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Theory of baryon resonances at large N_c

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At large number of colors, N_c quarks in baryons are in a mean field of definite space and flavor symmetry. We write down the general Lorentz and flavor structure of the mean field, and derive the Dirac equation for quarks in that field. The resulting baryon resonances exhibit a hierarchy of scales: the crude mass is $\mathcal{O}(N_c)$, the intrinsic quark excitations are $\mathcal{O}(1)$, and each intrinsic quark state entails a finite band of collective excitations that are split as $\mathcal{O}(1/N_c)$. We build a (new) theory of these collective excitations, where the full dynamics is represented by only a few constants. In a limiting (but unrealistic) case when the mean field is spherically and flavor symmetric, our classification of resonances reduces to the SU(6) classification of the old nonrelativistic quark model. Although in the real world N_c is only three, we obtain a good accordance with the observed resonance spectrum up to 2 GeV.

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I. INTRODUCTION

There are currently two main classes of baryon models: three-quark models with certain interactions between quarks, and models where baryons are treated as nonlinear solitons of bosonic fields, such that quarks are implicit. Both points of view have their strong and weak points. The main shortcoming of the three-quark models is that they ignore the phenomenologically important admixture of quark-antiquark pairs ($Q\bar{Q}$) in a baryon, and therefore they are essentially nonrelativistic. A consistent relativistic picture can be only field theoretic.

This is the advantage of the solitonic approach that stems from the Skyrme model [1,2] and includes its more recent incarnation in the holographic QCD [3,4]. Models of this kind effectively take care of $Q\bar{Q}$ pairs in baryons [5]. However, an obvious shortcoming is that they do not possess explicit quarks, and therefore it is difficult to address many important issues such as, for example, the quark and antiquark (parton) distributions in nucleons.

In this paper, we suggest an approach that is a bridge between the two models. On the one hand, we operate in terms of quarks and keep contact with the traditional three-quark models. On the other hand, we do not restrict ourselves to the valence quarks only but allow for an arbitrary amount of additional $Q\bar{Q}$ pairs. Our formalism is relativistically invariant. The key ingredient of our construction is the mean bosonic field for quarks, which, in fact, is nothing but the ''soliton'' of the second approach.

As in any other solitonic picture of baryons, we formally need to consider the number of colors N_c as a large algebraic parameter. When N_c is large the N_c quarks constituting a baryon can be considered in a mean (nonfluctuating) field that does not change as $N_c \to \infty$ [6]. Quantum fluctuations about a mean field are suppressed as $1/N_c$. While in the real world N_c is only three, we do not expect qualitative differences in the baryon spectrum from its large-N_c limit. The hope is that if one develops a clear picture at large N_c, and controls at least in principle $1/N_c$ corrections, its imprint will be visible at $N_c = 3$.

From the large-N_c viewpoint, baryons have been much studied in the past using general N_c counting rules and group-theoretic arguments; for reviews see Refs. [7–9] and references therein. In this framework, many relations for baryon resonances have been derived, with no reference to the underlying dynamics. The key new point of this paper is that we suggest a simple underlying physical picture that results in these relations and discloses the meaning of the otherwise free numerical coefficients therein. We also derive new relations that are valid in the large-N_c limit.

This work is in line with our previous chiral quark–soliton model [10], which successfully describes quark and antiquark distributions in nucleons [11] and other properties. In this paper we remove the previous limitation that the mean field is exclusively the pseudoscalar one, and focus on baryon resonances rather than on the ground state.

The advantage of the large-N_c approach is that at large N_c baryon physics simplifies considerably, which enables one to take into full account the important relativistic and field-theoretic effects that are often ignored. Baryons are not just three (or N_c) quarks but contain additional $Q\bar{Q}$ pairs, as is well known experimentally. The number of antiquarks in baryons is, theoretically, also proportional to N_c [5], which means that antiquarks cannot be obtained from adding one meson to a baryon: one needs $\mathcal{O}(N_c)$ mesons to explain $\mathcal{O}(N_c)$ antiquarks, implying in fact a classical mesonic field.

At the microscopic level quarks experience only color interactions, but gluon-field fluctuations are not suppressed if N_c is large; the mean field can be only ''colorless.'' An

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example how color interactions were originally Fierz-transformed into interactions of quarks with mesonic fields is provided by the instanton liquid model [12]. A non-fluctuating confining bag is another example of a "colorless" mean field. A more modern example of a mean field is given by the five- or six-dimensional "gravitational" and flavor background field in the holographic QCD models.

Since quarks inside baryons are generally relativistic, especially in excited baryons, we shall assume that quarks in the large-N_c baryon obey the Dirac equation in a background mesonic field. In fact, the Dirac equation for quarks may be nonlocal. All intrinsic quark Dirac levels in the mean field are stable in N_c. All negative-energy levels should be filled in by N_c quarks in the antisymmetric state in color, corresponding to the zero baryon-number state. Filling in the lowest positive-energy level by N_c "valence" quarks makes a baryon. Exciting higher quark levels or making particle-hole excitations produces baryon resonances. The baryon mass is $\mathcal{O}(N_c)$, and the excitation energy is $\mathcal{O}(1)$. When one excites one quark the change of the mean field is $\mathcal{O}(1/N_c)$, which can be neglected to the first approximation.

Moreover, if one replaces one light (u, d, or s) quark in light baryons by a heavy (c, b) one, as in charmed or bottom baryons, the change in the mean field is also $\mathcal{O}(1/N_c)$. Therefore, the spectrum of heavy baryons is directly related to that of light baryons. This fact is well known for low-lying multiples (see, e.g., Ref. [13]), and has been recently discussed for a more general situation in Ref. [14].

Our approach can be illustrated by the chiral quark–soliton model [10,15] or by the chiral bag model [16], but actually the arguments of this paper are much more general. We argue that the mean field in baryons of whatever nature has a definite symmetry, namely that it spontaneously breaks the symmetry under separate $\mathrm{SU}(3)_{\mathrm{flavor}}$ and $\mathrm{SO}(3)_{\mathrm{space}}$ rotations but does not change under simultaneous $\mathrm{SU}(2)_{\mathrm{iso+space}}$ rotations in ordinary space and a compensating rotation in isospace [14,17].

If the original symmetry (i.e., flavor and rotational symmetry) is spontaneously broken, it means that the ground state is degenerate: all states obtained by a rotation have the same energy. In quantum mechanics the rotations must be quantized, which leads to the splitting of the ground state—as well as all one-quark excitations—by the quantized rotational energy. It implies that each intrinsic quark state, be it the ground state or a one-quark excitation in the Dirac spectrum, generates a band of resonances appearing as collective rotational excitations of a given intrinsic state. The quantum numbers of these resonances, their total number, and their splittings are unequivocally dictated by the symmetry of the mean field. In this paper we present, for the first time, the theory of the rotational bands stemming from a given intrinsic one-quark excitation. Assuming the $\mathrm{SU}(2)_{\mathrm{iso+space}}$ symmetry of the mean field, we obtain the resonances observed in nature. Moreover, certain relations between resonance splittings that are satisfied with high accuracy are dynamics independent but follow solely from the particular symmetry of the mean field.

In this paper, we do not consider any specific dynamical model but concentrate mainly on symmetry. A concrete dynamical model would express the intrinsic relativistic quark spectrum in baryons. It may get it approximately correct, or altogether wrong. Instead of calculating the intrinsic Dirac spectrum of quarks from a model, we extract it from the experimentally known baryon spectrum by interpreting baryon resonances as collective excitations about the state and about the one-quark transitions. However, we show that the needed intrinsic quark spectrum can be obtained from a natural choice of the mean field satisfying the $\mathrm{SU}(2)_{\mathrm{iso+space}}$ symmetry.

In summary, we show that it is possible to obtain a realistic spectrum of baryon resonances up to 2 GeV, starting from the large-N_c limit. This means that we are able to find candidates for all rotational SU(3) multiplets generated around intrinsic quark levels and to check large-N_c relations between their masses. However, this is not the end of the story, as SU(3) multiplets in nature are split due to the nonzero mass of the strange quark. These splittings are not small and not all members of SU(3) mutliplets are known. For this reason even the contents of the lowest SU(3) multiplets is under discussion. We will follow the analysis of Ref. [18] (see also Ref. [19]).

We will assume that the mass of the strange quark is small enough and construct perturbation theory in m_s valid at large N_c. The question of its validity is under discussion as well (see, e.g., Ref. [20]), but we consider the success of relations following from this theory (classical Gell-Mann-Okubo or Guadagnini ones [21]) as an argument in favor of this approach. We derive a number of new relations valid at large N_c which are fulfilled with a good accuracy. We note that our approach gives essentially more relations than were derived in the framework of the approach used in Ref. [22]. (For mass relations and other aspects of broken SU(3) symmetry, see Ref. [23].) Also, we give the dynamical interpretation of the constants entering mass splittings and provide the formulas which allow one to calculate splittings provided that the underlying dynamical model is fixed.

The notion of baryon resonance implies that its width is small. For excited baryons at large N_c this is not granted: the width of the baryon is $O(1)$. The width is small compared to the total mass of the baryon [$O(N_c)$], but it appears to be of order the distance between quark levels. This width is due to transitions between different quark levels with the emission of mesons (e.g., one pion decay to the ground level), which is not suppressed by N_c. We will show below that in the leading order the width is universal for all rotational bands belonging to a given quark level. A nonzero width for the resonance also leads to some shift in its position. In spite of the fact that this shift is $O(1)$, it is also universal for rotational bands and does not ruin rotational spectra.

The mean-field approximation cannot be applied directly to unstable quark levels. The correct definition of the baryon resonance comes from the consideration of the meson-baryon scattering amplitude. The baryon resonance manifests itself as a pole in the complex plane of the energy with the imaginary part being half of the resonance width. The scattering amplitude at large N_c can be found from the meson quadratic form, which can be obtained by integrating out quark degrees of freedom.

We performed this program for exotic pentaquark states in Ref. [5] in the framework of the Skyrme model, but it looks to be too complicated for the general case of baryons considered in this paper. We will use the fact that only resonances with relatively small widths can be observed and neglect the widths of resonances in order to describe their positions. Moreover, as we do not consider any dynamical mechanism, the positions of quark levels play the role of phenomenological parameters. At this point our approach is close to that of the quark model which also neglects the influence of the resonance's width on its position.

The widths of baryon resonances also have some hierarchy in N_c. Decays with a transition from one quark level to another are $O(1)$, while decays inside the same rotational band are $O(1/N_c^2)$. In particular, the total widths of the baryons belonging to the rotational band of the ground state (like the Δ resonance) are only $O(1/N_c^2)$, while the total widths of all excited baryons are $O(1)$. These widths are the same for all excited baryons belonging to the given band up to corrections of $O(1/N_c)$, which can nevertheless be significant at $N_c = 3$ (to say nothing about corrections to the mass of the strange quark).

One can try two approaches for adjusting large-N_c limit results to the real $N_c = 3$ world. The calculation of physical quantities can be divided typically into two stages: translating the original quantity to some effective rotational operator, and calculating the matrix element of the effective operator with wave functions of rotational states representing the given baryon. The first stage requires the limit of large N_c in order to avoid the mess of strong interactions. The second is, in fact, trivial and leads to some SU(3) Clebsch-Gordan coefficients, which are calculable at any N_c. The approach pioneered in Ref. [22] requires the strict limit $N_c \to \infty$ at both stages. On the other hand, in Ref. [10] and subsequent papers we applied another approach: we used the limit $N_c \to \infty$ but replaced the Clebsch-Gordan coefficients by their values at $N_c = 3$. The same logic was, in fact, also used in the original paper [2]. This approach has at least the same accuracy as the first one but allows one to avoid large corrections related to the change of the Clebsch-Gordan coefficients from $N_c = \infty$ to $N_c = 3$. In this paper we will discuss both approaches, but we mainly use the second one.

The paper is organized as follows. In Sec. II we discuss the possible symmetry of the mean field and come to the conclusion that it should be a hedgehog one. This is one of the main features distinguishing our model from the quark model (which can also be considered at $N_c \to \infty$). The quark model assumes that the mean field inside the baryon has central symmetry. (In a majority of the versions it is just the confinement potential.) We believe that this assumption contradicts the data and that the mean meson field (e.g., mean field of the pion) is at least equally important.

In Sec III we derive Dirac equations in the general hedgehog meson field. One has to find intrinsic quark levels in this mean field. To determine the self-consistent meson field, it is necessary to also know the meson part of the Lagrangian. This can be done in the concrete dynamical model. We give the classification of quark levels and discuss the possible order of levels in the mean field.

In Sec. IV we construct the general theory of the rotational state around intrinsic quark levels. Previously this theory was discussed for the ground-state baryons; we extend it to arbitrary excited baryon states. We derive formulas for baryon masses and obtain the contents of the SU(3) multiplets entering the rotational bands. We also discuss their rotational and quark wave functions.

Section V considers the relation between the SU(3) multiplets at $N_c \to \infty$ and SU(6) multiplets of the quark model. We explain that there is a one-to-one correspondence between the quark model and one-quark excitations in the mean field at $N_c \to \infty$ for negative-parity baryons. This is not true for positive parity: SU(6) multiplets of the quark model correspond mainly to two-quark excitations. Meanwhile, one-particle excitations still exist and have the same structure as in the sector with negative parity. We prefer to use excitations of this type in order to describe experimental data as they should have smaller masses and be narrower than two-particle ones. We leave the quark-model picture for the positive-parity sector and arrive at a unified description for both parities: in order to describe experimental baryon spectra we need six levels with grand spin $K = 0^{\pm}, 1^{\pm}, 2^{\pm}$. We confront this simple picture of the data in Sec. VI and see that it can accommodate the experimental baryon spectra up to 2 GeV. At the same time it does not predict extra states, which are typical for the quark model. Section VI is devoted to the mass splitting inside SU(3) multiplets. This question can be important for identifying original SU(3) multiplets. We concentrate mainly on general relations that are model independent. We formulate our conclusions in Sec. VII.

We relegate a few important questions to a series of appendices. In Appendix A the simple exactly solvable model is considered. This model was already investigated in a number of papers; it helps us to illustrate the relations obtained in the main text at $N_c \to \infty$. Appendix B is related to the validity of the cranking approximation in the soliton picture; this validity was doubt in some papers. We discuss decays of baryon resonances in Appendix D. The full theory will be published elsewhere. Here we only give N_c counting of the baryon widths due to the different

decays and prove the universality of the width in the leading order for the given rotational band. Appendices C and E are devoted to some technical questions. In particular, in Appendix E we give the table of SU(3) Clebsch-Gordan coefficients conforming at $N_c = 3$ to the standard conventions.

II. SYMMETRY OF THE MEAN FIELD

In the mean-field approximation, justified at large N_c, one looks for the solutions of the Dirac equation for single-quark states in the background mean field. In a most general case the background field couples to quarks through all five Fermi variants. If the mean field is stationary in time, it leads to the Dirac eigenvalue equation for the u, d, s quarks in the background field, $H\psi = E\psi$, with the Dirac Hamiltonian being schematically described as

$$H = \gamma^0\Big(-i\partial_i\gamma^i + S(\mathbf{x}) + P(\mathbf{x})i\gamma^5 + V_\mu(\mathbf{x})\gamma^\mu + A_\mu(\mathbf{x})\gamma^\mu\gamma^5 + T_{\mu\nu}(\mathbf{x})\frac{i}{2}[\gamma^\mu\gamma^\nu]\Big), \tag{1}$$

where S, P, V, A, T are the scalar, pseudoscalar, vector, axial, and tensor mean fields, respectively; all are matrices in flavor. In fact, the one-particle Dirac Hamiltonian (1) is generally nonlocal; however, this does not destroy symmetries in which we are primarily interested. We include the current and the dynamically generated quark masses in the scalar term S.

The key issue is the symmetry of the mean field. We assume the chiral limit for u, d quarks, $m_u = m_d = 0$, which is an excellent approximation. We consider exact SU(3) flavor symmetry as a good starting point. It implies that baryons appear in degenerate SU(3) multiplets $\mathbf{8}, \mathbf{10}, \ldots$: the splittings inside SU(3) multiplets can be determined later on as a perturbation in m_s; see, e.g., Ref. [24] and Sec. VII.

A natural assumption, then, would be that the mean field is flavor symmetric and spherically symmetric. However, we know that baryons are strongly coupled to pseudoscalar mesons ($g_{\pi NN} \approx 13$). This means that there is a large pseudoscalar field inside baryons; at large N_c it is a classical mean field. There is no way of writing down the pseudoscalar field (it must change sign under inversion of coordinates) that would be compatible with the $\mathrm{SU(3)_{flav}} \times \mathrm{SO(3)_{space}}$ symmetry. The minimal extension of spherical symmetry is to write the "hedgehog" ansatz "marrying" the isotopic and space axes [25],

$$\pi^a(\mathbf{x}) = \begin{cases} n^a F(r),\ n^a = \frac{x^a}{r}, & a = 1, 2, 3, \\ 0, & a = 4, 5, 6, 7, 8. \end{cases} \tag{2}$$

This ansatz breaks the $\mathrm{SU(3)_{flav}}$ symmetry. Moreover, it breaks the symmetry under independent space $\mathrm{SO(3)_{space}}$ and isospin $\mathrm{SU(2)_{iso}}$ rotations, and only a simultaneous rotation in both spaces remains a symmetry, since a rotation in the isospin space labeled by a can be compensated by the rotation of the space axes. The ansatz (2) implies a spontaneous (as opposed to explicit) breaking of the original $\mathrm{SU(3)_{flav}} \times \mathrm{SO(3)_{space}}$ symmetry down to the $\mathrm{SU(2)_{iso+space}}$ symmetry. It is analogous to the spontaneous breaking of spherical symmetry by the ellipsoid form of many nuclei; there are many other examples in physics where the original symmetry is spontaneously broken in the ground state.

We list here all possible structures in the S, P, V, A, T fields, compatible with the $\mathrm{SU(2)_{iso+space}}$ symmetry and with the C, P, T quantum numbers of the fields [14,17]. The fields below are generalizations of the "hedgehog" ansatz (2) to mesonic fields with other quantum numbers.

Since SU(3) symmetry is broken, all fields can be divided into three categories:

(I) *Isovector fields acting on u, d quarks:*

$$\begin{aligned} \text{pseudoscalar: } & P^a(\mathbf{x}) = n^a P_0(r), \\ \text{vector, space part: } & V_i^a(\mathbf{x}) = \epsilon_{aik} n_k P_1(r), \\ \text{axial, space part: } & A_i^a(\mathbf{x}) = \delta_{ai} P_2(r) + n_a n_i P_3(r), \\ \text{tensor, space part: } & T_{ij}^a(\mathbf{x}) = \epsilon_{aij} P_4(r) + \epsilon_{bij} n_a n_b P_5(r). \end{aligned} \tag{3}$$

(II) *Isoscalar fields acting on u, d quarks:*

$$\begin{aligned} \text{scalar: } & S(\mathbf{x}) = Q_0(r), \\ \text{vector, time component: } & V_0(\mathbf{x}) = Q_1(r), \\ \text{tensor, mixed components: } & T_{0i}(\mathbf{x}) = n_i Q_2(r). \end{aligned} \tag{4}$$

(III) *Isoscalar fields acting on s quarks:*

$$\begin{aligned} \text{scalar: } & S(\mathbf{x}) = R_0(r), \\ \text{vector, time component: } & V_0(\mathbf{x}) = R_1(r), \\ \text{tensor, mixed components: } & T_{0i}(\mathbf{x}) = n_i R_2(r). \end{aligned} \tag{5}$$

The remaining fields and components are zero as they do not satisfy the $\mathrm{SU(2)_{iso+space}}$ symmetry and/or the needed discrete C, P, T symmetries. The 12 "profile" functions $P_{0,1,2,3,4,5}$, $Q_{0,1,2}$, and $R_{0,1,2}$ should eventually be found self-consistently from the minimization of the mass of the ground-state baryon. We shall call Eqs. (3)–(5) the hedgehog ansatz. However, even if we do not know these profiles, there are important consequences of this ansatz for the baryon spectrum.

III. u, d, s QUARKS IN THE "HEDGEHOG" FIELD

Given the $\mathrm{SU(2)_{iso+space}}$ symmetry of the mean field, the Dirac Hamiltonian for quarks actually splits into two parts: one for s quarks and the other for u and d quarks [17]. It should be stressed that the energy levels for u, d quarks on

the one hand and for s quarks on the other are completely different, even in the chiral limit $m_s \to 0$.

The energy levels for s quarks are classified by half-integer values J^P, where P is parity under space inversion, and $\mathbf{J} = \mathbf{L} + \mathbf{S}$ is the quark angular momentum; all levels are $(2J+1)$-fold degenerate. The energy levels for u and d quarks are classified by integer values K^P, where $\mathbf{K} = \mathbf{T} + \mathbf{J}$ is the "grand spin" (T is isospin) and all levels are $(2K+1)$-fold degenerate.

All energy levels, both positive and negative, are probably discrete owing to confinement. Indeed, a continuous spectrum would correspond to a situation where quarks are free at large distances from the center, which contradicts confinement. One can model confinement, e.g., by forcing the effective quark masses to grow linearly at infinity, $S(\mathbf{x}) \to \sigma r$.

The Dirac equation (1) for s quarks in the background field (5) takes the form of a system of two ordinary differential equations for two functions $f(r)$, $g(r)$ depending only on the distance from the center. The system of equations depends on the (half-integer) angular momentum and parity of the level under consideration. For s-quark levels with parity $P = (-1)^{J-\frac{1}{2}}$, e.g., for the levels $J^P = \frac{1}{2}^+, \frac{3}{2}^-, \frac{5}{2}^+, \ldots$, the system takes the form

$$\begin{cases} Ef = -g' - \frac{J+\frac{3}{2}}{r} g + R_0 f + R_1 f + R_2 g, \\ Eg = f' + \frac{-J+\frac{1}{2}}{r} f - R_0 g + R_1 g + R_2 f. \end{cases} \tag{6}$$

To find an s-quark energy level E with these quantum numbers, one has to solve Eq. (6) with the initial condition $f(r) \sim r^{J-\frac{1}{2}}$, $g(r) \sim r^{J+\frac{1}{2}}$ and with both functions decreasing at infinity.

For levels with opposite parity $P = (-1)^{J+\frac{1}{2}}$, e.g., $J^P = \frac{1}{2}^-, \frac{3}{2}^+, \frac{5}{2}^-, \ldots$, one has to solve another system,

$$\begin{cases} Ef = -g' - \frac{J+\frac{1}{2}}{r} g + R_0 f + R_1 f + R_2 g, \\ Eg = f' + \frac{-J+\frac{3}{2}}{r} f - R_0 g + R_1 g + R_2 f. \end{cases} \tag{7}$$

We note that in the absence of the $R_{1,2}$ fields the energy spectrum is symmetric under simultaneous sign changes of parity and energy.

The Dirac equation for u, d quarks in the background fields (3) and (4) is more complicated: one has here a system of four ordinary differential equations. These equations are direct generalizations of the Dirac equations in the "hedgehog" field of Ref. [10] and can be derived similarly to the method used in that reference.

The system of Dirac equations for the radial functions of the states with parity $(-1)^{K+1}$, namely $K^P = 1^+, 2^-, \ldots$, has the form

$$\begin{aligned} Ef = &-g' - \frac{1+K}{r} g + (Q_0 + Q_1 + P_2 + P_4) f + (Q_2 - P_1) g \\ &- \frac{P_0 - P_1}{2K+1} (g + b_K h) + \frac{P_3 + P_5}{2K+1} (f + b_K j), \end{aligned} \tag{8}$$

$$\begin{aligned} Eg = &f' - \frac{K-1}{r} f + (Q_1 - Q_0 - P_2 + P_4) g + (Q_2 - P_1) f \\ &- \frac{P_0 - P_1}{2K+1} (f + b_K j) + \frac{P_3 - P_5 + 2P_2 - 2P_4}{2K+1} (g + b_K h), \end{aligned} \tag{9}$$

$$\begin{aligned} Eh = &j' + \frac{2+K}{r} j + (Q_1 - Q_0 - P_2 + P_4) h \\ &+ (Q_2 - P_1) j + \frac{P_0 - P_1}{2K+1} (j - b_K f) \\ &- \frac{P_3 - P_5 + 2P_2 - 2P_4}{2K+1} (h - b_K g), \end{aligned} \tag{10}$$

$$\begin{aligned} Ej = &-h' + \frac{K}{r} h + (Q_0 + Q_1 + P_2 + P_4) j + (Q_2 - P_1) h \\ &+ \frac{P_0 - P_1}{2K+1} (h - b_K g) - \frac{P_3 + P_5}{2K+1} (j - b_K f), \end{aligned} \tag{11}$$

where $b_K = 2\sqrt{K(K+1)}$. The radial functions f, g, h, j refer to partial waves with $L = K-1$, K, K, $K+1$, respectively, and they behave at the origin as r^L. To find the energy levels for a given K^P, one has to solve these equations twice: once with the initial condition $f(r_{\min}) \sim r_{\min}^{K-1}$ with all the other functions set to zero at the origin, and another time with the initial condition $h(r_{\min}) \sim r_{\min}^{K}$ with all the other functions set to zero, $r_{\min} \to 0$. By numerically evolving the functions according to the equations up to some asymptotically large $r_{\max}$, one finds two sets of functions (f_1, g_1, h_1, j_1) and (f_2, g_2, h_2, j_2). The energy levels are found from the zeroes of two (equal) determinants, $f_1 h_2 - f_2 h_1 = g_1 j_2 - g_2 j_1$.

For states with parity $(-1)^K$, namely $K^P = 1^-, 2^+, \ldots$, the system of Dirac equations is

$$\begin{aligned} Ef = &-g' - \frac{1+K}{r} g + (Q_1 - Q_0 + P_2 - P_4) f \\ &- (Q_2 + P_1) g + \frac{P_0 + P_1}{2K+1} (g + b_K h) \\ &+ \frac{P_3 - P_5}{2K+1} (f + b_K j), \end{aligned} \tag{12}$$

$$\begin{aligned} Eg = &f' - \frac{K-1}{r} f + (Q_0 + Q_1 - P_2 - P_4) g \\ &- (Q_2 + P_1) f + \frac{P_0 + P_1}{2K+1} (f + b_K j) \\ &+ \frac{P_3 + P_5 + 2P_2 + 2P_4}{2K+1} (g + b_K h), \end{aligned} \tag{13}$$

$$\begin{aligned} Eh = &j' + \frac{2+K}{r} j + (Q_0 + Q_1 - P_2 - P_4) h \\ &- (Q_2 + P_1) j - \frac{P_0 + P_1}{2K+1} (j - b_K f) \\ &- \frac{P_3 + P_5 + 2P_2 + 2P_4}{2K+1} (h - b_K g), \end{aligned} \tag{14}$$

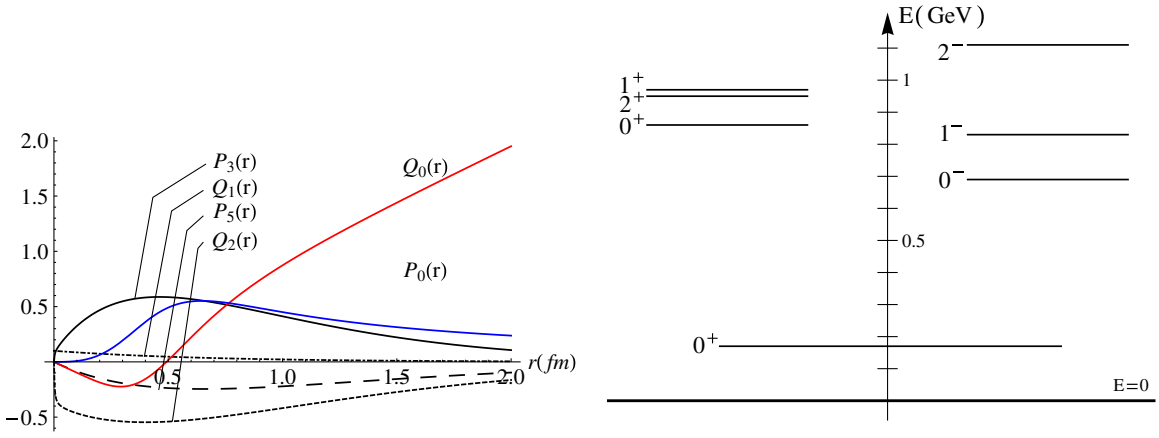

FIG. 1 (color online). An illustrative example of intrinsic quark levels with quantum numbers K^P (right) generated by the mean fields shown in the left panel.

$$Ej = -h' + \frac{K}{r}h + (Q_1 - Q_0 + P_2 - P_4)j - (Q_2 + P_1)h - \frac{P_0 + P_1}{2K+1}(h - b_K g) - \frac{P_3 - P_5}{2K+1}(j - b_K f), \quad (15)$$

where again $f \sim r^{K-1}$, $g \sim r^K$, $h \sim r^K$, $j \sim r^{K+1}$, and the levels are found by the same trick. The fields $Q_{1,2}$ and $P_{0,2,3}$ break symmetry with respect to simultaneous sign changes of parity and energy.

The case $K = 0$ is special since the angular momentum is restricted to only one value, $J = K + \frac{1}{2} = \frac{1}{2}$. This means that $g = f = 0$, and the system of equations (8)–(11) for the $K^P = 0^-$ level reduces to two equations,

$$Ej = -h' + (Q_0 + Q_1 + P_2 - P_3 + P_4 - P_5)j + (P_0 - 2P_1 + Q_2)h,$$
$$Eh = j' + \frac{2}{r}j + (-Q_0 + Q_1 - 3P_2 - P_3 + 3P_4 + P_5)h + (P_0 - 2P_1 + Q_2)j, \quad (16)$$

with $h \sim r^0$, $j \sim r^1$. Similarly, to find the $K^P = 0^+$ levels one has to solve only two equations,

$$Ej = -h' + (-Q_0 + Q_1 + P_2 - P_3 - P_4 + P_5)j - (P_0 + 2P_1 + Q_2)h,$$
$$Eh = j' + \frac{2}{r}j + (Q_0 + Q_1 - 3P_2 - P_3 - 3P_4 - P_5)h - (P_0 + 2P_1 + Q_2)j. \quad (17)$$

In Fig. 1 we show an example of quark levels obtained from a "natural" choice of the external fields Q_{0-2}, P_{0-5}. We take a confining scalar field $S(r) = \sigma r$ with a standard string tension $\sigma = (0.44\ \mathrm{GeV})^2$, and a topological chiral angle field $P(r) = 2\arctan(r_0^2/r^2)$ such that the profile functions introduced in Eqs. (3) and (4) are $Q_0(r) = S(r)\cos P(r)$, $P_0(r) = S(r)\sin P(r)$; the other profile functions decay exponentially at large distances. The external fields are shown in the left panel of Fig. 1, and the resulting quark levels with various K^P are shown in the right panel. These or similar levels dictate the masses of baryon resonances.

According to the Dirac theory, all *negative*-energy levels—for both s and u, d quarks—have to be fully occupied, corresponding to the vacuum. This means that there must be exactly N_c quarks antisymmetric in color occupying all degenerate levels with J_3 from $-J$ to J, or K_3 from $-K$ to K; they form closed shells. Filling in the lowest level with $E > 0$ by N_c quarks makes a ground-state baryon; see Fig. 2. A similar picture arises in the chiral bag model [16]. Excited baryons can be related to different one-, two-, and three-quark excitations to the other levels. We will try to advocate the point of view that known baryon resonances below 2 GeV are related to one-quark excitations only.

The mass of a baryon is the aggregate energy of all filled states, and being a functional of the mesonic field it is proportional to N_c since all quark levels are degenerate in color. Therefore quantum fluctuations of the mesonic field in baryons are suppressed as $1/N_c$ so that the mean field is indeed justified.

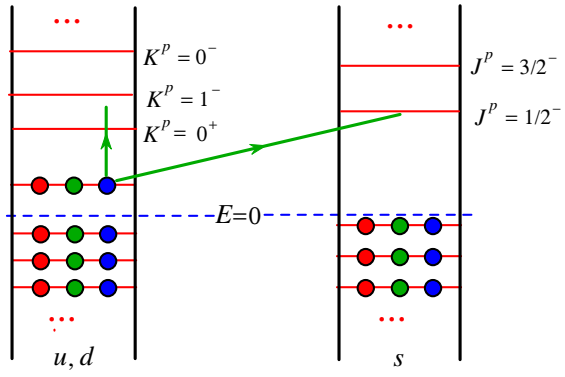

FIG. 2 (color online). Filling u, d, and s shells for the ground-state baryon. Excitations of one valence quark describe baryon resonances.

IV. ROTATIONAL BANDS ABOUT INTRINSIC QUARK LEVELS

Every intrinsic level is accompanied by the rotational band of the states. It appears as a result of the quantization of the slow rotations in both flavor and ordinary space. The theory of rotational bands over the ground state was developed years ago [10], but for excited states and for the general case of the mean field there are some specifics to consider.

The original symmetry of the theory in the chiral limit is $\mathrm{SU}(3)_{\mathrm{flav}} \times \mathrm{SO}(3)_{\mathrm{space}}$. Both symmetries are broken by the ''hedgehog'' ansatz of the mean field, so the soliton transforms under the space and flavor rotations nontrivially. However, the energy of the rotated soliton is the same as the original one. For this reason constant rotations are zero modes and should be taken into account exactly.

A. Ground states

Rotations that slowly depend on time split the energy level into the rotational band. It is convenient to describe this effect with an effective Lagrangian that depends on the collective coordinates, which are rotational matrices.

Let $R(t)$ be an SU(3) matrix describing a slow rotation in flavor space, and $\mathcal{S}(t)$ be an SU(2) matrix describing a slow spatial (and spin) rotation. They rotate quark wave functions $\phi^{\alpha i}(\boldsymbol{x})$ ($\alpha = 1\ldots3$ is flavor and $i = 1\ldots2$ are spin indices) in the given mean field as

$$\begin{aligned}\tilde{\phi}_n^{\alpha i}(x) &= R^{\alpha}_{\alpha'}(t)\mathcal{S}^{i}_{i'}(t)\phi_n^{\alpha' i'}(O(t)\boldsymbol{x}),\\ O_{ik}(t) &= \frac{1}{2}\,\mathrm{Tr}[\mathcal{S}^{+}(t)\sigma_k\mathcal{S}(t)\sigma_i].\end{aligned} \tag{18}$$

Then it is easy to see that the simultaneous transformation of the meson fields

$$\begin{aligned}\tilde{P}^{a}(\boldsymbol{x}) &= O_{ab}[R]P^{b}(O(\mathcal{S})\boldsymbol{x}),\\ \tilde{V}^{ai}(\boldsymbol{x}) &= O_{ab}[R]O_{ij}[\mathcal{S}]V^{bj}(O(\mathcal{S})\boldsymbol{x}),\\ \tilde{A}^{ai}(\boldsymbol{x}) &= O_{ab}[R]O_{ij}[\mathcal{S}]A^{bj}(O(\mathcal{S})\boldsymbol{x}),\end{aligned} \tag{19}$$

(and so on) leaves the Dirac equation invariant in the mean field provided that the matrices R and $\mathcal{S}$ are constant in time.

We now wish to integrate out the quarks. The effective action of the theory is a sum of the meson Lagrangian and the contribution of constituent quarks, which is the determinant of the Dirac equation in the mean field,

$$S_{\mathrm{eff}} = \int dt\mathcal{L}(M) - i\sum_c \mathrm{Sp}_{\mathrm{occup}}\mathrm{Log}\left\{i\frac{\partial}{\partial t} - \mathcal{H}[M]\right\}. \tag{20}$$

Here the sum is over color indices and Sp{...} is running over all *occupied* states. Since the meson field M and the Hamiltonian $\mathcal{H}$ are color blind, the sum over color indices produces the factor N_c for the ground state. For the one-particle excitations one term in this sum corresponds to a particular filling of the levels.

The slow rotations $\mathcal{S}(t)$, $R(t)$ are the particular cases of the quantum fluctuations of the general meson field M. Usually quantum fluctuations are suppressed in the limit of large N_c. Rotations are not suppressed as they are zero modes. Only their frequencies are small in N_c.

Let us parametrize the general meson field as $M = \bar{M} + \delta M$ [where $\bar{M}(\boldsymbol{x})$ is a time-independent mean field and $\delta M(\boldsymbol{x}, t)$ are quantum fluctuations] and calculate the effective action (20) on the set of slowly rotated states (18) and (19) [27],

$$\begin{aligned}S_{\mathrm{eff}} = &\int dt\mathcal{L}_{\mathrm{meson}}(\bar{M}+\delta M, \tilde{\Omega}, \tilde{\omega})\\ &- i\sum_c \mathrm{Sp}_{\mathrm{occup}}\mathrm{Log}\left\{i\frac{\partial}{\partial t} - \mathcal{H}[\bar{M}+\delta M] - \tilde{\Omega}_a t_a - \tilde{\omega}_i j_i\right\}.\end{aligned} \tag{21}$$

Here $\tilde{\Omega}_a$ and $\tilde{\omega}_i$ are flavor and angular frequencies in the body-fixed frame,

$$\tilde{\Omega}_a = -i\mathrm{Tr}[R^{+}\dot{R}\lambda_a], \qquad \tilde{\omega}_i = -i\mathrm{Tr}[\mathcal{S}^{+}\dot{\mathcal{S}}\sigma_i], \tag{22}$$

(λ_a are Gell-Mann flavor matrices and σ_i are Pauli spin matrices), and t_a and j_i are one-particle operators of flavor and total angular momenta,

$$t_a = \frac{1}{2}\lambda_a, \qquad j_i = s_i + l_i = \frac{1}{2}\sigma_i + i\varepsilon^{ikl}x_k\frac{\partial}{\partial x_l}. \tag{23}$$

Next we expand Eq. (21) in small δM, $\tilde{\Omega}$, $\tilde{\omega}$. The linear term should be absent, as the mean field $\bar{M}(\boldsymbol{x})$ is a solution of the equations of motion. There is a famous exclusion from this rule, namely the Witten-Wess-Zumino (WZW) term, which is linear in Ω_8 and proportional to the baryon charge B of the state,

$$\delta S^{(1)} = -\frac{N_c}{2\sqrt{3}}\int dt\tilde{\Omega}_8. \tag{24}$$

The second-order correction is, in general,

$$\begin{aligned}\delta S^{(2)} = &\frac{1}{2}\int d^4x\delta M W\delta M + \int d^4x(\delta M\mathcal{K}^{a}_{\Omega}\tilde{\Omega}_a\\ &+ \delta M\mathcal{K}^{i}_{\omega}\tilde{\omega}_i) - \frac{1}{2}\int dt[I^{(\Omega\Omega)}_{ab}\tilde{\Omega}_a\tilde{\Omega}_b\\ &+ I^{(\omega\omega)}_{ab}\tilde{\omega}_i\tilde{\omega}_j + I^{(\omega\Omega)}_{ai}\tilde{\Omega}_a\tilde{\omega}_i].\end{aligned} \tag{25}$$

Here the first term is a quadratic form for the quantum fluctuations that are not rotations, the second term describes the mixing of rotations and other quantum fluctuations, and the third term is a generic quadratic form for space and flavor rotations. All of the coefficients in Eq. (25) (W, $\mathcal{K}$, I) are proportional to N_c. Thus the quantum fluctuations are $\delta M = O(1/\sqrt{N_c})$. As for the frequencies $\tilde{\Omega}$, $\tilde{\omega}$, we will see that they are $\tilde{\Omega}$, $\tilde{\omega} = O(1/N_c)$.

We are interested in the collective rotational Lagrangian, i.e., the Lagrangian depending only on angular and flavor frequencies. There are two sources for such a Lagrangian. First, the Lagrangian comes from the immediate expansion

of the original action (21). Second, in the presence of mixing the rotation Lagrangian can arise as a result of an integration over nonrotational quantum fluctuations of the meson field δM. Indeed, in the second case the correction to the mean field,

$$\delta M = W^{-1}[\mathcal{K}^a_\Omega \tilde\Omega_a + \mathcal{K}^i_\omega \tilde\omega_i], \tag{26}$$

is of the first order in frequencies and should be accounted for in the leading-order rotational Lagrangian,

$$\begin{aligned} S^{(2)}_{\rm rot} &= -\int dt\Big[\frac{1}{2}\tilde\Omega_a I^{(\Omega\Omega)}_{ab}\tilde\Omega_b + \frac{1}{2}\tilde\omega_i I^{(\omega\omega)}_{ij}\tilde\omega_j \\ &\quad + \frac{1}{2}\tilde\Omega_a I^{(\omega\Omega)}_{ai}\tilde\omega_i\Big], \\ I^{(\Omega\Omega)}_{ab} &= I^{(\Omega\Omega)}_{ab} + \mathcal{K}^a_\Omega W^{-1}\mathcal{K}^b_\Omega, \\ I^{(\omega\omega)}_{ij} &= I^{(\omega\omega)}_{ij} + \mathcal{K}^i_\omega W^{-1}\mathcal{K}^j_\omega, \\ I^{(\omega\Omega)}_{ai} &= I^{(\omega\Omega)}_{ai} + \mathcal{K}^i_\omega W^{-1}\mathcal{K}^a_\Omega + \mathcal{K}^a_\Omega W^{-1}\mathcal{K}^i_\omega, \end{aligned} \tag{27}$$

i.e., the mixing leads to the renormalization of the moments of inertia. It is essential that the terms arising from mixing are of the same order in N_c [as $\mathcal{K} \sim O(N_c)$ and $W \sim O(N_c)$] and contribute to the collective action.

This phenomenon is well known from nuclear physics. The approximation wherein the mixing is neglected is called the *cranking* approximation [28]. The importance of the mixing was pointed out by Thouless-Valatin [29]. The mixing of the rotations and quantum fluctuations, however, is absent in many relativistic theories (at least for models based only on pions; see Appendix B). In such theories the cranking approximation is exact.

Cranked moments of inertia $I^{(\Omega\Omega)}_{ab}, I^{(\omega\omega)}_{ij}, I^{(\omega\omega)}_{ai}$ consist of two parts: a fermion part and a meson part. To obtain the meson part we substitute the rotated meson fields (19) into the mean field approximation of the meson Lagrangian. If the last Lagrangian contains some time derivative, we will get some terms that are quadratic in the frequencies $\tilde\Omega$, $\tilde\omega$ (one should neglect higher terms), which are the contributions to the moments of inertia.

The quark part of the moments of inertia can be obtained by expanding the fermion determinant of Eq. (21) in $\tilde\Omega$, $\tilde\omega$. The corresponding part of the moments of inertia is given by the well-known Inglis expression [28],

$$(I^{(\Omega\Omega)}_{ab})_{\rm q} = \frac{N_c}{2}\sum_{n,m}\frac{\langle n|t_a|m\rangle\langle m|t_b|n\rangle + \langle n|t_b|m\rangle\langle m|t_a|n\rangle}{\varepsilon_m - \varepsilon_n} \tag{28}$$

($|n\rangle$ are occupied one-quark states and $|m\rangle$ are nonoccupied states), with analogous expressions for $I^{(\omega\omega)}_{ij}$ and $I^{(\omega\Omega)}_{ij}$ given by replacing the flavor generator t_a with the total quark angular momentum j_i.

The hedgehog symmetry of the mean field leads to the following relations between different moments of inertia:

$$\begin{aligned} I^{(\Omega\Omega)}_{ab} &= \begin{cases} I_1\delta_{ab}, & a,b = 1\ldots3, \\ I_2\delta_{ab}, & a,b = 4\ldots7, \\ 0, & a,b = 8, \end{cases} \qquad I^{(\omega\Omega)}_{ai} = -2I_1\delta_{ai}, \\ I^{(\omega\omega)}_{ij} &= I_1\delta_{ij}, \end{aligned} \tag{29}$$

and hence the quadratic part of the rotational action reduces to

$$S^{(2)}_{\rm rot} = -\int dt\sum_{i=1}^{3}\frac{I_1}{2}(\tilde\Omega_i - \tilde\omega_i)^2 + \sum_{a=4}^{7}\frac{I_2}{2}\tilde\Omega_a^2. \tag{30}$$

This fact does not depend on the origin of the rotational Lagrangian. In particular, this result can be checked for the quark part (28) (see, e.g., Ref. [10]).

The complete rotational Lagrangian

$$\mathcal{L}_{\rm rot} = \sum_{i=1}^{3}\frac{I_1}{2}(\tilde\Omega_i - \tilde\omega_i)^2 + \sum_{a=4}^{7}\frac{I_2}{2}\tilde\Omega_a^2 + \frac{BN_c}{2\sqrt{3}}\tilde\Omega_8 \tag{31}$$

is a Lagrangian for some specific spherical top in both flavor and ordinary space. We then calculate operators of the angular momenta $\tilde{\boldsymbol{J}}$ and flavor momenta $\tilde{\boldsymbol{T}}$,

$$\begin{aligned} \tilde{\boldsymbol{J}} &= -\frac{1}{2}\,{\rm Tr}\Big[\mathcal{S}\boldsymbol{\sigma}\frac{\delta}{\delta\mathcal{S}}\Big] = \frac{\partial\mathcal{L}_{\rm rot}}{\partial\boldsymbol{\omega}} = I_1(\boldsymbol{\omega} - \boldsymbol{\Omega}), \\ \tilde T_a &= -\frac{1}{2}\,{\rm Tr}\Big[R\lambda_a\frac{\delta}{\delta R}\Big] = \frac{\partial\mathcal{L}_{\rm rot}}{\partial\Omega_a} \\ &= \begin{cases} I_1(\Omega_a - \omega_a), & a = 1\ldots3, \\ I_2\Omega_a, & a = 4\ldots7, \\ \frac{N_c}{2\sqrt{3}}, & a = 8. \end{cases} \end{aligned} \tag{32}$$

We see that the following quantization rules apply to the rotational bands of ground-state baryons:

$$\tilde{\boldsymbol{J}} + \tilde{\boldsymbol{T}} = 0, \qquad \tilde T_8 = \frac{N_c}{2\sqrt{3}}. \tag{33}$$

The second equation is the celebrated Witten quantization rule [2], which claims that the hypercharge in the body-fixed frame is $\tilde Y = \frac{2}{\sqrt{3}}\tilde T_8 = N_c/3$. It is completely the result of the hedgehog symmetry and the fact that N_c valence quarks with the hypercharge $\tilde Y = 1/3$ are put in some bound state in the sector of u and d quarks.

The Hamiltonian of rotations determined from Eq. (31) should be expressed in terms of the momenta $\tilde T$ and $\tilde J$,

$$\begin{aligned} \mathcal{H}_{\rm rot} &= \sum_{a=1}^{3}\frac{\tilde T_a^2}{2I_1} + \sum_{a=4}^{7}\frac{\tilde T_a^2}{2I_2} \\ &= \frac{c_2(r) - \tilde T(\tilde T+1) - \frac{3}{4}\tilde Y}{2I_2} + \frac{\tilde T(\tilde T+1)}{2I_1}. \end{aligned} \tag{34}$$

Here $c_2(r) = \sum_a \tilde T_a^2$ is the Casimir operator in the SU(3) representation r. It is also easy to determine the collective wave function, which is an eigenfunction of the Hamiltonian and the operators of momenta in the lab-fixed frame,

$$T_a = -\frac{1}{2}\mathrm{Tr}\left[\lambda_a R\frac{\delta}{\delta R}\right], \quad \boldsymbol{J} = -\frac{1}{2}\mathrm{Tr}\left[\boldsymbol{\sigma}\mathcal{S}\frac{\delta}{\delta \mathcal{S}}\right], \tag{35}$$

The wave function is a product of two Wigner $\mathcal{D}$ functions, one for the SU(3) group and one for the SU(2) group,

$$\begin{aligned}\Psi_{\mathrm{rot}}(R,\mathcal{S}) &= \sqrt{\dim(r)(2J+1)} \\ &\times \sum_{\tilde{T},\tilde{T}_3,\tilde{J}_3} C^{00}_{\tilde{T}\tilde{T}_3 J\tilde{J}_3}\mathcal{D}^{(r)}_{\tilde{Y}\tilde{T}\tilde{T}_3;YTT_3}(R^+)\mathcal{D}^{J}_{\tilde{J}_3;J_3}(\mathcal{S}^+) \\ &= \sqrt{\dim(r)}(-1)^{J+J_3}\mathcal{D}^{(r)}_{\tilde{Y}J,-J_3;YTT_3}(\mathcal{S}R^+).\end{aligned} \tag{36}$$

This function is an eigenfunction of spin $\boldsymbol{J}^2 = \tilde{\boldsymbol{J}}^2 = \tilde{\boldsymbol{T}}^2$, J_3, isospin $\boldsymbol{T}^2$ and T_3, and hypercharge Y; the index (r) labels the SU(3) representations with dimension $\dim(r)$. According to Eq. (33) the hypercharge $\tilde{Y} = N_c/3$. Finally, the Clebsch-Gordan coefficient $C^{00}_{\tilde{T}\tilde{T}_3 J\tilde{J}_3}$ causes the sum of the isospin $\tilde{\boldsymbol{T}}$ and the angular momentum $\tilde{\boldsymbol{J}}$ to be zero in order to obey the quantization rule (33). In fact, the rotational wave function depends only on the combination $R\mathcal{S}^+$. This is natural because—owing to the hedgehog symmetry—a flavor isospin rotation can be compensated by a space rotation.

B. One-quark excited states

Let us now proceed with one-quark excitations, i.e., excitations where only one quark is taken from the ground level and raised to some excited level. The effective Lagrangian (21) is only slightly changed: one term in the sum over N_c quarks has a different scheme of occupied levels. The other $N_c - 1$ terms, however, remain the same. This means that in the leading order in N_c the mean field does not change [the correction to the mean field is $O(1/N_c)$]. This is also true for the moments of inertia I_1 and I_2; they acquire $O(1)$ corrections, as compared to the leading-order $O(N_c)$. Therefore, the effective rotational Lagrangian (31) is also valid in this case. However, additional *linear* terms in the frequencies Ω and ω can appear. The reason is that the mean field is only a solution of the equations of motion for a ground state and not for excited states. Hence, there is no reason why linear terms in a perturbation (which is a rotation in this case) should be absent. The corresponding linear terms are of the form

$$\delta\mathcal{L}_{\mathrm{rot}} = \langle\mathrm{excited}|(\boldsymbol{\omega}\cdot\boldsymbol{j} + \boldsymbol{\Omega}\cdot\boldsymbol{t}) + \delta M|\mathrm{excited}\rangle, \tag{37}$$

where the last term accounts for a possible change in the contribution of the correction (26) to the mean field due to rotations. This correction should also only be calculated in the ground state (it is determined mainly by a rotation of other $N_c - 1$ quarks), and it is assumed that it is already known. It is also linear in the frequencies $\tilde{\omega}$ and $\tilde{\Omega}$.

Excited states are usually degenerate. Indeed, excitations in the s-quark sector have $2S+1$ degeneracy (where $S = \frac{1}{2}, \frac{3}{2}, \ldots$ is a total momentum of the state), and excitations in the u- and d-quark sectors have $2K+1$ degeneracy, where K is the grand spin of the state. Any of the degenerate states or their mixtures can be taken as an excitation. We define

$$|\mathrm{excited}\rangle = \sum \chi_{K_3}|KK_3JL\rangle \tag{38}$$

(we are now dealing with $K \neq 0$ excitations for definiteness). Here $|KK_3JL\rangle$ is the wave function of some excited state with grand spin K and projection K_3, and χ_{K_3} are amplitudes for different values of the projection. The energy does not depend on χ_{K_3}. Hence it is a new zero mode and should be considered as a collective coordinate together with S and R. The effective rotational Lagrangian for χ_{K_3} should slowly change with time; evidently, the complete Lagrangian is

$$\mathcal{L}_{\mathrm{excited}}[\chi,R,S] = \sum_{K_3}\chi^+_{K_3} i\frac{\partial}{\partial t}\chi_{K_3} + \mathcal{L}_{\mathrm{rot}} + \delta\mathcal{L}_{\mathrm{rot}}, \tag{39}$$

where $\mathcal{L}_{\mathrm{rot}}$ is the rotational Lagrangian for the ground state [Eq. (31)].

Plugging Eq. (38) into Eq. (37), we obtain

$$\begin{aligned}\delta\mathcal{L}_{\mathrm{rot}} = \sum_{K_3K_3'}\chi^+_{K_3'}\chi_{K_3}\Big[&\langle KK_3JL|(\boldsymbol{\omega}\cdot\boldsymbol{j} + \boldsymbol{\Omega}\cdot\boldsymbol{t})|KK_3'JL\rangle \\ &+ (\boldsymbol{\omega}-\boldsymbol{\Omega})\langle KK_3'JL|\frac{\partial\delta M}{\partial\boldsymbol{\omega}}|KK_3JL\rangle.\Big]\end{aligned} \tag{40}$$

Here we have used the fact that the fluctuation of the meson field should depend only on the difference of flavor and space frequencies due to the hedgehog symmetry of the ground state: $\delta M \sim \boldsymbol{\omega} - \boldsymbol{\Omega}$.

The one-quark flavor momentum $\boldsymbol{t}$ and angular momentum $\boldsymbol{j}$ are not conserved in the hedgehog field. Nevertheless, since they transform as vectors under simultaneous flavor and spin rotations their matrix elements should be proportional to the matrix elements of the conserved quantity, i.e., the grand spin $\boldsymbol{K}$,

$$\begin{aligned}\langle KK_3jl|\boldsymbol{t}|KK_3'jl\rangle &= a_K\langle KK_3jl|\boldsymbol{K}|KK_3'jl\rangle, \\ \langle KK_3jl|\boldsymbol{j}|KK_3'jl\rangle &= (1-a_K)\langle KK_3jl|\boldsymbol{K}|KK_3'jl\rangle, \\ \langle KK_3jl|\frac{\partial\delta M}{\partial\boldsymbol{\omega}}|KK_3'jl\rangle &= \zeta_K\langle KK_3jl|\boldsymbol{K}|KK_3'jl\rangle\end{aligned} \tag{41}$$

(the second relation follows from $\boldsymbol{j} + \boldsymbol{t} = \boldsymbol{K}$), where a_K and ζ_K are some constants that are specific to a given excited level. Using explicit expressions for the wave functions of the levels with a given K, one can derive the following relation for a_K:

$$\begin{aligned}a_K &= \frac{K+1-c_K(2K+1)}{2K(K+1)}, \\ c_K &= \frac{\int drr^2(h^2(r)+j^2(r))}{\int drr^2(h^2(r)+j^2(r)+g^2(r)+f^2(r))},\end{aligned} \tag{42}$$

where h, j, f, g are radial wave functions, i.e., solutions of the Dirac equation (8)–(11). The coefficient c_K $(0<c_K<1)$ measures the admixture of the state $j = K + \frac{1}{2}$ in the wave

function of the level with a given K (a complete wave function consists of two states, $j = K \pm \frac{1}{2}$; see Appendix C).

In general it is not possible to calculate the coefficient ζ_K. In particular, it depends on the form of the meson Lagrangian. The coefficient ζ_K renormalizes the value of a_K. Fortunately, the correction to the mean field δM is zero in a wide class of theories.

Collecting all the terms, we obtain the collective Lagrangian for one-quark excitations in the u- and d-quark sectors,

$$\mathcal{L}_K[\chi,R,S] = \sum_{K_3} \chi^+_{K_3} i\frac{\partial \chi_{K_3}}{\partial t} + \frac{N_c}{2\sqrt{3}}\tilde{\Omega}_8 + [(1-\tilde{a}_K)\boldsymbol{\omega} + \tilde{a}_K \boldsymbol{\Omega}] \times \sum_{K_3 K_3'} \chi^+_{K_3}\chi_{K_3'} \langle K K_3 jl|\boldsymbol{K}|K K_3' jl\rangle + \sum_{i=1}^{3}\frac{I_1}{2}(\tilde{\Omega}_i - \tilde{\omega}_i)^2 + \sum_{a=4}^{7}\frac{I_2}{2}\tilde{\Omega}_a^2,$$
$$\tilde{a}_K = a_K - \zeta. \tag{43}$$

The quantization of χ_{K_3} with the Lagrangian (43) is trivial. Due to the presence of the collective variable χ_{K_3} the quantity

$$\sum_{K_3 K_3'} \chi^+_{K_3}\chi_{K_3'} \langle K K_3 jl|\boldsymbol{K}|K K_3' jl\rangle = \hat{\boldsymbol{K}} \tag{44}$$

behaves as a quantum operator of the angular momentum K. Differentiating with respect to ω and Ω, we obtain the momenta in the body-fixed frame,

$$\tilde{\boldsymbol{J}} = I_1(\boldsymbol{\omega} - \boldsymbol{\Omega}) + (1-\tilde{a}_K)\hat{\boldsymbol{K}},$$
$$\tilde{\boldsymbol{T}} = I_1(\boldsymbol{\Omega} - \boldsymbol{\omega}) + \tilde{a}_K\hat{\boldsymbol{K}}, \qquad \tilde{T}_a = I_2\tilde{\Omega}_a, (a = 4\ldots 8),$$
$$\tilde{T}_8 = \frac{N_c}{2\sqrt{3}}. \tag{45}$$

This leads to the following quantization conditions [instead of Eq. (33)]:

$$\tilde{\boldsymbol{T}} + \tilde{\boldsymbol{J}} = \hat{\boldsymbol{K}}, \qquad \tilde{Y} = \frac{N_c}{3}. \tag{46}$$

Constructing the Hamiltonian from the Lagrangian (43), we obtain

$$\mathcal{H}_K = \frac{1}{2I_2}\sum_{a=4}^{7}(\tilde{T}_a)^2 + \frac{(\tilde{\boldsymbol{T}} - \tilde{a}_K\hat{\boldsymbol{K}})^2}{2I_1} = \frac{1}{2I_2}\sum_{a=4}^{7}(\tilde{T}_a)^2 + \frac{(\tilde{\boldsymbol{T}} - \tilde{a}_K(\tilde{\boldsymbol{J}} + \tilde{\boldsymbol{T}}))^2}{2I_1}. \tag{47}$$

The energy levels are

$$\mathcal{E}_K = \frac{c_2(r) - \tilde{T}(\tilde{T}+1) - \frac{3}{4}\tilde{Y}^2}{2I_2} + \frac{1}{2I_1}[\tilde{a}_K J(J+1) + (1-\tilde{a}_K)\tilde{T}(\tilde{T}+1) - \tilde{a}_K(1-\tilde{a}_K)K(K+1)]. \tag{48}$$

Here we have used the fact that $\tilde{J} = J$. The available spins are determined by the quantization rule (46): $J = |\tilde{T} - K|\ldots\tilde{T} + K$.

It is easy to construct the collective wave function. For this case it depends on $\mathcal{S}$, R, and χ_{K_3},

$$\Psi_K(R,\mathcal{S},\chi) = \sqrt{\frac{\dim(r)(2J+1)}{2K+1}} \times \sum_{\tilde{T},\tilde{T}_3,\tilde{J}_3} C^{K K_3}_{\tilde{T}\tilde{T}_3 J\tilde{J}_3}\mathcal{D}^{(r)}_{\tilde{Y}\tilde{T}\tilde{T}_3;YTT_3}(R^+)\mathcal{D}^{J}_{\tilde{J}_3;J_3}(\mathcal{S}^+)\chi_{K_3}. \tag{49}$$

This wave function is an eigenfunction of the hypercharge Y, the isospin T, and its projection T_3, as well as the spin J and its projection J_3. In fact, it is completely fixed by the symmetry and quantization requirements [Eq. (46)].

C. s-quark excited states

At last, let us describe excitations in the s-quark sector. We consider a one-quark excitation where one quark is taken from the ground state $K = 0$ and raised to the level of the s-quark with some total angular momentum S. The excited state is $2S + 1$-fold degenerate, and we take the mixture

$$|\text{excited}\rangle = \sum_{S_3}\chi_{S_3}|S_3\rangle, \tag{50}$$

where $|S_3\rangle$ are one-quark wave functions with different projections of $\boldsymbol{S}$. The calculation of the matrix elements gives [instead of Eq. (41)]

$$\langle S_3|\boldsymbol{j}|S_3'\rangle = \langle S_3|\boldsymbol{S}|S_3'\rangle, \quad \langle S_3|\frac{\partial \delta M}{\partial \boldsymbol{\omega}}|S_3'\rangle = \zeta_S\langle S_3|\boldsymbol{S}|S_3'\rangle, \tag{51}$$

and the matrix elements of $\boldsymbol{t}$ are zero since the s quark does not carry isospin. Thus these matrix elements coincide with Eq. (41) for $a_K = 0$. The subsequent steps are straightforward. The quantization rule (46) changes to

$$\tilde{\boldsymbol{T}} + \tilde{\boldsymbol{J}} = \hat{\boldsymbol{S}}, \qquad \tilde{Y} = \frac{N_c - 3}{3}. \tag{52}$$

The first rule repeats Eq. (46) with the evident replacement $\boldsymbol{K} \to \boldsymbol{S}$. The second rule appears because we substitute the quark with hypercharge $1/3$ (one of the u, d quarks in the ground state) with one s quark (in the excited state) with hypercharge $-2/3$. This rule can also be derived directly by calculating the coefficient in front of the Witten-Wess-Zumino term.

The energy levels for s-quark excitations are given by

$$\mathcal{E}_S = \frac{c_2(r) - \tilde{T}(\tilde{T}+1) - \frac{3}{4}\tilde{Y}^2}{2I_2} + \frac{1}{2I_1}[(1+\zeta_S)\tilde{T}(\tilde{T}+1) - \zeta_S J(J+1) + \zeta_S(1+\zeta_S)S(S+1)], \tag{53}$$

which is Eq. (48) with the substitution $K \to S$ and $a_K = 0$. The available spins are $J = |\tilde{T} - S|\ldots\tilde{T} + S$; the collective wave function is an analogue of Eq. (49),

$$\Psi_S(R,\mathcal{S},\chi) = \sqrt{\frac{\dim(r)(2J+1)}{2S+1}} \times \sum_{\tilde{T},\tilde{T}_3,S_3} C^{SS_3}_{\tilde{T}\tilde{T}_3 J\tilde{J}_3} \mathcal{D}^{(r)}_{\tilde{Y}\tilde{T}\tilde{T}_3;YTT_3}(R^+)\mathcal{D}^{J}_{\tilde{J}_3;J_3}(\mathcal{S}^+)\chi_{S_3}. \quad (54)$$

To summarize: rotational bands around a given excited intrinsic energy should be constructed in the following way. One has to choose SU(3) multiplets that contain states obeying the quantization rule for $\tilde{Y}$, read off the value of $\tilde{T}$ corresponding to this $\tilde{Y}$, and use formulas (34), (48), and (53) for their rotational energy.

Quark wave functions in the mean-field approximation are the product of one-particle wave functions of the filled levels. One has to rotate them according to Eq. (18) and then project to the collective wave functions obtained above [see Eqs. (36), (49), and (54)]. "Projection" means that one has to multiply the rotated quark wave function by the conjugated collective wave function and integrate in the matrices R and S. This will produce quark wave functions of the excited baryons with given quantum numbers.

V. ONE-QUARK EXCITATIONS IN THE MEAN FIELD AND THE QUARK MODEL

In the limit $N_c \to \infty$ the quark model becomes a particular case of the general soliton considered above. Indeed, the mean-field approximation should work for the quark model as well; any interquark potential can be considered in this approximation. Since we are only discussing the symmetry properties of the quark states, the details of the potential are not important.

The real difference between the quark model and the picture considered above is a symmetry of the mean field. The quark model insists on complete spherical symmetry and the only mean field in the quark model is the scalar field $S(\boldsymbol{x})$. The excitations of the quark model arise as SU(6) multiplets [to be more precise, multiplets of the SU(6) ⊗ O(3), with the O(3) group representing orbital angular momentum]. All splittings of such multiplets are considered as small perturbations. This should be compared to the soliton approach where it is assumed that the mean field has hedgehog symmetry and the departure from SU(6) is not considered to be small; it is of order of unity even at large N_c. Nevertheless we should be able to reproduce multiplets of the quark model by taking the spherically symmetrical mean field.

Due to the Witten quantization rule the SU(3) multiplets for large N_c become large as well. The classification of such multiplets was developed in Ref. [30].

Every SU(3) multiplet is characterized by two numbers p and q. However, this is inconvenient for our purposes. Let us label multiplets by (i) $\tilde{Y}$, the Witten condition fulfilled by this multiplet; (ii) $\tilde{T}$, the intrinsic isospin corresponding to this value; and (iii) the *exoticness* X, which is defined as a minimal number of quark-antiquark pairs in the wave function of the given baryon. The standard p and q numbers, the dimension of the multiplet, and the Casimir operator can be expressed in terms of these numbers as follows:

$$p = 2\tilde{T} - X \geq 0, \qquad q = \frac{3}{2}\tilde{Y} + 2X - \tilde{T} \geq 0, \quad (55)$$

and

$$\begin{aligned} \dim &= \frac{2\tilde{T}+1-X}{2}\left(\frac{3}{2}\tilde{Y}+1+2X-\tilde{T}\right) \\ &\quad \times \left(\frac{3}{2}\tilde{Y}+2+X+\tilde{T}\right), \\ c_2(r) &= \frac{3}{4}\tilde{Y}(\tilde{Y}+2)+\tilde{T}(\tilde{T}+1)+X\left(\frac{3}{2}\tilde{Y}+1-\tilde{T}\right)+X^2. \end{aligned} \quad (56)$$

Plugging these expressions into the general formula for the energy (48), we obtain

$$\begin{aligned} \mathcal{M}_K = \mathcal{M}_0 + \Delta\mathcal{E} &+ \frac{1}{2I_1}[\tilde{a}_K J(J+1) + (1-\tilde{a}_K)\tilde{T}(\tilde{T}+1) \\ &- \tilde{a}_K(1-\tilde{a}_K)K(K+1)] \\ &+ \frac{(1+X)(2+3\tilde{Y})}{2I_2}. \end{aligned} \quad (57)$$

We see that I_2 plays the role of the moment of inertia for exotic states; their spectrum is equidistant and distances between states are of order unity (we recall that $I_2 \sim N_c$ and $\tilde{Y} \sim N_c$). The moment of inertia I_1 governs ordinary excitation splittings [30]. We will not consider the *rotational* exotics here; exotic states have some specific properties related to the fact that their widths are $\sim O(1)$ [5,31]. Anyway, these states are separated from the normal rotational band by the interval $\sim O(1)$.

Every excited state has a restricted number of *nonexotic* states entering the rotational band with definite $\tilde{T}$. They are determined from the condition that $p \geq 0$ and $q \geq 0$. In particular, for excitations in the s-quark sector at $N_c = 3$, $\tilde{Y} = 0$ we get only one state—a singlet with spins $J = S \pm 1/2$ (where S is the spin of excited states)—and the other multiplets are exotic. At larger N_c s-quark excitations form nontrivial rotational bands have energies given by Eq. (57) and $\tilde{a}_K = -\zeta_s$ [see Eq. (53)].

Excitations in the u- and d-quark sectors have richer structures. For nonexotic states ($X = 0$) at $N_c = 3$ and $\tilde{Y} = 1$, it follows from Eq. (55) that we have only two possibilities: $\tilde{T} = \frac{1}{2}$ and $\tilde{T} = \frac{3}{2}$. In other words, they can only arise as octets or decuplets. At larger N_c other multiplets become nonexotic. At $K = 0$ we obtain the rotational band of $J = \tilde{T}$ with the different spins changing in the limits $\frac{1}{2} < J < \frac{N_c}{2}$. Their energies are given by the general formula (57) with $X = 0$, $\tilde{a}_K = 0$.

At $K = 1$ we have three possible series of rotational bands: $J = \tilde{T} - 1$, $\tilde{T}$, and $\tilde{T} + 1$; at $K = 2$ there are five series with $J = \tilde{T} - 2, \ldots \tilde{T} + 2$. The possible spins are again determined from Eq. (55).

Let us compare this picture to the quark model (see, e.g., Ref. [32]). The lowest state of the quark model is a 56-plet with orbital moment $L = 0$. This multiplet can be generalized to arbitrary N_c; its dimension is

$$\dim \text{“56”} = \frac{1}{120}(N_c + 1)(N_c + 2)(N_c + 3) \times (N_c + 4)(N_c + 5),$$
$$C_2(\mathrm{SU}(6)) = \frac{5}{12} N_c(N_c + 6). \tag{58}$$

The analogue of “56” in the mean-field picture is the rotational band around ground-state baryons. It is easy to check that the dimension of the full rotational band of ground-state baryons is

$$\sum_{J=\frac{1}{2}}^{N_c} (2J + 1) \dim\left(J, X = 0, \tilde{Y} = \frac{N_c}{3}\right) = \dim \text{“56”}.$$

In other words, the rotational band around the ground state completely coincides with the 56-plet. At $N_c = 3$ it reduces to the familiar $(\mathbf{56}) = (\mathbf{8}\,\frac{\mathbf{1}}{\mathbf{2}}) \oplus (\mathbf{10}\,\frac{\mathbf{3}}{\mathbf{2}})$.

The next SU(6) multiplet is **70**. Its dimension at arbitrary N_c is

$$\dim \text{“70”} = \frac{1}{24}(N_c - 1)(N_c + 1)(N_c + 2)(N_c + 3)(N_c + 4),$$
$$C_2(\mathrm{SU}(6)) = \frac{1}{12} N_c(5N_c + 18). \tag{59}$$

It consists of the following five series of SU(3) multiplets:

(1) Three series of multiplets with $\tilde{Y} = N_c/3$: $\tilde{T} = J - 1$ with $J = \frac{3}{2} \ldots \frac{N_c}{2}$, $\tilde{T} = J$ with $J = \frac{1}{2} \ldots \frac{N_c}{2} - 1$, and $\tilde{T} = J + 1$ with $J = \frac{1}{2} \ldots \frac{N_c}{2} - 1$.

(2) Two series of multiplets with $\tilde{Y} = (N_c - 3)/3$: $\tilde{T} = J - \frac{1}{2}$ with $J = \frac{1}{2} \ldots \frac{N_c}{2} - 1$ and $\tilde{T} = J + \frac{1}{2}$ with $J = \frac{1}{2} \ldots \frac{N_c}{2} - 2$.

It is easy to check using Eq. (56) that the total dimension of all five series is equal to $\dim \text{“70”}$.

The contents of the SU(3) multiplets entering the 70-plet imply that in the mean-field picture it consists of a one-quark excitation in the s-quark sector with spin $1/2$ and a $K = 1$ excitation in the u- and d-quark sectors. These states become degenerate in the case of a spherically symmetrical mean field. However, there are *more* states in the mean-field approximation: in this approximation *all* states with $\frac{1}{2} \le \tilde{T} \le \frac{N_c}{2}$ are present. We will see that absent states are spurious; they are forbidden by the Pauli principle which is not accounted for in the mean-field approximation.

The 70-plet is used in the quark model, e.g., in order to describe baryons with negative parity [32]. It is assumed that these baryons are described by the $(\mathbf{70}, \mathbf{3})$ representation of $\mathrm{SU}(6) \otimes \mathrm{O}(3)$: the baryons have a relative orbital angular moment of two quarks, $L = 1$. This provides negative parity and makes the total baryon wave function symmetrical (or antisymmetrical if one accounts for color) under a simultaneous exchange of spin, flavor indices [SU(6)], and coordinates. Adding $L = 1$ to the mean-field multiplets obtained above, we see that these baryons should be described by $K = 0$, 1, 2 excitations in the u- and d-quark sectors and $S = \frac{1}{2}, \frac{3}{2}$ excitations in the s-quark sector and their rotational bands. The difference with the quark model is that these five sets of states are split by $O(1)$, due to the hedgehog (not spherical) symmetry of the mean field from the very beginning.

Other multiplets of the quark model can also be analyzed in the same way and one can find their interpretation in terms of states appearing in the mean-field approximation.

The above series of five SU(3) multiplets—which at large N_c become the 70-plet of SU(6)—were first observed in the remarkable paper by Cohen and Lebed [33]. In this paper the approach is close in spirit to the approach of Manohar *et al.* [23]. Additionally, the authors of Ref. [33] proposed a relation between the masses of multiplets with different grand spin K, which in our notation reads

$$\Delta\mathcal{E}(0 \to 0) + 3\Delta\mathcal{E}(0 \to 1) + 5\Delta\mathcal{E}(0 \to 2) = 0. \tag{60}$$

This relation looks surprising since it cannot be fulfilled for an arbitrary external field. In particular, it is not fulfilled in the exactly solvable model, which we consider in Appendix A.

The situation in the sector with positive parity is completely different. The quark-model prediction for excited baryons in this sector consists of five $\mathrm{SU}(6) \otimes \mathrm{O}(3)$ multiplets [34]: $(\mathbf{56}, \mathbf{0})$ (radial excitation of the ground multiplet), $(\mathbf{56}, \mathbf{2})$, $(\mathbf{70}, \mathbf{0})$, $(\mathbf{70}, \mathbf{2})$, and $(\mathbf{20}, \mathbf{1})$. All these multiplets should be nearly degenerate in the quark model (they are degenerate in the oscillator limit of the model [34]). The vast majority of the states in these multiplets is not observed in nature.

Contrary to the situation with negative-parity baryons, some quark-model multiplets in the parity-plus sector have no interpretation as one-quark excitations in the mean field at $N_c \to \infty$. Instead they correspond to, at least, two-quark excitations when two quarks from the ground level are transferred to the excited levels (possibly with different K). This is especially clear for the $(\mathbf{20}, \mathbf{1})$ multiplet, where the SU(6) wave function is completely antisymmetric. This multiplet cannot fit the picture where $N_c - 1$ quarks are sitting on the same level (and thus have a completely symmetric wave function) and only the last quark carries angular momentum (to be more precise, nonzero K). The same statement is true for $(\mathbf{70}, \mathbf{0})$ and $(\mathbf{70}, \mathbf{2})$, which also have mixed symmetry. The one-quark excited multiplet should be symmetric in the SU(6) index of the excited

quark and hence it should be antisymmetric in the first two quarks sitting in the ground state.

One can consider the two-particle excitations in the mean field at $N_c \to \infty$ as well, keeping in mind, however, that the **70** and **20** multiplets should be generalized to large N_c in a different way than that used for the negative-parity baryons. We will not go into detail, but we mention that the 20-plet at arbitrary N_c has dimension $\frac{1}{12}(N_c-2)\times(N_c-1)(N_c+1)(N_c+2)(N_c+3)$ and corresponds to ten series of SU(3) multiplets with different spins [the SU(6) Casimir operator is $N_c(5N_c+6)/12$]. The dimension of the two-quark excited 70-plet with positive parity is $\frac{1}{3}(N_c-2)(N_c-1)N_c(N_c+2)(N_c+4)$ and it contains 40 series of SU(3) multiplets [the SU(6) Casimir operator is $3+N_c(5N_c+6)/12$]. Of course, most of the states are spurious at $N_c=3$. One can decompose these series to the state corresponding to a different value of K and check that they are, indeed, two-quark excitations (e.g., in the toy model considered in Appendix A).

However, in this paper we would like to insist that even positive-parity baryons can also be constructed as one-quark excitations in the mean field. We believe that baryons with two excited quarks are much heavier and have larger widths than one-quark excitations. In other words, we think that the quark model is misleading for the positive-parity excited baryons [in particular, the SU(6) group is completely broken], and its obvious success in the sector with negative parity is explained by the fact that negative-parity baryons *are* one-quark excitations. This point of view is supported by the fact that a significant portion of the quark-model states with positive parity has never been observed. We will adopt the idea that they are described by the same set of $K=0$, 1, 2 levels as for the negative-parity baryons. One can easily construct the wave functions of these states directly and check that there is only one spurious multiplet among them (for more details, see the next section). Our classification, as was already said, does not correspond to the quark model.

A different situation arises if one puts $N_c=3$ from the very beginning. Then the identification of one-particle and two-particle excitations becomes difficult. Indeed, after the separation of the center-of-motion movement, the wave functions are not the product of one-particle states. Moreover, one can impose the SU(6) limit (a central symmetrical scalar mean field) and then proceed to the oscillator limit of the interaction. Then all constructed $K=0$, 1, 2 states should fall into one of the five SU(6) multiplets mentioned above. (We recall that the mean-field approximation becomes exact for the oscillator model.) And they do: one can directly construct wave functions of these states with the help of the method described at the end of the previous section and compare them to the SU(6) wave functions built in Ref. [34] (Appendix B). Then one can see that the completely antisymmetric SU(6) multiplet **20** remains as a two-quark excitation even at $N_c=3$. The contents of $K=0$ and $K=2$ states is completely exhausted by the (**56**, **0**) and (**56**, **2**) multiplets. At last, all one-quark $K=1$ states are inside the multiplets of mixed symmetry, i.e., (**70**, **0**) and (**70**, **2**).

To summarize: one-particle excitations in the mean field at large N_c with the lowest $K=0$, 1, 2 states are only a small portion of the quark-model multiplets for positive parity. However, as we shall see in the next section, it is this portion which is observed in Nature (at least below 2 GeV).

VI. COMPARISON WITH THE EXPERIMENTAL SPECTRUM

We shall now look at the experimental spectrum of light baryon resonances up to around 2 GeV, trying to recognize among them the rotational bands about the one-quark excitations from the ground state to the intrinsic quark levels. We shall treat the quantities I_1, $\tilde{a}_K$, and $\Delta\mathcal{E}$ entering Eq. (57) as free parameters to be fitted from the known masses, although in principle they are calculable if the (self-consistent) mean field is known. Still, there are much more resonances than free parameters, and therefore the rotational bands are severely restricted by Eq. (57). As we shall see these restrictions are well supported by experimental observations, despite the fact that in the real world N_c is only three.

Since at this time we do not take into account the splittings inside SU(3) multiplets due to the nonzero m_s, Eq. (57) should be compared with the *centers* of multiplets. For the octet, the center is defined as $\mathcal{M}_8=(2m_N+2m_\Xi+3m_\Sigma+m_\Lambda)/8$, and for the decuplet it is $\mathcal{M}_{10}=(4m_\Delta+3m_{\Sigma^*}+2m_{\Xi^*}+m_\Omega)/10\approx m_{\Sigma^*}$. We take the phenomenological values for $\mathcal{M}_8$, $\mathcal{M}_{10}$ from the paper by Guzey and Polyakov [18] who have analyzed baryon multiplets up to 2 GeV.

A. Spurious states

Comparing the mean-field predictions (valid at large N_c) with the data, it should be kept in mind that certain rotational states are, in fact, spurious, as they are artifacts of the mean-field approximation where the spatial wave function is a product of one-particle wave functions. When averaging over the center of mass is taken into account [which is an $\mathcal{O}(1/N_c)$ effect] the baryon wave functions depend only on the differences of quark coordinates, which for some states may contradict the Pauli principle. This effect has been long known in both nuclear physics [35] and the nonrelativistic quark model [32]. The simplest way to identify spurious states is to continuously deform the mean field to the nonrelativistic oscillator potential where the wave functions are explicit. Again, they can be written directly by projecting the rotated mean-field quark wave functions with the collective wave functions constructed here. If some state is absent in this limit, it cannot appear

from a continuous deformation. An independent way to check for spurious states is to deform the problem at hand to the exactly solvable $(0+1)$-dimensional four-fermion interaction model [36] where the large-N_c approximation is also possible and reveals extra states (see Appendix A).

Specifically, in the parity-plus sector, the spurious state is $(\mathbf{10}, 1/2^+)$, arising from the rotational band about the $(0^+ \to 2^+)$ transition. Such a state also arises from the $(0^+ \to 1^+)$ transition, but in that case it is not spurious.

In the parity-minus sector there are more spurious states: the multiplets $(\mathbf{10}, 5/2^-)$ and $(\mathbf{10}, 7/2^-)$ stemming from the $(0^+ \to 2^-)$ transition are spurious, two out of three multiplets $(\mathbf{10}, 3/2^-)$ arising from $(0^+ \to 0^-, 1^-, 2^-)$ transitions are spurious, and one out of the two multiplets $(\mathbf{10}, 1/2^-)$ stemming from $(0^+ \to 1^-, 2^-)$ transitions is spurious as well. As was already said, the remaining negative-parity multiplets exactly coincide with octets and decuplets from the $(\mathbf{70}, \mathbf{1})$ multiplet of SU(6) ⊗ O(3) of the quark model.

Spurious rotational states should be removed from the consideration.

B. Parity-plus resonances

The two lowest multiplets, $(\mathbf{8}, 1/2^+, 1152)$ and $(\mathbf{10}, 3/2^+, 1382)$ (the last number in the parentheses is the center of the multiplet), form the rotational band about the ground-state filling scheme shown in Fig. 2. Fitting these masses by Eq. (57), we find $\mathcal{M}_0 + \frac{3}{4I_2} = 1090$ MeV, $1/I_1 = 153$ MeV.

Apart from the two lowest multiplets, there is another low-lying pair with the same quantum numbers: $(\mathbf{8}, 1/2^+, 1608)$ and $(\mathbf{10}, 3/2^+, 1732)$. Other parity-plus multiplets are essentially higher. One needs a $0^+ \to 0^+$ transition to explain this pair. Comparing the masses one finds that the second $K^P = 0^+$ intrinsic quark level must be 482 MeV higher than the ground-state 0^+ level, $\Delta\mathcal{E}(0^+ \to 0^+) = 482$ MeV. The moment of inertia appears to be considerably larger than for the ground-state multiplets, $1/I_1 = 83$ MeV. Although the difference is an $\mathcal{O}(1/N_c)$ effect, it may be enhanced if the radially excited 0^+ level has a much larger effective radius.

There is a group of five multiplets that are good candidates for the rotational band around the $0^+ \to 2^+$ transition: $(\mathbf{8}, 3/2^+, 1865)$, $(\mathbf{8}, 5/2^+, 1873)$, $(\mathbf{10}, 3/2^+, 2087)$, $(\mathbf{10}, 5/2^+, 2071)$, and $(\mathbf{10}, 7/2^+, 2038)$. Indeed, this is precisely the content of the rotational band for this transition [the spurious multiplet $(\mathbf{10}, 1/2^+)$ excluded], and a fit to the masses according to Eq. (57) gives a small $\sqrt{\chi^2} = 15$ MeV. It should be kept in mind, however, that not all members of all multiplets are well established [18], and those that are have an experimental uncertainty in the masses. It means that the "experimental" masses for the centers of multiplets are known at best to an accuracy of 20–40 MeV. We find from the fit $1/I_1 = 131$ MeV, $\Delta\mathcal{E}(0^+ \to 2^+) = 722$ MeV. Therefore, the intrinsic 2^+ level must be higher than the 0^+ one.

The only relatively well-established multiplet that is left in the range below 2 GeV is $(\mathbf{8}, 1/2^+, 1846)$. It can arise from the rotational band about the $0^+ \to 1^+$ transition; however, other parts of the band are poorly known. If one looks into the nonstrange baryons that are left, one finds $N(1/2^+, 1710^{***})$, $N(1/2^+, 1900^{**})$, $\Delta(1/2^+, 1910^{***})$, and $\Delta(5/2^+, 2000^{**})$, with $\Delta(3/2^+)$ missing. The quantum numbers and the masses of these supposed resonances fit (rather well) the hypothesis that they arise as a rotational band about the $0^+ \to 1^+$ transition; however, their low status prevents a definite conclusion from being drawn. The intrinsic 1^+ level must be approximately 60 MeV higher than the 2^+ quark level.

C. Parity-minus resonances

The situation here is similar to the parity-plus sector: one needs intrinsic quark levels with $K^P = 0^-, 1^-, 2^-$ to explain the resonances as belonging to rotational bands about these transitions. Given that several rotational states in the parity-minus sector are spurious, one expects to find the following multiplets stemming from these transitions: $(\mathbf{8}, 1/2^-) \times 2$, $(\mathbf{8}, 3/2^-) \times 2$, $(\mathbf{8}, 5/2^-)$, $(\mathbf{10}, 1/2^-)$, $(\mathbf{10}, 3/2^-)$. These are precisely the observed multiplets.

We know that all remaining multiplets are spurious, but we do not know how to assign a specific K to the observed one. We assign them according to Eq. (57), requiring that no mixing can happen ($\zeta_K = 0$). There is only one way to do this.

We assign the four lowest multiplets $[(\mathbf{8}, 1/2^-, 1592)$, $(\mathbf{8}, 3/2^-, 1673)$, $(\mathbf{10}, 1/2^-, 1758)$, and $(\mathbf{10}, 3/2^-, 1850)]$ to the rotational band of the $K = 1^-$ level. The fit tells us that the corresponding moment of inertia is $1/I_1 = 171$ MeV and the energy of the level $\Delta\mathcal{E}(0^+ \to 1^-) = 468$ MeV is close to $\Delta\mathcal{E}(0^+ \to 0^+)$. This does not look impossible.

The multiplet $(\mathbf{8}, 1/2^-, 1716)$ should be ascribed to a $0^+ \to 0^-$ transition and the two remaining multiplets $(\mathbf{8}, 3/2^-, 1896)$ and $(\mathbf{8}, 5/2^-, 1801)$ to a $0^+ \to 2^-$ transition.

These assignments produce reasonable values of the mixing coefficients $\tilde{a}_K$ which can be explained without mixing of rotations and other degrees of freedom in the effective meson Lagrangian. Some other information (mass splittings or resonance widths) should probably be used to finally fix the attribution of multiplets to the rotational bands. If the final scheme is different from the one assumed here, it will show the large role of mesons other than the pion mesons in the formation of negative-parity baryons.

To summarize, all parity-plus and parity-minus baryons around 2 GeV and below can be accommodated by the scheme, assuming that they all arise as rotational excitations about the $0^+ \to 0^+$, 1^+, 2^+ and $0^+ \to 0^-$, 1^-, 2^- transitions; see Table I. There are no unexplained resonances left, but there is an extra state $\Delta(3/2^+, \sim 1945)$ stemming from the $0^+ \to 1^+$ transition, which is so far unobserved, and therefore this state is a prediction.

TABLE I. Interpretation of all baryon resonances below 2 GeV as rotational excitations on top of intrinsic quark states.

quark levels	rotational bands	$(I_1)^{-1}$, MeV	$\tilde{a}_K$
$K^P = 0^+$, ground state	$(\mathbf{8}, 1/2^+, 1152)$ $(\mathbf{10}, 3/2^+, 1382)$	153	
$0^+ \to 0^+$ 482 MeV	$(\mathbf{8}, 1/2^+, 1608)$ $(\mathbf{10}, 3/2^+, 1732)$	83	
$0^+ \to 2^+$ 722 MeV	$(\mathbf{8}, 3/2^+, 1865)$ $(\mathbf{8}, 5/2^+, 1873)$ $(\mathbf{10}, 3/2^+, 2087)$ $(\mathbf{10}, 5/2^+, 2071)$ $(\mathbf{10}, 7/2^+, 2038)$	131	-0.050
$0^+ \to 1^+$ ~780 MeV	$N(1/2^+, 1710)$ $N(3/2^+, 1900)$ $\Delta(1/2^+, 1910)$ $\Delta(3/2^+, \sim 1945)$? $\Delta(5/2^+, 2000)$		
$0^+ \to 1^-$ 468 MeV	$(\mathbf{8}, 1/2^-, 1592)$ $(\mathbf{8}, 3/2^-, 1673)$ $(\mathbf{10}, 1/2^-, 1758)$ $(\mathbf{10}, 3/2^-, 1850)$	171	0.336
$0^+ \to 0^-$ 563 MeV	$(\mathbf{8}, 1/2^-, 1716)$	155(fit)	
$0^+ \to 2^-$ 730 MeV	$(\mathbf{8}, 3/2^-, 1896)$ $(\mathbf{8}, 5/2^-, 1801)$	155(fit)	-0.244
$0^+ \to \frac{1}{2}^-$ 254 MeV	$(\mathbf{1}, 1/2^-, 1405)$		
$0^+ \to \frac{3}{2}^-$ 379 MeV	$(\mathbf{1}, 1/2^-, 1520)$		

D. *s* quarks

As emphasized in Sec. III, *s* quarks are in a completely different external field than *u* and *d* quarks, even in the chiral limit. Only the confining forces which we model by a linear rising scalar field are the same for all quarks. The two excited levels for *s* quarks are shown in Fig. 3; they are needed to explain the singlet $\Lambda(1/2^-, 1405)$ and $\Lambda(3/2^-, 1520)$ resonances. The corresponding values of $\Delta\mathcal{E}$ are presented in the Table I. No more singlet Λ's are known below 2 GeV, and therefore there should be no intrinsic *s*-quark levels \with either positive or negative parity in this range.

Following the standard logic of the quark model, in this paper we describe all baryon resonances as excitations of *valence quarks* (see Fig. 2). This is not necessary, however. New resonances can appear due to transitions from the levels which belong to the Dirac continuum. The main configuration for such baryons will consist of five quarks (three valence quarks plus a quark-antiquark pair), but this does not mean that they should be exotic. In fact, it is just the opposite; most of them would be ordinary octets and decuplets.

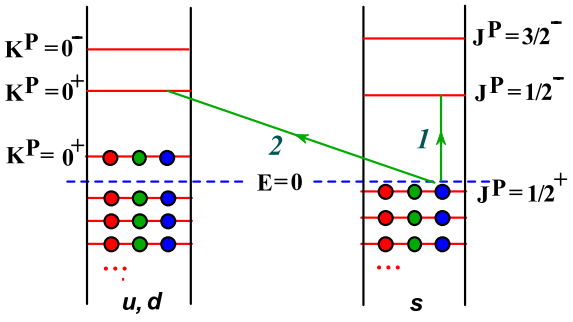

FIG. 3 (color online). (Color online) Possible transitions of the *s* quark from the highest filled *s* level to excited *s* levels (*1*), and to excited *u*, *d* levels (*2*).

For example, the intriguing question is the location of the highest *filled* level of *s* quarks. Presumably, it must be a level with quantum numbers $J^P = \frac{1}{2}^+$, as it possesses maximal symmetry. There can be one-quark excitations from that level to both the *s*-quark excited levels $\frac{1}{2}^-$ and $\frac{3}{2}^-$, and to the *u*, *d* excited levels $0^+, 0^-, \ldots$; see Fig. 3. Transitions of the first type generate a rotation band consisting of $(\mathbf{8}, 1/2^-) \times 2$, $(\mathbf{8}, 1/2^-)$, $(\mathbf{10}, 1/2^-)$, $(\mathbf{10}, 3/2^-) \times 2$, and $(\mathbf{10}, 5/2^-)$. Transitions of the second type are called—in the terminology of nuclear physics—Gamov-Teller transitions, and they generate the exotic *antidecuplet* $(\overline{\mathbf{10}}, 1/2^+)$, etc. [14]. It turns out that it is difficult, if not impossible, to move the highest filled level of *s* quarks $\frac{1}{2}^+$—which must satisfy Eq. (6) in a "realistic" mean field—more than ~700 MeV below the first excited level $\frac{1}{2}^-$. Therefore, the parity-minus resonances generated by the transition *1* in Fig. 3 must reveal themselves in the spectrum below 2 GeV. We note that such resonances will have a substantial five-quark component, $u(d)u(d)u(d)s\bar{s}$, since they require an *s* quark to be pulled out of the filled level and put into an excited level. The real-world resonances are probably certain mixtures of these excitations with the *u*, *d* excitations described in the previous section. This is a welcome feature as, for example, the well-known resonance $N(1/2^-, 1535)$ has a surprisingly large coupling to the η meson (see also Refs. [33,37]). The Gamov-Teller transition *2* gives a natural explanation of the exotic Θ^+ resonance [38] at exactly the position where it has been claimed by a number of experiments [14].

VII. MASS SPLITTINGS

The nonzero mass of the strange quark m_s breaks the SU(3) flavor group and splits SU(3) multiplets. Let us calculate these splittings. Inserting the quark mass m (a matrix in flavor) into the quark determinant (21) and expanding up to the first order in m, we obtain

$$\delta_m S = -i\sum_c \mathrm{Sp}_{\mathrm{occupied}}\left\{R^+ mR\gamma^0 \frac{1}{i\frac{\partial}{\partial t} - \mathcal{H}(M+\delta M) - \tilde{\Omega}t_a - \tilde{\omega}_i j_i}\right\}. \tag{61}$$

The mass of the strange quark has both a singlet and an octet part, $m = m_0\mathbf{1} + m_8\lambda_8$ and $m_8 = m_s/\sqrt{3}$, and splittings are determined only by the octet part m_8.

We want to calculate the mass splitting in the zeroth and the first order in the angular and flavor frequencies $\tilde{\omega}$ and $\tilde{\Omega}$. For ground-state baryons this calculation was carried out in many papers (see, e.g., Ref. [39]). The result reads

$$\begin{aligned}\delta_m S = \frac{m_s}{\sqrt{3}}\int dt\Bigg[&\sum_a\sum_{\mathrm{occup}} \mathcal{D}^{(8)}_{8a}(R)\langle n|\lambda_a\gamma^0|n\rangle \\ &+ 2K_1\sum_{i=1}^{3}\mathcal{D}^{(8)}_{8i}(R)(\tilde{\Omega}_i - \tilde{\omega}_i) \\ &+ 2K_2\sum_{a=4}^{7}\mathcal{D}^{(8)}_{8a}(R)\tilde{\Omega}_a\Bigg].\end{aligned} \tag{62}$$

Here the first term is of the zeroth order in frequencies, while the second and the third terms represent the first-order corrections. K_1 and K_2 are some constants analogous to the moments of inertia (28) (again, n runs over occupied levels and m runs over free levels in the mean field),

$$K_{ab} = N_c\sum_{n,m}\frac{\langle n|\gamma^0 t_a|m\rangle\langle m|\gamma^0 t_b|n\rangle + \langle n|\gamma^0 t_b|m\rangle\langle m|\gamma^0 t_a|n\rangle}{\varepsilon_m - \varepsilon_n} \tag{63}$$

(if there is a mixing of rotations with δM this expression should be modified). The tensor K_{ab} has a structure analogous to the structure of the moment of inertia,

$$K_{ab} = \begin{cases} K_1\delta_{ab} & a,b = 1\ldots 3, \\ K_2\delta_{ab} & a,b = 4\ldots 7, \\ 0, & a = b = 8. \end{cases} \tag{64}$$

Equation (62) is valid for rotational bands above the ground-state and one-particle excitations. The first term for the ground-state baryons is nonzero only at $a = 8$; it can be expressed through an experimentally known quantity called the Σ-term,

$$\begin{aligned}\sum_c\sum_{\mathrm{occup}}\langle n|\lambda_8\gamma^0|n\rangle &= \frac{1}{3}\frac{m_s}{m_u+m_d}\Sigma, \\ \Sigma &= (m_u+m_d)\frac{\partial\mathcal{M}}{\partial(m_u+m_d)}.\end{aligned} \tag{65}$$

Here we have used the fact that all valence levels are located in the u- and d-quark sectors. Below we will imply only this case. Indeed, we have seen that at $N_c = 3$ one-particle excitations in the s-quark sector (from the ground state) are singlets, so there is no mass splitting present for this type of excitation.

In the u- and d-quark sectors there is also another possibility, namely $a = 1, 2, 3$, for the excited level,

$$\langle\mathrm{excited}|\frac{\lambda_i}{2}\gamma^0|\mathrm{excited}\rangle = d_K\sum_{K_3,K_3'}\chi^+_{K_3'}\chi_{K_3}\langle K_3'|K_i|K_3\rangle, \tag{66}$$

where d_K is some constant which is determined by the wave function

$$d_K = \pm\int drr^2\left[\frac{g^2(r)-f^2(r)}{2K} - \frac{h^2(r)-j^2(r)}{2(K+1)}\right], \tag{67}$$

where a plus sign stands for the states with parity $(-1)^K$ and a minus sign for the states with parity $(-1)^{K+1}$. The calculation of d_K is analogous to the calculation of the coefficient c_K (see above).

The first-order terms in the frequencies in Eq. (62) can be simplified as well. We replace the frequencies with the operators $\tilde{T}_a$ according to Eq. (45). Using the relation

$$T_8 = \sum_{a=1}^{8}\mathcal{D}^{(8)}_{8i}(R)\tilde{T}_a,$$

one can express the last term in Eq. (62) in terms of the sum with $a = 1, 2, 3$ and hypercharge $Y = 2T_8/\sqrt{3}$. For the Hamiltonian we obtain

$$\begin{aligned}\mathcal{H}_m = \alpha\mathcal{D}^{(8)}_{88}(R) + \beta Y + \sqrt{3}\gamma\sum_{i=1}^{3}\mathcal{D}^{(8)}_{8i}(R)\tilde{T}_i \\ + \sqrt{3}\delta\sum_{i=1}^{3}\mathcal{D}^{(8)}_{8i}(R)\hat{K}_i,\end{aligned} \tag{68}$$

where

$$\begin{aligned}\alpha &= -\frac{2}{3}\frac{m_s}{m_u+m_d}\Sigma + m_s\frac{K_2}{I_2}, \qquad \beta = -m_s\frac{K_2}{I_2}, \\ \gamma &= \frac{2m_s}{3}\left(\frac{K_1}{I_1} - \frac{K_2}{I_2}\right), \qquad \delta = \frac{2m_s}{3}\left(d_K - \frac{K_1}{I_1}\tilde{a}_K\right).\end{aligned} \tag{69}$$

We see that mass splittings are determined by four possible structures. Only the last term is novel; the other three are known for ground-state baryons. Moreover, the constants α, β, γ up to corrections of order $1/N_c$ are the same for all levels. As for δ, it is determined by the properties of the excited level and is unique for a given level. Nevertheless, δ is the same for all rotational bands of the given level. [Note also that $\alpha \sim O(N_c)$, while β, γ, $\delta \sim O(1)$.]

Mass splittings are determined by the average of the Hamiltonian (68) in the collective wave functions (36) and (49). The resulting expressions, of course, respect the Gell-Mann-Okubo formula. We parametrize the masses of the particles in the octet as

TABLE II. Mass splittings for octet and decuplet particles for different values of K.

K	Rep	J	$\mu_1^{(8)}$	$\mu_2^{(8)}$	$\mu^{(10)}$
0	8	$\frac{1}{2}$	$-\frac{\alpha}{10}-\frac{3\gamma}{20}$	$-\frac{\alpha}{8}-\beta+\frac{5\gamma}{16}$	
	10	$\frac{3}{2}$			$-\frac{\alpha}{8}-\beta+\frac{5\gamma}{16}$
1	8	$\frac{1}{2}$	$-\frac{\alpha}{10}-\frac{11\gamma}{20}-\frac{3\delta}{5}$	$-\frac{\alpha}{8}-\beta+\frac{55\gamma}{48}-\frac{5\delta}{4}$	
		$\frac{3}{2}$	$-\frac{\alpha}{10}+\frac{\gamma}{20}+\frac{3\delta}{10}$	$-\frac{\alpha}{8}-\beta-\frac{5\gamma}{48}-\frac{5\delta}{8}$	
	10	$\frac{1}{2}$			$-\frac{\alpha}{8}-\beta+\frac{35\gamma}{48}+\frac{5\delta}{8}$
		$\frac{3}{2}$			$-\frac{\alpha}{8}-\beta+\frac{23\gamma}{48}+\frac{\delta}{4}$
		$\frac{5}{2}$			$-\frac{\alpha}{8}-\beta+\frac{\gamma}{16}-\frac{3\delta}{8}$
2	8	$\frac{3}{2}$	$-\frac{\alpha}{10}-\frac{3\gamma}{4}-\frac{9\delta}{10}$	$-\frac{\alpha}{8}-\beta+\frac{25\gamma}{16}+\frac{15\delta}{8}$	
		$\frac{5}{2}$	$-\frac{\alpha}{10}+\frac{\gamma}{4}+\frac{3\delta}{5}$	$-\frac{\alpha}{8}-\beta-\frac{25\gamma}{48}-\frac{5\delta}{4}$	
	10	$\frac{1}{2}$			$-\frac{\alpha}{8}-\beta+\frac{17\gamma}{16}+\frac{9\delta}{8}$
		$\frac{3}{2}$			$-\frac{\alpha}{8}-\beta+\frac{13\gamma}{16}+\frac{3\delta}{4}$
		$\frac{5}{2}$			$-\frac{\alpha}{8}-\beta+\frac{19\gamma}{48}+\frac{\delta}{8}$
		$\frac{7}{2}$			$-\frac{\alpha}{8}-\beta-\frac{3\gamma}{16}-\frac{3\delta}{4}$

$$\mathcal{M}_N = M_8 - \frac{7}{4}\mu_1^{(8)} - \mu_2^{(8)}, \quad \mathcal{M}_\Lambda = M_8 - \mu_1^{(8)},$$
$$\mathcal{M}_\Sigma = M_8 + \mu_1^{(8)}, \quad \mathcal{M}_\Xi = M_8 + \frac{3}{4}\mu_1^{(8)} + \mu_2^{(8)}, \tag{70}$$

and the masses of the decuplet particles as

$$\mathcal{M}_\Delta = M_{10} - \mu^{(10)}, \quad \mathcal{M}_\Sigma = M_{10},$$
$$\mathcal{M}_\Xi = M_{10} + \mu^{(10)}, \quad \mathcal{M}_\Omega = M_{10} + 2\mu^{(10)}. \tag{71}$$

This parametrization obeys the Gell-Mann-Okubo formula automatically. In Table II we give the values of μ in terms of α, β, γ, δ for different values of K and the spin of the multiplet J.

The expressions for the mass splittings give rise to the number of relations between the masses of the particles entering the same rotational band. These relations are similar to the well-known Guadagnini relation, which is valid for the ground-state octet and decuplet (see, e.g., Ref. [38]). Let us itemize these relations for the $K = 2$ rotational band of the baryons with positive parity: for octets and decuplets,

$$5\mu_2^{(8)}\left(\frac{3}{2}\right) + 9\mu^{(10)}\left(\frac{5}{2}\right) = 14\mu^{(10)}\left(\frac{3}{2}\right),$$
$$5\mu_2^{(8)}\left(\frac{5}{2}\right) + 11\mu^{(10)}\left(\frac{3}{2}\right) = 16\mu^{(10)}\left(\frac{5}{2}\right), \tag{72}$$

and for decuplets only

$$5\mu^{(10)}\left(\frac{7}{2}\right) + 7\mu^{(10)}\left(\frac{3}{2}\right) = 12\mu^{(10)}\left(\frac{5}{2}\right),$$
$$3\mu^{(10)}\left(\frac{5}{2}\right) + 5\mu^{(10)}\left(\frac{1}{2}\right) = 8\mu^{(10)}\left(\frac{3}{2}\right) \tag{73}$$

(we put in parenthesis the spin of the particles). All these relations work with an accuracy better than 10% and some even with an accuracy of 1–2%.

For $K = 1$ (negative parity) we get two relations,

$$7\mu^{(10)}\left(\frac{1}{2}\right) + 3\mu_2^{(8)}\left(\frac{3}{2}\right) = 10\mu^{(10)}\left(\frac{3}{2}\right),$$
$$5\mu^{(10)}\left(\frac{3}{2}\right) + 3\mu_2^{(8)}\left(\frac{1}{2}\right) = 8\mu^{(10)}\left(\frac{1}{2}\right). \tag{74}$$

While the first is fulfilled with an accuracy of 2%, the second is fulfilled only at the 10% level.

The last relation for $K = 0$ (which is precisely Guadagnini's one but for excited baryons) reads $\mu^{(10)}(\frac{3}{2}) = \mu_2^{(8)}(\frac{1}{2})$. This relation—which works rather well for the ground-state octet and decuplet—is broken surprisingly strongly for $K = 0^+$ excited state.

The situation changes in the strict limit $N_c \to \infty$, in the approach advocated in Ref. [22]. According to the last approach one should consider Clebsch-Gordan coefficients in the same limit $N_c \to \infty$. The required isoscalar factors are collected in Appendix E to make the calculations straightforward.

The recalculated results demonstrate different N_c counting. It appears that the mass splittings are not $O(m_s N_c)$, but rather only $O(m_s)$. Both constants α and β enter the leading term, while γ and δ appear in the $O(m_s/N_c)$ corrections. This picture is probably more satisfactory from the general point of view. Let us note that it coincides with the N_c counting developed in Refs. [22,23] and all the mass relations derived there are also automatically fulfilled.

The Gell-Mann Okubo relations still appear to be valid. This is not trivial, especially for "decuplets," where not one but two final states at arbitrary N_c are available (and therefore it is possible to talk about the F- and D-schemes for "decuplets"). However, at large N_c the Gell-Mann-Okubo relations are restored, up to the order $O(1/N_c)$

TABLE III. Isoscalar factors for $8_M \otimes$ "8"$_B \to$ "8"$_B$ at arbitrary $N_c = 2\nu + 1$. In the table two isoscalar factors for the antisymmetrical (F) and symmetrical (D) cases are presented. Any value from the table should be divided by a universal factor $\sqrt{5\nu^2 + 16\nu + 9}$, which we omit for brevity. The definitions of the isoscalar factors corresponds to the conventions of Ref. [46] and reduces to the usual ones at $N_c = 3$

	$\eta N \to N$	$\pi N \to N$	$K\Sigma \to N$	$K\Lambda \to N$	$\bar{K}N \to \Sigma$	$\eta\Sigma \to \Sigma$
F	$-\frac{3(\nu^2+3\nu+1)}{\sqrt{2(\nu+2)(\nu+4)}}$	$\frac{\nu^2+5\nu+9}{\sqrt{2(\nu+2)(\nu+4)}}$	$\frac{(11+4\nu)\sqrt{\nu}}{\sqrt{2(\nu+2)(\nu+4)}}$	$\frac{\sqrt{3}(2\nu+3)}{\sqrt{2(\nu+4)}}$	$-\frac{\sqrt{\nu}(11+4\nu)}{\sqrt{3(\nu+2)(\nu+4)}}$	$\frac{(1-\nu)(8+3\nu)}{\sqrt{3(\nu+2)(\nu+4)}}$
D	$\sqrt{\frac{\nu}{2}(\nu+2)}$	$-3\sqrt{\frac{\nu}{2}(\nu+2)}$	$3\sqrt{\frac{\nu}{2}+1}$	$-\sqrt{\frac{3\nu}{2}}$	$-\sqrt{3(2+\nu)}$	$(3-\nu)\sqrt{\frac{2\nu+3}{2\nu}}$
	$\pi\Sigma \to \Sigma$	$\pi\Lambda \to \Sigma$	$K\Xi \to \Sigma$	$\bar{K}N \to \Lambda$	$\eta\Lambda \to \Lambda$	$\pi\Sigma \to \Lambda$
F	$\frac{\sqrt{\nu+4}(\nu+5)}{\sqrt{3(\nu+2)}}$	$\frac{(1-\nu)\sqrt{\nu}}{\sqrt{6(\nu+4)}}$	$\frac{7\nu+8}{3\sqrt{\nu+4}}$	$\frac{\sqrt{3}(2\nu+3)}{\sqrt{\nu+4}}$	$\frac{3(2-\nu-\nu^2)}{\sqrt{2(\nu+2)(\nu+4)}}$	$\frac{(\nu-1)\sqrt{\nu}}{\sqrt{2(\nu+4)}}$
D	$(1-\nu)\sqrt{\frac{3(2+\nu)}{\nu}}$	$\sqrt{\frac{3}{2}}(\nu+1)$	$-\frac{2\nu+1}{\sqrt{\nu}}$	$-\sqrt{3\nu}$	$-\frac{(\nu+5)\sqrt{\nu}}{\sqrt{2(\nu+2)}}$	$-\sqrt{\frac{3}{2}}(\nu+1)$
	$K\Xi \to \Lambda$	$\bar{K}\Sigma \to \Xi$	$\bar{K}\Lambda \to \Xi$	$\eta\Xi \to \Xi$	$\pi\Xi \to \Xi$	
F	$\frac{5\sqrt{\nu(\nu+2)}}{\sqrt{\nu+4}}$	$\frac{7\nu+8}{\sqrt{6(\nu+4)}}$	$-\frac{5\sqrt{\nu(\nu+2)}}{\sqrt{2(\nu+4)}}$	$\frac{(8-3\nu)\sqrt{\nu+2}}{2\sqrt{\nu+4}}$	$\frac{(16-\nu)\sqrt{\nu+2}}{3\sqrt{2(\nu+4)}}$	
D	$\frac{3}{\sqrt{\nu+2}}$	$-\frac{(2\nu+1)\sqrt{3}}{\sqrt{2\nu}}$	$\frac{3}{\sqrt{2(\nu+2)}}$	$\frac{3-5\nu-\nu^2}{\sqrt{2\nu(\nu+2)}}$	$\frac{\nu^2+3\nu+5}{\sqrt{2\nu(\nu+2)}}$	

TABLE IV. Isoscalar factors for $8_M \otimes$ "10"$_B \to$ "10"$_B$ at arbitrary $N_c = 2\nu + 3$. Two factors for the antisymmetrical (F) and symmetrical (D) cases are given. The table values should be divided by a factor of $\sqrt{3\nu^2 + 16\nu + 15}$. The definitions of the isoscalar factors corresponds to the conventions of Ref. [46] and reduces to the usual ones at $N_c = 3$. In particular, at $N_c = 3$ only the symmetrical representation survives.

	$K\Sigma^* \to \Delta$	$\eta\Delta \to \Delta$	$\pi\Delta \to \Delta$	$K\Xi^* \to \Sigma^*$	$\pi\Sigma^* \to \Sigma$	$\eta\Sigma^* \to \Sigma^*$
F	$\frac{\sqrt{5(\nu+4)}(3\nu+2)}{2\sqrt{(\nu+2)(\nu+6)}}$	$-\frac{\sqrt{5}(\nu^2+5\nu+3)}{\sqrt{2(\nu+2)(\nu+6)}}$	$\frac{\nu^2+7\nu+15}{\sqrt{2(\nu+2)(\nu+6)}}$	$\frac{2\sqrt{5(\nu+3)}(2\nu+3)}{3\sqrt{(\nu+2)(\nu+6)}}$	$\frac{\sqrt{5}(\nu^2+5\nu+12)}{2\sqrt{3(\nu+2)(\nu+6)}}$	$-\frac{\sqrt{5}\nu(\nu+3)}{\sqrt{2(\nu+2)(\nu+6)}}$
D	$-\frac{\sqrt{\nu}}{2}$	$-\sqrt{\frac{\nu(\nu+4)}{2}}$	$\sqrt{\frac{5\nu(\nu+4)}{2}}$	$-\frac{2\sqrt{\nu(\nu+3)}}{3\sqrt{\nu+4}}$	$-\frac{(19+5\nu)\sqrt{\nu}}{2\sqrt{\nu+4}}$	$-\frac{(\nu+5)\sqrt{\nu}}{\sqrt{2(\nu+4)}}$
	$\bar{K}\Delta \to \Sigma^*$	$\pi\Xi^* \to \Xi^*$	$\eta\Xi^* \to \Xi^*$	$K\Omega \to \Xi^*$	$\bar{K}\Sigma^* \to \Xi^*$	$\bar{K}\Xi^* \to \Omega$
F	$\frac{\sqrt{5(\nu+4)}(2\nu+3)}{\sqrt{3(\nu+2)(\nu+6)}}$	$\frac{\sqrt{5}(\nu^2+3\nu+9)}{3\sqrt{2(\nu+2)(\nu+6)}}$	$\frac{\sqrt{5}(3-\nu-\nu^2)}{\sqrt{2(\nu+2)(\nu+6)}}$	$\frac{\sqrt{5}(2\nu+3)}{\sqrt{2(\nu+6)}}$	$\frac{\sqrt{10(\nu+3)}(2\nu+3)}{\sqrt{3(\nu+2)(\nu+6)}}$	$\frac{\sqrt{5}(2\nu+3)}{\sqrt{\nu+6}}$
D	$-\sqrt{\frac{\nu}{3}}$	$-\frac{(5\nu+18)\sqrt{\nu}}{3\sqrt{2(\nu+4)}}$	$-\frac{(\nu+6)\sqrt{\nu}}{\sqrt{2(\nu+4)}}$	$-\sqrt{\frac{\nu(\nu+2)}{2(\nu+4)}}$	$-\sqrt{\frac{2\nu(\nu+3)}{3(\nu+4)}}$	$-\sqrt{\frac{\nu(\nu+2)}{\nu+4}}$
	$\eta\Omega \to \Omega$					
F	$\frac{(3-\nu)\sqrt{5(\nu+2)}}{\sqrt{2(\nu+6)}}$					
D	$-\frac{(\nu+7)\sqrt{\nu}}{\sqrt{2(\nu+4)}}$					

inclusive (they are not exact in N_c!). To save space we will not fill up the complete table of masses analogous to the table at $N_c = 3$. Instead we write down only the mass relations that are independent of the concrete model (some portion of them were already known). For $K = 0$,

$$\mu^{(10)}\left(\frac{3}{2}\right) = \mu_2^{(8)}\left(\frac{1}{2}\right) - \frac{1}{4}\mu_1^{(8)}\left(\frac{1}{2}\right), \tag{75}$$

which replaces Guadagnini's relation derived at $N_c = 3$ (see above). We see that the accuracy of this relation is less than that of the original one. This is not surprising, as the continuation of the Clebsch-Gordan coefficient introduces a new source of inaccuracy. At $K = 1$ we have the following relations:

$$\begin{aligned} 12\mu_2^{(8)}\left(\frac{3}{2}\right) - 3\mu_1^{(8)}\left(\frac{3}{2}\right) + 14\mu^{(10)}\left(\frac{3}{2}\right) &= 26\mu^{(10)}\left(\frac{1}{2}\right), \\ 12\mu_2^{(8)}\left(\frac{1}{2}\right) - 3\mu_1^{(8)}\left(\frac{1}{2}\right) + 20\mu^{(10)}\left(\frac{3}{2}\right) &= 32\mu^{(10)}\left(\frac{1}{2}\right). \end{aligned} \tag{76}$$

And at last, for $K = 2$,

$$\begin{aligned} 20\mu_2^{(8)}\left(\frac{5}{2}\right) - 5\mu_1^{(8)}\left(\frac{5}{2}\right) + 44\mu^{(10)}\left(\frac{3}{2}\right) &= 64\mu^{(10)}\left(\frac{5}{2}\right), \\ 20\mu_2^{(8)}\left(\frac{3}{2}\right) - 5\mu_1^{(8)}\left(\frac{3}{2}\right) + 34\mu^{(10)}\left(\frac{3}{2}\right) &= 54\mu^{(10)}\left(\frac{5}{2}\right) \end{aligned} \tag{77}$$

(the relation for the decuplets is the same as at $N_c = 3$). These relations are valid in the linear order in m_s and up to

the order $O(1/N_c)$ inclusive. In general, they are not obeyed as well as the original ones at $N_c = 3$.

VIII. CONCLUSIONS

If the number of colors N_c is treated as a free algebraic parameter, baryon resonances are classified in a simple way. At large N_c all baryon resonances are basically determined by the intrinsic quark spectrum, which takes a certain limiting shape at $N_c \to \infty$. This spectrum is the same for light baryons ($q \ldots qq$ with N_c light quarks q) and for heavy baryons ($q \ldots qQ$ with $N_c - 1$ light quarks and one heavy quark Q), since the difference is a $1/N_c$ effect [13].

One can excite quark levels in various ways, called either one-particle or particle-hole excitations; in both cases the excitation energy is $\mathcal{O}(1)$. On top of each one-quark or quark-antiquark excitation there is generically a band of SU(3) multiplets of baryon resonances, which are rotational states of a baryon as a whole. Therefore, the splitting between multiplets is $\mathcal{O}(1/N_c)$. The rotational band is terminated when the rotational energy reaches $\mathcal{O}(1)$.

In reality N_c is only 3, and the above idealistic hierarchy of scales is somewhat blurred. Nevertheless, an inspection of the spectrum of baryon resonances reveals certain a hierarchy, schematically summarized as follows.

(i) Baryon mass: $\mathcal{O}(N_c)$, numerically 1200 MeV, the average mass of the ground-state octet.
(ii) One-quark and particle-hole excitations in the intrinsic spectrum: $\mathcal{O}(1)$, typically 400 MeV, for example the excitation of the Roper resonance.
(iii) Splitting between the centers of SU(3) multiplets arising as rotational excitations of a given intrinsic state: $\mathcal{O}(1/N_c)$, typically 133 MeV.
(iv) Splitting between the centers of rotational multiplets differing by spin, which are degenerate in the leading order: $\mathcal{O}(1/N_c^2)$, typically 44 MeV.
(v) Splitting inside a given multiplet owing to the nonzero strange quark mass: $\mathcal{O}(m_s N_c)$, typically 140 MeV.

In practical terms, we have shown that all baryon resonances up to 2 GeV made of light quarks can be understood as rotational excitations about certain transitions between intrinsic quark levels. The quantum numbers of the resonances and the splittings between multiplets belonging to the same rotational band are dictated by the quantum numbers of the intrinsic quark levels, and appear to be in good accordance with the data. The content and the splitting of the lowest charmed (and bottom) baryon multiplets are also in accordance with their interpretation as a rotational band about the ground-state filling scheme.

In this paper, we have concentrated on the algebraic aspect of the problem, leaving aside the dynamical aspects. Dynamical models should answer the question of why the intrinsic quark levels for u, d quarks with $K^P = 0^\pm, 1^\pm, 2^\pm$ and the s-quark levels with $J^P = \frac{1}{2}^\pm, \frac{3}{2}^\pm$, etc., have the particular energies summarized in Table I. However, we feel that it is anyway a step forward: instead of explaining two hundred resonances, one now needs to explain the positions of only a few intrinsic quark levels. Figure 1 illustrates that approximately the needed intrinsic spectrum can be achieved from a reasonable set of mean fields [40].

The proposed scheme for understanding baryon resonances has numerous phenomenological consequences that can be investigated even before real dynamics is considered. Namely, the fact that certain groups of SU(3) multiplets belong to the same rotational band related to one and the same one-quark transition implies relations between their couplings, form factors, splittings inside multiplets owing to the nonzero m_s, and so on.

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APPENDIX A: TOY MODEL

Our approach can be illustrated by the simple but still instructive model suggested in Ref. [36]. The model is one of the class considered first in the context of nuclear physics in Ref. [41] and is exactly solvable. It describes zero-dimensional quarks with spin, flavor, and color which interact by means of a four-fermion color-blind potential. We consider a specific case of the potential and write the Lagrangian of the model in an already bosonized form,

$$\mathcal{L} = \frac{\gamma}{2N_c}(\rho_i^a) + \psi^+(i\partial_t - \gamma\rho_i^a\lambda^a\sigma_i)\psi. \tag{A1}$$

Here σ_i are Pauli matrices acting on the spin indices of quarks and λ^a are Gell-Mann matrices from the $\mathrm{SU}(3)_{\mathrm{flavor}}$ group; the sum in color indices is implied. This model is of the type considered in the main text; in the limit $N_c \to \infty$ it can be considered in the mean-field approximation. The symmetry of ρ_i^a is analogous to that of the octet of vector mesons.

Integrating over ρ_i^a, we arrive at the Lagrangian with a four-fermion interaction,

$$\begin{aligned}\mathcal{L}_f &= \psi^+ i\partial_t\psi + \frac{\gamma}{2N_c}(\psi^+\lambda^a\sigma_i\psi)(\psi^+\lambda^a\sigma_i\psi),\\ \mathcal{H} &= -\frac{\gamma}{2N_c}(\psi^+\lambda^a\sigma_i\psi)(\psi^+\lambda^a\sigma_i\psi),\end{aligned} \tag{A2}$$

which has $\mathrm{SU}(3)_{\mathrm{flavor}} \otimes \mathrm{SU}(2)_{\mathrm{spin}}$ symmetry.

It is convenient to unite the spin and flavor indices in the one SU(6) index and classify the states of the model as SU(6) multiplets. However, these multiplets are split by the potential, which is not SU(6) symmetric. Let us introduce generators $\mathcal{T}_a$ of the SU(6) group and generators T_a (S) for the $\mathrm{SU(3)_{flavor}}$ [$\mathrm{SU(2)_{spin}}$] group,

$$\mathcal{T}_a = \frac{1}{2}\psi^+ \Lambda_a \psi, \quad T_a = \frac{1}{2}\psi^+ \lambda_a \psi, \quad S_i = \frac{1}{2}\psi^+ \sigma_i \psi, \tag{A3}$$

where Λ_a ($a = 1 \ldots 35$) are Gell-Mann matrices for SU(6). Using the Fierz identities for the SU(6), SU(3), and SU(2) groups, the Hamitonian (A2) can be identically rewritten as

$$\mathcal{H}_f = \frac{\gamma}{N_c}\left[4(\mathcal{T}_a)^2 - 2(T_a)^2 - \frac{4}{3}(S_i)^2\right]. \tag{A4}$$

The first term contains the Casimir operator for the SU(6) group, the second contains the operator for the $\mathrm{SU(3)_{flavor}}$ group, and the third contains the operator for the $\mathrm{SU(2)_{spin}}$ group.

According to the Pauli principle the allowed colorless states for N_c quarks should be completely symmetric under an exchange of SU(6) indices. There is only one SU(6) multiplet which obeys this condition: a symmetric spinor of N_cth rank. Its dimension is given by Eq. (58) in the main text (the 56-plet at $N_c = 3$). The $\mathrm{SU(3)_{flavor}} \otimes \mathrm{SU(2)_{spin}}$ contents of the 56-plet is discussed in the text and consists of the series of multiplets with $S = 1/2 \ldots N_c/2$. Their energies are given by Eq. (A4),

$$\mathcal{E}_{\text{“56”}} = \gamma\left(\frac{3N_c}{2} + 9 - \frac{10}{3N_c}S(S+1)\right) \tag{A5}$$

[where we have used Eq. (58) for C_2 and Eq. (56) for c_2 with $\tilde{Y} = N_c/3$, $X = 0$, $\tilde{T} = J$]. The first term here is the classical energy, the second is the quantum correction, and the third is the rotational energy. These formulas coincide with those obtained in Ref. [36] for a more general case.

The states (A5) exhaust the spectrum of the model [36] only because this model is too poor. One can easily generalize the model by adding to the quarks some internal parameters (indices) and assuming that the potential remains the same and does not depend on these ‘‘hidden’’ parameters. In this case the Pauli principle does not dictate a unique symmetry wave function. In a generalized model the flavor-spin wave function can have any symmetry as long as its product with a wave function of the ‘‘hidden’’ parameters is totally symmetric, as required.

The first excited state of the model corresponds to the SU(6) multiplet, with all SU(6) indices completely symmetric except one pair which is antisymmetric. Its dimension is determined by Eq. (59) of the text; at $N_c = 3$ it corresponds to the 70-plet. Using once more the expressions for the SU(6) and SU(3) Casimir operators, we obtain from Eq. (A4) that

$$\mathcal{E}^{(1-3)}_{\text{“70”}} = \gamma\left[\frac{3N_c}{2} + 5 - \frac{4}{3N_c}S(S+1) - \frac{2}{N_c}\tilde{T}(\tilde{T}+1)\right], \tag{A6}$$

where for three different series of rotational excitations $\tilde{T} = S - 1, S, S + 1$. These three series correspond, as we shall see, to excitations in the u- and d-quark sectors. There are also two series with energies

$$\mathcal{E}^{(4-5)}_{\text{“70”}} = \gamma\left[\frac{3N_c}{2} + 6 - \frac{4}{3N_c}S(S+1) - \frac{2}{N_c}\tilde{T}(\tilde{T}+1) + \frac{3}{2N_c}\right], \tag{A7}$$

where $\tilde{T} = S \pm \frac{1}{2}$. We recognize exactly the five series of $\mathrm{SU(3)} \otimes \mathrm{SU(2)}$ which were described in the text. Other excited states of the model can also be constructed.

Now we are going to reproduce Eqs. (A6) and (A7) in the mean-field approximation. We solve the Dirac equation and the consistency equation following from the Lagrangian (A1),

$$\rho_i^a \sigma_i \lambda^a \phi = \varepsilon\phi, \qquad \rho_i^a = \phi^+_{\text{ground}} \sigma_i \lambda^a \phi_{\text{ground}}. \tag{A8}$$

We are looking for the mean field as an SU(2) ‘‘hedgehog’’: $\rho_i^a = \bar{\rho}\delta_i^a$ for $a = 1 \ldots 3$ and zero otherwise. There are three solutions of the Dirac equation,

$$\phi_0^{\alpha i} = \begin{pmatrix} \varepsilon^{\alpha i} \\ 0 \end{pmatrix}, \qquad \phi_{1(a)}^{\alpha i} = \frac{1}{\sqrt{2}}\begin{pmatrix} \varepsilon^{\alpha j}(\sigma_a)^i_j \\ 0 \end{pmatrix},$$
$$\phi_{s(a)}^{\alpha i} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \delta^{ia} \end{pmatrix}. \tag{A9}$$

Here $\phi_0^{\alpha i}$ (where α is the isospinor index and i is the spinor index) is the wave function with $K = 0$ and energy $\varepsilon_0 = -3\gamma\bar{\rho}$, and $\phi_{1(a)}^{\alpha i}$ are (three degenerate) wave functions of the states with $K = 1$ and energy $\varepsilon_1 = \gamma\bar{\rho}$ in the u- and d-quark sectors. At last, $\phi_{s(a)}^{\alpha i}$ is a wave function of the (doubly degenerate) level in the s-quark sector with energy $\varepsilon_s = 0$.

We obtain the ground-state filling level ε_0 by N_c quarks; then, the consistency equation gives $\bar{\rho} = -1$. The classical part of the mass (proportional to N_c) is obtained by substituting the ground-state wave function into the Hamiltonian (A2). The $O(1)$ part of the energy implies the calculation of quantum corrections (it comes from the one-loop diagram in the quantum field ρ_i^a) and is not interesting for us (see Ref. [36]). However, the difference between the energy of the ground state and the excited ones is calculable, and indeed it is determined by the difference between one-quark levels,

$$\mathcal{E}^{(1-3)}_{\text{“70”}} - \mathcal{E}_{\text{‘56’}} = \varepsilon_1 - \varepsilon_0 + O\left(\frac{1}{N_c}\right) = -4\gamma,$$
$$\mathcal{E}^{(4-5)}_{\text{“70”}} - \mathcal{E}_{\text{‘56’}} = \varepsilon_s - \varepsilon_0 + O\left(\frac{1}{N_c}\right) = -3\gamma. \tag{A10}$$

(We have to choose $\gamma < 0$.)

Turning to the rotational energy [$O(1/N_c)$], we see that in all cases it is determined by Eq. (57) of the text with the moment of inertia

$$I_1 = -\frac{3}{20\gamma}N_c \tag{A11}$$

and the mixing coefficient $\tilde{a}_K = 2/5$ for the excitations in both the u- and d-quark sector [Eq. (A6)] and the s-quark sector [Eq. (A7)].

The moment of inertia I_1 does not coincide with the Inglis [28] moment of inertia $I_{\rm Inglis}$ obtained in the cranking approximation,

$$\mathcal{E}_{\rm rot} = \frac{N_c}{2}\sum_{m=\rm free}\frac{<0|\frac{1}{2}(\tilde{\boldsymbol{\Omega}}\cdot\boldsymbol{\lambda}+\tilde{\boldsymbol{\omega}}\cdot\boldsymbol{\sigma})|m><m|\frac{1}{2}(\tilde{\boldsymbol{\Omega}}\cdot\boldsymbol{\lambda}+\tilde{\boldsymbol{\omega}}\cdot\boldsymbol{\sigma})|0>}{\varepsilon_m-\varepsilon_0} = \frac{1}{2}I_{\rm Inglis}(\tilde{\boldsymbol{\Omega}}-\tilde{\boldsymbol{\omega}})^2, \quad I_{\rm Inglis} = -\frac{1}{8\gamma}N_c. \tag{A12}$$

This was noticed for the first time in Ref. [36]. Hence, in this model there is a nonzero mixing of rotations with other quantum fluctuations of the ρ_i^a field.

To account for the effect of this mixing in the rotational energy we have to find the correction $\delta\rho_i^a$ to the mean field. We solve the Dirac equation with the rotational correction and self-consistency equation,

$$\varepsilon\phi^{\alpha i} = \rho_l^a(\lambda^a)^\alpha_\beta(\sigma^l)^i_j\phi^{\alpha' i'} + \frac{1}{2}\tilde{\omega}_l(\sigma_l)^i_{i'}\phi^{\alpha i'} + \frac{1}{2}\tilde{\Omega}_l(\lambda_l)^\alpha_{\alpha'}\phi^{\alpha' i},$$
$$\rho_i^a = \phi^+\sigma_i\lambda^a\phi, \tag{A13}$$

in the leading order in spin frequency ω_l and flavor frequency Ω^a (we restrict ourselves to $a = 1\ldots3$, which are only relevant at the moment), and obtain

$$\delta\psi^{\alpha i} = -\frac{3}{20\sqrt{2}}(\tilde{\omega}_l(\sigma_l)^i_{i'}\varepsilon^{\alpha i'} + \tilde{\Omega}_a(\lambda_a)^\alpha_{\alpha'}\varepsilon^{\alpha' i}),$$
$$\delta\rho_i^8 = \frac{\sqrt{3}}{10\gamma}(\tilde{\Omega}_i - \tilde{\omega}_i). \tag{A14}$$

The last equation here gives the change of the mean field we are looking for. Due to the hedgehog symmetry $\delta\rho$ depends only on the difference of the flavor and spin frequencies. The correction to the energy due to rotation is (here the first term is a second-order expansion of the quark determinant in rotation and the change of mean field $\delta\rho$ and the second is the change of the pure meson Lagrangian)

$$\mathcal{E}_{\rm rot} = \frac{N_c}{2}\sum_{m=\rm free}\frac{\langle 0|\gamma\delta\rho_i^a\lambda^a\sigma_i + \frac{1}{2}(\tilde{\boldsymbol{\Omega}}\cdot\boldsymbol{\lambda}+\tilde{\boldsymbol{\omega}}\cdot\boldsymbol{\sigma})|m\rangle\langle m|\gamma\delta\rho_i^a\lambda^a\sigma_i + \frac{1}{2}(\tilde{\boldsymbol{\Omega}}\cdot\boldsymbol{\lambda}+\tilde{\boldsymbol{\omega}}\cdot\boldsymbol{\sigma})|0\rangle}{\varepsilon_m-\varepsilon_0} + \frac{\gamma N_c}{2}(\delta\rho_i^a)^2$$
$$= -\frac{3N_c}{40\gamma}(\tilde{\boldsymbol{\Omega}}-\tilde{\boldsymbol{\omega}})^2, \tag{A15}$$

which is, indeed, the rotational energy with moment of inertia I_1. Let us stress again that the fact that this expression is different from the cranking approximation (A12) is completely due to the mixing of rotational degrees of freedom with other quantum fluctuations. Let us note that this mixing is absent if the flavor group is SU(2). In general, the mixing appears because the model is nonrelativistic (see Appendix B).

There are two types of one-quark excitations: in the sector with u, d-quarks (wave function ϕ_1) and in the s-quark sector (wave function ϕ_s). The mixing coefficients a_K are determined by linear terms in the frequencies (see Sec. IV). We have

$$\delta\mathcal{S}_{\rm rot} = -\int dt\sum_{m=\rm excited}\Big[\langle m|\frac{1}{2}(\tilde{\boldsymbol{\Omega}}\cdot\boldsymbol{\lambda}+\tilde{\boldsymbol{\omega}}\cdot\boldsymbol{\sigma})|m\rangle + \langle m|\delta\rho_i^a\lambda^a\sigma_i|m\rangle\Big]$$
$$\equiv -\int dt[(1-a_K)\tilde{\boldsymbol{\omega}}_K\cdot\boldsymbol{K} + a_K\tilde{\boldsymbol{\Omega}}_K\cdot\boldsymbol{K}]. \tag{A16}$$

For the $K = 1$ level in the u- and d-quark sector we introduce the wave function of the excited level as $\phi^{\alpha i}_{\rm excited} = \sum\chi_{K_3}\phi^{\alpha i}_{1(K_3)}$; then,

$$\langle{\rm excited}|\frac{1}{2}(\tilde{\boldsymbol{\Omega}}\cdot\boldsymbol{\lambda}+\tilde{\boldsymbol{\omega}}\cdot\boldsymbol{\sigma})|{\rm excited}\rangle = \frac{1}{2}\sum_{K_3,K_3'}\langle K_3'|(\tilde{\boldsymbol{\Omega}}+\tilde{\boldsymbol{\omega}})\cdot\boldsymbol{K}|K_3\rangle\chi^+_{K_3'}\chi_{K_3}$$

and

$$\langle{\rm excited}|\delta\rho_i^a\sigma_i\lambda_a|{\rm excited}\rangle = -\frac{1}{10}\sum_{K_3,K_3'}\langle K_3'|(\tilde{\boldsymbol{\Omega}}-\tilde{\boldsymbol{\omega}})\cdot\boldsymbol{K}|K_3\rangle\chi^+_{K_3'}\chi_{K_3}. \tag{A17}$$

Comparing this with the Eq. (A16), we see that in this case $\tilde{a}_K = \frac{1}{2} - \frac{1}{10} = \frac{2}{5}$. In terms of Sec. IV, we can say that $c_K = 0$ (this is the consequence of the fact that in our theory there is no orbital momenta) and the mixing coefficient $\zeta = 1/10$.

For excitations of s-quarks the wave function is $\phi^{\alpha i}_{\rm excited} = \sum \chi_{j_3} \phi^{\alpha i}_{s(j_3)}$, where $j = \frac{1}{2}$ is the total momentum of the s-quark excitation. Then

$$\langle \text{excited}| \frac{1}{2}(\tilde{\boldsymbol{\Omega}} \cdot \boldsymbol{\lambda} + \tilde{\boldsymbol{\omega}} \cdot \boldsymbol{\sigma})|\text{excited}\rangle = \sum_{j_3, j_3'} \langle j_3'|\tilde{\boldsymbol{\omega}} \cdot \boldsymbol{j}|j_3\rangle \chi^+_{j_3'} \chi_{j_3},$$

$$\langle \text{excited}|\delta\rho_i^a \sigma_i \lambda_a|\text{excited}\rangle = \frac{2}{5} \sum_{j_3, j_3'} \langle j_3'|(\tilde{\boldsymbol{\Omega}} - \tilde{\boldsymbol{\omega}}) \cdot \boldsymbol{j}|j_3\rangle \chi^+_{j_3'} \chi_{j_3}. \quad \text{(A18)}$$

From the second of these expressions we obtain again the mixing coefficient $\tilde{a}_K = \frac{2}{5}$ (the role of $\boldsymbol{K}$ is played now by $\boldsymbol{j}$).

Thus the mean-field approximation correctly reproduces the exact formulas for energy [Eqs. (A6) and (A7)] in all cases.

APPENDIX B: MIXING OF THE SLOW ROTATIONS IN RELATIVISTIC THEORIES

Rotational degrees of freedom can mix with other quantum fluctuations. This phenomenon is well known in nuclear physics [29]. As we show below, this phenomenon is absent for solitons constructed within the mean-field approximation (at $N_c \to \infty$) with the help of a relativistically invariant meson Lagrangian. This statement is true, at least, for solitons made of pions, e.g., in the Skyrme model [2] or quark-soliton chiral model [10].

Indeed, let us consider in the Skyrme model a flavor rotation $R(t)$ together with some other quantum fluctuations $\delta\pi^a$. The general pion field is presented as

$$\pi_a(\boldsymbol{x}, t)\lambda_a = R(t)[\bar{\pi}^a(\boldsymbol{x}) + \delta\pi(\boldsymbol{x}, t)]\lambda_a R^+(t), \quad \text{(B1)}$$

where $\bar{\pi}_a(\boldsymbol{x})$ is the time-independent mean field. We have to substitute Eq. (B1) into the Skyrme action or the action obtained by an integration over quarks in the quark-soliton model and analyze the contribution of the fluctuations $\delta\pi$ to the effective Lagrangian.

A mixing of the rotations with other quantum fluctuations implies terms of the form $\delta\pi(x, t)\mathcal{K}\Omega$, where Ω is the flavor frequency and $\mathcal{K}$ is some operator. The appearance of Ω means that the mixing can arise only from those terms of the effective chiral Lagrangian which contain time derivatives. However, in the relativistic theory there is usually an even number of time derivatives and there are typically at least two of them. On the other hand, in the leading order in N_c we can consider the frequency Ω to be constant in time. Therefore, the second derivative should be applied to $\delta\pi$, so $\mathcal{K}$ is at least linear in time derivatives. However, all such terms are full derivatives and can be omitted.

The noticeable exclusion from this rule is the Witten-Wess-Zumino term which is linear in time derivatives. We have to apply this derivative to $R(t)$ in order to obtain the flavor frequency. Using the independence of Ω on the coordinates we arrive at (see, e.g., Ref. [10])

$$\delta L = \frac{N_c}{48\sqrt{3}\pi^2} \int dt \Omega_8 \int d^3x \varepsilon_{ijk} \mathrm{Tr}[\lambda_8((U^+\partial_i U) \times (U^+\partial_j U)(U^+\partial_k U)], \quad \text{(B2)}$$

where $U = \exp i(\bar{\pi}^a + \delta\pi^a(x, t))\lambda^a$ is the pion mean field together with quantum fluctuations. The quantity

$$Q_t = \frac{1}{24\pi} \int d^3x \varepsilon_{ijk} \mathrm{Tr}[\lambda_8(U^+\partial_i U)(U^+\partial_j U)(U^+\partial_k U)] \quad \text{(B3)}$$

is the topological charge of the field U and cannot be changed by the small fluctuations of the pion field. In other words,

$$\frac{\delta Q}{\delta\pi(x, t)} = 0. \quad \text{(B4)}$$

Hence the mixing of rotations with quantum fluctuations is absent in the Skyrme model.

The situation is similar in the quark-soliton model [10]. Mixing can appear only from the so-called "imaginary part" of the effective π-meson action. This part of the action starts with a Witten-Wess-Zumino term, but in principle it is an infinite series in gradients of the pion field. However all these terms are full derivatives and the sum reduces to the complete baryon charge of the state. This quantity is determined by the number of valence quarks and cannot be changed by the small fluctuations of the pion field.

This does not mean that the mixing is always zero. First, it can appear in more general meson Lagrangians. Second, it can arise if the frequencies of the rotations are not small. For example, the properties of rotational exotic states [at $\Omega_{4,5,6,7} \sim O(1)$] can be described as a mixing of rotations and quantum K-meson states belonging to the continuum spectrum. This means that the width of these states is $\sim O(1)$. However, the rotational theory in this case should be modified anyway, since the Wess-Zumino-Witten-term rotations in the strange directions turn into small oscillations (see, e.g., Refs. [5,20,42]).

APPENDIX C: MATRIX ELEMENTS OF ONE-PARTICLE OPERATORS

The Dirac equation in the u- and d-quark sector conserves the grand spin K. The angular part of the wave function of the state with a given K is a spherical spinor-isospinor,

$$\Xi^{\alpha i}_{KK_3 jl}(\boldsymbol{n}) = \sum_{j_3} C^{KK_3}_{jj_3,\frac{1}{2}\alpha} \Omega^i_{jj_3 l}(\boldsymbol{n}). \quad \text{(C1)}$$

Here α is the isospinor index, i is the spinor index, and Ω is a spherical spinor with total angular moment j (projection j_3) and orbital momentum l; the Clebsh-Gordan coefficient

$C^{\cdots}_{\cdots}$ joins j and the isospin $\boldsymbol{t}$ ($t = \frac{1}{2}$) into the grand spin $\boldsymbol{K}$. The total angular momentum can be $j = K \pm \frac{1}{2}$. The spherical spinor Ω^i is constructed from the spin of the quark $\boldsymbol{s}$ ($s = \frac{1}{2}$) and the orbital momentum $\boldsymbol{l}$,

$$\Omega^i_{jj_3l}(\boldsymbol{n}) = \sum_{j_3} C^{jj_3}_{ll_3,\frac{1}{2}i} Y_{ll_3}(\boldsymbol{n}), \tag{C2}$$

where $Y_{ll_3}(\boldsymbol{n})$ are usual spherical harmonics.

We are looking for the solution of the Dirac equation, $\varepsilon\Psi = \mathcal{H}\Psi$, with the Dirac Hamiltonian (1) which is a bispinor $\{\varphi, \chi\}$ in the form

$$\Psi^{\alpha i} = \begin{pmatrix} g(r)\Xi^{\alpha i}_{KK_3,K-\frac{1}{2},K} + h(r)\Xi^{\alpha i}_{KK_3,K+\frac{1}{2},K} \\ f(r)\Xi^{\alpha i}_{KK_3,K-\frac{1}{2},K-1} + j(r)\Xi^{\alpha i}_{KK_3,K+\frac{1}{2},K+1} \end{pmatrix}. \tag{C3}$$

This state has a "natural parity" $(-1)^K$. Indeed, the parity transformation is $\varphi(\boldsymbol{r}) \to \varphi(-\boldsymbol{r})$, $\chi(\boldsymbol{r}) \to -\chi(-\boldsymbol{r})$ and hence parity is determined by the value of l. The state with parity $(-1)^{K+1}$ comes from the exchange of φ and χ in this expression. At $K = 0$ the wave function is determined by only two functions: $h(r)$ and $j(r)$.

We want to calculate the matrix elements of $\boldsymbol{t}$. Since it acts only on isospinor indices α and the spherical spinors Ω are orthonormal, we obtain

$$\langle K_3'|\boldsymbol{t}|K_3\rangle = \sum_{\alpha\beta,j,j_3} (C^{KK_3'}_{jj_3\frac{1}{2}\alpha})^* \langle\alpha|\boldsymbol{t}|\beta\rangle C^{KK_3}_{jj_3\frac{1}{2}\beta} \times [(1 - c_K)\delta_{jK-\frac{1}{2}} + c_K\delta_{jK+\frac{1}{2}},] \tag{C4}$$

with c_K defined by Eq. (42) and $\langle\alpha|\boldsymbol{t}|\beta\rangle$ are the usual generators of isospin $\frac{1}{2}$.

Substituting the Clebsh-Gordan coeffients, we arrive at Eq. (41). In particular, if $\boldsymbol{t} \to t_3$ the matrix elements are diagonal in α, β and hence diagonal in K_3, K_3'. The matrix element

$$\langle K_3|t_3|K_3\rangle = \frac{1}{2}(|C^{KK_3}_{j,K_3-\frac{1}{2},\frac{1}{2},\frac{1}{2}}|^2 - |C^{KK_3}_{j,K_3+\frac{1}{2},\frac{1}{2},-\frac{1}{2}}|^2) \times [(1 - c_K)\delta_{jK-\frac{1}{2}} + c_K\delta_{jK+\frac{1}{2}},] \tag{C5}$$

and by using the expressions for the Clebsch-Gordan coefficients we obtain

$$\langle K_3|t_3|K_3\rangle = K_3\left(\frac{1 - c_K}{2K} - \frac{c_K}{2(1 + K)}\right). \tag{C6}$$

This is a particular case of Eq. (41).

APPENDIX D: DECAYS OF EXCITED BARYONS

Calculations of the widths of excited baryons are outside the scope of this paper; however, in this appendix we present only a general discussion of the baryon decays. The calculation procedure for the ground-state baryons is known: it was constructed for the Skyrme model in Ref. [2] and for the quark-soliton model in, e.g., Refs. [10,24,39]. Finally, in the limit $N_c \to \infty$ the decay constants in the approach of Ref. [22] were calculated in, e.g., Refs. [43,44] and in other, already cited papers.

Typical decays of excited baryons below 2 GeV are of the type $B_i \to B_f M$ with one emitted meson; at least, such decays always give the essential part of the width. To be specific, we will talk about decays into π mesons. Let us estimate the width in the limit of large N_c. [This estimate is already known: we are close here to the approach of J. L. Goity in Ref. [8] (see also Ref. [44]).] The one-pion decays of the excited baryons are described by the effective Lagrangian of the type

$$\mathcal{L}_{\text{eff}} = \frac{g_a}{F_\pi}\int d^3x \bar{\Psi}^{(f)}_B \gamma_\mu\gamma_5 \frac{\lambda^a}{2}\Psi^{(i)}_B \partial_\mu\pi. \tag{D1}$$

Here $\Psi^{(i)}_B$ and $\Psi^{(f)}_B$ are the fields of the initial and final baryons, π is the π-meson field with flavor a, λ_a is the corresponding Gell-Mann matrix, and g_a is the transitional axial coupling constant. The width Γ_{fi} of the partial decay to $B_f\pi$ is proportional to the coupling constant squared and the phase volume

$$\Gamma_{fi} \sim \frac{g_a^2}{8\pi F_\pi^2}\Delta^3 \tag{D2}$$

(see, e.g., Ref. [5]), where $\Delta = M_i - M_f$ is the difference of the mass of the initial and final baryons.

The coupling constant can be calculated as a matrix element of the corresponding quark operator between the mean-field initial and final states,

$$g_a(k) \sim \int d^3x \langle\text{fin}|\bar{\psi}\gamma_5\gamma_\mu\psi(x)|\text{in}\rangle e^{ikx}. \tag{D3}$$

The role of the quark operator is played by the axial current for decays with π mesons, the vector current for decays into ρ mesons, etc. Equation (D3) already implies the $N_c \to \infty$ limit, as baryons are considered to be heavy [with mass $O(N_c)$] nonrelativistic objects. [Equation (D2) is also written in this limit.] The plane wave e^{ikx} represents the wave function of an emitted meson, with k being its momentum, and $|\text{in}\rangle$ and $|\text{fin}\rangle$ are the mean-field approximations for the initial and final baryon quark wave functions, respectively. They are products of all one-quark wave functions—solutions of the Dirac equation in the mean field—for all filled levels. In general, here one has to write wave functions rotated by matrices R and S in order to take into account the degeneracy of the mean field. After the calculation of the matrix element (D3), we obtain some operator depending on the collective coordinates. Averaging this operator with the collective wave functions of the initial and final baryons, we obtain the coupling constant for some specific decay.

In fact, Eq. (D3) is only the first term of the expansion in the time derivatives of the collective coordinates. The next terms can be obtained in the same manner as was done for corrections to m_s in the main text. Due to the limit $N_c \to \infty$ all collective coordinates are slowly varying functions of coordinates, so the expansion in time derivatives is an

expansion in $1/N_c$, with Eq. (D3) being its leading term. For ground-state baryons and for decays into pions the corresponding formulas were presented in Ref. [39].

Decays of excited baryons are possible for either baryons belonging to the same rotational band or baryons which have a different filling of intrinsic quark levels (e.g., to ground-state baryons). In the first case the coupling constant is large [$O(N_c)$]. An example is the transitional axial constant $g_a(\Delta \to N\pi)$ [2]. In the second case the coupling constant is always smaller. This difference is clearly seen from Eq. (D3).

$$g_a(R,S) \sim \int d^3x \phi_f^*(\mathbf{x}) S^+ \gamma_3\gamma_5 S R^+ \frac{\lambda^a}{2} R \phi_i(\mathbf{x}) j_l(k|\mathbf{x}|) \mathcal{D}^l_{m_1,m_2}(S) Y_{lm_2}\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) \begin{cases} N_c & i,f = \text{sameband}, \\ \sqrt{N_c} & i = \text{excited}, f = \text{ground}, \\ 1 & i = \text{excited}, f = \text{excited}'. \end{cases} \tag{D4}$$

This expression is written a bit schematically. The wave functions ψ_i and ψ_f are the initial and final wave functions of the excited quark, R is a rotational matrix in flavor space and S is a rotational matrix in ordinary space, $\mathcal{D}^l_{m_1m_2}(S)$ is the Wigner function, Y_{lm} are ordinary spherical harmonics (a summation over all possible m_2's is implied), and $j_l(kr)$ is a spherical Bessel function. This appears (together with spherical harmonics) as a result of the expansion of a plane wave in Eq. (D3) into the set of spherical waves. If the momentum of an emitted meson is small, $ka \ll 1$ (a is the scale of the wave functions $\psi_{i,f}$, which coincides with the characteristic size of the baryon), it is sufficient to account for only the lowest angular momentum, $l = 0$ (the angular momentum of the emitted pion is 1).

The axial constant (D4) is an operator in the space of collective coordinates (derived in the leading order in N_c). To obtain the coupling constant responsible for the decay of a concrete baryon to another one, we have to average Eq. (D4) with the collective wave functions,

$$g_a(i \to f) = \int dR dS \psi_f^{(\mathrm{rot})*}(R,S) g_a(R,S) \psi_i^{(\mathrm{rot})}(R,S). \tag{D5}$$

Despite the fact that the coupling constants are smaller, the widths of the decays to the different quark levels are typically larger in N_c. The reason is that the phase volume in this case is always larger. The mass differences are $O(1/N_c)$ for decays inside the same rotational band, but they are $O(1)$ for transitions with a change in the intrinsic state of the excited quark. As a result the widths of the decays inside the rotational band are suppressed as $O(1/N_c^2)$, while the decays of excited baryons with a discharge of the excitation are always $O(1)$ (and decays to the other levels are suppressed). In particular, the total width of ground-state baryons (decuplet with spin $\frac{3}{2}$) is only $O(1/N_c^2)$, while all remaining baryons have a total width of $O(1)$.

In practical terms only decays to the ground octet or decuplet are observable. For all baryons they have partial widths independent of N_c up to corrections in $1/N_c$, which can still be essential at $N_c = 3$. Let us prove the theorem that the widths of all baryons belonging to the same rotational band are the same in the leading order in N_c.

Indeed, when the configuration of levels is the same for the initial and final stats the coupling constant is a sum of N_c one-particle matrix elements corresponding to all N_c quarks. If an excited quark changes its intrinsic state then only one of the N_c contributions would survive, namely the overlap one-particle matrix element between the initial and final states of the quark (all other contributions are zero due to the orthogonality of the wave functions). However, if the final state is the ground state an additional factor $\sqrt{N_c}$ appears, which is due to the different normalization of the initial and final wave functions,

Indeed, the mass differences of all baryons entering the same rotational band are the same in the leading order in N_c. Hence

$$\Gamma^{(i)}_{\mathrm{tot}} = \sum_f \Gamma(i \to f) = \frac{\Delta^3}{8\pi} \sum_f g_a^2(i \to f) = \Gamma_{\mathrm{level}}. \tag{D6}$$

However, the sum of axial constants squared over all possible final states does not depend on the initial state of the band. According to Eq. (D5) the axial constant squared contains two integrals in R, S and R', S'. The completeness of final baryon rotational functions,

$$\sum_f \psi_f^{(\mathrm{rot})*}(R,S) \psi_f^{(\mathrm{rot})}(R',S') = \delta(R-R')\delta(S-S'),$$

leads to $R = R'$ and $S = S'$. Then the sum in all possible flavors of pseudoscalar mesons and directions of axial current gives an expression which does not depend on R and S due to the Fiertz identities. The dependence on the matrices remains only in the initial wave function. The integral over R and S becomes the normalizing integral for the initial collective wave function and the dependence on the initial state disappears completely. The obtained total width has a sense of the complete width of the intrinsic quark level and is universal for the whole rotational band around it.

The theorem proved above is broken strongly in nature. There are many reasons for this, such as corrections in N_c and the mass of the strange quark m_s to the coupling constants, the mixing of multiplets, etc. Perhaps the strongest source of the deviations is simply the difference in the phase volumes [which is an $O(1/N_c)$ effect] for different baryons entering the same rotational band.

The fact that the widths of excited baryons are not suppressed in the large-N_c limit—as was mentioned in the

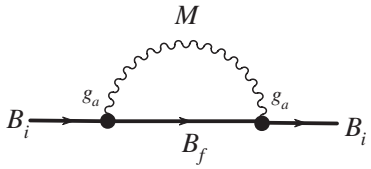

FIG. 4. Self energy correction to the excited baryon mass.

Introduction—prevents these baryons from being well defined. One can only count on the numerical smallness of the width not related to N_c. In such a situation baryon resonances can be defined only as poles in the complex plane of the meson-nucleon scattering amplitude. This approach was applied in Ref. [5] to the problem of the pentaquark (which also has a width that is independent of N_c) for the Skyrme model, but in the general case it looks too complicated. If the width is small one returns to the self-consistent field description presented here.

It seems that the width of the baryon, which is not suppressed at $N_c \to \infty$, poses the greatest danger to our approach in general. Indeed, due to unitarity a nonzero width implies not only the imaginary part of the pole but also a shift in the real part, i.e., it leads to a change in the baryon mass. It can be small numerically, but it is $O(1)$ in N_c. If it is different for the baryons entering the same rotational band, our formulas for the mass splittings inside the rotational band would become nonsensical. Fortunately, this is not the case.

Corrections to the mass due to decays into the π mesons are presented by a self-energy diagram in Fig. 4. The imaginary part of this diagram gives the width of $B_i \to B_f M$ decay, while the real part gives the shift in mass. The point is that the mass shift does not depend on the baryon B_i in the same rotational band,

$$\Delta M \sim \sum_f \int \frac{d^4k}{(2\pi)^4} \frac{g_a^2(i \to f)}{k^2(\Delta + k_0)},$$

as was proved above. Hence we arrive at the conclusion that mass shifts are universal for all baryons inside the rotational bands. It can be included in the general shift of the intrinsic level and it does not affect the mass relations in the $O(1/N_c)$ order derived in the text. Next-order corrections in N_c due to the finite width of the resonance also do not destroy these relations. However, they can renormalize the moment of inertia I_1. An example of such a situation is given by the pentaquark in the Skyrme model [5].

APPENDIX E: ISOSCALAR FACTORS FOR $N_c \to \infty$

Clebsch-Gordan coefficients for large N_c-baryon multiplets were calculated in a number of studies [22,31,45]. Two methods can be used for this calculation, based on either the decomposition the SU(3) spinor with a large number of indices [22,45] or the application of lowering and raising generators in the given representation to the state with the highest weight [31].

However, the tables of Clebsch-Gordan coefficients in the above references are usually incomplete and do not correspond to the conventions of Ref. [46] [which serves as a common standard for the SU(3) group], but rather differ from this standard by a unitary transformation. This is inconvenient and for this reason we give here the complete tables (Tables III and IV) for isoscalar factors at arbitrary N_c.

During the refereeing we were informed that the complete tables of isoscalar coefficients were presented in Ref. [47]. The tables presented here are analogous and differ only by the method of calculation. Nonetheless, we present here the tables for the completeness of the description. We thank the referee for drawing our attention to this fact.

Consulting with the tables, one can see that the change of Clebsch-Gordan coefficients from $N_c \to \infty$ to $N_c = 3$ is, indeed, rather large. In particular, one can note the cases when the isoscalar factor changes sign during this transition. Moreover, for the "decuplet" there are *two* possible final multiplets at $N_c \geq 5$, so one can discuss the F/D ratio for the "decuplet." The corresponding multiplet dies out as $N_c = 3$.

On the other hand, Clebsch-Gordan coefficients can be easily taken into account at any N_c and this source of inaccuracy can be avoided. For this reason we prefer to use isoscalar factors at $N_c = 3$.

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