Restoring general relativity in massive bigravity theory

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We study static spherically symmetric solutions of massive bigravity theory, free from the Boulware-Deser ghost. We show the recovery of general relativity via the Vainshtein mechanism, in the weak limit of the physical metric. We find a single polynomial equation determining the behavior of the solution for distances smaller than the inverse graviton mass. This equation is generically of the seventh order, while for a specific choice of the parameters of the theory it can be reduced to lower orders. The solution is analytic in different regimes: for distances below the Vainshtein radius (where general relativity is recovered), and in the opposite regime, beyond the Vainshtein radius, where the solution approaches the flat metric.

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I. INTRODUCTION

Modification of general relativity (GR) giving mass to the graviton started from the work of Fierz and Pauli [1]: they considered a linear theory of a single massive spin-2 field living in flat space-time. The first nonlinear realization of the massive graviton was presented much later [2], although in a completely different context. To extend at the nonperturbative level the action of Fierz and Pauli, adding to the Einstein-Hilbert action a nonderivative self-coupling for the metric g, it is required the introduction of an additional metric f that may be a fixed external field, or a dynamical one. When f is nondynamical we are in the framework of æther-like theories where diffeomorphism invariance can be restored by the introduction of a suitable set of Stuckelberg fields; on the other hand, if it is dynamical, we enter in the context of bigravity theories.

Unfortunately, for a generic potential the theory has a ghost propagating degree of freedom (dof) [3], the so-called Boulware-Deser ghost, associated with the Ostrogradski ghost in more general setup. Notably, the Fierz-Pauli theory was constructed so that it has 5 healthy propagating degrees of freedom, while the sixth mode is removed due to the specific choice of the coefficients in the mass term. When the theory is promoted to nonlinear level or considered around nonflat background, the sixth mode reappears leading to ghost instability. This problem was solved only recently by a careful choice of the massive gravity potential [4] such that, on the fully nonperturbative level, the theory propagates only 5 degrees of freedom [5] (see also [6–8]). We will refer to this theory as the de Rham-Gabadadze-Tolley (dRGT) model. Of course, one has to be cautious about the rest—the 5 propagating degrees of freedom—however, at least there is no *a priori* Ostrogradski instability associated with the sixth mode. The original dRGT construction of massive gravity takes the additional metric as a flat, nondynamical field; then it was extended in the bigravity context supplying an extra Einstein-Hilbert term for the second metric [9]. The bimetric approach to massive gravity is the main subject of our work. We would like to emphasize here that the bigravity formulation of massive gravity is not just a theoretical entertainment, but also cosmology calls for it. When the second metric is nondynamical and Minkowski there is no homogeneous spatially flat Friedmann-Robertson-Walker solution [10,11], on the contrary in the bigravity formulation flat Friedmann-Robertson-Walker homogeneous solutions do exist [12–14]. Moreover, the cosmological perturbations are far less problematic [15]: all the dof propagate at the linear level without ghost instabilities [16]. For more recent works see [17].

Another problem which arises in massive gravity models is generic for theories with extra propagating degrees of freedom. Since the graviton mass turns on (at least) 3 extra degrees of freedom, it is expected that the extra interaction change the Newtonian limit or/and the light deflection. This can be easily seen in the so-called decoupling limit—the scalar part of the graviton is directly coupled to the matter with approximately the same coupling constant of the helicity-2 piece. The extra scalar behaves similar to the Brans-Dicke field ruling out the theory on observational ground. Moreover, a naive way to recover GR sending the mass of the graviton to zero, does not solve the problem—the so-called van Dam-Veltman-Zakharov (vDVZ) discontinuity [18]—since the theory with arbitrary small (but nonzero) graviton mass contains the extra propagating scalar, absent in the massless theory (GR). A way to overcome this difficulty was proposed by Vainshtein [19] in 1972. Vainshtein noticed that the linear approximation breaks down at some distance far from the source (now called the Vainshtein radius) and therefore one cannot approximate the solution by linearizing it close to the source. On the contrary, he showed that a solution can be found by expanding around the GR solution in powers of the graviton mass. This construction indicates the possibility that the extra propagating scalar mode can be hidden close to the source by nonlinear effects. The question still remained if this close-to-GR solution could have been matched to an asymptotically flat solution [3]; it was in fact argued that it was not possible [20,21]. Only recently it has been realized that the Vainshtein solution close to the source matches the one obtained by linearization far from the source. Therefore GR is restored locally for asymptotically flat solutions, at least for some potentials in massive gravity [22,23][1] (see also [26] for a more recent work). This matching was shown for potentials giving rise to the sixth dangerous mode. Later it was found, both analytically in the so-called decoupling limit [27–29] and numerically [30], that GR is also restored in the dRGT model. In the framework of the same model, the Vainshtein mechanism was studied in the decoupling limit for asymptotically nonflat space-times in [31], where these solutions were shown to be the only stable ones (see also a related work on Galileons [32] and on Horndeski theory [33]); and in the quasidilaton extension in [34]. For a recent review on the Vainshtein mechanism see [35].[2] Other possible issues of massive gravity we are not going to discuss here include superluminality [37] (see, however, discussion in [38,39] on the relation between causality and superluminality), strong coupling problem [40,41], and instability of black holes [42].

For what concerns the bigravity formulation of the dRGT model, the numerical study of the Vainshtein mechanism was presented in [30], while the far-distance analytic expansion valid outside the Vainshtein radius was found in [43] and then studied up to the second order in [44]. Some estimates of the Vainshtein suppression have been put forward also in [45] in order to calculate the emission of gravitation waves. However, there is still a lack of an analytic analysis of the Vainshtein mechanism, which we fill in this work.

In this paper we analytically study the spherically symmetric solutions in the bigravity extension of the dRGT model and show that the Vainshtein mechanism indeed works also with a dynamical second metric. We make our analysis in the approximation of the weak gravitational field of the physical metric—the one coupled to the matter—which is always correct for (nonrelativistic) weak matter sources. This assumption allows us to treat the problem mostly analytically. The key equation we obtain is an algebraic polynomial equation, generically of the 7th order, for a function of the radius entering, in our ansatz, in the second metric. The other functions that parametrize the metrics are expressed in terms of this key function. In two regimes—inside and outside the Vainshtein radius—we can solve (approximately) the algebraic equation that gives different branches of the solution, then we identify the one which ensures the flat asymptotic behavior. We exhibit a solution featuring the GR behavior for the physical metric inside the Vainshtein radius and that matches the asymptotically flat (Yukawa decaying) solution of the linearized equations.

The paper is organized as follows. In Sec. II for completeness, we reanalyze the Vainshtein mechanism for the original dRGT model (with fixed reference metric) in a slightly different manner than in [27–29]. This approach will be generalized in the main part of the paper (Sec. III) for the bigravity extension, where we rigorously study the Vainshtein mechanism with the second metric dynamic. Our conclusions are formulated in Sec. IV.

II. DRGT MODEL

In this section we obtain static spherically symmetric solutions in the limit of weak gravitational field for the dRGT model. We reproduce the results already found in [27–29] for distances smaller than the Compton wavelength of the graviton, where the decoupling limit (DL) is a good approximation. In our approach however, originally introduced in [23] and called "weak-field approximation," the DL scheme is not used. The weak-field approximation allows to capture both the DL and the Yukawa part of the solution outside the Compton wavelength, where the DL ceases to operate.[3] In Sec. III, this scheme—with appropriate modifications—will be applied to the bigravity extension of the model.

A. Action and equations of motion

The action for the dRGT model can be written as follows [4]:

$$S = M_P^2 \int d^4x\sqrt{-g}\Big(\frac{R[g]}{2} + m^2\mathcal{U}[g,f]\Big) + S_m[g]. \qquad (1)$$

It is convenient to express the interaction potential $\mathcal{U}[g,f]$ in terms of the matrix $\mathcal{K}$, such that $\mathcal{K}^\mu_\nu = \delta^\mu_\nu - \gamma^\mu_\nu$, where the matrix γ^μ_ν is the square root of the product of the inverse physical metric $g^{\mu\alpha}$ and the fiducial metric $f_{\alpha\nu}$, i.e., $\gamma^\mu_\nu = \sqrt{g^{\mu\alpha}f_{\alpha\nu}}$, in the sense that $(\gamma^2)^\mu_\nu = \gamma^\mu_\alpha\gamma^\alpha_\nu = g^{\mu\alpha}f_{\alpha\nu}$. As it is often assumed we will consider the fiducial metric to be flat.[4] The potential $\mathcal{U}$ consists of three pieces,

$$\mathcal{U} = \mathcal{U}_2 + \alpha_3\mathcal{U}_3 + \alpha_4\mathcal{U}_4, \qquad (2)$$

each of them, in terms of $\mathcal{K}$, reads

[1]For the Dvali-Gabadadze-Porrati model [24] the cosmological version of the Vainshtein mechanism was found in [25].

[2]For completeness we mention that, very recently, a full class of new massive gravity potentials has been found in the Lorenz breaking scenario [36]; these models do not suffer of the vDVZ discontinuity and therefore do not need to rely on the Vainshtein mechanism to recover GR.

[3]In particular, we derive an ordinary differential equation for the gauge function which is valid at all radii, in the limit of weak source. For practical purposes this equation is not useful, however, it may show some important features, e.g., for the dRGT potential compared to a generic one see discussion in Sec. II B.

[4]We use the mostly positive signature $(-+++)$.

$$\mathcal{U}_2 = \frac{1}{2!}([\mathcal{K}]^2 - [\mathcal{K}^2]),$$
$$\mathcal{U}_3 = \frac{1}{3!}([\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3]),$$
$$\mathcal{U}_4 = \frac{1}{4!}([\mathcal{K}]^4 - 6[\mathcal{K}]^2[\mathcal{K}^2] + 3[\mathcal{K}^2]^2 + 8[\mathcal{K}^3][\mathcal{K}] - 6[\mathcal{K}^4]), \tag{3}$$

where we introduced the notations $[\mathcal{K}] \equiv \mathrm{tr}(\hat{K}) = \hat{\mathcal{K}}^\rho_\rho$ and $[\mathcal{K}^n] \equiv \mathrm{tr}(\hat{\mathcal{K}}^n) = (\hat{\mathcal{K}}^n)^\rho_\rho$.

Varying the action with respect to $g_{\mu\nu}$, one obtains

$$G_{\mu\nu} = m^2 T_{\mu\nu} + \frac{T^{(\mathrm{m})}_{\mu\nu}}{M_P^2},$$

where $G_{\mu\nu}$ is the Einstein tensor and on the right-hand side there are the contributions from the energy-momentum tensor for the matter, $T^{(m)}_{\alpha\beta} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\alpha\beta}}$, and for the interaction term with $f_{\mu\nu}$,

$$T_{\mu\nu} = \mathcal{U} g_{\mu\nu} - 2\frac{\delta\mathcal{U}}{\delta g^{\mu\nu}}.$$

The last can be computed and gives

$$T_{\mu\nu} = -g_{\mu\sigma}\gamma^\sigma_\alpha(\mathcal{K}^\alpha_\nu - [\mathcal{K}]\delta^\alpha_\nu) + \alpha_3 g_{\mu\sigma}\gamma^\sigma_\alpha(-\mathcal{U}_2\delta^\alpha_\nu - [\mathcal{K}]\mathcal{K}^\alpha_\nu + (\mathcal{K}^2)^\alpha_\nu) + \alpha_4 g_{\mu\sigma}\gamma^\sigma_\alpha(-\mathcal{U}_3\delta^\alpha_\nu + \mathcal{U}_2\mathcal{K}^\alpha_\nu + [\mathcal{K}](\mathcal{K}^2)^\alpha_\nu - (\mathcal{K}^3)^\alpha_\nu) + \mathcal{U} g_{\mu\nu}.$$

B. Static spherically symmetric solutions

In this section we study spherically symmetric solutions for the case where the nondynamical second metric $f_{\mu\nu}$ parametrizes a flat Minkowski space-time. The study of the Vainshtein mechanism in this case has been already done in a number of papers [28,29,46], in the decoupling limit. Here we reproduce these results; moreover, we will give some additional new upshots outside the DL regime. The procedure is to consider the full equations of motion and then make reasonable approximations valid for the regimes in which we are interested. Following the weak-limit approximation scheme [23], we take the following ansatz:

$$ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2 d\Omega^2,$$
$$df^2 = -dt^2 + (r + r\mu)'^2 dr^2 + (r + r\mu)^2 d\Omega^2. \tag{4}$$

This ansatz is not the most general, indeed we do not consider the case with one of the two metrics off diagonal, but (4) is where we find the Vainshtein mechanism at work.

Since we are interested in a recovery of GR solutions, we require for weak matter sources to have weak gravity, i.e., that the functions ν and λ are small' as well as their derivatives. So the first step is to consider

$$\{\lambda, \nu\} \ll 1, \qquad \{r\lambda', r\nu'\} \ll 1, \tag{5}$$

and to retain all the nonlinearities in μ and μ'. The tt, rr, and $\theta\theta$ components of the Einstein equations in this approximation read

$$-\frac{\lambda'}{r} - \frac{\lambda}{r^2} = m^2\Big(\frac{\lambda}{2} + \frac{1}{r^2}\Big\{r^3\Big(-\mu + \alpha\mu^2 - \frac{\beta}{3}\mu^3\Big)\Big\}'\Big) - \frac{\rho}{M_P^2}, \tag{6}$$

$$\frac{\nu'}{r} - \frac{\lambda}{r^2} = m^2\Big(\frac{\nu}{2} - 2\mu + \alpha\mu^2\Big), \tag{7}$$

$$-\frac{\lambda'}{2r} + \frac{1}{2}\Big(\nu'' + \frac{\nu'}{r}\Big) = m^2\Big(\frac{\nu + \lambda}{2} + \frac{1}{r}\Big\{r^2\Big(-\mu + \frac{\alpha}{2}\mu^2\Big)\Big\}'\Big), \tag{8}$$

where we introduced

$$\alpha = 1 + \alpha_3, \qquad \beta = \alpha_3 + \alpha_4.$$

The Bianchi identity, $\nabla_\mu T^\mu_r = 0$, gives

$$-\frac{\lambda}{r} + \frac{\nu'}{2} + \alpha\Big(\frac{\lambda}{r} - \nu'\Big)\mu + \frac{\beta}{2}\nu'\mu^2 = 0. \tag{9}$$

Note that the pressure in the right-hand side (rhs) of (7) and (8) disappears as a consequence of the conservation of the matter energy momentum tensor in the weak field regime (5). Of course, like in GR, the three Einstein equations and the Bianchi one are not all independent, so we can consider (6), (7), and (9) as our independent set to be solved.

From this set we are able to obtain one second order ordinary differential equation on μ only. Indeed we can solve (7) for λ and then, substituting into (6) and (9), we end up with two equations, one for ν', ν, μ, and the other for ν'', ν', ν, μ', μ. Taking the first equation and its first and second derivative, together with the second equation and its first derivative, we have a system of five equations in ν''', ν'', ν', ν, μ'', μ', μ. We can then solve algebraically four of them for ν and all its three derivatives and, substituting in the last equation, it will be a second order differential equation only on μ of the form

$$\mathcal{A}\mu'' + \mathcal{B}\mu'^3 + \mathcal{C}\mu'^2 + \mathcal{D}\mu' + \mathcal{E} = 0. \tag{10}$$

$\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ are functions of μ whose form is not particularly illuminating, so we can omit it here. It is worth stressing the difference with respect to the equivalent equation found in the ghosty massive gravity theories [23]. Our second order equation needs two initial conditions in order to be solved, fixing to one the number of degree of freedom that it describes. For other kinds of potentials [40] that exhibit the Boulware-Deser instability, the equation is of the fourth order [23], meaning four initial conditions and therefore two dof. One of these modes is absent for the dRGT potential.

Equation (10) is hard to solve, even numerically. Without solving it, however, the equation clearly indicates that the weak field approximation (5) gives the relevant features of the fields for all ranges of distance for nonrelativistic sources. In order to understand the behavior of the solutions, below we will consider various regimes.

1. Linear regime

Since we require asymptotically flat solutions, we expect that far away from the source also the field μ becomes small. Therefore, assuming $\mu \ll 1$ in (6), (7), and (9), as well as its derivative, we get the solutions

$$\mu = -\frac{Ce^{-mr}}{3m^2r^3}[1 + mr(1 + mr)], \tag{11}$$

$$\lambda = \frac{2Ce^{-mr}}{3r}(1 + mr), \tag{12}$$

$$\nu = -\frac{4Ce^{-mr}}{3r}, \tag{13}$$

where C is an integration constant that in the following we will see equal to the Schwarzschild radius r_S. Clearly, the gravitational potentials (12) and (13) exhibit the vDVZ discontinuity in the limit of $m \to 0$; instead the field μ shows a singularity in the same limit. Actually the linear regime is nowhere allowed in the vanishing mass limit; indeed this regime exists only for values of radii for which $\mu \ll 1$. For nonzero small m, as can be easily seen from (11), the linear regime is valid for $r \gg r_V$, where r_V is the Vainshtein radius,

$$r_V \equiv \left(\frac{r_S}{m^2}\right)^{1/3}.$$

In the limit $m \to 0$ we have $r_V \to \infty$, making unreliable the condition to be outside r_V. Hence, to look at the small mass limit, we need to consider a nonlinear regime in μ. Finally, it is remarkable that the weak field approximation is able to retain the asymptotically Yukawa decay at large distances from the source, as well as the Vainshtein crossover as we will see in the next paragraph.

2. Inside the Compton wavelength

In order to study the behavior of the solutions for which the $m \to 0$ limit is well defined and the Vainshtein mechanism operates, we need to consider the distances inside the Compton wavelength, i.e., $r \ll 1/m$. In this regime we can neglect terms $\sim m^2\lambda$ and $\sim m^2\nu$ in the rhs of Eqs. (6)–(8). In this approximation, we can integrate Eq. (6) to obtain

$$\lambda = \begin{cases} \frac{r_S}{r} + m^2r^2\left(\mu - \alpha\mu^2 + \frac{\beta}{3}\mu^3\right) & \text{for } r > R_\odot \\ \frac{\rho r^2}{3M_P^2} + m^2r^2\left(\mu - \alpha\mu^2 + \frac{\beta}{3}\mu^3\right) & \text{for } r < R_\odot, \end{cases} \tag{14}$$

where $R_\odot$ is the radius of the source, the Schwarzschild radius r_S reads

$$r_S = \frac{1}{M_P^2}\int_0^{R_\odot} \rho r^2 dr,$$

and the integration constant in (14) has been chosen to ensure the continuity of the solution at the surface of the star. In the following for simplicity we will consider only a constant density source. Neglecting $\sim m^2\lambda$ and $\sim m^2\nu$ in Eq. (7), one obtains

$$\frac{\nu'}{r} - \frac{\lambda}{r^2} = m^2(-2\mu + \alpha\mu^2), \tag{15}$$

while the integration of (8) gives (15) up to an integration constant. From (14) and (15) we find

$$r\nu' = \begin{cases} \frac{r_S}{r} - m^2r^2\left(\mu - \frac{\beta\mu^3}{3}\right) & \text{for } r > R_\odot \\ \frac{\rho r^2}{3M_P^2} - m^2r^2\left(\mu - \frac{\beta\mu^3}{3}\right) & \text{for } r < R_\odot. \end{cases} \tag{16}$$

Finally, combining (9), (14), and (16) we get a single algebraic equation on μ,

$$3\mu - 6\alpha\mu^2 + 2\left(\alpha^2 + \frac{2\beta}{3}\right)\mu^3 - \frac{\beta^2}{3}\mu^5 = \begin{cases} -\frac{r_S}{m^2r^3}(1 - \beta\mu^2) & \text{for } r > R_\odot \\ -\frac{\rho}{3m^2M_P^2}(1 - \beta\mu^2) & \text{for } r < R_\odot. \end{cases} \tag{17}$$

The last equation in (17) corresponds to the one found in the DL of the model [28,29,46]. This confirms that the approximations we made here correspond to the DL in the full equations of motion.

All the physics is hence enclosed in Eq. (17): once we have its solutions we can determine the gravitational potentials through Eqs. (14) and (16). Equation (17) is a fifth order algebraic equation and its solutions cannot be presented in a closed form. Choosing the parameters of the theory α and β, we can lower the degree of such an equation in order to get analytically solvable ones. E.g., for the minimal massive gravity potential with only $\mathcal{U}_2$—it corresponds to $\beta = 0$ and $\alpha = 1$—Eq. (17) becomes a third order algebraic equation. This special case was first studied in [27], where it was shown that the Vainshtein mechanism works properly reproducing GR inside the Vainshtein radius.

Since the solutions of the set of equations (17), (14), and (16) have been largely studied [28,29,46], we only report schematically the behavior of the solutions for different subregimes. To consider the most interesting case, for which the Vainshtein mechanism takes place, we set $\beta > 0$ so that Eq. (17) has two complex and three real solutions. Only one of the three real branches of the solution recovers GR and is asymptotically flat, so we give it in Eq. (18).

r	$r < R_\odot$	$R_\odot < r \ll r_V$	$r_V \ll r \ll 1/m$
μ	$-\frac{1}{\sqrt{\beta}} + \frac{m^2 R_\odot^3(\alpha^2+3\alpha\sqrt{\beta}+2\beta)}{\beta^2 r_S}$	$-\frac{1}{\sqrt{\beta}} + \frac{m^2 r^3(\alpha^2+3\alpha\sqrt{\beta}+2\beta)}{\beta^2 r_S}$	$-\frac{r_S}{3m^2 r^3}$
λ	$\frac{r_S r^2}{R_\odot^3} - \frac{m^2 r^2(3\alpha+4\sqrt{\beta})}{3\beta}$	$\frac{r_S}{r} - \frac{m^2 r^2(3\alpha+4\sqrt{\beta})}{3\beta}$	$\frac{2r_S}{3r}$
ν	$-\frac{3r_S}{2R_\odot} + \frac{r_S r^2}{2R_\odot^3} + \frac{m^2 r^2}{3\sqrt{\beta}}$	$-\frac{r_S}{r} + \frac{m^2 r^2}{3\sqrt{\beta}}$	$-\frac{4r_S}{3r}$

(18)

Inside the Vainshtein radius we find that $\mu = -1/\sqrt{\beta}$ at the leading order and the gravitational potentials are of the GR form plus small corrections. Clearly, the vanishing mass limit is well defined inside the Vainshtein radius and the corrections to GR smoothly vanish as $m \to 0$. Outside the Vainshtein radius we find the asymptotically flat weak field solution where also $\mu \ll 1$: this solution matches the one obtained in the linear regime (11)–(13) provided $C = r_S$ and $r \ll 1/m$.

To get some understanding of the solutions from Eq. (17), note the rhs of itself. The ratio $(r_V/r)^3$ that appears there becomes large inside the Vainshtein radius, and small otherwise. For $r \ll r_V$ we have either $|\mu| \gg 1$ retaining so the higher power of μ in the equation, or the leading order of μ cancels the rhs itself, i.e., $\mu = \pm 1/\sqrt{\beta}$. For $r \gg r_V$ instead we can neglect at the first order the rhs of (17), obtaining therefore three real constant values for μ: obviously only $\mu = 0$ gives asymptotically the flat metric.

The branch of the solution presented in (18) is unphysical for $\beta < 0$ due to the square root in μ. A complete description of the solutions for the whole range of the free parameters α and β can be found in [46].

III. BIGRAVITY

This section is devoted to the study of the Vainshtein mechanism in the Hassan-Rosen bigravity extension of the dRGT model [9]. In the bigravity approach the second metric $f_{\mu\nu}$, which was fixed before, now becomes dynamical. The weak-field approximation, which we applied in the previous section, is also useful here. The trick is to deform the ansatz (4) to include the dynamics of the metric $f_{\mu\nu}$. After writing down the action and the field equations in Sec. III A, we introduce the ansatz and identify the functions that can be treated linearly in the limit of weak sources, together with a fully nonperturbative function. In this way we are able to obtain again one algebraic polynomial equation, which captures the distances inside the Compton wavelength of the graviton.

A. Action and equations of motion

We consider the dRGT model where the additional metric $f_{\mu\nu}$ is dynamical thanks to its own Einstein-Hilbert term in the action [9],

$$S = M_P^2 \int d^4x\sqrt{-g}\Big(\frac{R[g]}{2} + m^2\mathcal{U}[g,f]\Big) + S_m[g] + \frac{\kappa M_P^2}{2}\int d^4x\sqrt{-f}\mathcal{R}[f]. \tag{19}$$

The interaction potential in (19) is given by the same expressions in (2) and (3).[5] Note an extra parameter κ in the action (19), which accounts for a possible difference in Planck masses for the two gravitational sectors. One can realize that the limit $\kappa \to \infty$ corresponds to the freezing of the metric f, therefore recovering the model with flat fiducial metric. Varying the action (19) with respect to $g_{\mu\nu}$, we obtain

$$G_{\mu\nu} = m^2 T_{\mu\nu} + \frac{T^{(\mathrm{m})}_{\mu\nu}}{M_P^2},$$

where $G_{\mu\nu}$ is the Einstein tensor associated with $g_{\mu\nu}$ and $T_{\mu\nu}$ reads

$$\begin{aligned} T_{\mu\nu} &= \mathcal{U}g_{\mu\nu} - 2\frac{\delta\mathcal{U}}{\delta g^{\mu\nu}} \\ &= -g_{\mu\sigma}\gamma^\sigma_\alpha(\mathcal{K}^\alpha_\nu - [\mathcal{K}]\delta^\alpha_\nu) \\ &\quad + \alpha_3 g_{\mu\sigma}\gamma^\sigma_\alpha(\mathcal{U}_2\delta^\alpha_\nu - [\mathcal{K}]\mathcal{K}^\alpha_\nu + (\mathcal{K}^2)^\alpha_\nu) \\ &\quad + \alpha_4 g_{\mu\sigma}\gamma^\sigma_\alpha(\mathcal{U}_3\delta^\alpha_\nu - \mathcal{U}_2\mathcal{K}^\alpha_\nu \\ &\quad + [\mathcal{K}](\mathcal{K}^2)^\alpha_\nu - (\mathcal{K}^3)^\alpha_\nu) + \mathcal{U}g_{\mu\nu}. \end{aligned}$$

On the other hand, the variation of the action with respect to $f_{\mu\nu}$ gives

$$\sqrt{-f}\mathcal{G}_{\mu\nu} = \sqrt{-g}\frac{m^2}{\kappa}\mathcal{T}_{\mu\nu},$$

where

$$\begin{aligned} \mathcal{T}_{\mu\nu} &= -2\frac{\delta\mathcal{U}}{\delta f^{\mu\nu}} \\ &= f_{\mu\sigma}\gamma^\sigma_\alpha(\mathcal{K}^\alpha_\nu - [\mathcal{K}]\delta^\alpha_\nu) - \alpha_3 f_{\mu\sigma}\gamma^\sigma_\alpha(\mathcal{U}_2\delta^\alpha_\nu \\ &\quad - [\mathcal{K}]\mathcal{K}^\alpha_\nu + (\mathcal{K}^2)^\alpha_\nu) - \alpha_4 f_{\mu\sigma}\gamma^\sigma_\alpha(\mathcal{U}_3\delta^\alpha_\nu \\ &\quad - \mathcal{U}_2\mathcal{K}^\alpha_\nu + [\mathcal{K}](\mathcal{K}^2)^\alpha_\nu - (\mathcal{K}^3)^\alpha_\nu). \end{aligned}$$

[5] Note that in principle we can add terms $\mathcal{U}_0 = 1$ and $\mathcal{U}_1 = [\mathcal{K}]$. These terms, however, account for cosmological terms for the g and f metric. Since we aim to find asymptotically flat solutions, we exclude those terms.

One can observe a useful relation when working with up-down indices, namely, for $T^\mu_\nu \equiv g^{\mu\alpha}T_{\alpha\nu}$ and $\mathcal{T}^\mu_\nu \equiv f^{\mu\alpha}\mathcal{T}_{\alpha\nu}$, we have $\mathcal{T}^\mu_\nu = -T^\mu_\nu + \mathcal{U}\delta^\mu_\nu$.

B. Static spherically symmetric solutions

Continuing the idea of the weak-field approximation scheme, we consider the following parametrization for the two metrics:

$$\begin{aligned} ds^2 &= -e^\nu dt^2 + e^\lambda dr^2 + r^2 d\Omega^2, \\ df^2 &= -e^n dt^2 + e^l (r + r\mu)'^2 dr^2 + (r + r\mu)^2 d\Omega^2, \end{aligned} \tag{20}$$

where ν, λ, n, l, and μ are the r-dependent functions that describe the spherically symmetric foliation of the space-time in a common coordinate system. Again, this ansatz is not the most general one, since we are considering bidiagonal metrics.

Let us note that our ansatz (20) is compatible with the biflat solution,

$$g = f = \eta = \mathrm{diag}(-1, 1, 1, 1), \tag{21}$$

obtained when one imposes $T = \mathcal{T} = 0$, and that will be the reference point for the asymptotic behavior.

1. Static spherically symmetric ansatz and equations of motion

Following the procedure used for the dRGT model, we consider a nonrelativistic matter source with constant density. This allow us to assume ν, λ, n, and l small, as well as their derivatives,

$$\{\lambda, \nu, l, n\} \ll 1, \qquad \{r\lambda', r\nu', rl', rn'\} \ll 1, \tag{22}$$

and to retain all the nonlinearities in the field μ and μ'. The tt, rr, and $\theta\theta$ components of the Einstein equations in this approximation are for the first metric,

$$-\frac{\lambda'}{r} - \frac{\lambda}{r^2} = m^2\left[\frac{1}{2}(\lambda - l) + \frac{1}{r^2}\left\{r^3\left(-\mu + \alpha\mu^2 - \frac{\beta}{3}\mu^3\right)\right\}'\right] - \frac{\rho}{M_P^2}, \tag{23}$$

$$\frac{\nu'}{r} - \frac{\lambda}{r^2} = m^2\left[\frac{1}{2}(\nu - n) - 2\mu + \alpha\mu^2\right], \tag{24}$$

$$-\frac{\lambda'}{2r} + \frac{1}{2}\left(\nu'' + \frac{\nu'}{r}\right) = m^2\left[\frac{1}{2}(\nu + \lambda - n - l) + \frac{1}{r}\left\{r^2\left(-\mu + \frac{\alpha}{2}\mu^2\right)\right\}'\right], \tag{25}$$

and for the second metric,

$$-(1 + \mu)\frac{l'}{r} - (r + r\mu)'\frac{l}{r^2} = \frac{m^2}{\kappa}\left[\frac{1}{2}(l - \lambda) + \frac{1}{r^2}\left\{r^3\left(\mu + (1 - \alpha)\mu^2 + \frac{1 - \alpha + \beta}{3}\mu^3\right)\right\}'\right], \tag{26}$$

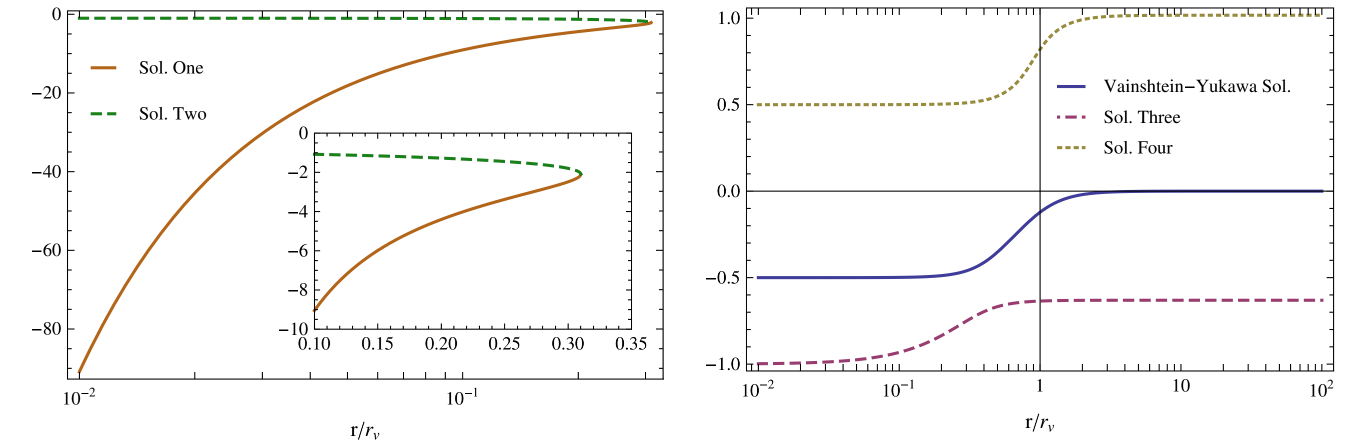
$$(1 + \mu)\frac{n'}{r} - (r + r\mu)'\frac{l}{r^2} = \frac{m^2}{\kappa}\left[\frac{1}{2}(n - \nu) + 2\mu + (1 - \alpha)\mu^2\right](r + r\mu)', \tag{27}$$

$$-\frac{l'}{2r} + \frac{1 + \mu}{2(r + r\mu)'}n'' + [1 + (r + r\mu)\{[(r + r\mu)']^{-1}\}']\frac{n'}{2r} = \frac{m^2}{\kappa}\left[\frac{1}{2}(l + n - \lambda - \nu) + \frac{1}{r}\left\{r^2\left(\mu + \frac{1 - \alpha}{2}\mu^2\right)\right\}'\right]. \tag{28}$$

Of course, these equations are not all independent; indeed a combination of (25) and (28) can be obtained taking a suitable combination of (24) and (27) and its derivative. The Bianchi identities, $\nabla^{(g)}_\mu T^\mu_r \propto \nabla^{(f)}_\mu(\sqrt{-g}\mathcal{T}^\mu_r/\sqrt{-f}) = 0$, give

$$\frac{1}{r}(r + r\mu)'(1 - \alpha\mu)(\lambda - l) - \frac{1}{2}(1 - 2\alpha\mu + \beta\mu^2)[(r + r\mu)'\nu' - n'] = 0. \tag{29}$$

Note that assuming $l = n = 0$ in (29) we get back (9), the Bianchi identity for the model with one dynamical metric. It is worth mentioning that we were not able to obtain an analogue of Eq. (10), because, applying the similar approach that we described there, we find a system of linear equations which is not linearly independent. Therefore, in order to find analytical solutions to this set of equations, we need to do other



(a) (b)

FIG. 1 (color online). Plot of the five branches (out of seven) of the solution of the function μ vs r/r_V, for $R_\odot < r < 1/m$. We take the following values for the parameters: $m \cdot r_V = 10^{-2}$, $\kappa = 1$, $\beta = 4$, and $\alpha = 1$. The name of each curve corresponds to the same name given in Appendix B where the analytic solutions are reported for the regime inside and outside the Vainshtein radius. The Vainshtein-Yukawa solution is the one that reproduces GR inside r_V and that gives the asymptotically flat solution outside r_V, as given in Eq. (41). (a) Solutions that, for $r/r_V = r_*/r_V \simeq 0.31$, join together in a complex conjugates pair. (b) The three everywhere real solutions.

approximations. This means to look at more specific regimes inside the weak field one (22).

2. Linear regime

Since asymptotically we want to find the biflat solutions (21), we expect that far away from the source also the field μ becomes small. Hence, assuming $\mu \ll 1$ and $r\mu' \ll 1$ in (23), (24), (26), (27), and (29), we find the solutions:

$$\mu = -\frac{C_2\kappa e^{-mr\sqrt{1+\frac{1}{\kappa}}}[\kappa + mr(mr(1+\kappa) + \sqrt{\kappa(1+\kappa)})]}{3m^4r^3(1+\kappa)}, \tag{30}$$

$$\lambda = \frac{C_1}{r} + \frac{2C_2\kappa e^{-mr\sqrt{1+\frac{1}{\kappa}}}[\kappa + mr\sqrt{\kappa(1+\kappa)}]}{3m^2r(1+\kappa)}, \tag{31}$$

$$\nu = -\frac{C_1}{r} - \frac{4C_2\kappa^2 e^{-mr\sqrt{1+\frac{1}{\kappa}}}}{3m^2r(1+\kappa)}, \tag{32}$$

$$l = \frac{C_1}{r} - \frac{2C_2 e^{-mr\sqrt{1+\frac{1}{\kappa}}}[\kappa + mr\sqrt{\kappa(1+\kappa)}]}{3m^2r(1+\kappa)}, \tag{33}$$

$$n = -\frac{C_1}{r} + \frac{4C_2\kappa e^{-mr\sqrt{1+\frac{1}{\kappa}}}}{3m^2r(1+\kappa)}, \tag{34}$$

where C_1 and C_2 are two integration constants that will be determined in the next paragraph to be

$$C_1 = \frac{r_S}{1+\kappa}, \qquad C_2 = \frac{m^2 r_S}{\kappa}. \tag{35}$$

It is important to stress that [once (35) is taken into account] taking the limit $\kappa \to \infty$, which freezes the dynamics of the second metric, we recover the solutions found in the same regime of the original dRGT model, i.e., (11)–(13) and $l = n = 0$.

For the $m \to 0$ limit, the vDVZ discontinuity appears with the divergence in μ, the same arguments as in the previous section apply here: the linear regime is nowhere allowed in the vanishing mass limit. Again, to properly consider this limit, we need to rely on nonlinearities in the field μ.

3. Inside the Compton wavelength

As before, in order to study the Vainshtein mechanism we consider distances inside the Compton wavelength, i.e., $r \ll 1/m$; this helps to avoid complications associated with the change of behavior at $r \sim 1/m$. Neglecting therefore $\sim m^2\lambda$ and $\sim m^2 l$ in the rhs of (23) and (26), we can integrate both the equations to obtain

$$\lambda = \begin{cases} \frac{r_S}{r} + m^2r^2\left(\mu - \alpha\mu^2 + \frac{\beta}{3}\mu^3\right) & \text{for } r > R_\odot \\ \frac{\rho r^2}{3M_P^2} + m^2r^2\left(\mu - \alpha\mu^2 + \frac{\beta}{3}\mu^3\right) & \text{for } r < R_\odot \end{cases} \tag{36}$$

and

$$l = -\frac{m^2r^2}{\kappa(1+\mu)}\left[\mu + (1-\alpha)\mu^2 + \frac{1}{3}(1-\alpha+\beta)\mu^3\right], \tag{37}$$

where the integration constants have been chosen requiring the continuity of the solutions at the surface of the star. Similarly, neglecting also $\sim m^2\nu$ and $\sim m^2 n$ in (24) and (27) one finds

$$r\nu' = \begin{cases} \frac{r_S}{r} - m^2 r^2\left(\mu - \frac{\beta\mu^3}{3}\right) & \text{for } r > R_\odot \\ \frac{\rho r^2}{3M_P^2} - m^2 r^2\left(\mu - \frac{\beta\mu^3}{3}\right) & \text{for } r < R_\odot \end{cases} \tag{38}$$

and

$$rn' = \frac{m^2 r^2 (r + r\mu)'}{\kappa(1+\mu)^2}\left(\mu + 2\mu^2 + \frac{2 - 2\alpha - \beta}{3}\mu^3\right), \tag{39}$$

while the integration of (25) and (28) does not give new equations. Finally, combining (29) and (36)–(39), we obtain a single algebraic equation on μ:

$$\begin{aligned} &3(\kappa+1)\mu + 6(\kappa+1)(1-\alpha)\mu^2 + \frac{1}{3}[6(\kappa+1)\alpha^2 - 2(18\kappa+17)\alpha + 4(\kappa+1)\beta + 9\kappa + 10]\mu^3 \\ &+ \frac{2}{3}[6(\kappa+1)\alpha^2 - (9\kappa+7)\alpha + 4(\kappa+1)\beta + 1]\mu^4 \\ &+ \frac{1}{3}[2(3\kappa+1)\alpha^2 - (\kappa+1)\beta^2 + 2(2\kappa+1)\beta - 4\alpha\beta - 2\alpha]\mu^5 - \frac{2}{3}\kappa\beta^2\mu^6 - \frac{1}{3}\kappa\beta^2\mu^7 \\ &= \begin{cases} -\frac{\kappa r_S}{m^2 r^3}(1+\mu)^2(1-\beta\mu^2) & \text{for } r > R_\odot \\ -\frac{\kappa\rho}{3m^2 M_P^2}(1+\mu)^2(1-\beta\mu^2) & \text{for } r < R_\odot. \end{cases} \end{aligned} \tag{40}$$

Notice that Eqs. (36) and (38) for λ and ν' are the same found for the dRGT model, see Eqs. (14) and (16). It is also important to stress that dividing Eq. (40) by $\kappa(1+\mu)^2$ and taking the limit $\kappa \to \infty$ that freezes f, we recover the master equation (17) of the previous section. The fields l and n' given in (37) and (39) vanish in the same limit.

Again, all the information is retained in an algebraic equation for μ only: for bigravity this equation is of the seventh order compared to the fifth order equation for the dRGT model. Generically Eq. (40) has seven solutions that can be real or complex.

For $\beta > 0$, it has three real and two complex conjugates solutions for all values of r; the remaining two solutions are real inside some radius r_* and join together in a complex conjugates pair for $r > r_*$, see Fig. 1. The value of r_* depends on the choice of the free parameters α and β. The three everywhere real solutions for μ, shown in Fig. 1, have different asymptotic behavior, however all three recover GR inside the Vainshtein radius. One of these three is asymptotically flat (Vainshtein-Yukawa solution), and the others (dashed) have nonflat asymptotics (solutions three and four in Appendix B). One of these last two solutions, (B4), may be of interest in the context of cosmology. The asymptotically nonflat solution and a possible match to a cosmological one, however, deserves a separate study and will be discussed elsewhere.

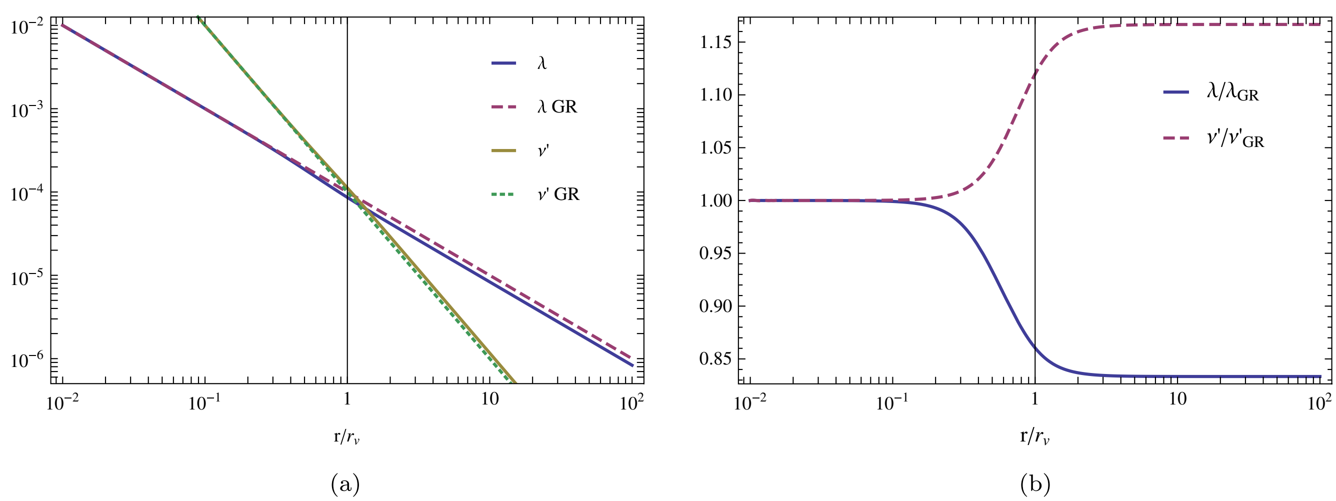


FIG. 2 (color online). Plot of the Vainshtein-Yukawa solution of the functions λ and ν' vs r/r_V, for $R_\odot < r < 1/m$. We take the following values for the parameters: $m \cdot r_V = 10^{-2}$, $\kappa = 1$, $\beta = 4$, and $\alpha = 1$. These solutions are plotted together with the corresponding ones of GR in (a) and in (b) is given their ratio for a better comparison. The analytic behavior for the regime well inside and outside the Vainshtein radius is given in Eq. (41).

For $\beta < 0$ two of the three solutions which are real for $\beta > 0$ become complex conjugates, therefore only one everywhere real solution is left. The other solutions show the same behavior as in the case $\beta > 0$. For $\beta < 0$ the only everywhere real solution is given by solution three of Appendix B and it does not show the asymptotically flat behavior nor the expected weak field solution inside the Vainshtein radius. Therefore, in the following, we will consider only the case $\beta > 0$. The special case $\beta = 0$, that for $\alpha = 1$ corresponds to the minimal massive gravity potential with only $\mathcal{U}_2$, will be presented in Appendix A.

However, we do not guarantee that solutions exist for the whole range of parameters space; i.e., we do not exclude that for some range of the parameters α and β the asymptotic solution might not match the solution inside the Vainshtein radius. This issue would require a complete analysis for each value of α and β that is beyond the scope of this work. Indeed, contrary to Eq. (17) of the dRGT model, where a symmetry allows one to study easily the whole range of parameters space [46], in Eq. (40) we were not able to find a similar strategy that facilitates the scan of solutions for general α and β.

Again, one can understand the behavior of the solutions analyzing two regimes: well inside and outside the Vainshtein radius. For $r \ll r_V$ the ratio $(r_V/r)^3$ in the rhs of (40) becomes large leaving us with two possibilities: either $|\mu| \gg 1$ in order to compensate the large rhs (so at the leading order the higher powers of μ dominate); or the leading order of μ cancels out the rhs, this happens for $\mu = -1$ (double root) and $\mu = \pm 1/\sqrt{\beta}$. For $r \gg r_V$, the ratio $(r_V/r)^3$ is small and it suggests to neglect at first order the rhs of (40): this determines the three real constant asymptotical values of μ. One of these values is obviously zero and it is the one approached by the linear solution (30).

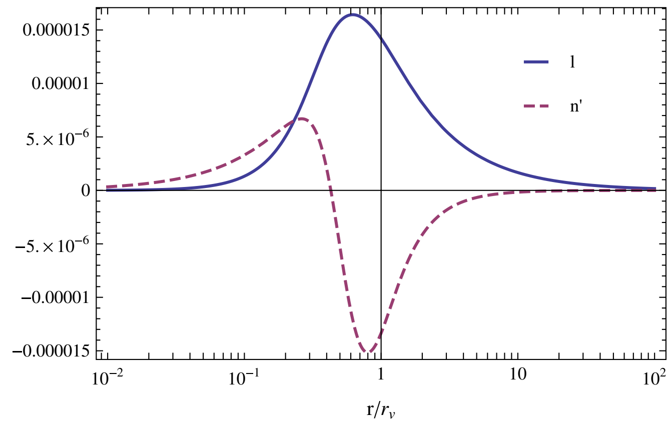


FIG. 3 (color online). Plot, for the Vainshtein-Yukawa branch, of the functions l and n' vs r/r_V, for $R_\odot < r < 1/m$. We take the following values for the parameters: $m \cdot r_V = 10^{-2}$, $\kappa = 1$, $\beta = 4$, and $\alpha = 1$. These numerical solutions show how the potentials l and n of the second metric get non-negligible values only in the intermediate regime around the Vainshtein radius. Indeed, as shown analytically in Eq. (41), for $r \ll r_V$ their values are of the same order of magnitude of the corrections respect to GR of the potentials of the first metric and, for $r \gg r_V$, the solutions are asymptotically flat.

In Eq. (41) we present the branch of the solution which features the Vainshtein recovery of GR and the Yukawa decay. The other branches are schematically presented in Appendix B. The way these regimes match together is understood from the numerical study that will be presented immediately after.

$$\begin{array}{cccc}
r & r < R_\odot & R_\odot < r \ll r_V & r_V \ll r \ll 1/m \\
\hline
\mu & -\frac{1}{\sqrt{\beta}} + \delta\mu \quad \underline{\delta\mu \ll 1} & -\frac{1}{\sqrt{\beta}} + \delta\mu \quad \underline{\delta\mu \ll 1} & -\frac{r_S\kappa}{3m^2r^3(1+\kappa)} \\
\lambda & \frac{r_S r^2}{R_\odot^3} - \frac{m^2r^2(3\alpha+4\sqrt{\beta})}{3\beta} & \frac{r_S}{r} - \frac{m^2r^2(3\alpha+4\sqrt{\beta})}{3\beta} & \frac{r_S(3+2\kappa)}{3r(1+\kappa)} \\
\nu & -\frac{3r_S}{2R_\odot} + \frac{r_S r^2}{2R_\odot^3} + \frac{m^2r^2}{3\sqrt{\beta}} & -\frac{r_S}{r} + \frac{m^2r^2}{3\sqrt{\beta}} & -\frac{r_S(3+4\kappa)}{3r(1+\kappa)} \\
l & -\frac{m^2r^2[1-\alpha-3(1-\alpha)\sqrt{\beta}+4\beta]}{3\beta\kappa(1-\sqrt{\beta})} & -\frac{m^2r^2[1-\alpha-3(1-\alpha)\sqrt{\beta}+4\beta]}{3\beta\kappa(1-\sqrt{\beta})} & \frac{r_S}{3r(1+\kappa)} \\
n & \frac{m^2r^2[1-\alpha-3\sqrt{\beta}+\beta]}{3\beta\kappa(1-\sqrt{\beta})} & \frac{m^2r^2[1-\alpha-3\sqrt{\beta}+\beta]}{3\beta\kappa(1-\sqrt{\beta})} & \frac{r_S}{3r(1+\kappa)}
\end{array} \tag{41}$$

Inside the Vainshtein radius, for the metric g the GR solution is recovered (plus small corrections) and for the second metric the potentials l and n are of the same order of the corrections to GR of the metric g. Outside the Vainshtein radius we find the asymptotically flat solution with $\mu \ll 1$: this solution matches the one obtained in the linear regime (30)–(34) for $r \ll 1/m$ and

$$C_1 = \frac{r_S}{1+\kappa}, \qquad C_2 = \frac{m^2 r_S}{\kappa},$$

as we already anticipated in (35). Note that inside the Vainshtein radius the vanishing mass limit, $m \to 0$, is well defined and gives exactly GR for the first metric while zero for l and n potentials of the second metric.

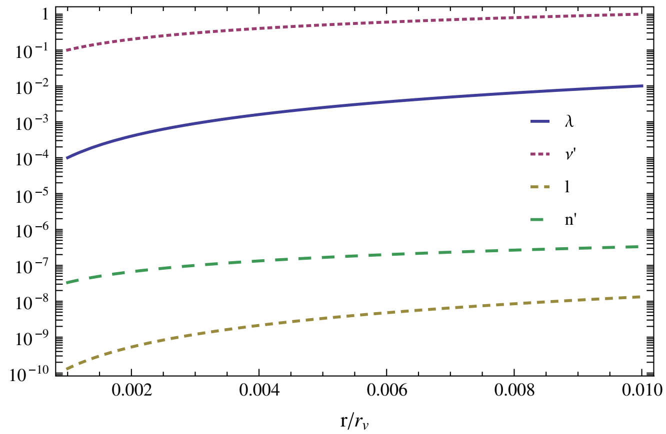


FIG. 4 (color online). Plot, for the Vainshtein-Yukawa branch, of the interior solution ($r < R_\odot$) for all the metrics functions (except μ which is constant) vs r/r_V. We take the following values for the parameters: $R_\odot = 10^{-2} r_V$, $m \cdot r_V = 10^{-2}$, $\kappa = 1$, $\beta = 4$, and $\alpha = 1$. The potentials of the first metric are indistinguishable from the analog in GR, instead the ones of the second metric are several orders of magnitude smaller. The analytic solutions are given in Eq. (41).

It is also important to stress that in the limit $\kappa \to \infty$ for which $f_{\mu\nu}$ is frozen, this branch of the solution reproduces the Vainshtein-Yukawa one of the dRGT model, namely Eq. (18).

In addition to Fig. 1, in Figs. 2–4, we depict correspondingly the potentials of the physical metric outside the source, the potentials of the second metric outside the source, and the potentials for both metrics inside the source, only for the Vainshtein-Yukawa solution.

IV. CONCLUSIONS

In this paper we studied the Vainshtein mechanism in the massive bigravity model with no Boulware-Deser ghost. To attack the problem, we applied the "weak-field approximation scheme" where the metric coefficients are separated into two parts. One contains functions of the radial coordinate which remain small (i.e., the quadratic and higher order terms are negligible in comparison to the linear ones) for nonrelativistic sources, even inside the Vainshtein radius; the other part is fully nonlinear, with nonlinearities crucial for the existence of a GR-like solution. This approach allows one to capture all the important features of the solutions: the Vainshtein regime, the linear regime, and the Yukawa decay.

In Sec. II we demonstrated how this scheme works for the original dRGT model—where the auxiliary metric is fixed to Minkowski. Inside the Compton wavelength, our results are in agreement with previous studies in the decoupling limit [27–29], see in particular (18). On the other hand, we can also describe the solution outside the Compton wavelength, which is beyond the validity of the decoupling limit. Moreover, for the function $\mu(r)$ which is introduced as the nonlinear piece of the metric (4), we derived a single ordinary differential equation of the second order (10) valid for all radii.

For bimetric massive gravity we modified the approach to incorporate the dynamics of the second metric. Notably, the ansatz we introduced (20), again separates the metric coefficients in the linearizable part and in the fully nonlinear part $\mu(r)$. If we additionally assume that far from the source also μ is in the linear regime, then we readily obtain the linearized solution that shows the Yukawa decay. Other regimes can be obtained assuming radii smaller than the Compton wavelength, i.e., $r \ll 1/m$. In this case, it is possible to derive one algebraic equation of the seventh order on μ, Eq. (40), while the other metric functions are given in terms of it. Using this master equation, we analyzed the behavior of the solutions in various subregimes and identified several branches of the solution. For $\beta > 0$ the only solution that has the desired behavior—asymptotic flatness—is presented in Eq. (41): this solution shows the recovery of GR inside the Vainshtein radius.

It is worth making a comment about the asymptotically nonflat solution (B4). Although we did not study (B4) in detail in the present work, since we concentrated on asymptotically flat solutions, this solution may have a physical meaning if matched to a cosmological solution at $r \to \infty$. The same comment applies to the choice of the potential $\alpha_3 = \alpha_4 = 0$, which we discuss in Appendix A. While for the dRGT model this simplest potential gives an asymptotically flat solution recovering GR inside the Vainshtein radius, in the case of the bimetric massive gravity, such an asymptotic solution does not exist and the solution featuring the Vainshtein behavior becomes asymptotically nonflat. The behavior and the physical meaning of these solutions, together with a possible match to cosmological ones, deserves a separate study.

To summarize, in the bigravity formulation of the dRGT massive gravity, with matter coupled to only one (physical) metric, we have found the recovery of GR for the physical metric inside the Vainshtein radius and the Yukawa decay outside. At the same time, the second metric is nontrivial because of the indirect coupling to matter via the interaction (mass) term; its deviation from flat space-time is highly suppressed and it reaches non-negligible values only around the Vainshtein radius.

ACKNOWLEDGMENTS

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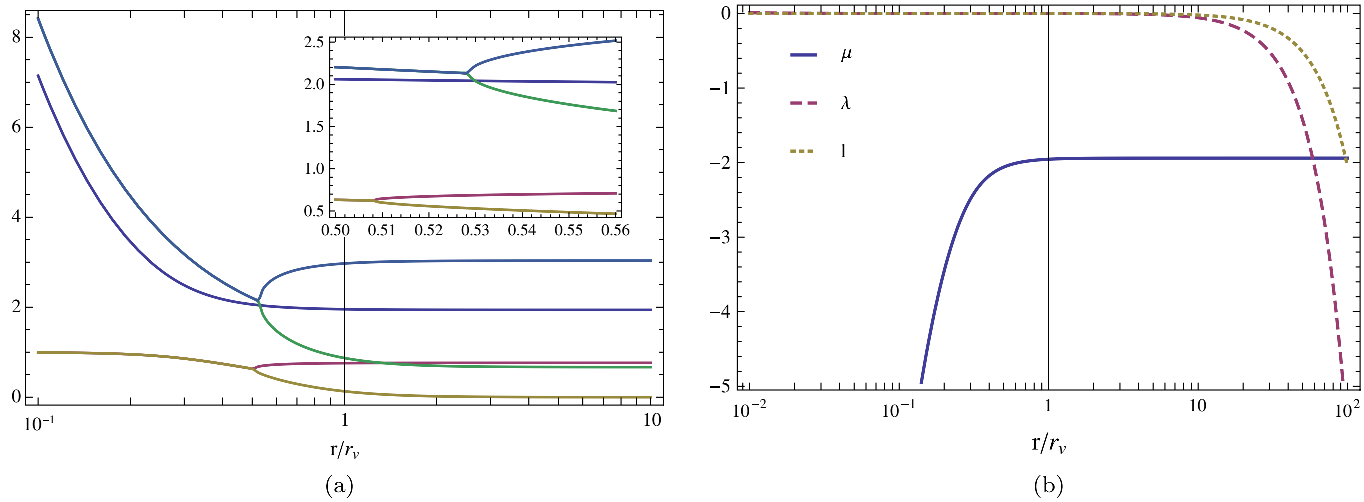


FIG. 5 (color online). (a) Plot of the absolute value of all five solutions of the function μ vs r/r_V, for $R_\odot < r < 1/m$. There is only one everywhere real solution. (b) Plot of the functions μ, λ, and l vs r/r_V, for $R_\odot < r < 1/m$. As shown analytically in Eq. (A1), the asymptotic behavior is not flat. We take for both plots the following values for the parameters: $m \cdot r_V = 10^{-2}$ and $\kappa = 1$.

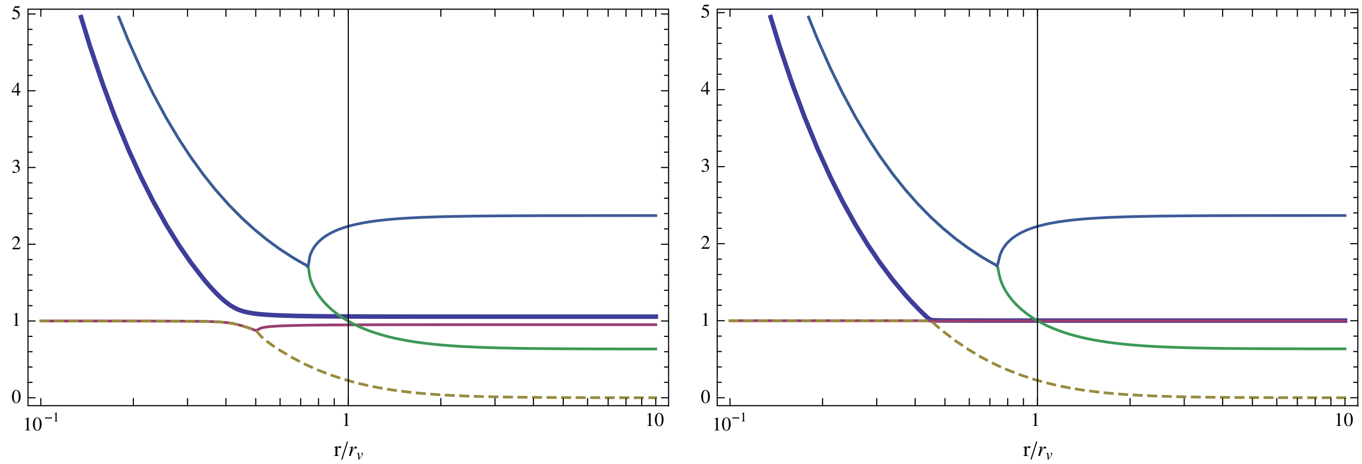
APPENDIX A: SIMPLEST MASSIVE GRAVITY POTENTIAL

Here we present the study of the simplest massive gravity potential with only $\mathcal{U}_2$ in (2). It corresponds to set $\beta = 0$ and $\alpha = 1$ in the equations of Sec. III B 3. This case deserves a particular analysis both since it was the first one studied in the framework of the dRGT model [27], showing a well working Vainshtein mechanism, and because the branch that realizes the Vainshtein-Yukawa solution (41) is not present for $\beta = 0$. For these values of the free parameters, the master equation (40) becomes a fifth degree equation with only one real solution for all the range of distances; the other four solutions start (as r increases) as two pairs of complex conjugate solutions and then divide into four real distinct ones, see Fig. 5. Let us thus concentrate on the everywhere real branch. Similar to the general case, we can analytically find solutions in different regimes, see Eq. (A1).

$$
\begin{array}{cccc}
r & r < R_\odot & R_\odot < r \ll r_V & r_V \ll r \ll 1/m \\
\hline
\mu & -\left(\frac{r_S}{2m^2}\right)^{1/3}\frac{1}{R_\odot} & -\left(\frac{r_S}{2m^2}\right)^{1/3}\frac{1}{r} & c_0 \\
\lambda & \frac{r_S r^2}{R_\odot^3} - \left(\frac{m r_S}{2}\right)^{2/3}\left(\frac{r}{R_\odot}\right)^2 & \frac{r_S}{r} - \left(\frac{m r_S}{2}\right)^{2/3} & \frac{r_S}{r} + m^2 r^2 (c_0 - c_0^2) \\
\nu & c_\nu + \frac{r_S r^2}{2R_\odot^3} + \left(\frac{m^4 r_S}{2}\right)^{1/3}\frac{r^2}{2R_\odot} & -\frac{r_S}{r} + \left(\frac{m^4 r_S}{2}\right)^{1/3} r & -\frac{r_S}{r} - \frac{1}{2} m^2 r^2 c_0 \\
l & -\frac{m^2 r^2}{\kappa} & -\frac{m^2 r^2}{\kappa} & -\frac{m^2 r^2 c_0}{\kappa(1+c_0)} \\
n & \frac{m^2 r^2}{\kappa} & \frac{m^2 r^2}{\kappa} & \frac{m^2 r^2 c_0 (1+2c_0)}{2\kappa(1+c_0)}
\end{array}
\tag{A1}
$$

c_ν in Eq. (A1) is an integration constant fixed by the matching condition to be: $c_\nu = -\frac{3r_S}{2R_\odot} + (\frac{m^4 r_S}{2})^{1/3}\frac{R_\odot}{2}$.
Inside the Vainshtein radius we find that $|\mu| \gg 1$ and, in contrast to the general case, it is this behavior of μ that gives a well working Vainshtein mechanism with GR-like solutions for the metric g and small potentials l and n for the other metric. For the dRGT model with the simplest potential a similar recovery of GR was found in [27], with flat asymptotic. For the bigravity model we find that, for $r \gg r_V$, μ asymptotically approaches a nonzero constant value c_0, giving nondecaying tails for all the other gravitational potentials. This solution does not match the asymptotically flat linear solution (30)–(34). It does not mean, however, that the solution (A1) is nonphysical; it may in fact match a nontrivial cosmological solution for large r. This possibility, however, deserves a separate study and goes beyond the scope of this paper. In Fig. 5 we show also the numerical study of this solution for all the range of distances inside the Compton wavelength.

The fact that the choice of parameters $\beta = 0$ does not allow for an asymptotically flat everywhere real solution, while with the nondynamical second metric such a



(a) (b)

FIG. 6 (color online). Plot of the absolute value of all five solutions of the function μ vs r/r_V, for $R_\odot < r < 1/m$. We take for (a) $\kappa = 10^2$ and for (b) $\kappa = 10^5$; both plots have $m \cdot r_V = 10^{-2}$.

solution exists, might seem surprising. There is however a simple explanation for this effect: when $\kappa \to \infty$ the outer part of the (nonphysical) asymptotically flat branch (dashed curve in Fig. 6) and the inner part of the everywhere real solution (thick curve in Fig. 6) join together to make one asymptotically flat solution recovering GR for small radii. This can be easily seen from Fig. 6, where the absolute value of the solutions of μ are plotted for bigger values of κ.

It is also instructive to compare our findings with the numerical work of Volkov [30], where he obtains an asymptotically flat solution for the simplest potential ($\alpha = 1$, $\beta = 0$). It looks as if our results contradict the ones in [30]. In fact, a possible explanation lies in some specific choice of m, r_S, and $R_\odot$: it seems that in [30] these parameters are chosen such that the size of the source $R_\odot$ is larger than r_*—the point below which the asymptotically flat solution (dashed curve in Fig. 6) becomes complex—therefore avoiding the problem since inside the source μ is constant. If the source is made more compact though still nonrelativistic, we expect that this problem of complex-valued solution comes back, although without complete numerical analysis of the full equations of motion we cannot prove this statement.

APPENDIX B: OTHER BRANCHES IN MASSIVE BIGRAVITY

In this Appendix we report the analytical solutions, for the regime inside and outside the Vainshtein radius, of the four branches of the massive bigravity solution that we omitted in the main text.

Solution one:

r	$r < R_\odot$	$R_\odot < r \ll r_V$	$r_V \ll r \ll 1/m$
μ	$-\left(\frac{3r_S}{\beta m^2}\right)^{1/3}\frac{1}{R_\odot}$	$-\left(\frac{3r_S}{\beta m^2}\right)^{1/3}\frac{1}{r}$	//
λ	$\mathcal{O}\left(\frac{1}{\mu}\right)^2$	$\mathcal{O}\left(\frac{1}{\mu}\right)^2$	//
ν	$\mathcal{O}\left(\frac{1}{\mu}\right)^2$	$\mathcal{O}\left(\frac{1}{\mu}\right)^2$	//
l	$-\frac{1-\alpha+\beta}{3^{1/3}\kappa}\left(\frac{mr_S}{\beta}\right)^{2/3}\left(\frac{r}{R_\odot}\right)^2$	$-\frac{1-\alpha+\beta}{3^{1/3}\kappa}\left(\frac{mr_S}{\beta}\right)^{2/3}$	//
n	$c_n + \frac{2-2\alpha-\beta}{2\times 3^{1/3}\kappa}\left(\frac{mr_S}{\beta}\right)^{2/3}\left(\frac{r}{R_\odot}\right)^2$	$-\frac{2-2\alpha-\beta}{3^{2/3}\kappa}\left(\frac{m^4 r_S}{\beta}\right)^{1/3} r$	//

(B1)

c_n in Eq. (B1) is an integration constant fixed by the matching condition to be: $c_n = -\frac{2-2\alpha-\beta}{2\times 3^{2/3}k}[3^{1/3}(\frac{mr_S}{\beta})^{2/3} + 2R_\odot(\frac{m^4 r_S}{\beta})^{1/3}]$.

Solution two:

r	$r < R_\odot$	$R_\odot < r \ll r_V$	$r_V \ll r \ll 1/m$
μ	$-1 + \delta\mu \quad \underline{\delta\mu \ll 1}$	$-1 + \delta\mu \quad \underline{\delta\mu \ll 1}$	//
λ	$\frac{r_S r^2}{R_\odot^3} - m^2 r^2\left(1 + \alpha + \frac{\beta}{3}\right)$	$\frac{r_S}{r} - m^2 r^2\left(1 + \alpha + \frac{\beta}{3}\right)$	//
ν	$-\frac{3r_S}{2R_\odot} + \frac{r_S r^2}{2R_\odot^3} + \frac{1}{2}m^2 r^2\left(1 - \frac{\beta}{3}\right)$	$-\frac{r_S}{r} + \frac{1}{2}m^2 r^2\left(1 - \frac{\beta}{3}\right)$	//
l	$\mathcal{O}\left(\frac{1}{\delta\mu}\right)$	$\mathcal{O}\left(\frac{1}{\delta\mu}\right)$	//
n	$\mathcal{O}\left(\frac{1}{\delta\mu}\right)$	$\mathcal{O}\left(\frac{1}{\delta\mu}\right)$	//

(B2)

Solution three:

r	$r < R_\odot$	$R_\odot < r \ll r_V$	$r_V \ll r \ll 1/m$
μ	$-1 + \delta\mu \quad \underline{\delta\mu \ll 1}$	$-1 + \delta\mu \quad \underline{\delta\mu \ll 1}$	c_1
λ	$\frac{r_S r^2}{R_\odot^3} - m^2 r^2\left(1 + \alpha + \frac{\beta}{3}\right)$	$\frac{r_S}{r} - m^2 r^2\left(1 + \alpha + \frac{\beta}{3}\right)$	$\frac{r_S}{r} + m^2 r^2\left(c_1 - \alpha c_1^2 + \frac{\beta}{3}c_1^3\right)$
ν	$-\frac{3r_S}{2R_\odot} + \frac{r_S r^2}{2R_\odot^3} + \frac{1}{2}m^2 r^2\left(1 - \frac{\beta}{3}\right)$	$-\frac{r_S}{r} + \frac{1}{2}m^2 r^2\left(1 - \frac{\beta}{3}\right)$	$-\frac{r_S}{r} - m^2 r^2 \frac{1}{2}\left(c_1 - \frac{\beta}{3}c_1^3\right)$
l	$\mathcal{O}\left(\frac{1}{\delta\mu}\right)$	$\mathcal{O}\left(\frac{1}{\delta\mu}\right)$	$-\frac{m^2 r^2}{\kappa(1+c_1)}\left[c_1 + (1-\alpha)c_1^2 + \frac{1}{3}(1-\alpha+\beta)c_1^3\right]$
n	$\mathcal{O}\left(\frac{1}{\delta\mu}\right)$	$\mathcal{O}\left(\frac{1}{\delta\mu}\right)$	$\frac{m^2 r^2}{2\kappa(1+c_1)}\left(c_1 + 2c_1^2 + \frac{2-2\alpha-\beta}{3}c_1^3\right)$

(B3)

Solution four:

r	$r < R_\odot$	$R_\odot < r \ll r_V$	$r_V \ll r \ll 1/m$
μ	$\frac{1}{\sqrt{\beta}} + \delta\mu \quad \underline{\delta\mu \ll 1}$	$\frac{1}{\sqrt{\beta}} + \delta\mu \quad \underline{\delta\mu \ll 1}$	c_2
λ	$\frac{r_S r^2}{R_\odot^3} - \frac{m^2 r^2(3\alpha - 4\sqrt{\beta})}{3\beta}$	$\frac{r_S}{r} - \frac{m^2 r^2(3\alpha - 4\sqrt{\beta})}{3\beta}$	$\frac{r_S}{r} + m^2 r^2\left(c_2 - \alpha c_2^2 + \frac{\beta}{3}c_2^3\right)$
ν	$-\frac{3r_S}{2R_\odot} + \frac{r_S r^2}{2R_\odot^3} - \frac{m^2 r^2}{3\sqrt{\beta}}$	$-\frac{r_S}{r} - \frac{m^2 r^2}{3\sqrt{\beta}}$	$-\frac{r_S}{r} - m^2 r^2 \frac{1}{2}\left(c_2 - \frac{\beta}{3}c_2^3\right)$
l	$-\frac{m^2 r^2[1-\alpha+3(1-\alpha)\sqrt{\beta}+4\beta]}{3\beta\kappa(1+\sqrt{\beta})}$	$-\frac{m^2 r^2[1-\alpha+3(1-\alpha)\sqrt{\beta}+4\beta]}{3\beta\kappa(1+\sqrt{\beta})}$	$-\frac{m^2 r^2}{\kappa(1+c_2)}\left[c_2 + (1-\alpha)c_2^2 + \frac{1}{3}(1-\alpha+\beta)c_2^3\right]$
n	$\frac{m^2 r^2[1-\alpha+3\sqrt{\beta}+\beta]}{3\beta\kappa(1+\sqrt{\beta})}$	$\frac{m^2 r^2[1-\alpha+3\sqrt{\beta}+\beta]}{3\beta\kappa(1+\sqrt{\beta})}$	$\frac{m^2 r^2}{2\kappa(1+c_2)}\left(c_2 + 2c_2^2 + \frac{2-2\alpha-\beta}{3}c_2^3\right)$

(B4)

Let us give some comments on these branches. The first two are given by the solutions of μ that at some point become complex conjugates (therefore the marks "//" for $r_V \ll r \ll 1/m$), so we can estimate their behavior only inside the Vainshtein radius. For solution one, $|\mu| \gg 1$ and this produces the screening of the gravitational potentials λ and ν already seen in the dRGT model with one dynamical metric [28]. For solutions two and three, at the leading order $\mu = -1$ giving GR-like solutions for the first metric potentials and very large values for the potentials of the second metric, i.e., l, $n \gg 1$. This is due to the fact that, for the ansatz (20), the inverse of the second metric is singular for μ strictly equal to -1. It should be stressed that these solutions violate the assumption of weak field approximation (22), therefore they are not viable. In branch three and four, outside the Vainshtein radius, μ approaches asymptotically to nonzero constant values c_1 and c_2: this gives nondecaying gravitational potentials. Finally, in branch four, for $r \ll r_V$ we find that the Vainshtein mechanism works properly, recovering GR as in the Vainshtein-Yukawa branch (41).

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