

Entropic cosmology for a generalized black-hole entropyNobuyoshi Komatsu^{1,*} and Shigeo Kimura²¹*Department of Mechanical Systems Engineering, Kanazawa University, Kakuma-machi, Kanazawa, Ishikawa 920-1192, Japan*²*The Institute of Nature and Environmental Technology, Kanazawa University, Kakuma-machi, Kanazawa, Ishikawa 920-1192, Japan*

(Received 23 July 2013; published 28 October 2013)

An entropic-force scenario, i.e., entropic cosmology, assumes that the horizon of the Universe has an entropy and a temperature. In the present paper, in order to examine entropic cosmology, we derive entropic-force terms not only from the Bekenstein entropy but also from a generalized black-hole entropy proposed by Tsallis and Cirto [Eur. Phys. J. C **73**, 2487 (2013)]. Unlike the Bekenstein entropy, which is proportional to area, the generalized entropy is proportional to volume because of appropriate nonadditive generalizations. The entropic-force term derived from the generalized entropy is found to behave as if it were an extra driving term for bulk viscous cosmology, in which a bulk viscosity of cosmological fluids is assumed. Using an effective description similar to bulk viscous cosmology, we formulate the modified Friedmann, acceleration, and continuity equations for entropic cosmology. Based on this formulation, we propose two entropic-force models derived from the Bekenstein and generalized entropies. In order to examine the properties of the two models, we consider a homogeneous, isotropic, and spatially flat universe, focusing on a single-fluid-dominated universe. The two entropic-force models agree well with the observed supernova data. Interestingly, the entropic-force model derived from the generalized entropy predicts a decelerating and accelerating universe, as for a fine-tuned standard Λ CDM (lambda cold dark matter) model, whereas the entropic-force model derived from the Bekenstein entropy predicts a uniformly accelerating universe.

DOI: [10.1103/PhysRevD.88.083534](https://doi.org/10.1103/PhysRevD.88.083534)

PACS numbers: 98.80.-k, 95.30.Tg, 98.80.Es

I. INTRODUCTION

Since the late 1990s, an accelerated expansion of the late universe has gradually been accepted as a new paradigm [1–6]. In order to explain this accelerated expansion, various cosmological models have been suggested [7–11] (see, e.g., Refs. [8–11] and the references therein). As one such model, Easson *et al.* [12,13] recently proposed an entropic-force scenario called “entropic cosmology.” In entropic cosmology, an extra driving term, i.e., an entropic-force term, should be added to the Friedmann-Lemaître acceleration equation, without introducing new fields [14]. The entropic-force term is derived from the usually neglected surface terms on the horizon of the Universe, assuming that the horizon has an entropy and a temperature due to the information holographically stored there [12].

Entropic cosmology has been extensively examined from various viewpoints [15–21]. As an entropy on the horizon, i.e., the Bekenstein entropy [22] is always used, substituting the horizon of the Universe for the event horizon of a black hole. In fact, the Bekenstein entropy is proportional to area (or horizon) and is additive. However, self-gravitating systems exhibit peculiar features [23,24], such as nonequilibrium thermodynamics and nonextensive statistical mechanics [25–41]. Accordingly, for example, the Tsallis entropy [26] and the Renyi entropy [27] have been proposed for nonadditive (nonextensive) generalized entropies and have been investigated from astrophysical

viewpoints [28–41]. In particular, Tsallis and Cirto recently suggested a generalized black-hole entropy proportional to its volume based on appropriate nonadditive generalizations [41]. Using the generalized entropy instead of the Bekenstein entropy will provide new insight into entropic cosmology. Therefore, it is important to derive entropic-force models from the two entropies and examine the properties of the two models in order to understand entropic cosmology more deeply. Note that power-law and logarithmic entropic corrections have been discussed [13,14,17–19,42,43].

In addition, in entropic cosmology, the entropy on the horizon of the Universe can increase during the evolution of the Universe [15], even if we consider a homogeneous and isotropic universe. However, a bulk viscosity of cosmological fluids [44–69] is usually the only thing that can generate an entropy in the homogeneous and isotropic universe [49]. (Such a cosmological model is referred to as bulk viscous cosmology.) Through the study of entropic cosmology, we may be able to discuss the classical entropy generated by bulk viscous stresses.

In this context, we examine entropic cosmology using a generalized black-hole entropy (proportional to its volume) proposed by Tsallis and Cirto [41]. In the present study, we derive entropic-force terms not only from the Bekenstein entropy but also from the generalized entropy. Moreover, using an effective description similar to bulk viscous cosmology, we formulate the modified Friedmann, acceleration, and continuity (conservation) equations for entropic cosmology. Note that the entropic force

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considered here is different from the idea that gravity itself is an entropic force [70,71]. Since we focus on background evolutions of the late universe, we do not discuss the inflation of the early universe.

The remainder of the present paper is organized as follows. In Sec. II, we present a brief review of three cosmological models, i.e., Λ CDM (lambda cold dark matter) cosmology, bulk viscous cosmology, and entropic cosmology. In Sec. III, we derive entropic-force terms from both the Bekenstein entropy and a generalized black-hole entropy. We also formulate the modified Friedmann, acceleration and continuity equations for two entropic-force models based on the obtained entropic-force terms. In Sec. IV, we examine a model that combines the two entropic-force models. In Sec. V, we discuss the evolution of the Universe in the two entropic-force models using solutions of the combined model. Finally, in Sec. VI, we present a discussion and our conclusions.

II. COSMOLOGICAL MODELS

In the present paper, we consider a homogeneous, isotropic, and spatially flat universe and examine the scale factor $a(t)$ at time t in the Friedmann-Lemaître-Robertson-Walker metric. In this section, we present a brief review of three cosmological models, i.e., Λ CDM, bulk viscous, and entropic cosmologies, focusing on the Friedmann, acceleration, and continuity equations. (A spatially nonflat universe is discussed in the last paragraph of in Sec. IV.)

A. Λ CDM cosmology

We first introduce the well-known Λ CDM models [7–9]. In the standard Λ CDM model, the Friedmann equation is given as

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = H(t)^2 = \frac{8\pi G}{3}\rho(t) + \frac{\Lambda}{3}, \quad (1)$$

and the acceleration equation is

$$\frac{\ddot{a}(t)}{a(t)} = \dot{H}(t) + H(t)^2 = -\frac{4\pi G}{3}\left(\rho(t) + \frac{3p(t)}{c^2}\right) + \frac{\Lambda}{3}. \quad (2)$$

In addition, the continuity equation is given by

$$\dot{\rho}(t) + 3\frac{\dot{a}(t)}{a(t)}\left(\rho(t) + \frac{p(t)}{c^2}\right) = 0, \quad (3)$$

where the Hubble parameter $H(t)$ is defined by

$$H(t) \equiv \frac{da/dt}{a(t)} = \frac{\dot{a}(t)}{a(t)}, \quad (4)$$

and G , Λ , c , $\rho(t)$, and $p(t)$ are the gravitational constant, a cosmological constant, the speed of light, the mass density of cosmological fluids, and the pressure of cosmological fluids, respectively. Equations (1) and (2) include the extra driving terms, i.e., $\Lambda/3$, which can explain the accelerated expansion of the late universe.

The continuity equation is consistent with the Friedmann and acceleration equations, because two of the three equations are independent [7]. In other words, if the Friedmann and acceleration equations are used, Eq. (3) is derived from these two equations. Note that Eq. (3) can be derived from the first law of thermodynamics as well, assuming adiabatic (isentropic) processes, without using the Friedmann and acceleration equations [7].

B. Bulk viscous cosmology

In bulk viscous cosmology, a bulk viscosity η of cosmological fluids is assumed [44–69]. Usually, the bulk viscosity is the only thing that can generate an entropy in the homogeneous and isotropic universe [49]. Such a cosmological model is referred to as bulk viscous cosmology. For example, in the 1980s, Barrow [46,47], Davies [48], and Lima *et al.* [49] investigated bulk viscous cosmology in order to discuss the inflation of the early universe. Recently, a number of studies have examined not only the inflation but also the accelerated expansion of the late universe, based on bulk viscous cosmology [50–69].

Generally, an effective pressure p' for bulk viscous cosmology is given by

$$p'(t) = p(t) - 3\eta H(t), \quad (5)$$

where, for simplicity, η is a constant. (It is possible to assume a variable bulk viscosity. For instance, see Refs. [64,67] and the references therein.) In bulk viscous cosmology, the Friedmann equation is given by

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho. \quad (6)$$

Unlike the Λ CDM model [Eq. (1)], Eq. (6) does not include an extra term such as a cosmological constant. Using the effective pressure p' , the acceleration equation for bulk viscous cosmology is given by

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3\frac{p'}{c^2}\right). \quad (7)$$

Substituting Eq. (5) into Eq. (7) and rearranging, we obtain the acceleration equation as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3\frac{p}{c^2}\right) + \frac{12\pi G}{c^2}\eta H. \quad (8)$$

The last term, $12\pi G\eta H/c^2$, corresponds to the extra driving term [15]. Accordingly, instead of the cosmological constant, the extra term due to the bulk viscosity can explain the accelerated expansion of the Universe. Note that the extra term considered here is proportional to the Hubble parameter $H(t)$. The continuity equation for bulk viscous cosmology is given by

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p'}{c^2}\right) = 0. \quad (9)$$

Substituting Eq. (5) into Eq. (9) and rearranging, we have

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = \frac{9\eta}{c^2}H^2. \quad (10)$$

Equation (10) has a nonzero right-hand side related to a classical entropy generated by bulk viscous stresses [15,46,47].

We now derive the continuity equation using a different approach. To this end, we use the generalized continuity equation obtained from the general Friedmann and acceleration equations [15], because two of the three equations are independent. The general Friedmann and acceleration equations are given, respectively, by

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + f(t), \quad (11)$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + g(t), \quad (12)$$

where $f(t)$ and $g(t)$ are general functions. Using Eqs. (11) and (12), we obtain the generalized continuity equation [15] given by

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = \frac{3}{4\pi G}H\left(-f(t) - \frac{\dot{f}(t)}{2H} + g(t)\right). \quad (13)$$

Comparing Eqs. (11) and (12) with Eqs. (6) and (8), we can set the general functions as $f(t) = 0$ and $g(t) = 12\pi G\eta H/c^2$. Substituting these functions into Eq. (13), we obtain

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = \frac{3}{4\pi G}H\left(\frac{12\pi G}{c^2}\eta H\right) = \frac{9\eta}{c^2}H^2. \quad (14)$$

Equation (14) is the same as Eq. (10). This indicates that Eq. (10) obtained from the effective pressure is consistent with Eq. (14) obtained from the Friedmann and acceleration equations. The consistency plays an important role in formulating the entropic-force models discussed in Sec. III.

C. Entropic cosmology

In the entropic cosmology suggested by Easson *et al.* [12,13], the horizon of the Universe is assumed to have an associated entropy and an approximate temperature. The entropy considered here is the Bekenstein entropy. In the present study, we refer to this cosmology as the standard entropic cosmology. (The Bekenstein entropy is discussed in detail in Sec. III A.)

In a study by Koivisto *et al.* [14], the modified Friedmann and acceleration equations are summarized as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \alpha_1 H^2 + \alpha_2 \dot{H}, \quad (15)$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \beta_1 H^2 + \beta_2 \dot{H}. \quad (16)$$

The four coefficients α_1 , α_2 , β_1 , and β_2 are dimensionless constants. The H^2 and \dot{H} terms with the dimensionless constants correspond to the extra driving terms, i.e., entropic-force terms. The entropic-force terms are derived, taking into account the entropy and temperature on the horizon of the Universe due to the information holographically stored there [12]. We can solve the two equations assuming the single-fluid-dominated universe [14,15]. Note that we neglect high-order terms for quantum corrections, because we do not discuss the inflation of the early universe in the present paper. (The dimensionless constants were expected to be bounded by $3/(2\pi) \lesssim \beta_1 \leq 1$ and $0 \leq \beta_2 \lesssim 3/(4\pi)$, and typical values for a better fitting were $\beta_1 = 3/(2\pi)$ and $\beta_2 = 3/(4\pi)$ [12]. It was argued that the extrinsic curvature at the surface was likely to result in something like $\alpha_1 = \beta_1 = 3/(2\pi)$ and $\alpha_2 = \beta_2 = 3/(4\pi)$ [13,14].)

As examined in Ref. [15], we can simplify the two modified Friedmann equations, assuming a nonadiabatic-like expansion of the Universe. The simple modified Friedmann and acceleration equations are summarized as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \alpha_1 H^2, \quad (17)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \beta_1 H^2. \quad (18)$$

Equations (17) and (18) do not include \dot{H} terms. In fact, Easson *et al.* first proposed that the entropic-force terms of the modified acceleration equation are H^2 or $\frac{3}{2\pi}H^2$; i.e., \dot{H} terms are not included in the entropic-force terms [12]. In other words, Eq. (18) is consistent with their original acceleration equation, as discussed in Sec. III A. In the present paper, we select Eqs. (17) and (18) as standard entropic-force models. (Note that it may be possible to neglect the entropic-force terms of the modified Friedmann equation, i.e., $\alpha_1 = \alpha_2 = 0$. In the standard entropic cosmology [14], the entropic-force terms are added not only with the acceleration equation but also with the Friedmann equation, as in the case of the Λ CDM models.)

In Ref. [12], Easson *et al.* considered that the continuity equation was given by $\dot{\rho} + 3(\dot{a}/a)[\rho + (p/c^2)] = 0$, assuming an adiabatic (isentropic) expansion. However, in entropic cosmology, since the entropy on the horizon is assumed, the entropy can increase during the evolution of the Universe. Therefore, in a previous study, we derived the continuity equation from the first law of thermodynamics, taking into account a nonadiabatic-like process caused by the entropy and the temperature on the horizon [15]. Consequently, the modified continuity equation was written as

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = -\gamma\left(\frac{3}{4\pi G}H\dot{H}\right). \quad (19)$$

Because of the nonzero right-hand side, Eq. (19) was likely consistent with the continuity equation obtained from the modified Friedmann and acceleration equations [15]. In the present paper, we do not use Eq. (19) as the modified continuity equation. Instead, in the next section, we derive the modified continuity equation from an effective pressure for entropic cosmology.

We point out that a similar nonzero right-hand side of the continuity equation appears not only in bulk viscous cosmology but also in “energy exchange cosmology,” in which the transfer of energy between two fluids is assumed [15,72], e.g., the interaction between matter and radiation [73], matter creation [74], interacting quintessence [75], the interaction between dark energy and dark matter [76], and dynamical vacuum energy [77–87]. In particular, cosmological equations for dynamical vacuum energy models are very similar to those for entropic-force models. Therefore, in the following paragraph, we discuss a similarity between the two models.

Instead of a cosmological constant, a variable cosmological term $\Lambda(t)$ is assumed in dynamical vacuum energy models, i.e., $\Lambda(t)$ CDM models [77–87]. Various $\Lambda(t)$ terms (e.g., H^2 , H , and constant terms) have been examined in those works. For example, see Refs. [79–83]. Recently, H^2 and \dot{H} terms have been investigated in Refs. [21,87]. [The H^2 and \dot{H} terms are the same as the entropic-force terms shown in Eqs. (15) and (16).] Consequently, the influence of \dot{H} terms was found to be similar to that of H^2 terms. In other words, essentially new properties of the late universe were not obtained from \dot{H} terms. This implies that the \dot{H} terms can be neglected as if Eqs. (17) and (18) were the approximated Friedmann and acceleration equations, respectively. In addition, in $\Lambda(t)$ CDM models, the continuity equation for matter “ m ” is arranged as $\dot{\rho}_m + 3(\dot{a}/a) \times (\rho_m + p_m/c^2) = -\dot{\Lambda}$ [82]. The continuity equation is equivalent to Eq. (19), when $\Lambda(t)$ is proportional to H^2 . In this way, cosmological equations for $\Lambda(t)$ CDM models are similar to those for entropic-force models. Accordingly, the cosmological equations and their solutions discussed in the present paper are similar to those for $\Lambda(t)$ CDM models, especially in the work of Basilakos *et al.* [82]. However, a theoretical background of the $\Lambda(t)$ CDM model is different from that of the entropic-force model.

Finally, we discuss a fundamental problem of the standard entropic cosmology which includes H^2 and \dot{H} terms [Eqs. (15) and (16)]. In fact, H^2 and \dot{H} terms of entropic-force terms cannot describe a decelerating and accelerating universe predicted by the standard Λ CDM model. This problem has been discussed in Refs. [21,85] and our previous paper [15]. In particular, Basilakos *et al.* have shown that it is not the H^2 and \dot{H} terms but extra constant terms that are important for describing the decelerating and

accelerating universe [21]. We emphasize that the standard entropic cosmology has such a fundamental problem, since entropic-force terms are usually considered to be H^2 and \dot{H} terms. However, a generalized black-hole entropy [41] is expected to solve the problem, as discussed later. (It should be noted that the above problem does not occur in $\Lambda(t)$ CDM models, as examined in Refs. [21,85]. This is because the extra constant term is naturally obtained from an integral constant of the renormalization group equation for the vacuum energy density. For details, see a summarized review [84] and a recent thorough review [86].)

III. DERIVATION OF ENTROPIC FORCE FOR ENTROPIC COSMOLOGY

In this section, in order to discuss entropic cosmology, we derive entropic-force terms from the Bekenstein entropy and a generalized black-hole entropy proposed by Tsallis and Cirto [41]. In Sec. III A, we derive the standard entropic-force term from the Bekenstein entropy, which is proportional to the surface area of a sphere with the Hubble radius. Moreover, using an effective description for pressure, we reformulate the entropic-force model. In Sec. III B, we assume the generalized entropy proportional to the volume. We derive an entropic-force term from the generalized entropy and propose a new entropic-force model. In the present paper, we use the Hubble radius as the preferred screen, because the apparent horizon coincides with the Hubble radius in the spatially flat universe [12].

A. Entropic force from the Bekenstein entropy

In the standard entropic cosmology, the modified Friedmann and acceleration equations include H^2 terms [Eqs. (15)–(18)] as entropic-force terms. We derive the entropic-force term from the Bekenstein entropy, according to the work of Easson *et al.* [12]. We also reformulate the entropic-force model, using an effective pressure for entropic cosmology.

1. Derivation of entropic force from the Bekenstein entropy

For entropic cosmology, we assume that the Hubble horizon has an approximate temperature T and an associated entropy S , where the Hubble horizon (radius) r_H is given by

$$r_H = \frac{c}{H}. \quad (20)$$

The temperature T on the Hubble horizon is given by

$$T = \frac{\hbar H}{2\pi k_B} \times \gamma = \frac{\hbar}{2\pi k_B} \frac{c}{r_H} \gamma, \quad (21)$$

where k_B and \hbar are the Boltzmann constant and the reduced Planck constant, respectively. The reduced

Planck constant is defined by $\hbar \equiv h/(2\pi)$, where h is the Planck constant. The temperature considered here is obtained by multiplying the horizon temperature, $\hbar H/(2\pi k_B)$, by γ [15]. In the present study, γ is a non-negative free parameter on the order of $O(1)$, typically $\gamma \sim \frac{3}{2\pi}$ or $\frac{1}{2}$. A similar parameter for the screen temperature has been discussed in Refs. [12,17–19]. (Note that the temperature on the horizon can be evaluated as 10^{-30} K, which is slightly lower than the temperature of our cosmic microwave background radiation, 2.73 K. We use the temperature on the horizon, assuming thermal equilibrium states based on a single holographic screen [12,13].)

As an associated entropy on the Hubble horizon, the Bekenstein entropy S is given as

$$S = \frac{k_B c^3}{\hbar G} \frac{A_H}{4}, \quad (22)$$

where A_H is the surface area of the sphere with the Hubble radius r_H . As shown in Eq. (22), the Bekenstein entropy is proportional to A_H . Substituting $A_H = 4\pi r_H^2$ into Eq. (22) and using $r_H = c/H$ as given in Eq. (20), we obtain

$$\begin{aligned} S &= \frac{k_B c^3}{\hbar G} \frac{A_H}{4} = \frac{k_B c^3}{\hbar G} \frac{4\pi r_H^2}{4} = \frac{k_B c^3}{\hbar G} \pi \left(\frac{c}{H}\right)^2 \\ &= \left(\frac{\pi k_B c^5}{\hbar G}\right) \frac{1}{H^2} = K \frac{1}{H^2}, \end{aligned} \quad (23)$$

where K is a positive constant [15] given by

$$K = \frac{\pi k_B c^5}{\hbar G}. \quad (24)$$

The entropic-force F_r can be given by

$$F_r = -\frac{dE}{dr} = -T \frac{dS}{dr} \left(= -T \frac{dS}{dr_H} \right), \quad (25)$$

where the minus sign indicates the direction of increasing entropy or the screen corresponding to the horizon [12]. Substituting Eqs. (21) and (22) into Eq. (25) and using $A_H = 4\pi r_H^2$, we have the entropic force as

$$\begin{aligned} F_r &= -T \frac{dS}{dr_H} = -\frac{\hbar}{2\pi k_B} \frac{c}{r_H} \gamma \times \frac{d}{dr_H} \left[\frac{k_B c^3}{\hbar G} \frac{4\pi r_H^2}{4} \right] \\ &= -\gamma \frac{c^4}{G}. \end{aligned} \quad (26)$$

Therefore, the pressure p_{e_B} derived from the Bekenstein entropy is given by

$$\begin{aligned} p_{e_B} &= \frac{F_r}{A_H} = -\gamma \frac{c^4}{G} \frac{1}{A_H} = -\gamma \frac{c^4}{G} \frac{1}{4\pi r_H^2} \\ &= -\gamma \frac{c^4}{G} \frac{1}{4\pi (c/H)^2} = -\gamma \frac{c^2}{4\pi G} H^2. \end{aligned} \quad (27)$$

The above derivation is based on the original idea of Easson *et al.* [12]. Note that Eq. (27) includes γ used in this study, where γ is a free parameter for the temperature.

In the following, in order to reformulate entropic cosmology, we consider an effective description similar to bulk viscous cosmology. In other words, we assume an effective pressure for entropic cosmology. We will discuss the reason for this in Sec. III B. We assume that the effective pressure p' based on the Bekenstein entropy is given by

$$p' = p + p_{e_B} = p - \gamma \frac{c^2}{4\pi G} H^2. \quad (28)$$

Using p' , the acceleration equation can be written as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left(\rho + \frac{3p'}{c^2} \right). \quad (29)$$

Accordingly, substituting Eq. (28) into Eq. (29), we have

$$\begin{aligned} \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3} \left(\rho + \frac{3(p - \gamma \frac{c^2}{4\pi G} H^2)}{c^2} \right) \\ &= -\frac{4\pi G}{3} \left(\rho + \frac{3p}{c^2} \right) + \gamma H^2. \end{aligned} \quad (30)$$

The γH^2 term is the entropic-force term, which can explain the accelerated expansion of the Universe. When $\gamma = 1$, Eq. (30) corresponds to the modified acceleration equation derived by Easson *et al.* [12].

We now examine the continuity equation for entropic cosmology. Using the effective pressure p' , the continuity equation is expected to be given by

$$\dot{\rho} + 3 \frac{\dot{a}}{a} \left(\rho + \frac{p'}{c^2} \right) = 0. \quad (31)$$

Substituting Eq. (28) into Eq. (31) and rearranging, we have

$$\dot{\rho} + 3 \frac{\dot{a}}{a} \left(\rho + \frac{p}{c^2} \right) = \gamma \frac{3}{4\pi G} H^3. \quad (32)$$

This is the modified continuity equation obtained from the effective pressure for entropic cosmology. In the present paper, we consider Eq. (32) as the modified continuity equation for entropic cosmology derived from the Bekenstein entropy. [Note that Eqs. (32) and (19) are consistent with each other when $\dot{H} = -H^2$. The universe for $\dot{H} = -H^2$ corresponds to the empty universe, as discussed in Sec. IV.]

We can determine two dimensionless constants α_1 and β_1 included in the modified Friedmann and acceleration equations [Eqs. (17) and (18)] using two continuity equations. The first continuity equation is Eq. (32), whereas the second continuity equation can be derived from the modified Friedmann and acceleration equations. In order to obtain the second continuity equation, we use the generalized continuity equation, i.e., Eq. (13). Comparing Eqs. (11) and (12) with Eqs. (17) and (18), we can set general functions as $f(t) = \alpha_1 H^2$ and $g(t) = \beta_1 H^2$.

Substituting these functions into Eq. (13) and rearranging, we have the second continuity equation, which is given by

$$\begin{aligned}\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) &= \frac{3}{4\pi G}H\left(-\alpha_1 H^2 - \frac{2\alpha_1 H\dot{H}}{2H} + \beta_1 H^2\right) \\ &= \frac{3}{4\pi G}H((\beta_1 - \alpha_1)H^2 - \alpha_1\dot{H}).\end{aligned}\quad (33)$$

We expect that the two modified continuity equations, i.e., Eqs. (32) and (33), are consistent with each other. Consequently, we obtain α_1 and β_1 as

$$\alpha_1 = 0 \quad \text{and} \quad \beta_1 = \gamma. \quad (34)$$

Therefore, we can neglect the entropic-force term $\alpha_1 H^2$ of the modified Friedmann equation shown in Eq. (17). This is because, in the present paper, we assume the effective pressure for entropic cosmology.

As mentioned previously, we consider γ as a free parameter for the temperature. Moreover, a free parameter for the entropy may be required for calculating TdS . However, we do not use the free parameter for the entropy, because we assume that the Bekenstein entropy is given by $S = (k_B c^3 A_H)/(4\hbar G)$ [Eq. (22)].

2. Entropic-force model for the Bekenstein entropy

In Sec. III A 1, we derived an entropic-force term from the Bekenstein entropy and obtained an entropic-force model. Consequently, the modified Friedmann, acceleration, and continuity equations are summarized as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \alpha_1 H^2, \quad (35)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \beta_1 H^2, \quad (36)$$

and

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = \gamma\frac{3}{4\pi G}H^3, \quad (37)$$

where α_1 and β_1 are given by

$$\alpha_1 = 0 \quad \text{and} \quad \beta_1 = \gamma. \quad (38)$$

Note that we leave $\alpha_1 H^2$ in Eq. (35) in order to discuss a combined model later.

B. Entropic force from a generalized entropy

Thus far, we have considered the Bekenstein entropy to discuss the standard entropic cosmology. Instead of the Bekenstein entropy, we now assume a generalized black-hole entropy proposed by Tsallis and Cirto [41].

1. Derivation of entropic force from a generalized entropy

Recently, Tsallis and Cirto [41] examined a black-hole entropy using appropriate nonadditive generalizations for d -dimensional systems and suggested a generalized black-hole entropy. In the following, we introduce the

generalized black-hole entropy, according to the work of Tsallis and Cirto [41]. In their study, a nonadditive entropy (for a set of W discrete states) is defined by

$$S_\delta = k_B \sum_{i=1}^W p_i \left(\ln \frac{1}{p_i}\right)^\delta \quad (\delta > 0), \quad (39)$$

where p_i is a probability distribution [24,41]. When $\delta = 1$, S_δ recovers the Boltzmann-Gibbs entropy given by

$$S_{\text{BG}} = -k_B \sum_{i=1}^W p_i \ln p_i. \quad (40)$$

If we compose two probabilistically independent subsystems A and B , the Boltzmann-Gibbs entropy S_{BG} is additive,

$$S_{\text{BG}}(A+B) = S_{\text{BG}}(A) + S_{\text{BG}}(B). \quad (41)$$

However, when $\delta \neq 1$, S_δ is nonadditive,

$$S_\delta(A+B) \neq S_\delta(A) + S_\delta(B). \quad (42)$$

This is because, for $\delta > 0$, $S_\delta(A+B)$ is given by

$$\frac{S_\delta(A+B)}{k_B} = \left(\left[\frac{S_\delta(A)}{k_B} \right]^{1/\delta} + \left[\frac{S_\delta(B)}{k_B} \right]^{1/\delta} \right)^\delta. \quad (43)$$

In Ref. [41], Tsallis and Cirto demonstrated that a generalized black-hole entropy can be written as

$$\frac{S_{\delta=3/2}}{k_B} \propto \left(\frac{S'}{k_B} \right)^{\frac{3}{2}}, \quad (44)$$

where the Bekenstein entropy S' is given by

$$S' = \frac{k_B c^3}{\hbar G} \frac{A'_H}{4}, \quad (45)$$

and A'_H is the event horizon area of a black hole. For details, see Ref. [41].

We now apply the generalized black-hole entropy to an entropy for entropic cosmology. To this end, substituting Eq. (45) into Eq. (44), replacing A'_H with A_H , and rearranging, the entropy S on the Hubble horizon can be evaluated as

$$S \propto A_H^{\frac{3}{2}} \propto r_H^3, \quad (46)$$

where A_H is the surface area of the sphere with the Hubble radius r_H , and, therefore, $A_H = 4\pi r_H^2$. Because of non-additive generalizations, the generalized entropy is proportional to r_H^3 , i.e., volume. Accordingly, from Eqs. (45) and (46), we assume a generalized entropy S_g given by

$$S_g = \frac{\pi k_B c^3}{\hbar G} \times \zeta r_H^3, \quad (47)$$

where ζ is a non-negative free parameter. Note that ζ is a dimensional constant. We hereinafter refer to S_g as the generalized entropy on the Hubble horizon. Substituting $r_H = c/H$ [Eq. (20)] into Eq. (47), we obtain

$$S_g = \frac{\pi k_B c^3}{\hbar G} \times \zeta r_H^3 = \frac{\pi k_B c^3}{\hbar G} \times \zeta \left(\frac{c}{H}\right)^3 = \left(\frac{\pi k_B c^3}{\hbar G}\right) c \zeta \frac{1}{H^3}$$

$$= K c \zeta \frac{1}{H^3}, \quad (48)$$

where K is $\pi k_B c^5 / (\hbar G)$ [Eq. (24)].

In order to calculate an entropic force, we use the generalized entropy [Eq. (47)] and the temperature T on the horizon [Eq. (21)]. Substituting Eqs. (21) and (47) into Eq. (25) and using $r_H = c/H$, we obtain the entropic force as

$$F_r = -T \frac{dS}{dr} = -T \frac{dS_g}{dr_H}$$

$$= -\frac{\hbar}{2\pi k_B} \frac{c}{r_H} \gamma \times \frac{d}{dr_H} \left[\frac{\pi k_B c^3}{\hbar G} \times \zeta r_H^3 \right]$$

$$= -\gamma \frac{c^4}{G} \left(\frac{3}{2} \zeta r_H \right) = -\gamma \frac{c^4}{G} \left(\frac{3c\zeta}{2} \frac{1}{H} \right). \quad (49)$$

Therefore, the pressure p_{e_g} derived from the generalized entropy is given as

$$p_{e_g} = \frac{F_r}{A_H} = -\gamma \frac{c^4}{G} \left(\frac{3c\zeta}{2} \frac{1}{H} \right) \frac{1}{4\pi r_H^2}$$

$$= -\gamma \frac{c^4}{G} \left(\frac{3c\zeta}{2} \frac{1}{H} \right) \frac{1}{4\pi (c/H)^2} = -\gamma \frac{c^2}{4\pi G} \frac{3c\zeta}{2} H. \quad (50)$$

In this equation, $3c\zeta/2$ is shown separately in order to clarify the difference between Eqs. (27) and (50). Similarly, $3c\zeta/2$ is shown separately in the following.

In the present study, we assume an effective pressure p' for entropic cosmology. Using Eq. (50), the effective pressure p' based on the generalized entropy is given by

$$p' = p + p_{e_g} = p - \gamma \frac{c^2}{4\pi G} \frac{3c\zeta}{2} H. \quad (51)$$

Substituting Eq. (51) into Eq. (29), we have

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left(\rho + \frac{3(p - \gamma \frac{c^2}{4\pi G} \frac{3c\zeta}{2} H)}{c^2} \right)$$

$$= -\frac{4\pi G}{3} \left(\rho + \frac{3p}{c^2} \right) + \gamma \frac{3c\zeta}{2} H. \quad (52)$$

This equation is the modified acceleration equation derived from the generalized entropy. The H term on the right-hand side corresponds to an extra driving term to explain the accelerated expansion of the Universe. The extra driving term is found not to be the H^2 term but rather to be the H term, unlike in the case of the standard entropic cosmology. The continuity equation is expected to be given by

$$\dot{\rho} + 3 \frac{\dot{a}}{a} \left(\rho + \frac{p'}{c^2} \right) = 0. \quad (53)$$

Substituting Eq. (51) into Eq. (53) and rearranging, we have

$$\dot{\rho} + 3 \frac{\dot{a}}{a} \left(\rho + \frac{p}{c^2} \right) = \gamma \frac{3}{4\pi G} \frac{3c\zeta}{2} H^2. \quad (54)$$

In this paper, we consider Eq. (54) as the modified continuity equation based on the generalized entropy [88].

Interestingly, the effective pressure, Eq. (51), is similar to Eq. (5), i.e., $p' = p - 3\eta H$ for bulk viscous cosmology. Therefore, Eqs. (52) and (54) are also similar to Eqs. (8) and (10) for bulk viscous cosmology. This is probably because the generalized entropy assumed here is proportional not to the surface area but to the volume. However, the similarity may be interpreted as a sign that the generalized entropy behaves as if it were a classical entropy generated by bulk viscous stresses. In other words, the generalized entropy may be related to a bulk viscosity of cosmological fluids through a holographic screen. In fact, the bulk viscosity is usually the only thing that can generate an entropy in the homogeneous and isotropic universe. Accordingly, this interpretation may help to explain the origin of the bulk viscosity of cosmological fluids. As an alternative interpretation, the bulk viscosity may be derived from an extra entropy that is proportional to the volume of the Universe.

2. Entropic-force model for a generalized entropy

In Sec. III B 1, H terms are derived from a generalized entropy on the horizon. Accordingly, we expect that the modified Friedmann and acceleration equations are summarized, respectively, as

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho + \hat{\alpha}_3 H, \quad (55)$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left(\rho + \frac{3p}{c^2} \right) + \hat{\beta}_3 H, \quad (56)$$

where $\hat{\alpha}_3$ and $\hat{\beta}_3$ are defined by

$$\hat{\alpha}_3 \equiv \alpha_3 H_0 \quad \text{and} \quad \hat{\beta}_3 \equiv \beta_3 H_0. \quad (57)$$

The two coefficients α_3 and β_3 are dimensionless constants, and H_0 is the present value of the Hubble parameter. As shown in Eq. (54), the modified continuity equation is given by

$$\dot{\rho} + 3 \frac{\dot{a}}{a} \left(\rho + \frac{p}{c^2} \right) = \gamma \frac{3}{4\pi G} \frac{3c\zeta}{2} H^2. \quad (58)$$

We can determine $\hat{\alpha}_3$ and $\hat{\beta}_3$, which are included in Eqs. (55) and (56), from two continuity equations. In this case, the first continuity equation is Eq. (58). The second continuity equation (derived from the modified Friedmann and acceleration equations) is calculated from the generalized continuity equation, i.e., Eq. (13). Comparing Eqs. (11) and (12) with Eqs. (55) and (56), we can set general functions as $f(t) = \hat{\alpha}_3 H$ and $g(t) = \hat{\beta}_3 H$. Therefore, substituting these functions into Eq. (13) and

TABLE I. Entropic-force terms $g_e(t)$ for the modified acceleration equation. The Bekenstein and Generalized columns indicate the information for the entropic-force models derived from the Bekenstein and generalized entropies, respectively.

| Parameter | Bekenstein | Generalized |
|-----------|--|--|
| Entropy | $\frac{\pi k_B c^3}{\hbar G} \times r_H^2$ | $\frac{\pi k_B c^3}{\hbar G} \times \zeta r_H^3$ |
| $g_e(t)$ | $\beta_1 H^2$ | $\hat{\beta}_3 H$ |

rearranging, we have the second continuity equation given by

$$\begin{aligned} \dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) &= \frac{3}{4\pi G}H\left(-\hat{\alpha}_3 H - \frac{\hat{\alpha}_3 \dot{H}}{2H} + \hat{\beta}_3 H\right) \\ &= \frac{3}{4\pi G}H\left((\hat{\beta}_3 - \hat{\alpha}_3)H - \hat{\alpha}_3 \frac{\dot{H}}{2H}\right). \end{aligned} \quad (59)$$

Since Eqs. (58) and (59) are expected to be consistent with each other, we obtain $\hat{\alpha}_3$ and $\hat{\beta}_3$ as

$$\hat{\alpha}_3 = 0 \quad \text{and} \quad \hat{\beta}_3 = \gamma \frac{3c\zeta}{2}. \quad (60)$$

Accordingly, the dimensionless constants α_3 and β_3 are given as

$$\alpha_3 = 0 \quad \text{and} \quad \beta_3 = \frac{\hat{\beta}_3}{H_0} = \gamma \frac{3c\zeta}{2H_0}. \quad (61)$$

As expected, we can neglect the entropic-force term $\hat{\alpha}_3 H$ of the modified Friedmann equation, Eq. (55). This is because, in the above discussion and in the present study, we assume an effective pressure for entropic cosmology. Note that we leave $\hat{\alpha}_3 H$ in Eq. (55) in order to discuss a combined model in the next section.

In this section, we derive entropic-force terms from the Bekenstein and generalized entropies, assuming that the temperature on the horizon is given by Eq. (21), i.e., $T = \gamma \hbar H / (2\pi k_B)$. Consequently, H^2 terms are derived from the Bekenstein entropy, whereas H terms are derived from the generalized entropy, which is proportional to volume. The two entropic-force terms for the modified acceleration equation are summarized in Table I. Interestingly, the H term is similar to an extra driving term for bulk viscous cosmology. Therefore, we assume an effective pressure for entropic cosmology. The modified acceleration equation is found to include the entropic-force terms, whereas the Friedmann equation does not include the entropic-force terms. Based on these results, we propose two entropic-force models. In the next section, we discuss solutions of the two entropic-force models.

IV. COMBINED MODEL

In the previous section, the Friedmann equation was found not to include entropic-force terms, because we use an effective pressure for entropic cosmology.

Moreover, two entropic-force terms, i.e., H^2 and H terms, are discussed separately, as shown in Secs. III A and III B. This is because the H^2 term is derived from the Bekenstein entropy, whereas the H term is derived from a generalized entropy. However, in this section, in order to obtain general solutions that are widely used, we consider a combined model in which the Friedmann and acceleration equations include both H^2 and H terms as the entropic-force terms. In order to solve the equations, we extend our solution method, which is discussed in Ref. [15]. Note that the two extra driving terms for the acceleration equation, i.e., H^2 and H terms, have been examined in bulk viscous cosmology. For example, see the work of Avelino and Nucamendi [64]. Of course, similar extra driving terms have been discussed using a variable cosmological term. For instance, Basilakos *et al.* [82] have examined H^2 and H terms in detail. Several types of variable cosmological terms, which include constant terms, are closely investigated in Ref. [82]. For variable Λ cosmologies, see Ref. [78], a recent review [86], and the references therein. We point out that cosmological equations and their solutions discussed in this section are similar to those examined in the above previous works.

A. Formulations for the combined model

For a combined model, we consider both H^2 and H terms as extra driving terms. Accordingly, the modified Friedmann equation is given as

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi G}{3}\rho + \alpha_1 H^2 + \hat{\alpha}_3 H, \quad (62)$$

and the modified acceleration equation is given as

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{4\pi G}{3}(1+3w)\rho + \beta_1 H^2 + \hat{\beta}_3 H, \quad (63)$$

where w is given by

$$w = \frac{p}{\rho c^2}. \quad (64)$$

$\hat{\alpha}_3$ and $\hat{\beta}_3$ are defined by

$$\hat{\alpha}_3 \equiv \alpha_3 H_0 \quad \text{and} \quad \hat{\beta}_3 \equiv \beta_3 H_0, \quad (65)$$

and w represents the equation-of-state parameter for a generic component of matter. For nonrelativistic matter (or the matter-dominated universe) w is 0, and for relativistic matter (or the radiation-dominated universe) w is 1/3. The four coefficients α_1 , β_1 , α_3 , and β_3 are dimensionless constants. For entropic-force models for the Bekenstein and generalized entropies, three dimensionless constants are set as $\alpha_1 = \alpha_3 = \beta_3 = 0$, and $\alpha_1 = \alpha_3 = \beta_1 = 0$, respectively.

Coupling $(1+3w) \times$ Eq. (62) with $2 \times$ Eq. (63) and rearranging, we obtain

$$\dot{H} = \frac{dH}{dt} = -C_1 H^2 + \hat{C}_3 H, \quad (66)$$

where

$$C_1 = \frac{3(1+w) - \alpha_1(1+3w) - 2\beta_1}{2}, \quad (67)$$

and

$$\hat{C}_3 = \frac{\hat{\alpha}_3(1+3w) + 2\hat{\beta}_3}{2}. \quad (68)$$

Equation (66) includes not only the H^2 term but also the H term. (In our previous study [15], the H term was not considered.) Here, we point out that C_1 is a dimensionless parameter, whereas \hat{C}_3 is a dimensional parameter. Substituting Eq. (65) into Eq. (68), we obtain a dimensionless parameter C_3 given by

$$C_3 = \frac{\hat{C}_3}{H_0} = \frac{\alpha_3(1+3w) + 2\beta_3}{2}. \quad (69)$$

From Eq. (66), $(dH/da)a$ is calculated as

$$\begin{aligned} \left(\frac{dH}{da}\right)a &= \left(\frac{dH}{dt}\right)\frac{dt}{da}a = (-C_1 H^2 + \hat{C}_3 H)\frac{a}{\dot{a}} \\ &= (-C_1 H^2 + \hat{C}_3 H)\frac{1}{H} = -C_1 H + \hat{C}_3. \end{aligned} \quad (70)$$

We can rearrange Eq. (70) as

$$\frac{dH}{dN} = -C_1 H + \hat{C}_3, \quad (71)$$

where N is defined by

$$N \equiv \ln a \quad \text{and therefore} \quad dN = \frac{da}{a}. \quad (72)$$

Because of the \hat{C}_3 term, Eq. (71) is slightly more complicated than the equation examined in Ref. [15]. We can solve Eq. (71), as discussed in the next subsection.

B. Solutions for the combined model in the single-fluid-dominated universe

In the previous subsection, we obtain Eq. (71) for the combined model. We can solve Eq. (71) when C_1 and \hat{C}_3 are constant. In fact, C_1 and \hat{C}_3 are constant when five parameters, i.e., α_1 , β_1 , $\hat{\alpha}_3$, $\hat{\beta}_3$, and w , are constant. Therefore, we assume that these five parameters are constant. This indicates that we assume a single-fluid-dominated universe. For example, w is 0 for the matter-dominated universe and $1/3$ for the radiation-dominated universe. In the following, we consider C_1 , \hat{C}_3 , and C_3 as non-negative free parameters. For simplicity, we also assume $C_1 > 0$.

When C_1 and \hat{C}_3 are constant, Eq. (71) is integrated as

$$\int \frac{dH}{-C_1 H + \hat{C}_3} = \int dN. \quad (73)$$

Solving this integral, and using $N = \ln a$, we have

$$C_1 H - \hat{C}_3 = D a^{-C_1}, \quad (74)$$

and dividing this equation by C_1 gives

$$H - \frac{\hat{C}_3}{C_1} = \frac{D}{C_1} a^{-C_1}, \quad (75)$$

where D is an integral constant. Dividing Eq. (75) by $H_0 - (\hat{C}_3/C_1) = (D/C_1)a_0^{-C_1}$, we have

$$\frac{H - (\hat{C}_3/C_1)}{H_0 - (\hat{C}_3/C_1)} = \left(\frac{a}{a_0}\right)^{-C_1}, \quad (76)$$

where a_0 is the present value of the scale factor. Rearranging Eq. (76), and substituting $C_3 = \hat{C}_3/H_0$ [Eq. (69)] into the resulting equation, we obtain

$$\begin{aligned} \frac{H}{H_0} &= \left(1 - \frac{1}{H_0} \frac{\hat{C}_3}{C_1}\right) \left(\frac{a}{a_0}\right)^{-C_1} + \frac{1}{H_0} \frac{\hat{C}_3}{C_1} \\ &= \left(1 - \frac{C_3}{C_1}\right) \left(\frac{a}{a_0}\right)^{-C_1} + \frac{C_3}{C_1}. \end{aligned} \quad (77)$$

Equation (77) indicates that C_1 and C_3 play an important role in the combined model. We can determine C_1 and C_3 from Eqs. (67) and (69), respectively. Since C_3 is related to the H terms, Eq. (77) is somewhat complicated. For the case in which $C_3 = 0$, typical results have been discussed in our previous study [15]. For example, when $C_3 = 0$, H/H_0 for $C_1 = 2, 1.5, 1$, and 0 are consistent with H/H_0 for the radiation-dominated, matter-dominated, empty, and Λ -dominated universes, respectively, [15].

1. Scale factor a

We examine the scale factor $a(t)$, using Eq. (77). For this purpose, Eq. (77) is rearranged as

$$\tilde{H} = (1 - A)\tilde{a}^{-C_1} + A, \quad (78)$$

where \tilde{H} , \tilde{a} , and A are defined by

$$\tilde{H} \equiv \frac{H}{H_0}, \quad \tilde{a} \equiv \frac{a}{a_0}, \quad A \equiv \frac{C_3}{C_1}. \quad (79)$$

Multiplying Eq. (78) by \tilde{a} , we obtain

$$\tilde{H} \tilde{a} = \tilde{a}[(1 - A)\tilde{a}^{-C_1} + A]. \quad (80)$$

On the other hand, we can calculate $\tilde{H} \tilde{a}$ as

$$\tilde{H} \tilde{a} = \frac{H}{H_0} \frac{a}{a_0} = \frac{\dot{a}/a}{H_0} \frac{a}{a_0} = \frac{\dot{a}}{H_0 a_0} = \frac{a_0 \frac{d}{dt} \left(\frac{a}{a_0}\right)}{H_0 a_0} = \frac{1}{H_0} \frac{d\tilde{a}}{dt}. \quad (81)$$

Therefore, substituting Eq. (81) into Eq. (80), we have

$$\frac{1}{H_0} \frac{d\tilde{a}}{dt} = \tilde{a}[(1-A)\tilde{a}^{-C_1} + A], \quad (82)$$

As examined in Ref. [64], integrating Eq. (82), we obtain

$$\int_1^{\tilde{a}} \frac{da'}{a'[(1-A)a'^{-C_1} + A]} = \int_{t_0}^t H_0 dt' = H_0(t - t_0), \quad (83)$$

where t_0 represents the present time. Solving this integral yields

$$\frac{1}{AC_1} \ln[1 - A + A\tilde{a}^{C_1}] = H_0(t - t_0). \quad (84)$$

Moreover, solving Eq. (84) for \tilde{a} and substituting Eq. (79) into the result, we finally have

$$\frac{a}{a_0} = \left[1 + \frac{C_1}{C_3} (\exp[C_3 H_0(t - t_0)] - 1) \right]^{\frac{1}{C_1}}, \quad (85)$$

where $C_1 > 0$ and $C_3 > 0$ are assumed. Avelino and Nucamendi discussed this type of equation for bulk viscous cosmology in detail [64]. Note that, unlike the combined model of the present study, the Friedmann equation for bulk viscous cosmology does not have extra terms. When $C_3 = 0$, the normalized scale factor a/a_0 is given by

$$\frac{a}{a_0} = \begin{cases} (C_1 H_0 t)^{\frac{1}{C_1}} & (C_1 \neq 0), \\ \exp[H_0(t - t_0)] & (C_1 = 0), \end{cases} \quad (86)$$

where t_0 is set to be $1/(C_1 H_0)$ [15].

We now examine the deceleration parameter q_0 , which is used to discuss the expansion of the Universe [7–9]. The deceleration parameter is defined by

$$q_0 \equiv -\left(\frac{\ddot{a}}{aH^2}\right)_{t=t_0}. \quad (87)$$

Substituting Eq. (66) into Eq. (63), we obtain

$$\begin{aligned} \frac{\ddot{a}}{a} &= \dot{H} + H^2 = (-C_1 H^2 + \hat{C}_3 H) + H^2 \\ &= (1 - C_1)H^2 + \hat{C}_3 H, \end{aligned} \quad (88)$$

and rearranging the result yields

$$-\frac{\ddot{a}}{aH^2} = C_1 - \frac{\hat{C}_3}{H} - 1. \quad (89)$$

This equation is a temporal deceleration parameter $q(t)$. Even if C_1 and \hat{C}_3 are constant, $q(t)$ can vary during the evolution of the Universe depending on the \hat{C}_3/H term. Substituting Eq. (89) into Eq. (87) and using $\hat{C}_3 = C_3 H_0$ [Eq. (69)], the deceleration parameter q_0 is given as

$$q_0 = C_1 - 1 - \frac{\hat{C}_3}{H_0} = C_1 - C_3 - 1. \quad (90)$$

When $C_1 - C_3 < 1$, q_0 is negative corresponding to a positive acceleration. This will be discussed in Sec. V.

Note that, in calculating q_0 , we do not assume the single-fluid-dominated universe.

The temporal deceleration parameter $q(t)$ [Eq. (89)] is a negative constant when $C_1 < 1$ and $\hat{C}_3 = 0$. In other words, the expansion of the Universe uniformly accelerates when $C_1 < 1$ and $\hat{C}_3 = 0$. Therefore, entropic-force models for the Bekenstein entropy cannot describe a decelerating and accelerating universe. As discussed in Sec. II C, a similar fact has been examined in Refs. [15,21,85], based on the standard entropic cosmology which includes H^2 and \dot{H} terms [Eqs. (15) and (16)].

2. Luminosity distance d_L

The luminosity distance is an important parameter for investigating the accelerated expansion of the Universe. Therefore, we examine the luminosity distance d_L of the single-fluid-dominated universe in the combined model. The luminosity distance [89] is generally given as

$$d_L(z) = \frac{c(1+z)}{H_0} \int_1^{1+z} \frac{dy}{F(y)}, \quad (91)$$

where the integrating variable y and the function $F(y)$ are given by

$$y = \frac{a_0}{a}, \quad (92)$$

and

$$F(y) = \frac{H}{H_0}, \quad (93)$$

and z is the redshift defined by

$$1 + z \equiv y = \frac{a_0}{a}. \quad (94)$$

For the combined model, substituting Eq. (77) into Eq. (93), and using $y = a_0/a$, we obtain $F(y)$ as

$$\begin{aligned} F(y) &= \frac{H}{H_0} = \left(1 - \frac{C_3}{C_1}\right) \left(\frac{a}{a_0}\right)^{-C_1} + \frac{C_3}{C_1} \\ &= \left(1 - \frac{C_3}{C_1}\right) y^{C_1} + \frac{C_3}{C_1}. \end{aligned} \quad (95)$$

Accordingly, substituting Eq. (95) into Eq. (91), we have

$$\left(\frac{H_0}{c}\right) d_L = (1+z) \int_1^{1+z} \frac{dy}{\left(1 - \frac{C_3}{C_1}\right) y^{C_1} + \frac{C_3}{C_1}}, \quad (96)$$

where $C_1 > 0$ is assumed. We can calculate the luminosity distance d_L from Eq. (96). As a typical example [15], the luminosity distance for $C_3 = 0$ is given by

$$\left(\frac{H_0}{c}\right) d_L = \begin{cases} \frac{1+z}{C_1-1} [1 - (1+z)^{-C_1+1}] & (C_1 \neq 1), \\ (1+z) \ln(1+z) & (C_1 = 1). \end{cases} \quad (97)$$

In this section, we obtain solutions for the combined model, assuming the single-fluid-dominated universe. The solutions can be widely used because the modified

Friedmann and acceleration equations simultaneously include both H^2 and H terms. In the next section, using the obtained solutions, we discuss two entropic-force models derived from the Bekenstein and generalized entropies.

In the present study, in order to examine entropic cosmology, we consider a spatially flat universe ($k = 0$), where k is a curvature constant. However, in a spatially nonflat universe ($k \neq 0$), the apparent horizon, $r_A = c/\sqrt{H^2 + (k/a^2)}$, does not coincide with the Hubble horizon, $r_H = c/H$, because of $k \neq 0$ [12]. Accordingly, we use the apparent horizon as the preferred screen rather than the Hubble horizon, when we consider the spatially nonflat universe. Of course, in the standard entropic cosmology, it was argued that the extrinsic curvature at the surface was likely to result in an expression such as $\alpha_1 = \beta_1 = \frac{3}{2\pi}$ and $\alpha_2 = \beta_2 = \frac{3}{4\pi}$ [13,14]. Similarly, the influence of the curvature may be included in four coefficients $\alpha_1, \beta_1, \alpha_3$, and β_3 . Note that in the present paper, we consider a spatially flat universe. (Bulk viscous cosmology in the spatially nonflat universe has been examined in detail. See, e.g., Refs. [47,64].)

V. EVOLUTION OF THE UNIVERSE IN THE TWO ENTROPIC-FORCE MODELS

As entropic-force terms, H^2 terms are derived from the Bekenstein entropy, whereas H terms are derived from a generalized entropy. Thus, we propose two entropic-force models derived from the Bekenstein and generalized entropies. We hereinafter refer to these models as the Bekenstein entropic-force model [Eqs. (35)–(38)] and the generalized entropic-force model [Eqs. (55)–(61)], respectively. In this section, we examine the evolution of the Universe in the two entropic-force models.

For the two entropic-force models, we consider the matter-dominated universe given by

$$w = 0. \quad (98)$$

In Sec. IV, we obtained solutions for the combined model using four dimensionless constants, i.e., $\alpha_1, \alpha_3, \beta_1$, and β_3 . In other words, by setting the four constants, we can obtain the solution of each model. The four constants for each model are listed in Table II.

For the Bekenstein entropic-force model, α_1, α_3 , and β_3 are 0, as described in Sec. III A 2 (and shown in Table II). Therefore, substituting these values and $w = 0$ into Eq. (69), we have $C_3 = 0$. On the other hand, β_1 is set to be $3/4$, in order to obtain $C_1 = 3/4$. This is because, as reported in Ref. [15], when we consider the equation $dH/dN = -C_1H$, the solution with $C_1 = 3/4$ was consistent with the observed supernova data. Here, $C_1 = 3/4$ was calculated from the Hawking temperature description [12], without using a fitting with the supernova data [15]. Of course, we accept that β_1 should be a free parameter. For example, C_1 is approximated as 0.780, if a fitting

TABLE II. Dimensionless constants for the two entropic-force models. The Bekenstein and Generalized columns indicate the information for the entropic-force models derived from the Bekenstein and generalized entropies, respectively. For details, see the text.

| Parameter | Bekenstein | Generalized |
|------------|------------|-------------|
| α_1 | 0 | 0 |
| α_3 | 0 | 0 |
| β_1 | 0.75(=3/4) | 0 |
| β_3 | 0 | 0.884 |
| C_1 | 0.75(=3/4) | 1.5(=3/2) |
| C_3 | 0 | 0.884 |

method discussed later herein is used to determine C_1 . The value of 0.780 is close to $3/4$, or, more specifically, to $\frac{3}{2}(1 - \frac{3}{2\pi}) = 0.7838 \dots$. In Ref. [15], $C_1 = \frac{3}{2}(1 - \frac{3}{2\pi})$ was calculated from the anticipated surface term order [12]. We have confirmed that the properties of the Universe for $C_1 = \frac{3}{2}(1 - \frac{3}{2\pi})$ or $C_1 = 0.780$ are similar to the properties for $C_1 = 3/4$. Accordingly, in the present paper, we select $3/4$ as C_1 for the Bekenstein entropic-force model.

In contrast, for the generalized entropic-force model, α_1, α_3 , and β_1 are 0, as examined in Sec. III B 2 (and shown in Table II). Substituting these values and $w = 0$ into Eq. (67), we obtain $C_1 = 3/2$. (In the standard cosmology, the universe for $C_1 = 3/2$ corresponds to the matter-dominated universe [7–9].) On the other hand, β_3 is a free parameter. Therefore, β_3 is determined through fitting with a fine-tuned standard Λ CDM model based on the Planck 2013 results [6]. To this end, we use the luminosity distance.

For the two entropic-force models, we calculate the luminosity distance d_L from Eqs. (96) and (97) and Table II. For the standard Λ CDM model, the luminosity distance of the spatially flat universe is given as

$$\left(\frac{H_0}{c}\right)d_L = (1+z) \int_0^z dz' [(1+z')^2(1+\Omega_m z') - z'(2+z')\Omega_\Lambda]^{-1/2}, \quad (99)$$

where $\Omega_m = \frac{\rho_m}{\rho_c} = \frac{8\pi G \rho_m}{3H_0^2}$ and $\Omega_\Lambda = \frac{\Lambda}{3H_0^2}$ [90]. Ω_m and Ω_Λ represent the density parameters for matter and Λ , respectively. Moreover, ρ_c represents the critical density, and ρ_m is the density of matter, which includes baryon and dark matter. The universe in which $(\Omega_m, \Omega_\Lambda) = (0.315, 0.685)$ is a fine-tuned standard Λ CDM model, which takes into account the recent Planck 2013 best fit values [6]. Note that we assume $\Omega_{\text{total}} = \Omega_m + \Omega_\Lambda = 1$ and neglect the density parameter Ω_r for the radiation [15,16].

In the present study, β_3 for the generalized entropic-force model is determined through fitting with the fine-tuned standard Λ CDM model. In fact, substituting $\alpha_3 = 0$ into Eq. (69), C_3 is equal to β_3 . Accordingly, instead of β_3 ,

we determine C_3 through fitting with the fine-tuned standard Λ CDM model, minimizing the following function [36] given by

$$\chi^2(C_1, C_3) = \sum_{i=0}^{N_z} \left[\frac{d_{L,\Lambda}(z) - d_L(z; C_1, C_3)}{d_{L,\Lambda}(z)} \right]^2, \quad (100)$$

where $d_{L,\Lambda}(z)$ and $d_L(z; C_1, C_3)$ are the luminosity distances for the fine-tuned standard Λ CDM model and the combined (entropic-force) model, respectively. For the generalized entropic-force model, C_1 is set to be 1.5. In order to determine C_3 , the redshift is varied from $z = 0$ to $z = 2$ in increments of $\Delta z = 0.01$. In other words, z is given by $z = i\Delta z$, where i is varied from 0 to $N_z = 200$. Through fitting, C_3 is approximately determined to be 0.884, as shown in Table II. Since $\beta_3 = C_3$, β_3 is set to be 0.884. (It may be possible to consider C_1 and C_3 as free parameters simultaneously. In this case, C_1 and C_3 are approximately 2.08 and 1.58, respectively. We do not discuss such conditions in the present paper.)

We now observe the luminosity distance d_L . As shown in Fig. 1, the two entropic-force models agree well with supernova data points and the fine-tuned standard Λ CDM model. For the generalized entropic-force model, $C_3 = 0.884$ is determined through fitting with the fine-tuned standard Λ CDM model, as mentioned previously. Therefore, it is not so surprising that the generalized entropic-force model agrees well with the standard Λ CDM model. For the Bekenstein entropic-force model, $C_1 = 0.75$ is selected, where the value of 0.75 was calculated based on the description of the Hawking

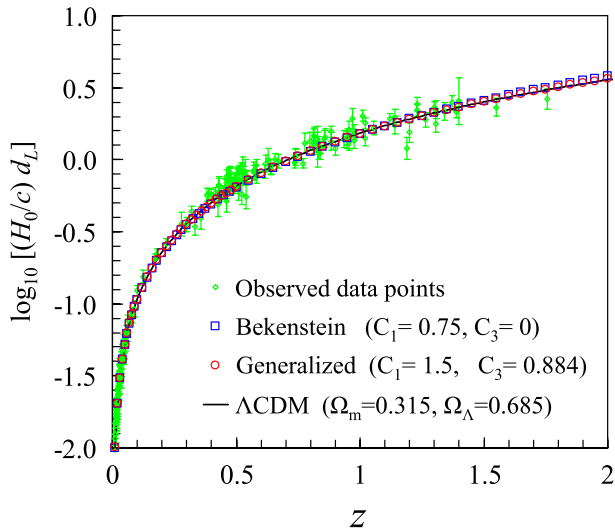


FIG. 1 (color online). Dependence of luminosity distance d_L on redshift z . Here, Bekenstein and Generalized indicate the information for the entropic-force models derived from the Bekenstein and generalized entropies, respectively. The open diamonds with error bars are supernova data points taken from Ref. [3]. For supernova data points, H_0 is set to be 67.3 km/s/Mpc based on the Planck 2013 results [6].

temperature [15]. However, it is demonstrated that the two entropic-force models can describe the present accelerating universe without adding the cosmological constant or dark energy.

As shown in Fig. 1, the difference between the two entropic-force models is not clear when we observe the luminosity distance at late times. Therefore, we examine the time evolutions of the normalized scale factor a/a_0 for longer time ranges. For the two entropic-force models, we calculate a/a_0 from Eqs. (85) and (86) and Table II. For the standard Λ CDM model, we use the following equation [7,8]:

$$H_0(t-t_0) = \int_1^{a/a_0} \frac{da'}{\sqrt{\frac{\Omega_r}{a'^2} + \frac{\Omega_m}{a'} + \Omega_\Lambda a'^2 + (1 - \Omega_{\text{total}})}}. \quad (101)$$

We integrate Eq. (101) numerically in order to obtain the time evolution of the normalized scale factor a/a_0 . In the present study, for a spatially flat universe, we assume $\Omega_{\text{total}} = \Omega_r + \Omega_m + \Omega_\Lambda = 1$ and neglect the influence of radiation, i.e., $\Omega_r = 0$ [15].

Figure 2 shows the time evolutions of the normalized scale factor a/a_0 . For the Bekenstein entropic-force model, a/a_0 increases uniformly, because a/a_0 is given by $(C_1 H_0 t)^{1/C_1}$ [Eq. (86)] when $C_1 \neq 0$ and $C_3 = 0$. In fact, a temporal deceleration parameter $q(t)$ is given by $-\ddot{a}/(aH^2) = C_1 - (\hat{C}_3/H) - 1$, as shown in Eq. (89). Accordingly, $q(t)$ is constant, when C_1 is constant and $\hat{C}_3 = C_3 = 0$. In contrast, for the generalized entropic-force model, $q(t)$ can vary when $\hat{C}_3 \neq 0$, even if C_1 is constant. Therefore, we can expect that a/a_0 for the generalized model does not increase uniformly, unlike that for the Bekenstein model. In order to observe this phenomenon, we focus on the generalized entropic-force model.

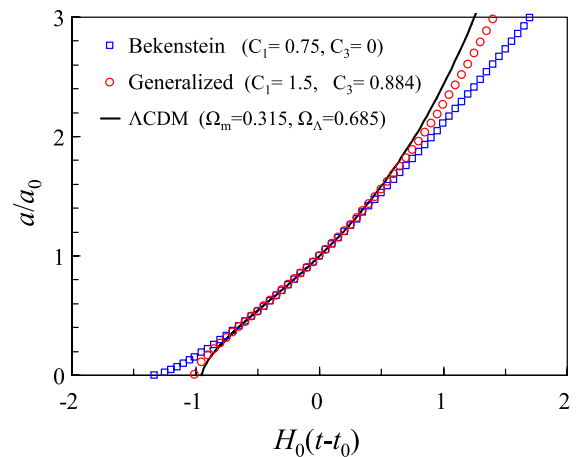


FIG. 2 (color online). Time evolution of normalized scale factor a/a_0 . The horizontal axis is normalized as $H_0(t-t_0)$. Here, Bekenstein and Generalized indicate the information for the entropic-force models derived from the Bekenstein and generalized entropies, respectively.

As shown in Fig. 2, the increase in a/a_0 tends to become gradually slower for $H_0(t - t_0) \leq -0.5$. However, for $H_0(t - t_0) \geq -0.5$, the increase in a/a_0 tends to become gradually faster. In other words, the generalized entropic-force model predicts a decelerated and accelerated expansion of the Universe, which is similar to a/a_0 for the fine-tuned standard Λ CDM model. Interestingly, the generalized entropic-force model can describe the decelerated and accelerated expansion, without adding a constant term such as a cosmological constant.

As discussed in Sec. II C, the standard entropic cosmology cannot describe a decelerating and accelerating universe [21,85]. However, the standard entropic cosmology can describe the decelerating and accelerating universe if the modified Friedmann and acceleration equations include extra constant terms, as shown in Ref. [21]. In the present study, we do not discuss such entropic-force models that include constant terms, because the origin of the constant terms is not clear. (In $\Lambda(t)$ CDM models, the constant terms are naturally obtained from an integral constant of the renormalization group equation for the vacuum energy density [86].) Of course, bulk viscous cosmology can describe the decelerated and accelerated expansion [64] because the modified acceleration equation includes the H term as an extra driving term. However, in the generalized entropic-force model, the H term is derived from the generalized entropy, without using a bulk viscosity of cosmological fluids.

We now examine a transition time t_{trans} [64], which represents the time of transition from the decelerated expansion epoch to the accelerated expansion epoch. Calculating the second derivative of Eq. (85), equating to 0, and solving the result, we obtain the transition time for the combined model as

$$H_0(t_{\text{trans}} - t_0) = \frac{1}{C_3} \ln(C_1 - C_3) \quad (C_1 > C_3 > 0). \quad (102)$$

Substituting $C_1 = 1.5$ and $C_3 = 0.884$ into Eq. (102), we have $H_0(t_{\text{trans}} - t_0) \approx -0.55$, for the generalized entropic-force model. Note that t_{trans} does not exist for the Bekenstein entropic-force model, because a/a_0 increases uniformly when $C_3 = 0$.

As shown in Fig. 2, the two entropic-force models are consistent with the fine-tuned standard Λ CDM model at the present time. In order to discuss this more closely, we examine the deceleration parameter q_0 . For example, we can estimate q_0 for the fine-tuned standard Λ CDM model as $q_0 \approx -0.53$, from $q_0 = (\Omega_m - 2\Omega_\Lambda + 2\Omega_r)/2$ [8], where $(\Omega_m, \Omega_\Lambda, \Omega_r)$ is set to be (0.315, 0.685, 0). For the two entropic-force models, q_0 is calculated from $q_0 = C_1 - C_3 - 1$, as shown in Eq. (90). Consequently, for the Bekenstein and generalized entropic-force models, we have $q_0 = -0.25$ and $q_0 \approx -0.38$, respectively. Therefore, at the present time, the acceleration for the

two entropic-force models is slightly slower than that for the fine-tuned standard Λ CDM model.

Finally, we observe the evolutions of the Bekenstein and generalized entropies. In order to calculate the entropy, $S = K/H^2$ [Eq. (23)] is used for the Bekenstein entropic-force model, whereas $S = Kc\zeta/H^3$ [Eq. (48)] is used for the generalized entropic-force model. In the following, the two entropies are normalized as $(H_0^2/K)S$ and $(H_0^3/(Kc\zeta))S$, respectively. We numerically calculate $(H/H_0)^{-2}$ and $(H/H_0)^{-3}$ of the fine-tuned standard Λ CDM model, in order to compare the two entropic-force models with the Λ CDM model. This is because $(H/H_0)^{-2}$ and $(H/H_0)^{-3}$ correspond to $(H_0^2/K)S$ and $(H_0^3/(Kc\zeta))S$, respectively.

Figure 3 shows the evolutions of the Bekenstein and generalized entropies. We first focus on the entropies for the fine-tuned standard Λ CDM model. For $a/a_0 \leq 1$, the entropies for the standard Λ CDM model increase rapidly, whereas, for $a/a_0 \geq 1$, the increase in the entropies tends to become gradually slower. Since the standard Λ CDM model has been widely examined, similar results have been reported [91–97]. We now examine the two entropic-force models. For $a/a_0 \leq 1$, the entropy for the Bekenstein entropic-force model is consistent with $(H_0^2/K)S$ for the standard Λ CDM model. Similarly, for $a/a_0 \leq 1$, the entropy for the generalized entropic-force model agrees well with $(H_0^3/(Kc\zeta))S$ for the Λ CDM model. This indicates that the evolutions of the late universe for the two entropic-force models are consistent with the evolution for the Λ CDM model. We have confirmed this from Figs. 1 and 2 as well. However, for $a/a_0 \geq 1$, the entropy for the Bekenstein entropic-force

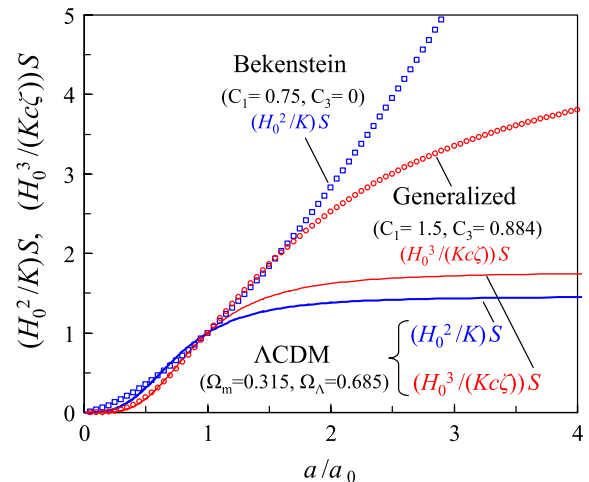


FIG. 3 (color online). Evolutions of the Bekenstein and generalized entropies. The vertical axis represents $(H_0^2/K)S$ and $(H_0^3/(Kc\zeta))S$ for the Bekenstein and generalized entropic-force models, respectively. The solid lines represent $(H_0^2/K)S$ and $(H_0^3/(Kc\zeta))S$ for the fine-tuned standard Λ CDM model and are numerically calculated from $(H/H_0)^{-2}$ and $(H/H_0)^{-3}$.

model increases uniformly, whereas the increase in the entropy for the generalized entropic-force model tends to become gradually slower [Fig. 3]. Therefore, for $a/a_0 \gtrsim 1$, the entropy for the Bekenstein entropic-force model is clearly different from $(H_0^2/K)S$ for the standard Λ CDM model. On the other hand, for $a/a_0 \gtrsim 1$, the entropy for the generalized entropic-force model increases more rapidly than $(H_0^3/(Kc\zeta))S$ for the Λ CDM model. However, the increase in the entropy for the generalized entropic-force model tends to become gradually slower. Accordingly, the evolution of the entropy for the generalized entropic-force model is similar to the evolution for the Λ CDM model. This is because the generalized entropic-force model can describe a decelerating and accelerating universe.

VI. DISCUSSION AND CONCLUSIONS

In order to examine entropic cosmology, we have derived extra entropic-force terms not only from the Bekenstein entropy but also from a generalized entropy, assuming that the horizon of the Universe has an entropy and a temperature. Because of nonadditive generalizations, the generalized entropy is proportional to volume, unlike for the Bekenstein entropy. Consequently, as entropic-force terms, H terms are derived from the generalized entropy, whereas H^2 terms are derived from the Bekenstein entropy.

Interestingly, the H term is similar to an extra driving term for bulk viscous cosmology. This similarity may be interpreted as a sign that the generalized entropy behaves as if it were a classical entropy generated by bulk viscous stresses. This is probably because the generalized entropy considered here is proportional not to area but rather to volume. However, as one interpretation, the generalized entropy may be related to a bulk viscosity of cosmological fluids, through a holographic screen. This interpretation does not contradict the following description; i.e., the bulk viscosity is considered to be the only thing that can generate an entropy in the homogeneous and isotropic universe. In the present study, we assume an entropy on the horizon of the Universe, using the holographic principle. However, if we assume another entropy that is proportional to the volume of the Universe, its entropic force can likely explain the accelerated expansion of the Universe. In other words, the concept of the entropic force may be applied to the bulk of the Universe, without using the holographic principle.

In the present paper, based on effective pressure, we have formulated the modified Friedmann, acceleration, and continuity equations for two entropic-force models derived from the Bekenstein and generalized entropies. The Friedmann equation is shown not to include the entropic-force term, whereas the continuity equation has a nonzero right-hand side. In order to examine the properties of the

Universe in the two entropic-force models, we have considered a combined model, in which the Friedmann and acceleration equations include both H^2 and H terms simultaneously as the entropic-force terms. We have obtained solutions of the combined model, assuming a homogeneous, isotropic, spatially flat universe, focusing on the single-fluid-dominated universe.

We have confirmed that the two entropic-force models can describe the present accelerating universe, without adding the cosmological constant or dark energy. The Bekenstein entropic-force model is found to predict a uniformly accelerating universe, whereas the generalized entropic-force model predicts a decelerating and accelerating universe, as in the case for a fine-tuned standard Λ CDM model. Similar properties of the universe with the extra driving terms, i.e., H^2 and H terms, have been examined and discussed in not only bulk viscous cosmology but also $\Lambda(t)$ CDM models with a variable cosmological term $\Lambda(t)$. However, in the present study, the extra driving terms are derived from the Bekenstein and generalized entropies, without using a bulk viscosity of cosmological fluids and dark energy.

The present study has revealed fundamental properties of the expanding universe in entropic cosmologies based on the Bekenstein and generalized entropies. In particular, the entropic force derived from the generalized entropy will help in discussing the accelerating universe in other cosmological models. For example, when an entropy is assumed to be proportional to r_H^4 , the obtained entropic-force term is a constant, as if it were a cosmological constant. Although such an entropy has probably not yet been suggested, we can discuss cosmological models from various entropic-force viewpoints. In the present study, in order to examine entropic cosmology, we assume an entropy on the horizon of the Universe. At least in principle, it is possible to assume an entropy that is proportional to the volume of the Universe in order to discuss entropic-force models.

In the present study, we have not examined cosmological fluctuations, since we have focused on background evolutions of the late universe. However, through linear density perturbations, it has been discussed that bulk viscous models are difficult to reconcile with astronomical observations of structure formations [60]. In addition, as shown in Ref. [82], $\Lambda(t)$ CDM models similar to the combined model are not consistent with the structure formation data. (The structure formation in $\Lambda(t)$ CDM models has been examined in detail [83].) The previous works imply that the present entropic-force models are difficult to reconcile with the structure formation data. We expect that another type of entropy, e.g., $S \propto r_H^4$, helps to overcome the difficulty. This task is left for future research.

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