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Potential for general relativity and its geometry

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The unique ghost-free mass and nonlinear potential terms for general relativity are presented in a diffeomorphism and local Lorentz invariant vierbein formalism. This construction requires an additional two-index Stückelberg field, beyond the four scalar fields used in the metric formulation, and unveils a new local SL(4) symmetry group of the mass and potential terms, not shared by the Einstein-Hilbert term. The new field is auxiliary but transforms as a vector under two different Lorentz groups, one of them the group of local Lorentz transformations, the other an additional global group. This formulation enables a geometric interpretation of the mass and potential terms for gravity in terms of certain volume forms. Furthermore, we find that the decoupling limit is much simpler to extract in this approach; in particular, we are able to derive expressions for the interactions of the vector modes. We also note that it is possible to extend the theory by promoting the two-index auxiliary field into a Nambu-Goldstone boson nonlinearly realizing a certain spacetime symmetry, and show how it is "eaten up" by the antisymmetric part of the vierbein.

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I. INTRODUCTION AND SUMMARY

Einstein's gravity is the theory that describes the 2 degrees of freedom of the massless helicity-2 representation of the Poincaré group, and their two derivative self-interactions. One may ask whether it is possible to alter the interactions of the graviton beyond those dictated by the Einstein-Hilbert (EH) action. At the lowest, zero-derivative level, such a deformation would correspond to adding a potential for the metric perturbation. An obvious example is the potential described by the cosmological constant (CC) term, $\mathcal{L}_0 \sim \sqrt{-g}\Lambda$. This changes neither the number of propagating degrees of freedom of general relativity (GR), nor the consistency of the theory, but necessarily alters the background spacetime.

The CC is the only such term—other potentials inevitably change the number of degrees of freedom. The Fierz-Pauli term [1] is the unique consistent quadratic potential that gives rise to 5 degrees of freedom, as required by the massive spin-2 representation of the Poincaré group. Adding a generic potential to the EH action, however, leads to the loss of all four Hamiltonian constraints of GR, and thus a total of six propagating degrees of freedom, one of which is necessarily a ghost [2].

Nevertheless, there exists a special class of mass and potential terms (the often-called de Rham, Gabadadze, Tolley (dRGT) terms [3,4], see [5] for a review) that make the graviton massive, while retaining one of the four Hamiltonian constraints. This remaining constraint projects out the ghostly sixth degree of freedom [6,7], see also [8–10].

In addition to the CC term, the dRGT construction allows for 3 free parameters. One combination is the graviton mass, m, and the other two independent combinations, α_3 and α_4, set the strength of the nonlinear potential. The theory can be formulated by using four spurious diffeomorphism scalars, $\phi^{\bar{a}}$—first introduced in an earlier proposal for massive gravity [11]—to allow for a manifestly diffeomorphism-invariant description. Adopting these four scalars, and following [4], one can define a matrix with components $\mathcal{K}^\mu{}_\nu = \delta^\mu_\nu - \sqrt{g^{\mu\alpha}\partial_\alpha\phi^{\bar{a}}\partial_\nu\phi^{\bar{b}}\eta_{\bar{a}\bar{b}}}$, that can be used to build invariants supplementing the EH action by the graviton mass as well as zero-derivative interactions that guarantee 5 degrees of freedom on an arbitrary background. One such term is given by [4]

$$\mathcal{L}_2 \sim \frac{M_{\rm Pl}^2 m^2}{2}\sqrt{-g}\varepsilon_{\mu_1\mu_2\bullet\bullet}\varepsilon^{\nu_1\nu_2\bullet\bullet}\mathcal{K}^{\mu_1}{}_{\nu_1}\mathcal{K}^{\mu_2}{}_{\nu_2}. \quad (1)$$

The remaining two possible terms $\mathcal{L}_{3,4}$, cubic and quartic in $\mathcal{K}$ respectively, can be obtained by the higher order generalization of (1),[1]

$$\mathcal{L}_3 \sim \alpha_3 M_{\rm Pl}^2 m^2\sqrt{-g}\varepsilon_{\mu_1\mu_2\mu_3\bullet}\varepsilon^{\nu_1\nu_2\nu_3\bullet}\mathcal{K}^{\mu_1}{}_{\nu_1}\mathcal{K}^{\mu_2}{}_{\nu_2}\mathcal{K}^{\mu_3}{}_{\nu_3}, \quad (2)$$

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[1]The ϵ's here are the epsilon symbols, with no factors of $\sqrt{-g}$. Moreover, the linear term $\mathcal{L}_1 \sim \sqrt{-g}\varepsilon\varepsilon\mathcal{K}$ can be expressed—up to a total derivative—through a linear combination of $\mathcal{L}_{2,3,4}$ and the CC.

$$\mathcal{L}_4 \sim \alpha_4 M_{\rm Pl}^2 m^2 \sqrt{-g}\,\varepsilon_{\mu_1\mu_2\mu_3\mu_4}\varepsilon^{\nu_1\nu_2\nu_3\nu_4}\mathcal{K}^{\mu_1}{}_{\nu_1} \times \mathcal{K}^{\mu_2}{}_{\nu_2}\mathcal{K}^{\mu_3}{}_{\nu_3}\mathcal{K}^{\mu_4}{}_{\nu_4}. \quad (3)$$

In addition to being invariant under the global Poincaré subgroup, $ISO(3,1)_{\rm GCT}$, of the group of general coordinate transformations (GCT), the theory is invariant under an additional, global internal Poincaré group, $ISO(3,1)_{\rm INT}$, realized on the "flavor" indices of the scalars, as first pointed out by Siegel in an earlier context [11]

$$\phi^{\bar a} \to L^{\bar a}{}_{\bar b}\phi^{\bar b} + c^{\bar b}. \quad (4)$$

Generation of the graviton mass occurs in the phase defined by the vacuum expectation value of the order parameter $\langle \partial_\mu \phi^{\bar a}\rangle = \delta^{\bar a}_\mu$. This results in the spontaneous symmetry breaking pattern of the global symmetry group

$$ISO(3,1)_{\rm GCT} \times ISO(3,1)_{\rm INT} \to ISO(3,1)_{\rm ST}. \quad (5)$$

The unbroken $ISO(3,1)_{\rm ST}$ group guarantees that the resulting theory is invariant under the ordinary spacetime (ST) Poincaré transformations. Three of the four auxiliary scalars $\phi^{\bar a}$ are "eaten" by the graviton to form a massive spin-2 representation of the latter group, while the fourth, potentially ghostly scalar is made nondynamical by the single remaining Hamiltonian constraint of massive GR, originating from the specific structure of the dRGT terms $\mathcal{L}_{2,3,4}$.

The dRGT theory gets rid of the sixth ghostly mode, and also guarantees that the remaining 5 are unitary degrees of freedom at low energies and on nearly Minkowski backgrounds (i.e., the backgrounds with typical curvature smaller than the graviton mass square). However, the theory does not guarantee that for more general backgrounds the 5 physical modes are healthy. In fact, some of their kinetic terms may change signs around certain cosmological backgrounds. Moreover, for a large region of the α_2, α_3 parameter space, the potential is known to violate the null energy condition and one often gets kinetic and gradient terms that give rise to superluminal group and phase velocities. Most of the above issues stem from one and the same source: the dRGT theory is strongly coupled at the energy/momentum scale $\Lambda_3 \equiv (M_{\rm Pl}m^2)^{1/3}$ [3,4]. As a result, a typical curvature of order m^2 produces order 1 corrections to the kinetic terms for fluctuations, often giving rise to vanishing or negative kinetic terms, or superluminal group and phase velocities (for brief comments on the current state of affairs on all these issues, see Sec. VI).

As for any strongly coupled theory, an extension above the scale Λ_3 is desirable.[2] However, it is hard to think of such an extension since the Lagrangian contains square roots of the longitudinal modes (represented by the $\phi^{\bar a}$'s). This inconvenience might be mitigated by using the vierbeins, which are square roots of the metric. The goal of the present work is to rewrite the theory in terms of the vierbeins in a GCT and local Lorentz transformation (LLT) invariant form. The hope is that this form of the theory might make it easier to find a weakly coupled completion. Also, irrespectively of that, the vierbein formulation itself merits a separate consideration.

A vierbein reformulation of the theory was given by one of us and R. A. Rosen[3] [9]. That work focused on a unitary gauge description, which for a single massive graviton is not GCT or LLT invariant. In the present work, we give a GCT- and LLT-invariant action for a massive graviton.

We find that such a formulation requires a new two-index Stückelberg field, $\lambda^a{}_{\bar a}$, in addition to the four scalar fields $\phi^{\bar a}$ used in the metric description. The new field is auxiliary and enters the action algebraically. To recover dRGT, this field should transform as a vector under two different Lorentz groups, $\lambda^a{}_{\bar a} \to Q^a{}_b(x)\lambda^b{}_{\bar a}$, and $\lambda^a{}_{\bar a} \to L^{\bar b}{}_{\bar a}\lambda^a{}_{\bar b}$, where $Q(x)$ belongs to $SO(3,1)_{\rm LLT}$, while the constant matrix L belongs to the global group $SO(3,1)_{\rm INT}$. Moreover, we note that the mass and potential terms—once written in the GCT- and LLT-invariant form—are amenable to an extension with $\lambda \in SL(4)$ and unveil a new local symmetry with respect to simultaneous transformations, $e_\mu{}^a \to Q^a{}_b(x)e_\mu{}^b$ and $\lambda^a{}_{\bar a} \to Q^a{}_b(x)\lambda^b{}_{\bar a}$, where $Q(x) \in SL(4)$. Thus, the enhanced symmetry group of the mass and potential terms, $SL(4) \times G_{\rm GCT}$, is larger than the symmetry group of the EH action. This observation suggests an extension of the theory by additional fields [see Sec. II and III, for a $GL(4)$ symmetric extension].

As we will discuss in Sec. III, the vierbein formulation enables one to give a geometric interpretation to the mass and potential terms—they can be expressed in terms of certain volume forms.

There are other benefits as well: we find that the decoupling limit is much simpler to extract in this approach. The original results of [3] can be obtained with significantly less effort. Moreover, it is straightforward to derive closed-form expressions for the vector modes, which have not been obtained in complete generality before.

We also note that the field $\lambda^a{}_{\bar a} \in SO(3,1)$ can be represented as $\lambda^a{}_{\bar a} = \exp(v^a{}_{\bar a}/f)$, where v is an antisymmetric field (once indices are lowered with η) and f is some dimensionful constant. Then, v can be promoted into a dynamical Nambu-Goldstone field parametrizing a coset $(SO(3,1)_{\rm GCT} \times SO(3,1)_{\rm INT})/SO(3,1)_{\rm Diag}$. We show that these six bosons are "eaten up" by the antisymmetric

[2]Using the particle physics terminology, dRGT is a theory with no "radial mode," i.e., the graviton gets a mass via the Anderson mechanism, as opposed to the Higgs mechanism. What may be needed is an extension to include putative "radial mode(s)" that would ensure weakly coupled behavior above Λ_3.

[3]Reference [12] has extracted the square root in dRGT using vierbeins; however, we disagree with the main conclusion of that work on the Boulware-Deser degree of freedom. See also Refs. [13,14] for earlier interesting works on the vierbein formulation of bigravity.

part of the vierbein. This extends ghost-free massive gravity to a theory where the six antisymmetric components of the vierbein become dynamical.

II. VIERBEIN FORMULATION

The formulation of massive GR, as well as its extensions, is significantly simplified in the vierbein formalism [9]. Introducing the vierbein field ${e_\mu}^a$, $g_{\mu\nu} = {e_\mu}^a {e_\nu}^b \eta_{ab}$ with $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$, the cosmological constant term can be written as $d^4x\mathcal{L}_0 \sim d^4x\sqrt{-g}\Lambda \sim \Lambda\varepsilon_{abcd}e^a \wedge e^b \wedge e^c \wedge e^d$, where the one form e^a is defined as $e^a \equiv {e_\mu}^a \mathrm{d}x^\mu$. The ghost-free interactions of the vierbein perturbations can be represented in a similar fashion; e.g. in the unitary gauge, one such term is given by

$$d^4x\mathcal{L}_2 \sim M_{\mathrm{Pl}}^2 m^2 \varepsilon_{abcd} e^a \wedge e^b \wedge (e^c - \mathbf{1}^c) \wedge (e^d - \mathbf{1}^d),$$

where $\mathbf{1}^a \equiv {\delta_\mu}^a \mathrm{d}x^\mu$ represents a unit vierbein. The two contributions $\mathcal{L}_{0,2}$ to the potential can be supplemented by the two other independent terms $\mathcal{L}_{3,4}$, involving respectively three and four powers of $(e - \mathbf{1})$, contracted with the ε symbol in a similar fashion,[4]

$$d^4x\mathcal{L}_3 \sim M_{\mathrm{Pl}}^2 m^2 \varepsilon_{abcd} e^a \wedge (e^b - \mathbf{1}^b) \wedge (e^c - \mathbf{1}^c) \wedge (e^d - \mathbf{1}^d), \tag{6}$$

$$d^4x\mathcal{L}_4 \sim M_{\mathrm{Pl}}^2 m^2 \varepsilon_{abcd} (e^a - \mathbf{1}^a) \wedge (e^b - \mathbf{1}^b) \wedge (e^c - \mathbf{1}^c) \wedge (e^d - \mathbf{1}^d). \tag{7}$$

The above terms together with the Einstein-Hilbert term define an action for the 16 variables in the vierbein which is neither GCT nor LLT invariant, whereas the metric formulation is an action for 10 metric variables (plus four scalars in the Stückelberg formulation). Nevertheless, both formulations are dynamically equivalent. Following [9], we first show that the vierbein action is dynamically equivalent to the same action only with the additional constraint that the vierbein is symmetric (with respect to the Minkowski metric). In matrix notation,

$$e\eta = \eta e^T. \tag{8}$$

We parametrize the general vierbein as a constrained vierbein $\hat{e}$ satisfying (8), times a Lorentz transformation, parametrized as the exponential of a matrix $\hat{B}$ (which is antisymmetric with respect to η),[5]

$$e = \hat{e} e^{-\hat{B}}, \qquad \eta\hat{B} = -\hat{B}^T\eta. \tag{9}$$

The $\hat{B}$'s do not enter the Einstein-Hilbert term, since this term is invariant under local Lorentz transformations. Thus, the 6 variables in $\hat{B}$ appear only in the mass and potential terms (which in the metric formulation depend on the inverse metric g^{-1} through the matrix $\mathcal{K} = \mathbf{1} - \sqrt{g^{-1}\partial\phi\partial\phi}$). These fields therefore appear without derivatives—they are auxiliary fields. We now vary with respect to $\hat{B}$ and look at the equations of motion, in powers of $\hat{B}$. The lowest-order terms contain no powers of $\hat{B}$ (other than the variation $\delta\hat{B}$). Therefore, the only terms that appear at lowest order are the ones containing traces of one power of $\delta\hat{B}$ along with powers of $\hat{e}^{-1}$. Because $\hat{e}^{-1}$ is symmetric and $\delta\hat{B}$ antisymmetric, and because $\delta\hat{B}$ appears only linearly, the terms in the equations of motion linear in $\hat{B}$ all vanish. This means that the equations of motion of $\hat{B}$ start linearly in $\hat{B}$, and are solved by $\hat{B} = 0$. Plugging this solution back into the action, we see that the action with unconstrained vierbeins is dynamically equivalent to the action with symmetric vierbeins.

To relate the potential with symmetric vierbeins to the potential in the metric formulation we use the matrix representation $g = e\eta e^T$,

$$g^{-1}\eta = (e^{-1})^T\eta^{-1}e^{-1}\eta. \tag{10}$$

Using the parametrization (9) and the symmetry property of $\hat{e}^{-1}$:

$$\sqrt{g^{-1}\eta} = (\hat{e}^{-1})^T. \tag{11}$$

Thus in the unitary gauge, $\partial_\mu\phi^{\bar{a}} = \delta^{\bar{a}}_\mu$, we can write

$$\mathcal{L}_{2,3,4}\left(\sqrt{g^{-1}\eta}\right) = \mathcal{L}_{2,3,4}(\hat{e}^{-1}). \tag{12}$$

Due to the presence of the unit vierbein, in the form presented above, the first-order theory lacks invariance under both the GCT and LLT, characteristic of general relativity. Both of the symmetries however can be restored via corresponding Stückelberg fields. For this, one introduces the auxiliary scalars $\phi^{\bar{a}}$, analogous to those of the metric description of massive GR, as well as the "link" field ${\lambda^a}_{\bar{a}}$. The latter transforms as a contravariant vector under the local Lorentz group, ${\lambda^a}_{\bar{a}} \to {Q^a}_b(x){\lambda^b}_{\bar{a}}$, where $Q(x) \in SO(3,1)_{\mathrm{LLT}}$, and as a covariant vector under the global group $SO(3,1)_{\mathrm{INT}}$, ${\lambda^a}_{\bar{a}} \to {L^{\bar{b}}}_{\bar{a}}{\lambda^a}_{\bar{b}}$. Using these fields, the mass and potential terms can be rewritten in a manifestly GCT × LLT-invariant form via the "k vierbein," ${k_\mu}^a \equiv {e_\mu}^a - {\lambda^a}_{\bar{a}}\partial_\mu\phi^{\bar{a}}$,

$$\mathcal{L}_2 \sim M_{\mathrm{Pl}}^2 m^2 \varepsilon^{\mu\nu\alpha\beta}\varepsilon_{abcd}{e_\mu}^a{e_\nu}^b{k_\alpha}^c{k_\beta}^d, \tag{13}$$

$$\mathcal{L}_3 \sim \alpha_3 M_{\mathrm{Pl}}^2 m^2 \varepsilon^{\mu\nu\alpha\beta}\varepsilon_{abcd}{e_\mu}^a{k_\nu}^b{k_\alpha}^c{k_\beta}^d, \tag{14}$$

$$\mathcal{L}_4 \sim \alpha_4 M_{\mathrm{Pl}}^2 m^2 \varepsilon^{\mu\nu\alpha\beta}\varepsilon_{abcd}{k_\mu}^a{k_\nu}^b{k_\alpha}^c{k_\beta}^d. \tag{15}$$

(As before, the ϵ's here are the epsilon symbols, i.e. there are no factors of $\sqrt{-g}$.) In the unitary gauge defined by

[4] As in the second-order case, the remaining possible term $\mathcal{L}_1$, linear in $(e - \mathbf{1})$, can be expressed as a combination of the rest of the terms.

[5] See [15] for more on this condition and its relation to the square roots of the metric formulation.

$\lambda^{\bar{a}}{}_a = \delta^{\bar{a}}_a$ and $\partial_\mu \phi^{\bar{a}} = \delta^{\bar{a}}_\mu$, one recovers the $\mathcal{L}_{2,3,4}$ of (1). Away from this gauge, the theory acquires invariance under GCT, as well as under LLT, realized on the vierbein and the link fields as follows:

$$e_\mu{}^a \to Q(x)^a{}_b e_\mu{}^b, \qquad \lambda^a{}_{\bar{a}} \to Q(x)^a{}_b \lambda^b{}_{\bar{a}}. \quad (16)$$

The transformations (16) with $Q(x) \in SO(3,1)_{\mathrm{LLT}}$ represent a symmetry of the entire action, the potentials (13)–(15) and the Einstein-Hibert term. However, the potential terms themselves, (13)–(15), without the EH term, can have a larger symmetry. To see this, we first note that these potentials are invariant under the formal field redefinition (16) with $Q(x) \in SL(4)$. Now, we defined λ to be a $SO(3,1)$ matrix and therefore, such transformations with $SL(4)$ matrices would take them outside of $SO(3,1)$. This observation suggests that in the theory where the EH term is absent, the λ can be promoted to a $SL(4)$-valued field. The resulting terms, (13)–(15), will have a local $SL(4)$ symmetry, in addition to being invariant under GCTs. This extended local $SL(4)$ symmetry is the defining property of the mass and potential terms.

However, the EH term does not respect the $SL(4)$. Therefore, there are two ways to combine the EH term with the potentials (13)–(15): (1) To define a theory where λ is an $SO(3,1)$-valued field, (2) alternatively, to define a theory with $\lambda \in SL(4)$. In this paper we chose the former case because that is the theory of a single massive graviton. The latter choice gives a theory with $9 = \dim SL(4) - \dim SO(3,1)$ additional fields, and might be an interesting model to look at in the future.

Thus, for $\lambda \in SO(3,1)$, in the unitary gauge, $\lambda = \mathbf{1}$, with $\phi^{\bar{a}}$ kept unfixed, one recovers the GCT-invariant but LLT noninvariant formulation of massive GR.[6] The relevant symmetry breaking pattern, corresponding to this case,

$$SO(3,1)_{\mathrm{INT}} \times SO(3,1)_{\mathrm{LLT}} \to SO(3,1)_{\mathrm{DIAG}}, \quad (17)$$

involves six broken generators, while the remaining six correspond to the diagonal part of LLT and internal Lorentz groups. The equation of motion for λ, evaluated in the unitary gauge, gives precisely the constraint (8), needed for the theory to reduce to massive GR.

As already remarked above, it is useful to represent the vierbein as $e_\mu{}^a = \exp(\hat{B}^a{}_b)\hat{e}_\mu{}^b$ and the λ field as $\lambda^a{}_{\bar{a}} = \exp(v^a{}_{\bar{a}}/f)$, where both B and v are antisymmetric fields (once indices are lowered by η). Under LLT, both of these fields shift by a coordinate dependent gauge function, so one or the other of them may be gauged away, but not both. One linear combination of $\hat{B}$ and v is invariant under LLT. This combination has no kinetic term in our construction, and it is algebraically determined by classical equations of motion guaranteeing, by the same arguments given earlier. Only the five helicities of the graviton are propagating degrees of freedom in the theory.

An interesting alternative is to give dynamics to the gauge-invariant combination by regarding it as a Nambu-Goldstone field parametrizing the coset corresponding to the symmetry breaking pattern (17). The kinetic term for this field also breaks the local $SL(4)$ of the potential down to the group of LLTs. We will discuss this possibility in Sec. V. Before then we will stay in the framework of massive gravity and the gauge invariant part of the λ field will be regarded as nondynamical.

III. GEOMETRIC INTERPRETATION AND GENERALIZATIONS

In this section, we will give this formulation of the theory a geometric interpretation. Let us consider two manifolds of the same dimension,[7] and a smooth mapping between them ϕ: $\mathcal{M} \to E$. When a set of coordinates is given, the mapping ϕ consists of 4 smooth functions which we denote by $\phi^a(x)$. (We ignore any possible topological obstructions at the moment. Such a smooth mapping always exists locally within certain patches of both $\mathcal{M}$ and E.)

We denote, at each point of $x \in \mathcal{M}$ and $\phi(x) \in E$, the cotangent spaces $T^*_{\mathcal{M}}(x)$ and $T^*_E(\phi)$ respectively. A set of vierbeins $e^a = \mathrm{d}x^\mu e_\mu{}^a$ is defined for every $T^*_{\mathcal{M}}(x)$ which endow $\mathcal{M}$ with a metric $g_{\mu\nu} \equiv e_\mu{}^a e_\nu{}^b \eta_{ab}$. Usually, for the mapping ϕ between $\mathcal{M}$ and E to be compatible with their Riemannian structures, one must assume that the metric on $\mathcal{M}$ coincides with the metric pulled back from E through the functions $\phi^a(x)$, i.e. both manifolds share identical Riemannian geometries and the mapping ϕ represents nothing other than a simple coordinate transformation. Physically, the two are indistinguishable.

If, on the other hand, we insist that the manifold E should stay flat, we may choose to define the vierbeins in each $T^*_E(\phi(x))$ as $\theta^a = \mathrm{d}\phi^a$. Together with the torsion-free condition, such a choice guarantees that the curvature tensor on E vanishes. But, such a construction of θ^a does not respect the local Lorentz symmetry of $T^*_E(\phi(x))$, leaving only the global version intact.[8]

[6]In this gauge $k_\mu{}^a = e_\mu{}^a - \delta^a_{\bar{a}}\partial_\mu \phi^{\bar{a}}$, and the global Siegels $ISO(3,1)_{\mathrm{INT}}$ symmetry (4) gets enhanced to a symmetry with respect to the global $ISL(4)$ transformations of the $\phi^{\bar{a}}$ fields, if the vierbein is also transformed under the global $SL(4)$. The existence of this enhanced global symmetry of the mass and potential terms had been pointed out by W. Siegel [16], and has recently made us realize that in the LLT invariant theory the potentials (13)–(15) can be promoted to the local $SL(4)$ symmetric form.

[7]We assume both manifolds to be dimension 4 for brevity in the current discussion, but this can be straightforwardly generalized, including to manifolds of different dimension, giving theories with new scalar degrees of freedom, along the lines of [17].

[8]Formally, one may fix this by introducing another set of flat spin connections on E and write $\theta^a = \mathrm{D}\phi^a$ instead, where D is the covariant exterior derivative. It is not necessary for the current discussion and we choose not to pursue this direction here.

Now that the two manifolds $\mathcal{M}$ and E are endowed with totally different Riemannian structures, there is no natural way to mix the cotangent vectors living in $T^*_{\mathcal{M}}(x)$ and those living in $T^*_E(\phi(x))$. Indeed, if we just write terms such as $e^a - \mathrm{d}\phi^a$, they violate invariance with respect to the LLTs. The link fields $\lambda^a{}_{\bar{a}}(x)$ are introduced to remedy this. Due to the specific transformation of $\lambda^a{}_{\bar{a}}$ under the two Lorentz groups, we are able to map the forms in one cotangent space to the other and introduce mixing via the k vierbein $e^a - \lambda^a{}_{\bar{a}}\mathrm{d}\phi^{\bar{a}}$, where $\mathrm{d}\phi^{\bar{a}} = \partial_\mu \phi^{\bar{a}} dx^\mu$. We write the mass and potential in terms of the forms

$$
\begin{aligned}
d^4x\mathcal{L}_1 &\sim \epsilon_{abcd}(e^a - \lambda^a{}_{\bar{a}}\mathrm{d}\phi^{\bar{a}}) \wedge e^b \wedge e^c \wedge e^d,\\
d^4x\mathcal{L}_2 &\sim \epsilon_{abcd}(e^a - \lambda^a{}_{\bar{a}}\mathrm{d}\phi^{\bar{a}}) \wedge (e^b - \lambda^b{}_{\bar{b}}\mathrm{d}\phi^{\bar{b}}) \wedge e^c \wedge e^d,\\
d^4x\mathcal{L}_3 &\sim \epsilon_{abcd}(e^a - \lambda^a{}_{\bar{a}}\mathrm{d}\phi^{\bar{a}}) \wedge (e^b - \lambda^b{}_{\bar{b}}\mathrm{d}\phi^{\bar{b}})\\
&\quad \wedge (e^c - \lambda^c{}_{\bar{c}}\mathrm{d}\phi^{\bar{c}}) \wedge e^d,\\
d^4x\mathcal{L}_4 &\sim \epsilon_{abcd}(e^a - \lambda^a{}_{\bar{a}}\mathrm{d}\phi^{\bar{a}}) \wedge (e^b - \lambda^b{}_{\bar{b}}\mathrm{d}\phi^{\bar{b}})\\
&\quad \wedge (e^c - \lambda^c{}_{\bar{c}}\mathrm{d}\phi^{\bar{c}}) \wedge (e^d - \lambda^d{}_{\bar{d}}\mathrm{d}\phi^{\bar{d}}).
\end{aligned} \tag{18}
$$

As discussed in the previous section, these expressions manifestly respect the local Lorentz symmetry on $\mathcal{M}$, defined by

$$
e^a \to Q^a{}_b e^b \qquad \lambda^a{}_{\bar{b}} \to Q^a{}_c \lambda^c{}_{\bar{b}} \qquad \phi^{\bar{a}} \to \phi^{\bar{a}},
$$

at the cost of introducing the Stückelberg fields $\lambda^a{}_{\bar{a}}$.

Notice that we can equally well write terms by multiplying $\lambda_a{}^{\bar{b}}$—which we define to be the inverse matrix of $\lambda^a{}_{\bar{b}}$—onto e^a instead of $\mathrm{d}\phi^{\bar{a}}$. So we could write, as an example,

$$
\begin{aligned}
d^4x\mathcal{L}_2 \sim \epsilon_{\bar{a}\bar{b}\bar{c}\bar{d}}&(\lambda_a{}^{\bar{a}}e^a - \mathrm{d}\phi^{\bar{a}}) \wedge (\lambda_b{}^{\bar{b}}e^b - \mathrm{d}\phi^{\bar{b}})\\
&\wedge \lambda_c{}^{\bar{c}}e^c \wedge \lambda_d{}^{\bar{d}}e^d.
\end{aligned}
$$

This formulation however is equivalent to the one in (18). In the new form, the invariance under LLT is manifestly visible since $\mathrm{d}\phi^{\bar{a}}$ are invariant, and the LLT transformations of e^a are simply compensated by the opposite rotation for $\lambda_a{}^{\bar{a}}$ so the combination $\lambda_a{}^{\bar{a}}e^a$ remains invariant automatically. Note that in this latter formulation one can directly extend λ to a $GL(4)$-valued field, and then have the mass and potential terms invariant under local $GL(4)$, instead of $SL(4)$ discussed in Sec. II. The $GL(4)$ invariant form can also be achieved in the original formulation, if the mass and potential terms (13)–(15) are multiplied by $\det(\lambda^{-1})$, with $\lambda \in GL(4)$.

The terms in (18) are quite reminiscent of the CC term in GR—these terms strongly resemble some sort of volume forms. In particular, one linear combination of the four terms gives the CC term (up to a total derivative). If, for the moment, we imagine that the fields $\phi^{\bar{a}}$ are the embedding coordinates of the manifold $\mathcal{M}$ into a higher dimensional flat manifold (so that $\bar{a}$ takes the values of $1, 2, \ldots, D$ where $D > 4$), a term $\mathcal{L} \sim \lambda^a{}_{\bar{a}}\lambda^b{}_{\bar{b}}\lambda^c{}_{\bar{c}}\lambda^d{}_{\bar{d}}\mathrm{d}\phi^{\bar{a}} \wedge \mathrm{d}\phi^{\bar{b}} \wedge \mathrm{d}\phi^{\bar{c}} \wedge \mathrm{d}\phi^{\bar{d}}\varepsilon_{abcd}$, with a *fixed* matrix $\lambda^a{}_{\bar{a}}$ that projects the D-dimensional tangent vectors down to the tangent space of $\mathcal{M}$, is the volume form for the surface $\mathcal{M}$ as embedded in E.

Here, in our formulation, there are two major differences. First of all, we are dealing with the mixing terms among the vierbeins of two different manifolds, $\mathcal{M}$ and E, with different geometries but an identical dimensionality. Secondly, we must integrate with respect to all possible embeddings parametrized by $\lambda^a{}_{\bar{b}}$ to make a comparison between different volume forms meaningful. Both differences complicate the geometrical identification of these mixing terms. However, for any fixed $\lambda^a{}_{\bar{b}}$, each term in (18) can be given a geometric interpretation in terms of a difference between certain volume forms of the two different manifolds.

Consider the simplest example, $d^4x\mathcal{L}_1 \sim (e^a - \lambda^a{}_{\bar{a}}\mathrm{d}\phi^{\bar{a}}) \wedge e^b \wedge e^c \wedge e^d\epsilon_{abcd}$. Apart from the volume form of $\mathcal{M}$, it contains the term $\lambda^a{}_{\bar{a}}\mathrm{d}\phi^{\bar{a}} \wedge e^b \wedge e^c \wedge e^d\epsilon_{abcd}$. If we choose the gauge $\lambda^a{}_{\bar{a}} = \delta^a{}_{\bar{a}}$, and focus only on the term $(a, b, c, d) = (1, 2, 3, 4)$, we recognize this as the volume form of $\mathcal{M}^3 \times R$, where $\mathcal{M}^3$ denotes a 3-dimensional submanifold spanned out by the cotangent vectors e^2, e^3, and e^4, and R denotes the "flat dimension" parametrized by $\phi^1(x)$. So, $\mathcal{L}_1$ gives a difference between the two types of volume forms: the one of $\mathcal{M}$ and another from those of $M^3 \times R$, with M^3 now representing a 3-dimensional submanifold of $\mathcal{M}$ spanned by any three of the four vierbeins e^a. Individually, each such term depends on the arbitrary choice of e^a, $\phi^{\bar{a}}$, as well as the embedding matrix $\lambda^a{}_{\bar{a}}$, but when all the indices are contracted and the fields are integrated over, we obtain a well-defined notion of a relative volume forms of the two manifolds.

Figure 1 gives an illustration to this. The left figure represents the original volume form of $\mathcal{M}$, with the 4th dimension suppressed, and the right one depicts the volume form obtained when the direction along that of e^1 is "straightened." The difference between the two volume forms is $d^4x\mathcal{L}_1$.

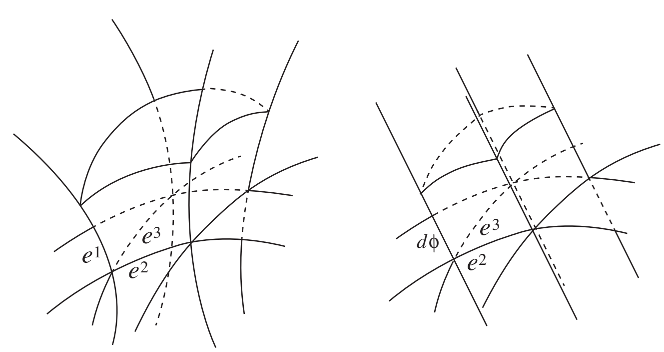

FIG. 1. Illustration of various volume forms, appearing in the graviton potential in massive GR.

Likewise, we may interpret terms $\lambda^a{}_{\bar a}\lambda^b{}_{\bar b}\mathrm{d}\phi^{\bar a}\wedge\mathrm{d}\phi^{\bar b}\wedge e^c\wedge e^d\epsilon_{abcd}$ as the volume form for various different $M^2\times R^2$, where M^2 denotes the 2-dimensional submanifolds spanned by an arbitrary pair of e^a and e^b. A linear combination of all the four terms in (18)—that is a most general potential for GR that includes the CC term—can be thought as linear combination of all possible departures of the volume forms of $\mathcal{M}$ from those of $\mathcal{M}^{4-n}\times R^n$, with $n=1,2,3,4$ denoting the number of dimensions that have been "straightened out."

The principles outlined here allow one to consider various generalizations. For example, the dimensionality of E does not have to coincide with the dimensionality of $\mathcal{M}$. If the dimensionality of E is D, the index $\bar a$ takes values from 1 to D in the vector representation of $SO(D-1,1)$, while the auxiliary fields $\lambda^a{}_{\bar b}$ transform as bivector of $SO(3,1)_{\mathrm{LLT}}$ and $SO(D-1,1)_{\mathrm{INT}}$ respectively. If $D>4$, the extra coordinates will correspond to extra physical scalar fields with a Galileon-like symmetry. The construction remains consistent, in the sense that a Boulware-Deser-like ghost will not be introduced. Such an extension of dRGT was already considered in [17]. Its vierbein formulation was given in [18] and was used to show ghost freedom. The present formalism provides the LLT-invariant vierbein formulation of this theory.

In the extreme case where $D=1$, $\phi^{\bar a}$ reduces to a single scalar ϕ (the index $\bar a$ takes only one value) and $\lambda^a{}_{\bar b}$ reduces to a single Lorentz vector $v^a(x)$ subjected to the condition $v^2=1$. Following the discussion given above, one finds that one of the natural interaction terms to consider is

$$L\sim v^a\mathrm{d}\phi\wedge e^b\wedge e^c\wedge e^d\epsilon_{abcd}, \tag{19}$$

which, after integrating out v^a, gives rise to an action of the Cuscuton type [19]

$$\mathcal{L}\sim\sqrt{-g}\sqrt{|g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi|}. \tag{20}$$

Last but not least, one may consider an even more general class of theories where the internal global symmetry does not have to be the Lorentz symmetry but is instead described by an arbitrary Lie group G. As long as $\phi^{\bar a}$ is in some representation R of G and the field $\lambda^a{}_{\bar b}$ is in the bi-representation of R and the Lorentz group, we may consider interactions of $\phi^{\bar a}$ with gravity described by the Lagrangians given in (18). If one further gauges this internal symmetry, one arrives at a broader class of theories, which includes the bigravity theories considered in [20].

IV. THE DECOUPLING LIMIT IN THE FIRST-ORDER FORMULATION

In this section, we will illustrate the advantages of the first-order formalism for the analysis of the decoupling limit (DL) of massive GR. In addition to reproducing very easily the already well-known scalar-tensor interactions that arise in this limit, we will derive an all-orders expression for the DL interactions involving the vector helicity of the massive graviton. To the best of our knowledge, the vector interactions have previously been unknown in closed form, though partial results are available [21–23].

We start by decomposing the vierbein field as before

$$e_\mu{}^a=(\exp\hat B)^a{}_b\hat e_\mu{}^b, \tag{21}$$

where $\hat B^a{}_b\equiv B^a{}_b/M_{\mathrm{Pl}}^{1/2}$ is an antisymmetric generator of LLT, $\hat B_{ab}=\eta_{ac}\hat B^c{}_b=-\hat B_{ba}$, while $\hat e$ is the vierbein, symmetric on its lower indices, $\hat e_{\mu\nu}\equiv\hat e_\mu{}^b\eta_{b\nu}=\hat e_{\nu\mu}$. The symmetric vierbein and the auxiliary scalars are decomposed into background values and their perturbations as

$$\hat e_\mu{}^a=\delta^a_\mu+\frac{S_\mu{}^a}{M_{\mathrm{Pl}}},\qquad \phi^{\bar a}=\delta^{\bar a}_\mu x^\mu-\pi^{\bar a}, \tag{22}$$

where $\pi^{\bar a}=\eta^{\bar a\mu}(\partial_\mu\pi/\Lambda_3^3+mA_\mu/\Lambda_3^3)$ and $\Lambda_3\equiv(M_{\mathrm{Pl}}m^2)^{1/3}$. The scalings for various perturbation fields have been chosen so as to recover the correct quadratic terms in the decoupling limit of the theory. In the ghost-free theories at hand, this limit is $m\to 0$, $M_{\mathrm{Pl}}\to\infty$, with Λ_3 held finite [3,24,25].

Concentrating first on the S-π interactions that result from the Lagrangian (13), one can easily see that only terms with a single S and a certain number of π's survive in the decoupling limit

$$\mathcal{L}_2^{\mathrm{d.l.}}\sim S_\mu{}^a\bigg(\varepsilon^{\mu\nu\bullet\bullet}\varepsilon_{ab\bullet\bullet}\partial^b\partial_\nu\pi+\frac{1}{\Lambda_3^3}\varepsilon^{\mu\nu\alpha\bullet}\varepsilon_{abc\bullet}\partial^b\partial_\nu\pi\partial^c\partial_\alpha\pi\bigg),$$

where the indices on the ε symbols are contracted with the help of the unit vierbein. At linear order, the vierbein and metric perturbations are related as $2S_\mu{}^a\eta_{a\nu}=h_{\mu\nu}$; therefore, the above scalar-tensor interactions are nothing but the well-known ghost-free DL interactions of the helicity-0 and helicity-2 gravitons in massive GR [3]. Including the independent interactions $\mathcal{L}_{3,4}$ with three and four powers of k, one equally easily reproduces the remaining $h(\partial^2\pi)^3$ interaction of the decoupling limit of massive GR.

As a next step, we use the above formalism to derive a closed-form expression for the vector-scalar interactions in the DL. To illustrate, we will start with the case when the two free parameters of dRGT are chosen so that all the scalar-tensor nonlinear interaction at the scale Λ_3 identically vanish [3]. For this parameter choice a linear combination of $\mathcal{L}_2$, $\mathcal{L}_3$ and $\mathcal{L}_4$ can be expressed, up to a total derivative, in terms of $\mathcal{L}_1$ and a CC term with a tuned value [26]; the resulting theory was dubbed "the minimal model." In the GCT- and LLT-invariant vierbein formalism the minimal model takes the form:

$$d^4x\mathcal{L}_{\mathrm{min}}=M_{\mathrm{Pl}}^2m^2\varepsilon_{abcd}(e^a\wedge e^b\wedge e^c\wedge e^d-4e^a\wedge e^b\wedge e^c\wedge k^d), \tag{23}$$

where the one form k is defined in the usual way $k^d = dx^\beta k_\beta{}^d$ using the k vierbein $k_\mu{}^a \equiv e_\mu{}^a - \lambda^a{}_{\bar{a}}\partial_\mu \phi^{\bar{a}}$. In spite of the absence of the nonlinear helicity-0 interactions with helicity-2 at the scale Λ_3, the minimal model has nonlinear interaction terms of the vector mode with the helicity-0 at the scale[9] Λ_3.

As can be straightforwardly checked, the potentially diverging contributions, e.g. of the form $\varepsilon\varepsilon B\partial^2\pi$, in fact vanish due to the symmetry properties of the B field [this is precisely what allows us to consistently set the scaling of the field B to be $(M_{\rm Pl})^{-1/2}$]. Keeping all finite terms involving B in the decoupling limit and expanding the wedge product in (23), one obtains[10]

$$\mathcal{L}^{\rm d.l.}_{\rm min} \supset 12(\Lambda_3^3 B^{\mu\nu}B_{\mu\nu} - B^{\mu\alpha}B_\alpha{}^\nu(\partial_\mu\partial_\nu\pi - \eta_{\mu\nu}\Box\pi) - 2\Lambda_3^{3/2}B^{\mu\nu}\partial_\mu A_\nu). \tag{24}$$

This is the simplest all-orders expression. It involves the auxiliary field B. We may, if we like, integrate it out to obtain an expression involving only the physical fields π and A, at the cost of generating an infinite number of terms. In matrix notation (all indices are understood to be contracted with the help of the flat metric), the equation of motion for B yields

$$P^{\alpha\beta}_{\mu\nu}(\pi)B_{\alpha\beta} = F_{\mu\nu}, \tag{25}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ denotes the field strength for the vector mode, and P is a tensor of the schematic form $(\eta\eta + \eta\partial\partial\pi)$ appropriately antisymmetrized.[11] When substituted back into the action, the last equation gives the closed-form expression for the vector-scalar interactions in the decoupling limit of the "minimal" massive GR

$$\mathcal{L}^{\rm d.l.}_{\rm min} \supset 6\,{\rm Tr}[P^{-1}\cdot F\cdot \partial A]. \tag{26}$$

The lowest-order term in the expansion of the latter Lagrangian in powers of $\partial\partial\pi$ yields the (correct-sign) kinetic term for the vector, while higher order terms give its interactions with the scalar helicity.

Moving away from the minimal model, for the most general form of the potential the Lagrangian has the following schematic form in the decoupling limit:[12]

$$\mathcal{L}^{\rm d.l.} \sim \Lambda_3^4\Big[\frac{BB}{\Lambda_3}\Big(1 + \frac{\partial^2\pi}{\Lambda_3^3} + \frac{(\partial^2\pi)^2}{\Lambda_3^6} + \frac{(\partial^2\pi)^3}{\Lambda_3^9}\Big) + \frac{B\partial A}{\Lambda_3^{5/2}}\Big(1 + \frac{\partial^2\pi}{\Lambda_3^3} + \frac{(\partial^2\pi)^2}{\Lambda_3^6}\Big)\Big]. \tag{27}$$

Varying with respect to the nondynamical field B yields an expression for it in terms of π and A, that can be substituted back into the action, recovering the complete decoupling limit form of the vector-scalar interactions. These interactions are derived in [28]. The resulting expressions can be readily used for studying dynamics of the given sector of the theory on various background solutions.

V. DYNAMICAL ANTISYMMETRIC FIELD

While in pure massive gravity the link fields $\lambda^a{}_{\bar{a}}$ are nondynamical, one can go further and consider a generalization with dynamical link fields, nonlinearly realizing the symmetry breaking pattern (17). Given the symmetries at hand, the most general Lagrangian at low energy can be written as a function of the fields with definite transformation properties under GCT $\times$ LLT $\times$ $ISO(3,1)_{\rm INT}$,

$$S = \int d^4x \mathcal{L}(\lambda^a{}_{\bar{a}}, \phi^{\bar{a}}, e_\mu{}^a, D_\mu). \tag{28}$$

The covariant derivative D_μ acts on the LLT indices through the standard expression $D_\mu\lambda^a{}_{\bar{a}} = \partial_\mu\lambda^a{}_{\bar{a}} + \omega_\mu{}_b{}^a\lambda^b{}_{\bar{a}}$, where the spin connection $\omega_\mu{}_b{}^a$ can be expressed in terms of the vierbein and its derivatives in a torsion-free theory. Being a Lorentz matrix-valued field, λ is most conveniently expressed in terms of the antisymmetric generator, $\lambda = \exp(v/f)$, where f denotes the "decay constant" of v. The decay constant f is an adjustable parameter of the theory.

The lowest-order nontrivial invariant that one can form from these fields can be written as follows:

$$\mathcal{L} = -f^2(D_\mu\lambda)^2 = -\eta_{ab}\eta^{\bar{a}\bar{b}}\partial_\mu v^a{}_{\bar{a}}\partial^\mu v^b{}_{\bar{b}} + \cdots,$$

and includes the kinetic term for the 6 degrees of freedom present in $v^a{}_{\bar{a}}$. Note that the kinetic term for the λ field, alongside with the EH term, breaks the local $SL(4)$ symmetry of the mass and potential terms if we were to promote the λ to a $SL(4)$-valued field. We will write $f \sim \hat{f}(M_{\rm Pl}\Lambda_3)^{1/2}$, where $\hat{f}$ is dimensionless. The Λ_3 decoupling limit remains intact as long as $\hat{f}$ remains fixed in this limit, i.e. does not depend parametrically on any other scales.

Supplementing the action by the ghost-free potential terms, for example Eq. (13), one obtains a set of interactions of v with the rest of the fields present in the theory (for the moment, we choose the LLT gauge defined by $B = 0$.) At the linearized order, (13) yields the mass term, as well as a mixing with the vector mode in the decoupling limit (we disregard the distinction between the LLT and spacetime indices for notational simplicity)

[9] It also has vector-scalar-tensor interaction terms at higher scales, such as $\Lambda_2 = (M_{\rm Pl}m)^{1/2}$ and/or at scales formed by products of the Λ_2^2 and Λ_3^3 scales. These nonlinear terms, up and including quartic order, were calculated by L. Berezhiani and G. Chkareuli [27].

[10] For the sake of simplicity, we will not make distinction between the Lorentzian and spacetime indices of the B field in the decoupling limit, since both are contracted with the flat metric.

[11] We thank the authors of [28] for pointing out a sloppy treatment of P^{-1} in version 1 of this work. Explicit expressions are obtained in [28].

[12] We discard explicit vector-scalar interactions of the form $\partial A\partial^2\pi$, $\partial A\partial^2\pi\partial^2\pi$, $\partial A\partial A\partial^2\pi$ because these turn out to be total derivatives.

$$\mathcal{L}^{\rm d.l.} = v^{\mu\nu}\left(\Box + \frac{2\Lambda_3^2}{\hat{f}^2}\right)v_{\mu\nu} + \frac{2\Lambda_3}{\hat{f}}F^{\mu\nu}v_{\mu\nu}. \quad (29)$$

A shift in the v field, $v_{\mu\nu} \to \hat{v}_{\mu\nu} - \frac{F_{\mu\nu}}{\frac{\Lambda_3}{\hat{f}}(2+\frac{\hat{f}^2\Box}{\Lambda_3^2})}$ diagonalizes the action, bringing it to the following form:

$$\mathcal{L}^{\rm d.l.} = \hat{v}^{\mu\nu}\left(\Box + \frac{2\Lambda_3^2}{\hat{f}^2}\right)\hat{v}_{\mu\nu} - F^{\mu\nu}\frac{1}{2+\frac{\hat{f}^2\Box}{\Lambda_3^2}}F_{\mu\nu}. \quad (30)$$

Another peculiar feature of the above action is that $\hat{v}$ acquires a mass $|m_v^2| \sim \Lambda_3^2/\hat{f}^2$. This is below the cutoff of the effective theory to the extent that $\hat{f} \gg 1$. Note that, in order to reproduce the correct sign of the vector kinetic term at low energies, m_v^2 has to be tachyonic; however, one could expect higher powers of v (e.g. v^4) to also be present, and these could stabilize the v potential. Likewise, the kinetic term of the vector acquires a modification. In the regime, $\hat{f}^2\Box/\Lambda_3^2 \ll 1$, the modification is irrelevant and A_μ propagates the usual two vector polarizations of the massive graviton. Note that the residue of the vector particle propagator vanishes at the position of the pole of the v field.

One can give the above generation of the mass m_v an Anderson-mechanism-like interpretation. Indeed, both of the antisymmetric fields, B and v, nonlinearly realize the local Lorentz invariance. One can always choose a gauge in which either of the two, e.g. B, is frozen to be zero; however, one combination of these is gauge invariant (at the linear level, the invariant combination is simply $\hat{f}\Lambda_3^{1/2}B - v$). Then, the gauge-invariant combination (which reduces to v in the $B = 0$ gauge) acquires a mass due to the spontaneous breaking of LLT.

Finally, we comment on ghost freedom of the interactions of the antisymmetric field v with the rest of the modes, present in the decoupling limit Lagrangian. The object $k_a{}^\mu$ is decomposed (excluding the symmetric vierbein perturbation) in the $B = 0$ gauge as follows:

$$k_a{}^\mu = \frac{\partial_\mu\partial^a\pi}{\Lambda_3^3} + \frac{\partial_\mu A^a}{M_{\rm Pl}^{1/2}\Lambda_3^{3/2}} - \frac{v^a{}_\mu}{\hat{f}(M_{\rm Pl}\Lambda_3)^{1/2}} + \frac{v^a{}_b\partial_\mu\partial^b\pi}{\hat{f}M_{\rm Pl}^{1/2}\Lambda_3^{7/2}} + \frac{v^a{}_b\partial_\mu A^b}{\hat{f}M_{\rm Pl}\Lambda_3^2} - \frac{v^a{}_b v^b{}_\mu}{2\hat{f}^2 M_{\rm Pl}\Lambda_3} + \frac{v^a{}_b v^b{}_c\partial_\mu\partial^c\pi}{2\hat{f}^2 M_{\rm Pl}\Lambda_3^4} + \cdots.$$

Most of the terms, that follow from the expansion of (13) are easily checked to be safe from more than two derivatives acting on fields in the resulting equations of motion—either on the basis of antisymmetry of v, or due to the presence of the ε symbols in the corresponding expressions.

The only two interactions for which this property is not apparent are of the $vv\partial\partial\pi\partial\partial\pi$-type. The first of these is $\varepsilon^{\mu\nu\bullet\bullet}\varepsilon_{ab\bullet\bullet}v^a{}_\rho\partial_\mu\partial^\rho\pi v^b{}_\sigma\partial_\nu\partial^\sigma\pi$. The only potentially dangerous, three-derivative term arises in the equation of motion for π (all other similar terms vanish by antisymmetrization), and has the following form:

$$\varepsilon^{\mu\nu\bullet\bullet}\varepsilon_{ab\bullet\bullet}\partial_\mu(v^a{}_\rho v^b{}_\sigma)\partial^\rho\partial^\sigma\partial_\nu\pi.$$

Now, antisymmetrization in the a and b indices tells us that the object in the parentheses is antisymmetric in the (ρ, σ) pair. Contracted with $\partial^\rho\partial^\sigma$ on the scalar, the term at hand vanishes. Likewise, a potentially dangerous term in the Lagrangian $\varepsilon^{\mu\nu\bullet\bullet}\varepsilon_{ab\bullet\bullet}\partial_\mu\partial^a\pi v^b{}_\rho v^\rho{}_\sigma\partial_\nu\partial^\sigma\pi$ yields an apparently ghostly contribution to the π-equation of motion

$$\varepsilon^{\mu\nu\bullet\bullet}\varepsilon_{ab\bullet\bullet}[\partial_\mu(v^b{}_\rho v^\rho{}_\sigma)\partial^a\partial_\nu\partial^\sigma\pi + \partial_\nu(v^b{}_\rho v^\rho{}_\sigma)\partial^a\partial_\mu\partial^\sigma\pi].$$

However, the object in the square parentheses in this expression is manifestly symmetric under $\mu \to \nu$. Contracted with the antisymmetric $\varepsilon^{\mu\nu\bullet\bullet}$, this again yields zero. Of course, although this is a nice consistency check, such a vanishing of the three-derivative terms in the equations of motion is by no means surprising and follows automatically from the inherent ghost freedom of the potential (13).

VI. BRIEF COMMENTS ON THE LITERATURE

In this section, we briefly discuss the status of massive gravity as applied to the real world. In this approach, the graviton mass is taken to be of the order of the present day Hubble parameter, $m \sim H_0 \sim 10^{-33}$ eV (for phenomenological bounds on the graviton mass see [29]). Although this is a very small parameter as compared to the Planck scale, such smallness is robust—the mass parameter does not get renormalized by large quantum corrections [24,30]; this is unlike the cosmological constant which does receive large renormalizations. Therefore, it is appealing to describe the observed cosmic acceleration as an effect due to a nonzero graviton mass.

Massive gravitons can produce a state with the stress tensor mimicking dark energy (the so-called self-accelerated solutions [31–36]). Massive gravity dark energy is expected to have a slightly different predictions from those of CC based cosmology, and the differences may be tested observationally. These solutions produce dark energy with the equations of state identical to that of CC, but different fluctuations. Unfortunately, certain fluctuations about these solutions are problematic—some of the physical 5 degrees of freedom have vanishing kinetic terms, destabilizing the background [37]. Extensions of dRGT by additional scalars [33,38] or bi- and multigravity [9,20], or further extensions [39,40], also exhibit self-accelerated solutions. Recently, an extension by scalars has been proposed by De Felice and Mukohyama [41] and shown to have a self-accelerated solution with stable fluctuations—a first example of this kind.

Spherically symmetric solutions and black holes in massive GR have been studied in [32,42]. A general issue in dRGT is that it is a strongly coupled theory at the distance scale $(\Lambda_3)^{-1}$, which for the above value of the graviton mass is $\sim$1000 km. This scale is background dependent, and decreases for realistic backgrounds [43], but never enough for one to feel comfortable with it. The higher dimensional operators—that best manifest themselves in the decoupling limit—are suppressed by this scale. Moreover, on realistic backgrounds these operators give rise to order 1 or larger classical renormalization of the kinetic terms of fluctuations. That is how some of these kinetic terms vanish or flip their signs on the self-accelerated backgrounds. Therefore, dRGT needs an extension beyond the strong coupling scale in order for it to be potentially applicable to the real world. This extension is unknown at present, but for it to work it should introduce new states at or below the scale Λ_3. Therefore, many properties of the backgrounds and fluctuations sensitive to scales above Λ_3 can get modified in an extended theory.[13]

Furthermore, in the decoupling limit dRGT gets related [3] to the Galileons [45]. The latter are known to exhibit superluminal propagation on nontrivial backgrounds. So does dRGT for a large portion of the α_3, α_4 parameter space. For theories satisfying the Froissart bound, this has been argued [46] to preclude a standard UV completion by a local, Lorentz invariant field or string theory, however, theories with long-range fields do not necessarily obey this bound; moreover, there is no claim to rule out a possible Lorentz-violating, nonlocal or intrinsically higher dimensional completion. Furthermore, there is an exception for some special values of α_3, α_4, where subluminality for a spherically symmetric solution is achieved at the expense of not having an asymptotically flat background[14] [43,50].

The question of whether superluminality can lead to prohibitive acausality is entangled with the strong coupling issue [51]. The conclusion of acausality of massive gravity [52,53] that has been reached by constructing superluminal shock waves and characteristics is, in the context of a low energy theory, not warranted without a further nuanced study. A well-known counterexample is the following: quantum electrodynamics (QED) in an external gravitational field, at energies below the electron mass, gives rise to dimension 6 operators, one of which yields superluminal characteristics for a photon propagating in a given nontrivial gravitational background [54]. However, this superluminality—which appears within the effective theory—does not mean that QED supplemented by GR is an acausal theory. In spite of a large body of literature on the issue of superluminality vs acausality, some with split views, we believe that the low energy effective field theory understanding of systematic criteria for potential harms, or their absence, of superluminal low energy group and phase velocities is still to be precisely formulated.

[13]For related recent developments see Ref. [44].

[14]Although not directly related to massive gravity, cosmological solutions with subluminal spectra in dilatation invariant theories of Galileons have been found in [47–49].

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Note added:—Reference [28] has studied the decoupling limit of dRGT using the vierbein formalism. This work, even though it appeared later than v1 of the present work, should be considered as concurrent on the main idea of studying the decoupling limit in this formalism; moreover the results of [28] on the decoupling limit are superior to ours in their completeness. The remarkable work [55], appearing in 2006, introduced almost all of the ingredients of massive gravity, including the Stückelbergs for the LLTs (but not the ϕ^a fields). Unfortunately, Ref. [55] adopts an incorrect conclusion regarding the existence of the Boulware-Deser ghost. We thank Andrew Tolley for bringing this to our attention.

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