Box\phi] \\ &+ \int d^4x\sqrt{-g}\Big[G_5(\phi,X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{1}{6}((\Box\phi)^3 \\ &- 3\Box\phi(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3)\Big] + S_m[\tilde{g},\psi], \end{aligned} \tag{45}$$

where $G_4(\phi, X)$ is given by the previous expressions, while $G_5(\phi, X)$ is left totally free. With the disformal transformation (13), we can eliminate the conformal coupling to gravity, leaving only a NMC via the Einstein tensor and matter fields propagating along disformal geodesics.

We conclude this section recalling that the invariance of the Horndeski action under disformal transformations and field rescaling holds true for the full Horndeski theory (1). Possible restrictions on the shape and functional dependencies of the free functions of the theory are to be ascribed only to physical motivations, e.g., the requirement of an Einstein frame, or to classification aims, e.g., identifying equivalent models, but not to constraints imposed by the invariance itself.

V. CONCLUSIONS

The gravitational interaction has been the first one studied in a systematic way, and its modern formulation is encoded in the theory of general relativity. Despite its successes, GR is nowadays challenged both at the theoretical and experimental level, leading to several proposals for alternative theories of gravity. However, the lack of an axiomatic procedure for the construction of such theories and the limited regime for which we have highly constraining observational data make it hard to reduce the number of alternative theories and to find their mutual relations.

A major tool in physics is represented by symmetries. This is a clean and precise way to order models, find their simplest formulations, and identify the minimal set of degrees of freedom required to fully define a theory. In the context of standard scalar-tensor theories, this has been systematically investigated, and the discovery of the invariance of such theories under conformal metric transformations and field rescaling has made it possible to identify the minimal number of functions required to describe the theory and shown the mutual relations between apparently different representations.

Along this line of reasoning, we have investigated in this paper the symmetries of the Horndeski action and found that it is invariant under a more general metric transformation than a conformal one, the so-called disformal transformation, as well as under field rescalings. These transformations contain free functions and hence can, in principle, be used to constrain the coefficient functions that define the Horndeski action. However, we have shown that the most general disformal transformation (6) cannot be used for this purpose, as the Horndeski action is not invariant under transformations induced by it. We have hence circumscribed our investigation to a subset of disformal transformations (13), in which the two free functions needed to define it only depend on the scalar field. We have shown that the Horndeski action is actually invariant under such a class of disformal transformations, albeit the generality of the Horndeski action does not allow for an efficient fixing of the coefficient functions.

For this reason, we looked to the constraints that one has to impose on the Horndeski coefficient functions in order to have a theory that admits an Einstein frame. We discovered that this is a quite constraining request, as, in fact, the full Horndeski action is reduced to the action (33), in which only a conformal nonminimal coupling is present. This allowed us to investigate the existence of equivalent frames, in an analogous way to what is done for standard scalar-tensor theories. We found that, apart from the well-known Einstein and Jordan frames, the invariance under disformal transformations allows for the definition of two more equivalent frames: the so-called Galileon and disformal frames. We further extend our analysis to frames that do not admit an Einstein frame and showed that, even without this requirement, one can find physically relevant frames connected by disformal transformations.

In conclusion, with this work, we have found a new class of scalar-tensor theories of gravity that admits disformally equivalent frames, which are related by disformal transformation and field rescaling, thus generalizing the previous results obtained in the context of standard scalar-tensor theories. This may have important consequences in cosmological context as it may allow us to identify a large class of models in different representations of the same theory. We hope that these issues will be further investigated in the near future.

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APPENDIX A: KEEPING SECOND-ORDER FIELD EQUATIONS

In this section we show how a metric transformation induced by the general disformal relation (6) spoils the property of the Horndeski action of producing second-order field equations.[6]

Our proof consists of a direct calculation of the modifications that the disformal transformation has onto a particular term of the full Lagrangian, namely, $\mathcal{L}_4$, when the disformal functions depend only on the kinetic term of the scalar field ϕ. Despite the fact that this does not represent a formal proof of our statement, it is nonetheless general enough to discard any kinetic term dependence in the disformal transformation if second-order field equations are to be preserved. We leave the formal proof of this for further work, but we stress that the result obtained here holds in general. Our calculations make use of Ref. [20], in which a general procedure on how to build actions for a metric and a scalar field that keep the equation of motion second order was put forward. We will shortly review it for what concerns us, referring the interested reader to the original paper.[7]

In flat space-times, consider the Lagrangian

$$\mathcal{L} = \mathcal{T}_{(2n)}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n}\nabla_{\mu_1}\nabla_{\nu_1}\phi\ldots\nabla_{\mu_n}\nabla_{\nu_n}\phi, \qquad \text{(A1)}$$

where

[6] We want to stress that this result holds on both curved backgrounds as well as on flat backgrounds with the exception that, on flat space-times, there exist subcases that give second-order field equations even after a disformal transformation.

[7] Notice that, in our work, we have the following correspondences: $\pi \to \phi$, $\pi_\mu \to \phi_\mu$, $\pi_{\mu\nu} \to \nabla_\mu\nabla_\nu\phi$, and $X \to 2X$.

$$\mathcal{T} = \mathcal{T}(\phi, \phi_\alpha), \qquad \mathcal{L} = \mathcal{L}(\phi, \phi_\mu, \nabla_\mu \nabla_\nu \phi); \tag{A2}$$

then, the following lemma holds.

Lemma 1.—A sufficient condition for the field equations derived from the Lagrangian (A1) to remain second order or less is that $\mathcal{T}_{(2n)}^{\mu_1 \ldots \mu_n \nu_1 \cdots \nu_n}$ is totally antisymmetric in its first indices μ_i as well as (separately) in its last indices ν_i.

Notice that this is a sufficient condition. However, the opposite statement has been proven, and a uniqueness condition exists so that the condition is both necessary and sufficient.

When one moves to curved space-times and covariantizes, promoting partial derivatives to covariant derivatives, third-order derivatives of the metric are produced. It has been shown that adding a suitable finite number of non-minimally coupled terms to the Lagrangian is enough to eliminate the higher-than-second derivatives from the equations of motion in both the scalar field and in the metric. As a final result, the authors of Ref. [20] gave the form of the Lagrangian that preserves the second-order equations,

$$\mathcal{L}_n\{f\} = \sum_{p=0}^{\lfloor n/2 \rfloor} \mathcal{C}_{n,p} \mathcal{L}_{n,p}\{f\}, \tag{A3}$$

where $\lfloor n/2 \rfloor$ indicates the integer part, while the graph bracket indicates that $\mathcal{L}$ is a functional of f, which is, in general, different for any n and for which

$$\mathcal{L}_{n,p}\{f\} = P_{(p)}^{\mu_1 \ldots \mu_n \nu_1 \cdots \nu_n} \nabla_{\mu_1} R_{(p)} S_{(q \equiv n-2p)}, \tag{A4}$$

$$R_{(p)} = \prod_{i=1}^{p} R_{\mu_{2i-1}\mu_{2i}\nu_{2i-1}\nu_{2i}}, \tag{A5}$$

$$S_{(q \equiv n-2p)} = \prod_{i=0}^{q-1} \nabla_{\mu_{n-i}} \nabla_{\nu_{n-i}} \phi, \tag{A6}$$

while

$$P_{(p)}^{\mu_1 \ldots \mu_n \nu_1 \ldots \nu_n} = \int_{X_0}^{X} dX_1 \cdots \int_{X_0}^{X_{p-1}} dX_p \mathcal{T}_{(2n)}^{\mu_1 \ldots \mu_n \nu_1 \ldots \nu_n}(\phi, X_1), \tag{A7}$$

while the coefficients are given by

$$\mathcal{C}_{n,p} = \left(-\frac{1}{8}\right)^p \frac{n!}{(n-2p)!p!}. \tag{A8}$$

Using the Lagrangian (A1) and the rules reported above, it is possible to construct all covariant theories that give second-order field equations, and, in particular, in four dimensions, we have that the Horndeski action is a linear combination of the following terms:

$$\mathcal{L}_{0,0} = X f_0(\phi, X), \qquad \mathcal{C}_{0,0} = 1, \tag{A9}$$

$$\mathcal{L}_{1,0} = X f_1(\phi, X) A_2^{\mu\nu} \nabla_\mu \nabla_\nu \phi, \qquad \mathcal{C}_{1,0} = 1, \tag{A10}$$

$$\mathcal{L}_{2,0} = X f_2(\phi, X) A_4^{\mu_1\mu_2\nu_1\nu_2} \nabla_{\mu_1} \nabla_{\nu_2} \phi \nabla_{\mu_2} \nabla_{\nu_2} \phi, \quad \mathcal{C}_{2,0} = 1, \tag{A11}$$

$$\mathcal{L}_{3,0} = X f_3(\phi, X) A_6^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3} \times \nabla_{\mu_1} \nabla_{\nu_2} \phi \nabla_{\mu_2} \nabla_{\nu_2} \phi \nabla_{\mu_3} \nabla_{\nu_3} \phi, \quad \mathcal{C}_{3,0} = 1, \tag{A12}$$

$$\mathcal{L}_{2,1} = P_{(1)}^{\mu_1\mu_2\nu_1\nu_2} R_{\mu_1\mu_2\nu_1\nu_2}, \quad P_{(1)} = \int dX_1 \mathcal{A}_4^{\mu_1\mu_2\nu_1\nu_2} X_1 f_{(2)}(\phi, X_1), \quad \mathcal{C}_{2,1} = -\frac{1}{4}, \tag{A13}$$

$$\mathcal{L}_{3,1} = P_{(1)}^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3} R_{\mu_1\mu_2\nu_1\nu_2} \nabla_{\mu_3} \nabla_{\nu_3} \phi, \quad P_{(1)} = \int dX_1 \mathcal{A}_6^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3} X_1 f_{(3)}(\phi, X_1), \quad \mathcal{C}_{3,1} = -\frac{3}{4}, \tag{A14}$$

where we have redefined the form function $\mathcal{T}_{2n}(\phi, X) = X f_n(\phi, 2X) \mathcal{A}_{2n}$ in such a way as to separate the field dependences (ϕ, X) from the structure term $\mathcal{A}(g_{\alpha\beta}, \phi_\alpha)$. Notice that the terms (2, 0) and (2, 1) as well as (3, 0) and (3, 1) are coupled terms for which joint presence is required in order to cancel the unwanted higher-order derivatives. The Horndeski action can be rephrased in these terms with the following identifications[8]:

$$K(\phi, X) = X f_{(0)}(\phi, X), \tag{A15}$$

$$G_3(\phi, X) = X f_{(1)}(\phi, X), \qquad \mathcal{A}_{(2)}^{\mu\nu} = g^{\mu\nu}, \tag{A16}$$

$$G_4(\phi, X) = \int [X_1 f_{(2)}(\phi, X_1) dX_1], \tag{A17}$$

$$\mathcal{A}_{(4)}^{\mu\alpha\nu\beta} = g^{\mu\beta} g^{\nu\alpha} - g^{\mu\nu} g^{\alpha\beta}, \tag{A18}$$

$$G_5(\phi, X) = \int [X_1 f_{(3)}(\phi, X_1) dX_1], \tag{A19}$$

$$\mathcal{A}_{(6)}^{\mu\sigma\alpha\nu\rho\beta} = g^{\alpha\nu}[g^{\beta\mu} g^{\sigma\rho} - g^{\beta\rho} g^{\sigma\mu}] + g^{\alpha\sigma}[-g^{\beta\mu} g^{\nu\rho} + g^{\beta\rho} g^{\nu\mu}] + g^{\alpha\beta}[g^{\sigma\mu} g^{\nu\rho} - g^{\sigma\rho} g^{\nu\mu}]. \tag{A20}$$

To prove our statement, we need to show how a kinetic-dependent metric transformation spoils the antisymmetric

[8] Notice that, compared with the convention used in the definition of the kinetic term X in Ref. [20], there are factors $1/2$ that have been reabsorbed into the definition of the function $f_{(n)}$.

structure of the model. Before entering the calculations, we note that the effects of a disformal metric transformation on the coefficient functions $f_{(n)}$ only redefines its functional dependence, while the structure functions $\mathcal{A}$ are again only redefined with no modifications on their antisymmetric structure. Hence, to check the breaking of the antisymmetric structure, we only need to compute the effects of metric transformations on second covariant derivatives of the field and on the Riemann tensor. To do this in a simple way, we will look at the effects of the kinetic dependence of the disformal functions $A(X)$ and $B(X)$ by applying separately a conformal transformation and a purely disformal one on the terms corresponding to $\mathcal{L}_4$ in the rephrased Horndeski action (A14).

1. Conformal transformation

Consider a conformal transformation of the kind

$$\bar{g}_{\mu\nu} = A(X) g_{\mu\nu}. \tag{A21}$$

After the conformal transformation (A21) is performed, the original $\mathcal{L}_4$ Lagrangian is mapped into

$$\begin{aligned} &A^2 G_4 R - A^2 G_{4,X}[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2] \\ &\quad - 6AA'G_4(\nabla_\mu\nabla_\nu\phi)^2 - 6AA'G_4\phi^\alpha\Box\nabla_\alpha\phi \\ &\quad - 2AA'G_{4,X}\left[\Box\phi + \frac{A'}{A}\phi^\mu\phi^\nu\nabla_\mu\nabla_\nu\phi\right]\phi^\alpha\phi^\beta\nabla_\alpha\nabla_\beta\phi \\ &\quad + \phi^\mu\phi^\nu\nabla_\mu\nabla_\alpha\phi\nabla_\nu\nabla^\alpha\phi[-4AA'G_{4,X} \\ &\quad + AA'^2X - 6A''AG_4]. \end{aligned} \tag{A22}$$

From this expression, it is clear that the first two terms are not dangerous, as they have the same structure as those in the original Lagrangian. To better understand the others, we proceed in rewriting them in the form $\mathcal{A}^{\mu\alpha\nu\beta}\nabla_\mu\nabla_\nu\phi\nabla_\alpha\nabla_\beta\phi$. Any antisymmetry violating term will then directly lead to higher derivatives in the equations of motion. After some manipulation, we arrive at the expression

$$\begin{aligned} &\sim[-6AA'G_4(g^{\alpha\mu}g^{\beta\nu}) + (-2G_{4,X}AA' + 6A'^2G_4 + AA''G_4 \\ &\quad + AA'G_{4,X})g^{\mu\nu}\phi^\alpha\phi^\beta + (-4G_{4,X}AA' + 4G_{4,X}A'^2X \\ &\quad - 6AA''G_4)g^{\nu\beta}\phi^\alpha\phi^\mu - 2G_{4,X}A'^2\phi^\mu\phi^\nu\phi^\alpha\phi^\beta] \\ &\quad \times\nabla_\mu\nabla_\nu\phi\nabla_\alpha\nabla_\beta\phi, \end{aligned} \tag{A23}$$

where the symbol $\sim$ indicates that only the dangerous terms have been considered, and notice that we have added a surface term to rewrite the third-order derivative. As can be easily seen, antisymmetry breaking terms have appeared in the Lagrangian. We can then conclude that the generalized conformal transformation (A21) spoils the antisymmetric structure of the Horndeski action and hence gives equation of motion for the fields that are higher than second order.

2. Disformal transformations

Consider now a metric transformation of the form

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + B(X)\phi_\mu\phi_\nu. \tag{A24}$$

Using the same procedure of the previous section, we can write the transformed $\mathcal{L}_4$ part of the Lagrangian and see whether or not it is possible to recover the antisymmetric structure. The dangerous terms of the transformed Lagrangian read

$$\begin{aligned} \sim&\left[g^{\mu\nu}\phi^\alpha\phi^\beta\left(\frac{2G_{4,X}}{(1+2XB)^{1/2}}(B'X+B) - \frac{2G_4}{(1+2XB)^{3/2}}(B^2 - B'(1+BX)) - 2\frac{G_4}{(1+2XB)^{1/2}}R_{\alpha\mu\beta\nu} - \frac{G_4B'}{(1+2XB)^{1/2}}\right.\right. \\ &\left.\left. + \left(\frac{2B'G_4}{(1+2XB)^{1/2}} + \frac{2XB''G_4}{(1+2XB)^{1/2}} + \frac{2B'G_{4,X}}{(1+2XB)^{1/2}} - \frac{2B'G_4}{(1+2XB)^{3/2}}(B'X+B)\right)\right)\right] \\ &+ g^{\mu\alpha}\phi^\beta\phi^\nu\left[-\frac{G_{4,X}}{(1+2XB)^{1/2}}(B - 2XB'(-1X^2B')) + \frac{G_4}{(1+2XB)^{3/2}}(B^2 - B' + B'^2X^2 - XB''(1+2XB))\right. \\ &\left. + \frac{G_4B'}{(1+2XB)^{1/2}} - 2\frac{G_{4,X}B'X}{(1+2XB)^{1/2}} - 2\frac{G_4B'}{(1+2XB)^{1/2}}\right] + \phi^\mu\phi^\nu\phi^\alpha\phi^\beta\left[\frac{G_{4,X}B'}{(1+2XB)^{1/2}}(1-2X^2B')\right. \\ &+ \frac{G_4}{(1+2XB)^{3/2}}(-XB'^2 + B''(1+2XB) - BB') - \frac{G_{4,X}B'}{(1+2XB)^{1/2}} - \frac{G_4B''}{(1+2XB)^{1/2}} \\ &\left. + \frac{G_4B'}{(1+2XB)^{3/2}}(B'X+B)\right]\nabla_\mu\nabla_\nu\nabla_\alpha\phi\nabla_\beta\phi, \end{aligned} \tag{A25}$$

which, again, contains terms that are not antisymmetric in the couples (α, β) and (μ, ν), hence giving rise to higher derivatives in the equations of motion.

In conclusion, even if a formal proof of this result would be desirable, our result clearly states that, if one wants to preserve second-order field equations, then the most general disformal transformation that can be used is the one reported in Eq. (13), in which the disformal functions A and B only depend on the scalar field ϕ.

APPENDIX B: TRANSFORMATION PROPERTIES OF GEOMETRICAL QUANTITIES

We provide here the transformation rules for geometric quantities when the metric undergoes a disformal transformation of the kind

$$\bar{g}_{\mu\nu} = A(\phi)g_{\mu\nu} + B(\phi)\phi\mu\phi\nu, \tag{B1}$$

where both metrics g and $\bar{g}$ are well-defined metrics that can be equally used to raise and lower indices. The transformed inverse is

$$\bar{g}^{\mu\nu} = \frac{1}{A(\phi)}g^{\mu\nu} - \frac{B(\phi)}{A(\phi)^2(1+2XB/A)}\phi^\mu\phi^\nu, \tag{B2}$$

while the volume element changes (see Appendix C of Ref. [48]) as $\sqrt{-\bar{g}} = A(\phi)^2(1+2XB/A)^{1/2}\sqrt{-g}$.

From these definitions, one can express all the barred curvature quantities in the function of the unbarred metric and the scalar field ϕ. We list these below.

Connection coefficient

$$\bar{\Gamma}^\alpha{}_{\mu\nu} = \Gamma^\alpha{}_{\mu\nu} + \frac{B}{A(1+2XB/A)}\phi^\alpha\nabla_\mu\nabla_\nu\phi + \frac{A'}{2A}(\delta^\alpha_\nu\phi_\mu + \delta^\alpha_\mu\phi_\nu) + \frac{1}{2}\frac{\phi^\alpha}{A^2(1+2XB/A)}(-AA'g_{\mu\nu} + (AB' - 2A'B)\phi_\mu\phi_\nu). \tag{B3}$$

Ricci tensor

$$\begin{aligned}
\bar{R}_{\alpha\beta} = R_{\alpha\beta} &+ \frac{[AB(1+2XB/A)\Box\phi - B^2\phi^\mu\nabla_\mu X - AA'(1+2XB/A) + (AB' - A'B)X]}{A^2(1+2XB/A)^2}\nabla_\alpha\nabla_\beta\phi \\
&+ \frac{[-A^2A'(1+2XB/A)\Box\phi + AA'B\phi^\mu\nabla_\mu X - 2A'X^2(A'B - AB') - 2A^2A''X(1+2XB/A)]}{2A^3(1+2XB/A)^2}g_{\alpha\beta} \\
&+ \Bigg[\frac{3A^2A'^2 + 6A'^2B^2X^2 + 2A'A^2B'X + (A^3B' - 4AA'B^2X)\Box\phi - 2A^3A''}{2A^4(1+2XB/A)^2} \\
&+ \frac{2AB(A'B\nabla^\mu\nabla_\mu X + 5A'^2X + 3A'B'X^2) - 6A''BX^2}{2A^4(1+2XB/A)^2}\Bigg]\phi_\alpha\phi_\beta \\
&+ \Bigg[\frac{-2AB(1+2XB/A)R_{\alpha\mu\beta\nu}\phi^\mu\phi^\nu - 2AB(1+2XB/A)\nabla_\alpha\nabla_\lambda\phi\nabla_\beta\nabla^\lambda\phi + 2B^2\nabla_\alpha X\nabla^\alpha X}{2A^2(1+2XB/A)^2} \\
&+ \frac{(A'B - AB')(\phi_\alpha\nabla_\beta X + \phi_\beta\phi_\alpha) - \phi_\alpha\phi_\beta\phi^\mu\phi\nabla_\mu X + 2\phi_\alpha\phi_\beta\phi\Box\phi(A' - B'X) + 10XA''\phi_\alpha\phi_\beta\phi}{2A^2(1+2XB/A)^2}\Bigg].
\end{aligned} \tag{B4}$$

Ricci scalar

$$\begin{aligned}
\bar{R} = R &- \frac{2B}{A^2(1+2XB/A)}R_{\alpha\beta}\phi^\alpha\phi^\beta + \frac{B}{A^2(1+2XB/A)}[(\Box\phi)^2 - (\nabla_\alpha\nabla_\beta\phi)^2] \\
&+ \frac{2B}{A^3(1+2XB/A)^2}[\nabla^\alpha X\nabla_\alpha X - \phi^\alpha\nabla_\alpha X\Box\phi] - \frac{8A'BX + A(3A' - 2B'X)}{A^3(1+2XB/A)^2}\Box\phi \\
&+ \frac{4A'B - AB'}{A^3(1+2XB/A)^2}\phi^\alpha\nabla_\alpha X + \frac{3A'X(A' + 2B'X)}{A^2(1+2XB/A)^2} - \frac{6A''X}{A^2(1+2XB/A)}.
\end{aligned} \tag{B5}$$

Notice that both functions A and B are intended as general functions of the scalar field ϕ.

APPENDIX C: TRANSFORMATION PROPERTIES OF THE HORNDESKI ACTION UNDER DISFORMAL TRANSFORMATIONS

We explored the consequences on the Horndeski action when the metric is transformed via a disformal transformation,

$$\bar{g}_{\mu\nu} = A(\phi)g_{\mu\nu} + B(\phi)\phi_\mu\phi_\nu, \tag{C1}$$

through a direct calculation. Our results show that, after this transformation is performed, the new action can be recast into the same initial Horndeski form given that all the effect of the transformation are absorbed into the rescaling of the free coefficient functions. As a consequence, we can say that the Horndeski action is formally invariant under this class of disformal transformation. We report below the transformations properties of the Horndeski Lagrangian coefficient functions. The new Lagrangian is

$$\bar{\mathcal{L}} = \sum_i \bar{\mathcal{L}}_i, \tag{C2}$$

where

$$\bar{\mathcal{L}}_2 = \bar{K}(\phi, X), \tag{C3}$$

$$\bar{\mathcal{L}}_3 = \bar{G}_3(\phi, X)\Box\phi, \tag{C4}$$

$$\bar{\mathcal{L}}_4 = \bar{G}_4(\phi, X)R - \bar{G}_{4,X}(\phi, X)[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2], \tag{C5}$$

$$\bar{\mathcal{L}}_5 = \bar{G}_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi + \frac{\bar{G}_{5,X}(\phi, X)}{6}[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\nu\nabla_\mu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3], \tag{C6}$$

where

$$\begin{aligned}\bar{K}(\phi, X) = {}& (1 + 2XB/A)^{1/2}K(\phi, X_D) + 2X\left[\frac{G_3(\phi, X_D)AA'}{(1 + 2XB/A)^{1/2}} + \frac{G_3(\phi, X_D)(A'B)X}{(1 + 2XB/A)^{3/2}} + H_{3,\phi}(\phi, X)\right] \\ & + 3X\frac{G_4(\phi, X_D)[A' + 2A'B'X - 2AA' - 4A'BX]}{A(1 + 2XB/A)^{3/2}} + 12X\frac{G_{4,X}(\phi, X_D)X[A'^2BX - AA'B'X]}{A^2(1 + 2XB/A)^{1/2}} \\ & - 2XH_{4,\phi}(\phi, X)\frac{3G_5(\phi, X_D)X^2A'}{A^4(1 + 2XB/A)^{5/2}}[-A'^2BX + 2A^2A''(1 + 2XB/A) - A(2A'^2 + 3A'B'X)] \\ & - 2XH_{5,\phi}(\phi, X) + \frac{2G_{5,X}(\phi, X_D)X^3}{A^4(1 + 2XB/A)^{3/2}}(A'^3BX + AA'(A' + 3B'X)),\end{aligned} \tag{C7}$$

$$\begin{aligned}\bar{G}_3(\phi, X) = {}& \left[\frac{AG_3(\phi, X_D)}{(1 + 2XB/A)^{1/2}} + H_3(\phi, X)\right] + \left[\frac{G_4(\phi, X_D)(4AA'B + ABB'X + A'B^2X)}{A^2(1 + 2XB/A)^{3/2}}\right. \\ & \left. + \frac{BG_{4,\phi}(\phi, X_D)}{(1 + 2XB/A)^{1/2}} + \frac{G_{4,X}(AA'BX - 2A^2A' + 2A^2B'X)}{A^2(1 + 2XB/A)^{1/2}} - H_4(\phi, X)\right] \\ & + \left[X\left(-2(H_{\Box,\phi}(\phi, X) - H_{R,\phi\phi}(\phi, X)) + \frac{G_5(\phi, X_D)}{A^3(1 + 2XB/A)^{5/2}}\left(5A'^2BX - A\left(\frac{A'^2}{2} + 6A'B'X\right)\right)\right.\right. \\ & \left.\left. + \frac{2G_5(\phi, X_D)}{A(1 + 2XB/A)^{3/2}}A''\frac{G_{5,X}XA'}{A^3(1 + 2XB/A)^{3/2}}(AA' - 2A'BX + 4AB'X)\right) - H_5(\phi, X)\right],\end{aligned} \tag{C8}$$

$$\bar{G}_4(\phi, X) = A(1 + 2XB/A)^{1/2}G_4(\phi, X_D) - \left(\frac{G_5(\phi, X_D)X^2}{A^2(1 + 2XB/A)^{3/2}}(A'B - AB') + H_{R,\phi}(\phi, X)X\right), \tag{C9}$$

$$\bar{G}_5(\phi, X) = \frac{G_5(\phi, X_D)}{(1 + 2XB/A)^{1/2}} + H_R(\phi, X), \tag{C10}$$

where the explicit form of the functions H_i are

$$H_4(\phi, X) = \int dX\left[\frac{G_4(\phi, X_D)(4AA'B + ABB'X + A'B^2X)}{A^2(1 + 2XB/A)^{3/2}}\right], \qquad H_3(\phi, X) = B\int dX\frac{G_3(\phi, X_D)}{(1 + 2XB/A)^{3/2}}, \tag{C11}$$

$$\begin{aligned}H_5(\phi, X) = \int dX\Bigg[& H_{\Box,\phi}(\phi, X) - H_{R,\phi\phi}(\phi, X) + \frac{G_5(\phi, X_D)}{2A^3(1 + 2XB/A)^{5/2}}(-5A'BX - 2A^2A''(1 + 2XB/A) \\ & + A(5A'^2 + 6A'B'X)) + \frac{G_{5,X}(\phi, X_D)}{A^2(1 + 2XB/A)^{3/2}}(-A'BX + 2A(A' + B'X))\Bigg],\end{aligned} \tag{C12}$$

$$H_\Box(\phi, X) = G_5(\phi, X_D)\frac{AA' + (AB' - A'B)X}{A^2(1 + 2XB/A)^{3/2}}, \qquad H_R(\phi, X) = \frac{B}{A}\int dX\frac{G_5(\phi, X_D)}{(1 + 2XB/A)^{3/2}}, \tag{C13}$$

while $X_D = X/[A(1 + 2BX/A)]$, and, again, the functions A and B depend on the scalar field ϕ. The most relevant conclusion is that the effect of the disformal transformation on the Horndeski action can be recast into renormalization of the coefficient functions, exactly as in the case of conformal transformations for standard scalar-tensor theories, which, we stress, are a subcase of our result. Then, notice that, if one starts with a only a subset of the

Lagrangians, a disformal transformation will in general produce contributions at all sub-Lagrangians in a hierarchical way. Said in other words, the corrections propagate from higher derivatives down to lower-derivative terms.

APPENDIX D: INVARIANCE UNDER FIELD RESCALING

Besides the previously analyzed invariance under disformal transformation, it can be proven that the Horndeski action is also invariant under the rescaling of the scalar field

$$\phi = s(\psi)\psi. \tag{D1}$$

In fact, the effects of this transformation can be again reabsorbed into redefinitions of the Horndeski coefficient functions, which become

$$\bar{K}(\psi, \bar{X}) = K(\psi, \bar{X}) + 2YG_3(\psi, \bar{X})(2s' + \psi s'') + -2YH_{4,\psi}(\psi, \bar{X}) + 2YH_{\Box,\psi}, \tag{D2}$$

$$\bar{G}_3(\psi, \bar{X}) = (s'\psi + s)G_3(\psi, \bar{X}) - (4YG_{4,Y}(\psi, \bar{X}) + -2G_4(\psi, \bar{X}))\frac{2s' + s''\psi}{s + \psi s'} + 2YH_{5,\psi} - H_\Box, \tag{D3}$$

$$\bar{G}_4(\psi, \bar{X}) = G_4(\psi, \bar{X}) - Y(2s' + \psi s'')G_5(\psi, \bar{X}), \tag{D4}$$

$$\bar{G}_5(\psi, \bar{X}) = (2s' + s''\psi)G_5(\psi, \bar{X}), \tag{D5}$$

where

$$H_4(\psi, \bar{X}) = G_4(\psi, \bar{X})\frac{2s' + s''\psi}{s + \psi s'}, \tag{D6}$$

$$H_5(\psi, \bar{X}) = (2s' + \psi s'')G_5\frac{2s' + s''\psi}{s + \psi s'}, \tag{D7}$$

$$H_\Box(\psi, \bar{X}) = \int d\bar{X} H_{5,\psi}(\psi, \bar{X}), \tag{D8}$$

where $\bar{X} = (s'(\psi)\psi + s(\psi))^2 Y$, with $Y = \psi^\mu \psi_\mu/2$, and where a prime denotes the derivative with respect to ψ.

The field transformation is, in principle, arbitrary. However, as can be seen from, e.g., the $\bar{G}_3$ coefficient, infinities may be generated if $s + \psi s' = 0$. This amounts to saying that the solution $s(\psi) = \psi^{-1}$ is excluded from the set of admissible rescaling. This fact is in some sense obvious because it is equivalent to the limit of having no scalar field. A second remark concerns the possibility to eliminate the NMC with the Einstein tensor with a field redefinition. In fact, the transformed G_5 coefficient is proportional to $2s'(\psi) + \psi s''(\psi)$. This equation can be integrated once giving $s(\psi) = -\psi s'(\psi)$, for which the solution is excluded by the previous requirement. We conclude that it is not possible to eliminate the NMC with the Einstein tensor with a field redefinition.

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