Model independent electromagnetic corrections in hadronic τ decays

F. V. Flores-Baez* and J. R. Morones-Ibarra†

Facultad de Ciencias Físico-Matemáticas, Universidad Autónoma de Nuevo León (UANL), Avenue Universidad S/N Ciudad Universitaria, San Nicolás de los Garza, Nuevo León 66451, Mexico

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In this article, the long-distance correction to the total decay width of the $\tau^\pm \to K^0\pi^\pm\nu$ decay is calculated in a model-independent approach, in which a discrimination of photons in the bremmstrahlung process is assumed. This correction is completely free of ultraviolet and infrared singularities, and, moreover, it satisfies electromagnetic gauge invariance. The result of this work can be applied on the tau decays: $\tau^\pm \to \pi^\pm\pi^0\nu$, $K^\pm\pi^0\nu$.

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I. INTRODUCTION

It is well known that hadronic τ decays are an ideal laboratory to obtain information about the fundamental parameters within the Standard Model and also some properties of QCD at low energies [1,2]. In particular, the $\tau \to K\pi\nu$ decay has been studied in the past by ALEPH [3] and OPAL [4] and recently at B factories [5,6], where the high statistic measurements provide excellent information about the structure of the spectral functions, parameters of the intermediate states, and the total hadronic spectral function. It is also possible to determine the product $|V_{us}|F_+^{K^0\pi^+}(0)$ from this decay, although the best determination comes from semileptonic kaon decays [7].

On the theoretical side, these sorts of processes have a nice feature: the decay amplitude can be factorized into a pure leptonic part and a hadronic spectral function [8] in such a way that the differential decay distribution reads

$$\frac{d\Gamma_{K\pi}}{d\sqrt{s}} = \frac{G_F^2(|V_{us}|F_+^{K\pi}(0))^2 m_\tau^3}{32\pi^3 s} S_{\mathrm{EW}}\left(1-\frac{s}{m_\tau^2}\right)^2 \times\left[\left(1+\frac{2s}{m_\tau^2}\right)q_{K\pi}^3|\tilde{F}_+^{K\pi}(s)|^2 + \frac{3\Delta_{K\pi}^2 q_{K\pi}|\tilde{F}_0^{K\pi}(s)|^2}{4s}\right], \tag{1}$$

where a sum over the two possible decays $\tau^+ \to K^+\pi^0\nu$ and $\tau^+ \to K^0\pi^+\nu$ has been done, isospin symmetry is assumed, and the reduced vector and scalar form factors have been normalized to 1 at the origin

$$\tilde{F}_+^{K\pi}(s) = \frac{F_+^{K\pi}(s)}{F_+^{K\pi}(0)}, \qquad \tilde{F}_0^{K\pi}(s) = \frac{F_0^{K\pi}(s)}{F_+^{K\pi}(0)}. \tag{2}$$

In this expression, $\Delta_{K\pi} \equiv m_K^2 - m_\pi^2$, and the kaon momentum in the rest frame of the hadronic system reads

$$q_{K\pi} = \frac{1}{2\sqrt{s}}\sqrt{(s-(m_K+m_\pi)^2)(s-(m_K-m_\pi)^2)} \times \theta(s-(m_K+m_\pi)^2). \tag{3}$$

A theoretical description [9] of the vector and scalar form factors has been done in the resonance chiral theory (RχT) framework [10], providing a successful representation of the data. It is worth mentioning that Eq. (1) includes the short-distance correction S_{EW} [11,12]; however, the long-distance correction is not included neither as a global factor nor inside the form factors. This long-distance correction is the electromagnetic correction considering the decay as a punctual 4-body interaction with form factors. In the B factories [5,6], an improved experimental precision can be achieved in the future, which makes it mandatory to have a theoretical analysis of the long-distance correction effects.

It is well known that, in all decays with a charged particle, the emission of photons is always present, altering the dynamics of the decay. The approximative next-to-leading-order algorithms [13] are used to simulate the correction due to soft photons for which the virtual corrections (one loop) are reconstructed numerically up to the leading logarithms from the real photon corrections. An improved algorithm [14] can be applied if a phenomenological model is used to describe the behavior of the invariant amplitude.

On the other hand, the first attempt to describe the lepton-hadron interaction used a simple effective interaction approach; nonetheless, once the electromagnetic correction is computed, an unfortunate feature appears: it depends on the cutoff energy [15] that controls the UV singularity. Nevertheless, in the chiral perturbation theory (χPT) framework [16], it is possible to describe the interactions between the lightest multiplet of pseudoscalar mesons and the lightest leptons at low energy [below the $\rho(770)$ resonance region] including also real and virtual photons [17,18], and, due to the character of χPT, the UV singularity is cancelled by adding a finite number of appropriated counterterms. At the end, one does not have to deal with a UV cutoff but with the finite pieces of the

*fflores@fcfm.uanl.mx
†rmorones@fcfm.uanl.mx

coupling constants of the effective theory. A nice example of this type of electromagnetic correction treatment was done in K_{l3} [19].

Another alternative for the analysis of electromagnetic corrections is given by a model-independent (MI) approach [20]. In this method, the invariant amplitude of the radiative correction is separated into external and internal contribution. The external radiative correction contains the MI correction and a model-dependent (MD) piece, and it is obtained by considering Feynman diagrams in which the emission and absorption of real or virtual photons occur in external lines. The internal radiative correction corresponds to diagrams in which a photon is attached to an internal line and clearly depends on the precise details of the process; in other words, this part is MD. The aim of this technique is to procure an electromagnetic correction, evaluated once and for all, which is gauge invariant, free of UV singularities, and contains the dominant logarithms that come after the infrared singularity cancellation.

The MI technique has been applied in the analysis of the electromagnetic corrections to the Dalitz plot of semileptonic decays of hyperons [21], in the calculation of radiative correction to leptonic decays of a pseudoscalar meson [22], and also recently in the radiative corrections analysis to the Dalitz plot of the K_{l3} decay [23]. It is worth mentioning that the authors in Ref. [23] have quoted that their result in K_{l3} has been compared with the universal electromagnetic correction given in Ref. [19] with a very small quantitative difference. That comparison also helps us to identify the model-dependent part in the MI scheme with the corresponding form factor computed within χPT. In the effective approach, the form factors can be separated into two parts: the first one contains the universal electromagnetic correction that is linked with the MI electromagnetic correction, and the second one can be seen as the corresponding MD electromagnetic form factor, which in the context of χPT is made of hadronic loop contributions, finite local terms, and electromagnetic pieces that are left aside once the universal correction is defined. It is worth mentioning that the definition of the universal correction is not unique; meanwhile, the correction computed in the MI approach is very well defined.

On the other hand, in hadronic τ decays, the energy transferred $\sqrt{s}$ to the hadronic system goes from a threshold $(m_1 + m_2)$ up to m_τ, which means that the expansion parameter of χPT is no longer valid in all the region. In this respect, we cannot apply consistently the work done in k_{l3} in computing the electromagnetic corrections in hadronic tau decays. In this paper, we are not committed to giving a description of the form factors; our aim is to present the MI electromagnetic corrections to the $\tau \to K\pi\nu$ decay following the techniques given in Ref. [20]. In Sec. II, we describe briefly the basis of the hadronic tau decay and general effects of electromagnetic correction. In Sec. III, the MI electromagnetic corrections for virtual and real photons are computed. In Sec. IV, we present the MI corrections to the $\tau^+ \to K^0\pi^+\nu$ decay width, and in Sec. V, we give a brief discussion about our result.

II. τ DECAY AND THE ELECTROMAGNETIC INTERACTION

A. Basic elements on τ decays

Here, we do a brief review of the very well-known elements of the $\tau^\pm \to P^\pm P^0 \nu_\tau$ decay for which P is a pseudoscalar meson. We start with the invariant amplitude without electromagnetic effects expressed [8] as

$$\mathcal{M}_\tau^{(0)} = C_{\rm CG} G_F V_{\rm CKM} l_\mu h^\mu, \tag{4}$$

where G_F is the Fermi constant, $V_{\rm CKM}$ is the Cabibbo–Kobayashi–Maskawa (CKM) matrix element, the Clebsch–Gordon coefficient $C_{\rm CG} = (-\frac{\sqrt{2}}{2}, \frac{1}{2}, 1)$ depends on the hadronic final state $(\pi^+ K^0, \pi^0 K^+, \pi^+\pi^0)$, and the leptonic current is given by

$$l_\mu = \bar{u}_{\nu_\tau}(q)\gamma_\mu(1-\gamma^5)u_\tau(p_\tau); \tag{5}$$

meanwhile, the hadronic part h^μ that contains the form factors $F_+(s)$, $F_-(s)$ is written as

$$h^\mu = F_+(s)(p^+ - p^0)^\mu + F_-(s)(p^+ + p^0)^\mu, \tag{6}$$

where $p_+(p_0)$ is the 4-momenta of the charged (neutral) pseudoscalar meson and $s = (p_+ + p_0)^2$. The density function at tree level is given by

$$\rho^{(0)}(s,u) = |F_+(s)|^2\mathcal{D}(s,u) + 2F_+(s)F_-^\dagger(s)\mathcal{D}_3(s,u) + |F_-(s)|^2\mathcal{D}_2(s), \tag{7}$$

and the $\mathcal{D}$ functions are given as follows:

$$D(s,u) = 2u^2 + 2u(s - m_\tau^2 - m_+^2 - m_0^2) + \frac{m_\tau^2}{2}(m_\tau^2 - s) + 2m_+^2 m_0^2, \tag{8}$$

$$D_2(s) = \frac{m_\tau^2}{2}(m_\tau^2 - s), \tag{9}$$

$$D_3(s,u) = \frac{m_\tau^2}{2}(2m_0^2 + m_\tau^2 - s - 2u). \tag{10}$$

The differential decay width can be written in a compact equation,

$$\frac{d\Gamma^{(0)}_{P^+P^0}}{dsdu} = \frac{|C_{\rm CG} G_F V_{\rm CKM}|^2}{(4\pi)^3 m_\tau^3}\rho^{(0)}(s,u), \tag{11}$$

where $u = (p_\tau - p^+)^2$. After doing the u integration, it is straightforward to get

$$\frac{d\Gamma^{(0)}_{P^+P^0}}{ds} = \frac{|C_{\rm CG}G_F V_{\rm CKM}|^2 m_\tau^3 q_{+0}(1-\frac{s}{m_\tau^2})^2}{3(4\pi)^3\sqrt{s}} \times \left\{|F_+(s)|^2\left[\left(1+\frac{2s}{m_\tau^2}\right)\frac{4q_{+0}^2}{s}+\frac{3\Delta_+^2}{s^2}\right] + 3|F_-(s)|^2 + 6F_+(s)F_-^\dagger(s)\frac{\Delta_+}{s}\right\}, \tag{12}$$

where $\Delta_+ = m_+^2 - m_0^2$ and the mass of the charged (neutral) meson is denoted by $m_+(m_0)$. The momentum q_{+0} in the rest frame of the hadronic system reads

$$q_{+0} = \frac{1}{2\sqrt{s}}\sqrt{(s-(m_+ + m_0)^2)(s-(m_+ - m_0)^2)} \times \theta(s-(m_+ + m_0)^2). \tag{13}$$

It is more familiar and elegant to write Eq. (12) in terms of the scalar form factor $F_0(s)$ defined by

$$F_-(s) = \frac{\Delta_+}{s}[F_0(s) - F_+(s)]. \tag{14}$$

With the help of Eq. (14), the differential decay width reads

$$\frac{d\Gamma^{(0)}_{P^+P^0}}{d\sqrt{s}} = \frac{|G_F V_{\rm CKM}|^2 m_\tau^3}{32\pi^3 s}\left[1-\frac{s}{m_\tau^2}\right]^2 q_{+0}^3 A_{p^+p^{(0)}} \times \left[\frac{3\Delta_+^2}{4sq_{+0}^2}|F_0(s)|^2 + |F_+(s)|^2\left(1+\frac{2s}{m_\tau^2}\right)\right], \tag{15}$$

where $A_{p^+p^{(0)}} = \frac{4}{3}C_{\rm CG}^2$ and its value is equal to $\frac{1}{3}(\frac{2}{3})$ for the hadronic final state $K^+\pi^0$, $(K^0\pi^+)$. It is easy to see that summing both final $K\pi$ states, we get Eq. (1). In the case of the $\pi^+\pi^0$-final state, $A_{\pi\pi} = 4/3$, and the scalar form factor contribution vanishes due to isospin symmetry.

The energy-momenta conservation defines a set of allowed values for the cinematic variables (u, s) that can be shown in a Dalitz plot (See Fig. 1), for which the borders are given by

$$(m_+ + m_0)^2 \le s \le m_\tau^2, \qquad u_-(s) \le u \le u_+(s), \tag{16}$$

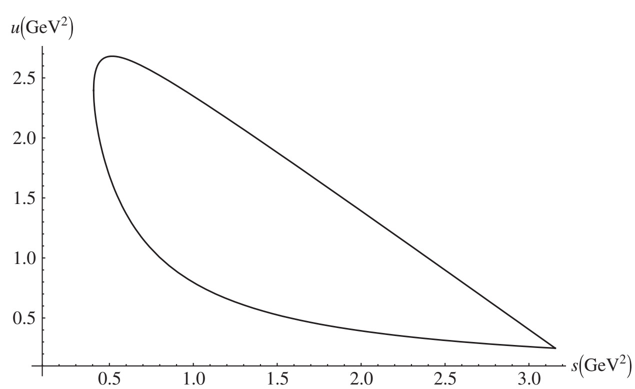

FIG. 1. The Dalitz plot without electromagnetic corrections for the $\tau^\pm \to K^0\pi^\pm\nu$ decay. The region is defined by Eqs. (16) and (17).

where

$$u_\pm(s) = \frac{1}{2s}\left\{(s-m_\tau^2)(s+m_+^2-m_0^2) + 2s(m_\tau^2+m_0^2-s) \pm (m_\tau^2 - s)\sqrt{(s-(m_+^2+m_0^2))^2 - 4m_+^2 m_0^2}\right\}. \tag{17}$$

Considering the hadronic final state $K^0\pi^+$, the integrated decay width is found to be

$$\Gamma^{(0)}_{K\pi} = |F_+(0)V_{\rm CKM}|^2 G_F^2 C_{\rm CG}^2 \mathcal{N}_i I^{(0)}_{K\pi}, \tag{18}$$

where $\mathcal{N}_i = [\frac{m_\tau^5}{48\pi^3}]$ and the integrated density $I^{(0)}_{K\pi}$ is defined as follows:

$$I^{(0)}_{K\pi} = \int \frac{[1-\frac{s}{m_\tau^2}]^2 ds}{m_\tau^2 s\sqrt{s}}\left[|\tilde{F}_+^{K\pi}(s)|^2\left(1+\frac{2s}{m_\tau^2}\right)q_{K\pi}^3 + \frac{3\Delta_+^2 q_{K\pi}}{4s}|\tilde{F}_0^{K\pi}(s)|^2\right]. \tag{19}$$

In order to compute $I^{(0)}_{K\pi}$, one needs a theoretical description of the normalized vector and scalar form factors $\tilde{F}_+^{K\pi}(s)$ and $\tilde{F}_0^{K\pi}(s)$. This task is achieved by a fit to the measured distributions to $\tau^\pm \to K_s\pi^\pm\nu$. There is a strong effort in computing this integral [24] with a dispersive representation of the form factors, but for illustrative purpose, we consider the parametrized vector and scalar form factors as given by the Belle Collaboration [6] as

$$F_V(s) = F_+(s) = \frac{1}{1+\beta}[\mathrm{BW}_{K^*(892)}(s) + \beta\mathrm{BW}_{K^*(1410)}(s)], \tag{20}$$

where β is the fraction of the $K^*(1410)$ resonance contribution and $\mathrm{BW}_R(s)$ is a relativistic Breit–Wigner function

$$\mathrm{BW}_R(s) = \frac{M_R^2}{M_R^2 - s - \imath M_R\Gamma_R(s)}, \tag{21}$$

and $\Gamma_R(s)$ is the s-dependent total width of the resonance,

$$\Gamma_R(s) = \Gamma_{0,R}\frac{s}{M_R^2}\left(\frac{\sigma_{K\pi}(s)}{\sigma_{K\pi}(m_R^2)}\right)^3, \tag{22}$$

$$\sigma_{K\pi}(s) = \frac{2q_{K\pi}(s)}{\sqrt{s}}. \tag{23}$$

For the scalar form factor $F_0(s)$, we take a description that includes only the $K_0^*(800)$ resonance,

$$F_0(s) = \kappa\frac{s}{M^2_{K_0^*(800)}}\mathrm{BW}_{K_0^*(800)}(s), \tag{24}$$

where κ is a complex constant and represents the fraction of the scalar resonance contribution.

The masses and widths of $K^*(892)$, $K^*(1410)$ are fixed from Ref. [25]; meanwhile, the parameters of $K_0^*(800)$ are taken from Ref. [26]. With the values of Table 3 given in Ref. [6], it is found $I^{(0)}_{K_S\pi^\pm} = 0.384221$. Moreover, assuming that

$$\begin{aligned}\mathcal{B}(\tau^\pm \to K^0\pi^\pm\nu) &= \mathcal{B}(\tau^\pm \to K_S\pi^\pm\nu) + \mathcal{B}(\tau^\pm \to K_L\pi^\pm\nu)\\ &= 2\mathcal{B}(\tau^\pm \to K_S\pi^\pm\nu),\end{aligned} \tag{25}$$

then we get the raw number

$$I^{(0)}_{K^0\pi^\pm} = 2I^{(0)}_{K_S\pi^\pm} \sim 0.768. \tag{26}$$

B. Overview on electromagnetic corrections

Here, we address general effects of the electromagnetic corrections. First, we assume the simplest and easiest scenario: there is just one form factor that is not affected by the one-loop integration; in other words, its dependence on transferred energy is negligible, and we denote this by writing $F_+(s) = F_V$. In this case, the tree-level amplitude for the $\tau^+ \to P^+P^0\nu$ decay reads

$$\mathcal{M}^{(0)} = G_F V_{\mathrm{CKM}} C_{\mathrm{CG}} F_V l_\mu (p_+ - p_0)^\mu. \tag{27}$$

The amplitude for the one-loop electromagnetic correction with a pointlike meson-photon interaction is found to be

$$\begin{aligned}\mathcal{M}^v = G_F V_{\mathrm{CKM}} C_{\mathrm{CG}} F_V l_\mu \frac{\alpha}{4\pi}[f_+^{e.m.}(u)(p^+ - p^0)^\mu\\ + f_-^{e.m.}(u)(p^+ + p^0)^\mu_+].\end{aligned} \tag{28}$$

The details of the one-loop functions are not necessary for giving our remarks. It is well known that $f_+^{e.m.}(u)$ encloses UV and IR singularities; meanwhile, $f_-^{e.m.}(u)$ contains just a UV singularity. The IR divergence is cancelled by taking into account the real photon emission; meanwhile, the UV problem can be solved with a cutoff in the one-loop integration that replaces the singularity. In short, this method consists of making the next replacement

$$\Delta_{\mathrm{UV}} \to \ln\left[\frac{\Lambda^2}{m^2}\right], \tag{29}$$

where Δ_{UV} encloses the UV singularity in some regularization method, the cutoff Λ can be some resonance mass, and m is a characteristic mass of the process.

The replacement makes sense because it represents that something is hidden in the effective 4-body interaction encoded in Eq. (27). To make an analogy, we recall the one-loop corrections (at short distances) computed [27] within the Standard Model for the process $\tau \to \nu_\tau d\bar{u}$, where the pure QED correction in the Fermi model is UV divergent, but if all the electroweak corrections are included (Z and W virtual exchange), then the ultraviolet cutoff ($\ln[\frac{\Lambda^2}{m_\tau^2}]$) is replaced naturally by a large logarithm, namely, $\ln[m_Z/m_\tau]$ [11]. In this case, the hidden objects in the Fermi model are Z and W bosons. There are other prescriptions for the treatment of the UV problem [28], but that is out of the scope of this work.

On the other hand, when the electromagnetic corrections are computed in the MI approach, the discussion about the value of the cutoff Λ is left aside. For instance, consider we compute the electromagnetic corrections to Eq. (27) following the work of Ref. [20]; in this case, the one loop invariant matrix can be written as

$$\begin{aligned}\mathcal{M}^v = \mathcal{M}^{(0)}\frac{\alpha}{4\pi}f^{e.m.}_{m.i.}(u,\lambda^2) + \mathcal{M}_+\frac{\alpha}{4\pi}f^r_{m.i.}(u)\\ + \mathcal{M}^{(0)}\frac{\alpha}{4\pi}f_+^{m.d}(u),\end{aligned} \tag{30}$$

where

$$\begin{aligned}\mathcal{M}_+ = G_F V_{\mathrm{CKM}} C_{\mathrm{CG}} F_V \bar{u}_{\nu_\tau}(q)\gamma_\mu(1-\gamma^5)\\ \times (p^+ - p^0)^\mu \not{p}^+ u_\tau(p_\tau).\end{aligned} \tag{31}$$

The first and second pieces of Eq. (30) are independent of structure effects, come from the convection and spin term, are gauge invariant, are free of UV divergences, and all the IR singularity is located in these terms. The last piece gathers, in the function $f_+^{m.d}(u)$, the electromagnetic effects on the vector form factor, details of strong interactions, and the model-dependent assumptions to cancel the UV divergences. Notice that, in assuming the vector form factor is the dominant one, we are taking into account only the effects at $\mathcal{O}(\alpha)$ to this form factor. Adding Eqs. (27) and (30), we get

$$\mathcal{M} = \mathcal{M}^{(0)\prime}\left[1 + \frac{\alpha}{4\pi}f^{e.m.}_{m.i.}(u,\lambda^2)\right] + \mathcal{M}'_+\frac{\alpha}{4\pi}f^r_{m.i.}(u), \tag{32}$$

where the vector form factor has been redefined at order α as follows:

$$F'_V(u) = F_V + \frac{\alpha}{4\pi}f_+^{m.d.}(u). \tag{33}$$

This fact is indicated by a prime on the amplitudes of Eq. (32). As a consequence, it is assumed that $F'_V(u)$ is extracted from the experiment and hence gives information about the structure dependence and also helps to select the best model or theory that describes this form factor and its complications.

The same procedure is applied when the tree-level amplitude depends on two form factors; in this case, the MD amplitude can be written as follows:

$$\begin{aligned}\mathcal{M}^v_{md} = G_F V_{\mathrm{CKM}} C_{\mathrm{CG}} l_\mu \frac{\alpha}{4\pi}[F_+^{m.d}(s,u)(p^+ - p^0)^\mu\\ + F_-^{m.d}(s,u)(p^+ + p^0)^\mu].\end{aligned} \tag{34}$$

In this general case, the form factors are redefined as follows:

$$F'_+(s,u) = F_+(s) + \frac{\alpha}{4\pi}F_+^{m.d.}(s,u),$$
$$F'_-(s,u) = F_-(s) + \frac{\alpha}{4\pi}F_-^{m.d.}(s,u). \tag{35}$$

III. MODEL-INDEPENDENT RADIATIVE CORRECTIONS

A. Virtual photons

The one-loop electromagnetic corrections are computed from the diagrams shown in Fig. 2, with the tree-level amplitude given in Eq. (4). The amplitude of the first external diagram Fig. 2(a) reads

$$\mathcal{M}_a = \bar{u}_\nu(q)G\frac{-\imath e^2}{(2\pi)^4}\int dk^4[F_+(s')(\not{p}^+ - \not{p}^0 + \not{k}) + F_-(s')(\not{p}^+ + \not{p}^0 + \not{k})]\gamma^7\frac{\not{p}_\tau + \not{k} - m_\tau}{k^2[(p_\tau+k)^2 - m_\tau^2]} \times \frac{[2\not{p}^+ + \not{k}]}{(p^+ + k)^2 - m_+^2}u_\tau(p_\tau), \tag{36}$$

where now $s' = (p^0 + p^+ + k)^2$ and here we define $G = G_F V_{\mathrm{CKM}} C_{\mathrm{CG}}$ and $\gamma^7 = (1-\gamma^5)$ in order to avoid long expressions. According to our aim and following the result given in Ref. [20], the amplitude is separated in one piece that depends only on general QED properties and a second piece that depends on the description of the structure.

This assumption allows us to write the amplitude $\mathcal{M}_a$ in the form

$$\mathcal{M}_a = (\mathcal{M}_a^{cc} + \mathcal{M}_a^{st}) + \mathcal{M}_a^{md}, \tag{37}$$

where

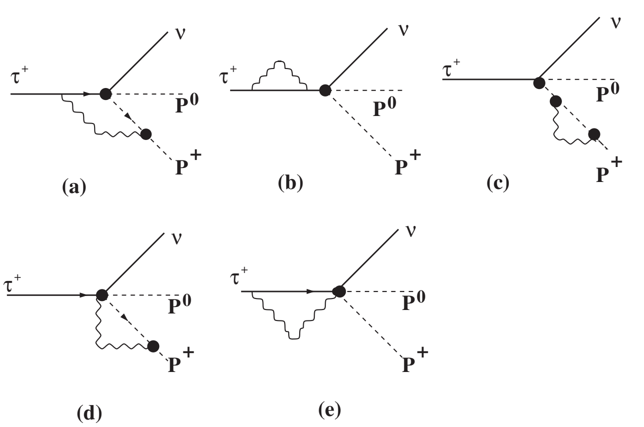

FIG. 2. The one-loop external radiative correction is constructed with diagrams (a), (b), and (c), where $P^+(P^0)$ denotes the charged (neutral) scalar meson. Diagrams (d) and (e) correspond to internal radiative corrections.

$$\mathcal{M}_a^{cc} = \mathcal{M}_\tau^{(0)}\frac{-\imath e^2}{(2\pi)^4}\int\frac{(2p_\tau + k)\cdot(2p^+ + k)}{k^2[(p_\tau+k)^2 - m_\tau^2]} \times \frac{dk^4}{[(p^+ + k)^2 - m_+^2]} \tag{38}$$

is known as the convection term and contains the infrared singularity, some UV divergences, and the major finite correction,

$$\mathcal{M}_a^{st} = G\bar{u}_\nu(q)h_\nu\gamma^\nu\gamma^7\frac{\imath e^2}{(2\pi)^4}\int\frac{\frac{1}{2}[\gamma^\alpha,\not{k}](2p^+ + k)_\alpha}{k^2[(p_\tau+k)^2 - m_\tau^2]} \times \frac{dk^4}{[(p^+ + k)^2 - m_+^2]}u_\tau(p_\tau), \tag{39}$$

is the spin term that is UV and IR finite and also gauge invariant by itself. The MI correction piece of this amplitude is equal to $\mathcal{M}_a^{cc} + \mathcal{M}_a^{st}$; the MD piece denoted as $\mathcal{M}_a^{md}$ is written as indicated in Eq. (34).

The convection term is written in terms of the tree-level amplitude times the electromagnetic correction as

$$\mathcal{M}_a^{cc} = \mathcal{M}_\tau^{(0)}\frac{\alpha}{4\pi}[4p_\tau\cdot p^+ C0[m_+^2, u, m_\tau^2, \lambda^2] + B0[m_\tau^2, 0, m_\tau^2] - B0[u, m_+^2, m_\tau^2] + B0[m_+^2, 0, m_+^2]], \tag{40}$$

where λ is the fictitious mass of the photon used in order to control the IR divergence. The integration is done using the FEYNCALC [29] package for MATHEMATICA, and in order to have the certainty of an exact IR singularity cancellation, we use the analytical expression for the scalar one-loop function (see the Appendix). The spin term is found to be

$$\mathcal{M}_a^{st} = \mathcal{M}_\tau^{(0)}\frac{\alpha}{4\pi}(2p_\tau\cdot p^+)f^r + G\bar{u}_{\nu_\tau}(q)h_\nu\gamma^\nu\gamma^7\not{p}^+u_\tau(p_\tau)\frac{\alpha}{4\pi}[2m_\tau f^r], \tag{41}$$

where the function f^r reads

$$f^r = \frac{1}{m_+^2(y_\tau^2 - 4r_\tau)}\left[\frac{m_+^2(2-y_\tau)}{u}\left[\frac{2-y_\tau}{2}\ln[r_\tau] - \frac{x_\tau}{1-x_\tau^2}\frac{\ln[x_\tau]}{\sqrt{r_\tau}}(y_\tau^2 - 4r_\tau)\right] - \ln[r_\tau^2]\right] \tag{42}$$

and we have used the same notation given in Ref. [30].

The amplitude of Fig. 2(b) is the very well-known lepton self-energy, which is entirely MI and reads

$$\mathcal{M}_b = \frac{\mathcal{M}_\tau^{(0)}}{2}\frac{\alpha}{4\pi}\left[-\Delta_{\mathrm{UV}} + 2\ln\left[\frac{m_\tau^3}{\mu\lambda^2}\right] - 4\right], \tag{43}$$

where $\Delta_{\mathrm{UV}} = \frac{2}{4-D} - \gamma_E + \ln[4\pi]$ in dimensional regularization, μ is the well-known mass parameter, and γ_E is the Euler constant.

The amplitude of Fig. 2(c) represents the scalar meson self-energy, and it is split into two pieces:

$$\mathcal{M}_c = \mathcal{M}_c^{cc} + \mathcal{M}_c^{md}. \quad (44)$$

The MI contribution is given by (see Ref. [20])

$$\mathcal{M}_c^{cc} = \mathcal{M}_\tau^{(0)} \frac{1}{2} \frac{4\imath\pi\alpha}{(2\pi)^4} \int \frac{dk^4 (2p^+ + k)\cdot(2p^+ + k)}{k^2[(p^+ + k)^2 - m_+^2]^2}, \quad (45)$$

and the MD piece ($\mathcal{M}_c^{md}$) is written in the form given in Eq. (34). The addition of the amplitudes corresponding to last diagrams [Fig. 2(d) and 2(e)] is MD; we denote this as follows:

$$\mathcal{M}_{de}^{md} = \mathcal{M}_d^{md} + \mathcal{M}_e^{md}. \quad (46)$$

The total convection amplitude $\mathcal{M}_a^{cc} + \mathcal{M}_b + \mathcal{M}_c^{cc}$ is free of UV singularities and is electromagnetic gauge invariant, which can be easily checked by adding to the photon propagator the term $\xi k^\mu k^\nu / k^2$, where ξ is an arbitrary parameter, and then seeing that ξ-dependent contributions from the lepton self-energy and MI meson self-energy cancel the respective terms coming from Eq. (40).

The one-loop electromagnetic correction is the sum of Eqs. (37), (43), (44), and (46) and reads

$$\mathcal{M}^v = \mathcal{M}_\tau^{(0)} \frac{\alpha}{4\pi}[2(p_\tau \cdot p^+) f^r + f_{\text{m.i.}}^v] + G\bar{u}_{\nu_\tau}(q) h_\nu \gamma^\nu \gamma^7 \not{p}^+ u(p_\tau) \frac{\alpha}{4\pi}[2m_\tau f^r] + \mathcal{M}_T^{md}, \quad (47)$$

where the total MD contribution is

$$\mathcal{M}_T^{md} = \mathcal{M}_a^{md} + \mathcal{M}_c^{md} + \mathcal{M}_{de}^{md}, \quad (48)$$

and $f_{\text{m.i.}}^v$ embraces the total convection effect and reads

$$f_{\text{m.i.}}^v = 2\ln\left[\frac{m_+ m_\tau}{\lambda^2}\right] - 1 + \frac{m_+ m_\tau}{u}\left[\frac{1}{x_t} - x_t\right]\ln[x_t] + 2m_+^2 y_t C0[m_+^2, u, m_\tau^2, \lambda^2] - \frac{m_+^2}{2u}(1 - r_t)\ln[r_t]. \quad (49)$$

The first and second pieces of Eq. (47) are the MI one-loop correction, and the total MD contribution is written as indicated in Eq. (34). From here, we assume that the model-dependent part is included in the redefinition of the form factors as given in Eq. (35).

The MI one-loop correction to the differential tau decay width is written as

$$\frac{d\Gamma_{P^+P^0}}{dsdu} = \frac{|G|^2}{(4\pi)^3 m_\tau^3}\Big[\rho^{(0)}(s,u) + \frac{\alpha}{2\pi}[\rho^{(0)}(s,u) \times [f_{\text{m.i.}}^v + 2p_\tau \cdot p^+ f^r] + m_\tau^2 f^r\{|F_+(s)|^2 \mathcal{G}(s,u) + |F_-(s)|^2 \mathcal{E}(s,u) + 2F_+^\dagger(s) F_-(s) \mathcal{H}(s,u)\}]\Big], \quad (50)$$

where

$$\mathcal{G}(s,u) = s(m_+^2 - u) + m_+^2(4u - 4m_0^2) + m_\tau^2(m_0^2 - m_+^2), \quad (51)$$

$$\mathcal{E}(s,u) = s(m_+^2 - u) + m_\tau^2 m_0^2 - m_+^2 m_\tau^2, \quad (52)$$

$$\mathcal{H}(s,u) = s(m_+^2 + u) - m_\tau^2 m_0^2 - m_+^2 m_\tau^2. \quad (53)$$

The IR singularity enclosed only in $f_{m.i.}^v$ is cancelled after taking into account the real-photon emission [31].

In the case of two pions in the hadronic final state where the vector form factor is dominant, Eq. (50) yields

$$\frac{d\Gamma_{\pi\pi}}{dsdu} = \frac{\|GF_V(s)\|^2}{(4\pi)^3 m_\tau^3}\Big[\mathcal{D}(s,u) + \frac{\alpha}{2\pi}[(f_{m.i.}^v + 2p_\tau \cdot p^+ f^r)\mathcal{D}(s,u) + m_\tau^2 f^r \mathcal{G}(s,u)]\Big]. \quad (54)$$

B. Real-photon emission

To be consistent with the work done in the one-loop correction, the model-independent part of the radiative process must be defined, a task that is achieved by using the low-energy theorems [32,33].

According to the Low theorem [32], the radiative invariant amplitude denoted as $\mathcal{M}^\gamma$ can be expanded in powers of the photon energy k for small k as

$$\mathcal{M}^\gamma = \frac{\mathcal{M}}{k} + \mathcal{M}_1 k^0 + k\mathcal{M}_2 + \cdots, \quad (55)$$

where the dots symbolize terms with powers of order ≥ 2 in k. The first piece $\mathcal{M}$ (Low term) and $\mathcal{M}_1$ can be calculated completely from the nonradiative invariant amplitude; meanwhile, $\mathcal{M}_2$ and the next elements of the series depend on the theoretical model that describes the details of the photon emission from either hadronic external lines or an internal hadronic vertex. This means that Eq. (55) establishes the definition of the model-independent and model-dependent terms in the radiative amplitude.

On the other hand, it was shown in Ref. [33] that the unpolarized and squared amplitude of the radiative process can be split into two parts, one element of order $1/k^2$ that comes entirely from the Low term and the rest that contains contributions of order $k^0, k^1, \ldots$, as the equation

$$\sum_{\text{spins}} |\mathcal{M}^\gamma|^2 = \sum_{\text{spins}} |\mathcal{M}^{(0)} e|^2 \left[\frac{p^+ \cdot \epsilon}{p^+ \cdot k} - \frac{p_\tau \cdot \epsilon}{p_\tau \cdot k}\right]^2 + \sum_{\text{spins}} |\mathcal{M}'|^2 k^0 + \sum_{\text{spins}} |\mathcal{M}''|^2 k^1 + \cdots \quad (56)$$

shows, where $\epsilon = \epsilon(k)$ is the photon polarization vector and $\sum_{\text{spins}}$ indicates an average over initial spin states and a sum over final spin states, except over the photon degrees of freedom. The first piece is the Low term, which is precisely the convection term in the MI scheme and encloses the appropriated terms that cancel the IR singularity in Eq. (49). The term of order k^0 includes contributions from the interference between the model-independent and the model-dependent terms; therefore, it is model dependent. In this work, we consider only the Low term for the radiative amplitude.

According to the previous lines, our gauge-invariant MI radiative amplitude for the $\tau^+ \to K^0\pi^+\nu\gamma$ decay reads

$$\mathcal{M}_\tau^\gamma = \mathcal{M}_\tau^{(0)'} e\left[\frac{p^+\cdot\epsilon}{p^+\cdot k} - \frac{p_\tau\cdot\epsilon}{p_\tau\cdot k}\right]. \tag{57}$$

The soft photon approximation [34] was computed in a previous work [35] with a careful handling of the infrared singularity [36], and it was shown that the radiative correction depends on a cutoff energy ω_0. However, in this work, we consider an alternative procedure [15,37,38] that proposes a separation of the Dalitz-plot region in such a way that the uncomfortable dependence is avoided.

In computing the well-known invariant integrals for the real-photon correction, a novel approach was done in the work of A. Martínez *et al.* [23], obtaining the same result as Ginsberg [37]; however, we follow the technique of the latter. In the radiative process, a new variable arises known as the invariant mass of the undetected particles denoted by $x = (q+k)^2$ and is bounded as

$$x_-(s,u) \le x \le x_+(s,u), \tag{58}$$

where $k = (k_0, \mathbf{k})$ are the energy and the momentum of the photon and q is the neutrino 4-momentum. The maximal and minimal values of x are given in the Appendix. To compute the real-photon contribution, some assumptions have to be considered; in this respect, we adopt those given in Ref. [23], which means the following:

(i) the allowed kinematical region of all values for (u, s) that satisfy the relation of the three-body space phase, given by Eqs. (16) and (17);

(ii) the values of the Lorentz-invariant x consistent with the first point.

In other words, we consider the set of values of u, s, and x that define a region for which the borders are given as follows:

$$D_{III} = \{u_-(s) \le u \le u_+(s), (m_{K^0}+m_{\pi^+})^2 \le s \le m_\tau^2, \\ \lambda^2 \le x \le x_+(s,u)\}. \tag{59}$$

It is important to point out that the IR divergence is precisely within this region. Here, the minimal value of x is written in terms of λ, the same regulator used in handling the IR singularity in Sec. II A. As a consequence of these remarks, we assume a discrimination of real photons that could be done in an experimental setup by means of an analysis of the 4-body radiative Dalitz plot (see Fig. 3) [39].

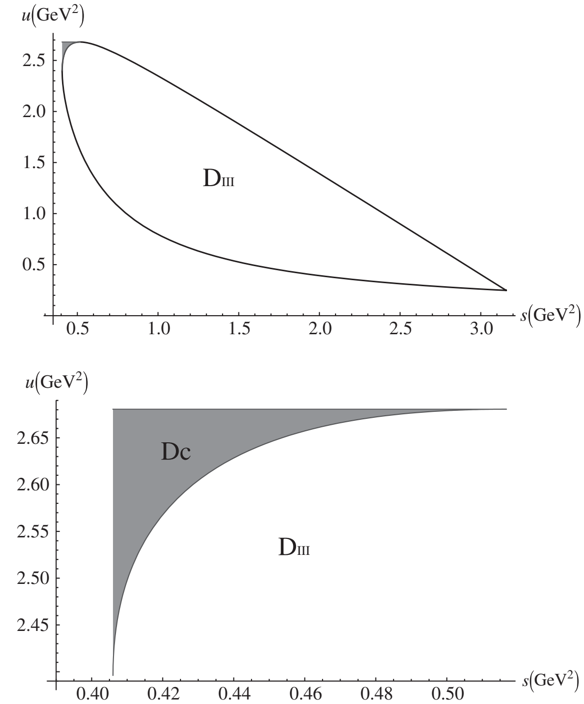

FIG. 3. The upper plot shows the projection of the 4-body radiative Dalitz plot [see Eq. (59)] onto the u–s plane, where it can be seen that the 3-body Dalitz plot is inside this region. The lower plot shows an amplification of the projected complementary region D_c [see Eq. (60)] accessible only to the radiative process.

The discriminated photons are inside the complementary region D_c, only accessible to the radiative process, where the energy of the photon is never zero, and as a consequence, the radiative amplitude is free of IR singularity. This region is defined for the following set of values:

$$D_c = \left\{u_+ \le u \le (m_\tau - m_{\pi^+})^2, x_- \le x \le x_+, \\ s_{\min} \le s \le \frac{m_\tau(m_\tau m_{\pi^+} + m_{K^0}^2 - m_{\pi^+}^2)}{m_\tau - m_{\pi^+}}\right\}. \tag{60}$$

Once our assumptions have been clarified, we can calculate the contribution of the Low term, Eq. (57), with the corresponding phase space described in Eq. (59). Notice that we are computing Eq. (18) given in Ref. [37], but adapted to our process. The differential radiative decay width reads

$$\frac{d\Gamma_{K\pi}^\gamma}{dsdu} = \frac{G_F^2|V_{us}F_+^{K\pi}(0)|^2 C_{CG}^2 \rho(s,u)}{m_\tau^3(4\pi)^3}\frac{\alpha}{\pi}(I_{1,1}+I_{2,0}+I_{0,2}). \tag{61}$$

The very well-known invariant integrals are given in Ref. [37], and details of the computation and the notation are presented in the appendix; here, we just write our results as follows:

$$I_{2,0}=\ln\left[\frac{m_\tau^2-s}{x_+(s,u)}\right]+\ln\left[\frac{\lambda}{m_\tau}\right],$$

$$I_{0,2}=\ln\left[\frac{m_\tau^2-s-u+m_{K^0}^2}{x_+(s,u)}\right]+\ln\left[\frac{\lambda}{m_{\pi^+}}\right],$$

$$\begin{aligned}I_{1,1}=\frac{x_\tau y_\tau}{\sqrt{r_\tau}(1-x_\tau^2)}\Big\{&Li_2\left[\frac{-a^2}{4r_+}\right]-Li_2\left[\frac{-4r_-}{a^2}\right]\\&-\ln[x_\tau]\ln\left[\frac{y_\tau^2-4r_\tau}{\sqrt{r_\tau}}\right]+\ln[x_\tau]\ln\left[\frac{\lambda^2}{m_\tau m_{\pi^+}}\right]\\&-\ln[x_\tau]\ln\left[\frac{x_\tau x_+^2(s,u)}{4r_+}\right]\Big\}.\end{aligned}\tag{62}$$

IV. CORRECTIONS TO THE $\tau^+\to K^0\pi^+\nu$ DECAY

The results of the previous sections are applicable to the final hadronic modes ($K^0\pi^+$, $K^+\pi^0$, $\pi^0\pi^+$); however, we are interested only in the first mode. Consequently, we present here the MI electromagnetic corrections to the differential width decay due to virtual photons, Eq. (50), and real photons, Eq. (61), computed within the region given in Eq. (59),

$$\begin{aligned}\frac{d\Gamma_{K^0\pi^+}}{dsdu}=\frac{G_F^2|V_{us}F_+^{K\pi}(0)|^2C_{CG}^2}{(4\pi)^3m_\tau^3}\{&|\tilde F_+(s)|^2\tilde{\mathcal{D}}(s,u)\\&+2\tilde F_-(s)\tilde F_+(s)\tilde{\mathcal{D}}_3(s,u)+|\tilde F_-(s)|^2\tilde{\mathcal{D}}_2(s,u)\},\end{aligned}\tag{63}$$

where $C_{CG}^2=1/2$ and the $\tilde{\mathcal{D}}$ functions corrected are given by

$$\tilde{\mathcal{D}}(s,u)=\mathcal{D}(s,u)+\frac{\alpha}{2\pi}[\mathcal{D}(s,u)f_I^{m.i.}+\mathcal{G}(s,u)f_{II}^{m.i.}],\tag{64}$$

$$\tilde{\mathcal{D}}_2(s,u)=\mathcal{D}_2(s)+\frac{\alpha}{2\pi}[\mathcal{D}_2(s)f_I^{m.i.}+\mathcal{E}(s,u)f_{II}^{m.i.}],\tag{65}$$

$$\tilde{\mathcal{D}}_3(s,u)=\mathcal{D}_3(s,u)+\frac{\alpha}{2\pi}[\mathcal{D}_3(s,u)f_I^{m.i.}+\mathcal{H}(s,u)f_{II}^{m.i.}],\tag{66}$$

where

$$f_I^{m.i.}=2[I_{2,0}+I_{0,2}+I_{1,1}]+f_{m.i.}^v+m_{\pi^+}^2y_\tau f^r,\tag{67}$$

$$f_{II}^{m.i.}=m_\tau^2f^r.\tag{68}$$

The simple form of Eq. (63) gives a hand to identify the effect of the electromagnetic corrections:

(i) The density function, Eq. (7), is corrected at $\mathcal{O}(\alpha)$ by the MI corrections.

(ii) The form factors include, for definition in Eq. (35), the MD corrections at $\mathcal{O}(\alpha)$.

Integrating Eq. (63), we obtain the total decay width corrected by the MI electromagnetic corrections,

$$\Gamma_{K^0\pi^+}=\Gamma_{K^0\pi^+}^{(0)}[1+\delta_{\rm EM}^{m.i.}],\tag{69}$$

where the electromagnetic correction function reads

$$\begin{aligned}\delta_{\rm EM}^{m.i.}=\frac{[\alpha/2\pi]}{m_\tau^8I_{K\pi}^{(0)}}\frac{3}{4}\int\{&|\tilde F_+(s)|^2[\mathcal{D}(s,u)f_I^{m.i.}+\mathcal{G}(s,u)f_{II}^{m.i.}]\\&+|\tilde F_-(s)|^2[\mathcal{D}_2(s)f_I^{m.i.}+\mathcal{E}(s,u)f_{II}^{m.i.}]\\&+2\tilde F_-(s)\tilde F_+(s)[\mathcal{D}_3(s,u)f_I^{m.i.}\\&+\mathcal{H}(s,u)f_{II}^{m.i.}]\}dsdu.\end{aligned}\tag{70}$$

To estimate the electromagnetic correction, we follow the theoretical description for the form factors as given in Sec. II A; hence, we obtain $\delta_{\rm EM}^{m.i.}=-0.127\%$. The aim of this work has been achieved by writing Eq. (70); however, the Dalitz-plot corrections can be obtained straightforward by using Eqs. (64)–(66).

On the other hand, Antonelli *et al.* [40] have given an estimate of the long-distance electromagnetic corrections to $\tau^+\to K^0\pi^+\nu$, where the structure-dependent effects have been neglected. Their approach relies in the analysis already done in Kl_3 and $\tau\to\pi\pi\nu$. They found $\delta_{em}^{\bar K^0\pi^+}=(-0.15\pm0.2)\%$, where the uncertainty they assigned is due to the unknown structure-dependent effects.

V. CONCLUSIONS

Summarizing the work done, we provide the MI electromagnetic corrections to the $\tau^+\to K^0\pi^+\nu$ decay following the procedure described in Ref. [20], considering both virtual and real photons within the 3-body phase space region given in Eq. (59). As it has been pointed out, this correction is electromagnetic gauge invariant, free of IR singularities, free of UV singularities, and, most importantly, it does not have an UV cutoff. This approach considers that all structure dependence is included inside the form factors $\tilde F_\pm(s)$ after an appropriated redefinition. In this respect, our work is not focused in the theoretical description of those form factors or the corresponding solution to the remaining UV problem.

On the other hand, it is important to accentuate that Eq. (70) can be used for the hadronic final states $\pi^0K^\pm$ and $\pi^\pm\pi^0$ with the corresponding replacements.

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APPENDIX: INVARIANT INTEGRALS

The description of the radiative process [Sec. III B] requires the introduction of the Mandelstam variable

$$x = (q + k)^2, \tag{A1}$$

where q is the neutrino 4-momenta and k is the photon 4-momenta. The maximum and minimum values of x are given as

$$\begin{aligned} x_\pm(s,u) = \frac{1}{2m_\pi^2}[&u(m_+^2 - m_0^2) - m_+^2(m_+^2 - m_0^2) \\ &+ m_\tau^2(m_+^2 + m_0^2) - s(m_\tau^2 - m_+^2 - u) \\ &\pm \lambda^{1/2}(u, m_+^2, m_\tau^2)\lambda^{1/2}(s, m_+^2, m_0^2)], \end{aligned} \tag{A2}$$

where $m_+(m_0)$ is the mass of the charged (neutral) pseudoscalar and the well-known Källen function reads

$$\lambda(a,b,c) = a^2 + b^2 + c^2 - 2(ab + ac + bc). \tag{A3}$$

According to the definition given in Ref. [37], the radiative contribution requires the evaluation of the functions I_{11}, I_{20}, and I_{02}. These types of integrals were already studied in Ref. [37]; here, we present brief details about computing them for our situation. The easiest one is written as

$$I_{2,0} = -\lim_{\lambda\to 0}\int_{\lambda^2}^{x_+(s,u)} p_\tau^2 i_{2,0}\,dx \tag{A4}$$

$$i_{2,0} = \frac{1}{2\pi}\int \frac{d^3\mathbf{k}\,d^3\mathbf{q}}{k_0 q_0}\frac{\delta(\mathbf{P} - \mathbf{k} - \mathbf{q})\delta(E_\nu - k_0 - q_0)}{[(p_\tau - k)^2 - m_\tau^2]^2}, \tag{A5}$$

where $P = p_\tau - p_+ - p_0$ and $E_\nu = E_\tau - E_+ - E_0$. The regulator for the IR singularity is chosen to be the same as the one used in the one-loop correction. To simplify the evaluation of Eq. (A5), we chose a frame in which $\mathbf{P} = 0$. After integrating over the photon variable, one delta function is canceled, and we are left with

$$\begin{aligned} i_{2,0} = \frac{1}{8\pi}\int &\frac{d^3\mathbf{q}\,\delta\Big(E_\nu - \sqrt{q_0^2 + \lambda^2} - q_0\Big)}{q_0\sqrt{q_0^2 + \lambda^2}} \\ &\times \frac{1}{\Big[E_\tau\sqrt{q_0^2 + \lambda^2} + \mathbf{p}_\tau\cdot\mathbf{q} - \lambda^2/2\Big]^2}. \end{aligned} \tag{A6}$$

Equation (A6) is evaluated in spherical coordinates, computing first over the azimuth angle ϕ and then over q_0, so we get

$$\begin{aligned} i_{2,0} = \int_0^\pi &\frac{(E_\nu^2 - \lambda^2)\sin\theta d\theta}{2} \\ &\times \frac{1}{[E_\tau(E_\nu^2 + \lambda^2) + |\mathbf{p}_\tau|(E_\nu^2 - \lambda^2)\cos\theta - \lambda^2 E_\nu]^2}. \end{aligned} \tag{A7}$$

After doing the integration in θ, Eq. (A7) is found to be

$$i_{2,0} = \frac{(x - \lambda^2)}{x^2 m_\tau^2 + 4\lambda^2(P\cdot p_\tau)^2 - 2\lambda^2 p_\tau^2 x}, \tag{A8}$$

where we have thrown away contributions of $\mathcal{O}(\lambda^4)$ and greater. Finally, the x integration is done in Eq. (A4), and at the end of the procedure, we maintain only the logarithm term in λ. We thus find

$$I_{2,0} = \ln\left[\frac{m_\tau^2 - s}{x_+(s,u)}\right] + \ln\left[\frac{\lambda}{m_\tau}\right]. \tag{A9}$$

The computation of $I_{0,2}$ is quite similar, and here we just present the result:

$$I_{0,2} = \ln\left[\frac{m_\tau^2 - u - s + m_0^2}{x_+(s,u)}\right] + \ln\left[\frac{\lambda}{m_+}\right]. \tag{A10}$$

The third integral is written as follows:

$$I_{1,1} = \lim_{\lambda\to 0}\int_{\lambda^2}^{x_+(s,u)} 2p_\tau\cdot p_+ i_{1,1}\,dx \tag{A11}$$

$$\begin{aligned} i_{1,1} = \frac{1}{2\pi}\int &\frac{d^3\mathbf{q}\,d^3\mathbf{k}\,\delta(\mathbf{k} + \mathbf{q})}{4q_0k_0[E_\tau k_0 - \mathbf{p}_\tau\cdot\mathbf{k} - \lambda^2/2]} \\ &\times \frac{\delta(E_\nu - k_0 - q_0)}{[E_+k_0 - \mathbf{p}_+\cdot\mathbf{k} + \lambda^2/2]}. \end{aligned} \tag{A12}$$

After integrating the delta function $\delta(\mathbf{k} + \mathbf{q})$, we get

$$\begin{aligned} i_{1,1} = \frac{1}{2\pi}\int &\frac{d^3\mathbf{q}}{4q_0k_0[E_\tau k_0 + \mathbf{p}_\tau\cdot\mathbf{q} - \lambda^2/2]} \\ &\times \frac{\delta(E_\nu - k_0 - q_0)}{[E_+k_0 - \mathbf{p}_+\cdot\mathbf{k} + \lambda^2/2]}, \end{aligned} \tag{A13}$$

where $k_0 = \sqrt{q_0^2 + \lambda^2}$. The Feynman trick allows us to combine propagators as

$$\begin{aligned} \frac{1}{ab} &= \frac{1}{[E_\tau k_0 + \mathbf{p}_\tau\cdot\mathbf{q} - \lambda^2/2][E_+k_0 + \mathbf{p}_+\cdot\mathbf{q} + \lambda^2/2]} \\ &= \int_0^1 \frac{dz}{[c + \mathbf{q}\cdot\mathbf{p}]^2}, \end{aligned} \tag{A14}$$

where

$$c = E_+k_0 + z((E_\tau - E_+)k_0 - \lambda^2) + \lambda^2/2, \tag{A15}$$

$$\mathbf{p} = \mathbf{p}_\tau z + \mathbf{p}_+(1 - z). \tag{A16}$$

Combining propagators, using the properties of the delta function, and computing in spherical coordinates, Eq. (A13) is reduced to

$$\begin{aligned} i_{1,1} &= \frac{1}{2\pi}\frac{E_\nu^2 - \lambda^2}{8E_\nu^2}\int_0^1\int_0^{2\pi}\int_0^\pi \frac{\sin\theta d\theta d\phi dz}{\Big[\bar{c} + \frac{E_\nu^2 - \lambda^2}{2E_\nu}|\mathbf{p}|\cos\theta\Big]^2} \\ &= \frac{E_\nu^2 - \lambda^2}{4E_\nu^2}\int_0^1 \frac{dz}{\bar{c}^2 - \Big(|\mathbf{p}|\frac{E_\nu^2 - \lambda^2}{2E_\nu}\Big)^2}, \end{aligned} \tag{A17}$$

where

$$\bar{c} = E_+ \frac{E_\nu^2 + \lambda^2}{2E_\nu} + z\left[(E_\tau - E_+)\frac{E_\nu^2 + \lambda^2}{2E_\nu} - \lambda^2\right] + \lambda^2/2. \tag{A18}$$

To do the z integration, the denominator is written as a polynomial function on z,

$$\bar{c}^2 - \left(|\mathbf{p}|\frac{E_\nu^2 - \lambda^2}{2E_\nu}\right)^2 = \frac{1}{4E_\nu^2}[\alpha z^2 + 2\beta z + \gamma]. \tag{A19}$$

The coefficients are expressed in terms of scalar products as follows:

$$\begin{aligned}\alpha &= (x-\lambda^2)^2(m_\tau^2 + m_+^2 - 2(p_\tau\cdot p_+)) \\ &\quad + 4\lambda^4 x + 4\lambda^2(p_\tau\cdot P - p_+\cdot P) \\ &\quad \times((p_\tau\cdot P - p_+\cdot P) - (x+\lambda^2)),\end{aligned} \tag{A20}$$

$$\begin{aligned}\beta &= (x-\lambda^2)^2(p_\tau\cdot p_+ - m_+^2) + 2\lambda^4\{P\cdot p_\tau - 3P\cdot p_+\} \\ &\quad + 4\lambda^2\{P\cdot p_\tau P\cdot p_+ - (P\cdot p_+)^2\} \\ &\quad - 2\lambda^4 x + (x-\lambda^2)\lambda^2\{P\cdot p_\tau - 3P\cdot p_+\},\end{aligned} \tag{A21}$$

$$\gamma = (x-\lambda^2)^2 m_+^2 + \lambda^2(2P\cdot p_+)(2P\cdot p_+ + x + \lambda^2) + \lambda^4 x. \tag{A22}$$

It is an easy matter to find that, after some algebraic manipulation, Eq. (A17) reads

$$\begin{aligned}i_{1,1} &= \int_0^1 \frac{(x-\lambda^2)dz}{\alpha z^2 + 2\beta z + \gamma} \\ &= \frac{(x-\lambda^2)}{2\sqrt{\beta^2 - \alpha\gamma}} \ln\left[\frac{\beta + \gamma + \sqrt{\beta^2 - \alpha\gamma}}{\beta + \gamma - \sqrt{\beta^2 - \alpha\gamma}}\right].\end{aligned} \tag{A23}$$

To proceed more, we write Eqs. (A20)–(A22) in terms of Maldenstam variables (s, u, x) with the help of the following relations:

$$P\cdot p_\tau = \frac{1}{2}(x + m_\tau^2 - s), \tag{A24}$$

$$P\cdot p_+ = \frac{1}{2}(m_\tau^2 + m_0^2 - s - u), \tag{A25}$$

$$p_\tau\cdot p_+ = \frac{1}{2}(m_\tau^2 + m_+^2 - u). \tag{A26}$$

Finally, we reduce Eq. (A11) in the conventional form given by Ginsberg (Eq. (25) in Ref. [37]),

$$\begin{aligned}I_{1,1} &= \lim_{\lambda\to 0}\int_{\lambda^2}^{x_+(s,u)} dx \frac{E_\tau}{\bar{\delta}} \\ &\quad \times \ln\left[\frac{x^2 E_\tau + 2\lambda^2\epsilon_\nu(s - m_\tau^2) + (x-\lambda^2)\bar{\delta}}{x^2 E_\tau + 2\lambda^2\epsilon_\nu(s - m_\tau^2) - (x-\lambda^2)\bar{\delta}}\right],\end{aligned} \tag{A27}$$

where

$$\begin{aligned}&\bar{\delta} = \sqrt{x^2|\mathbf{p}_\tau|^2 + \lambda^2 a^2}, \qquad E_\tau = \frac{m_\tau^2 + m_+^2 - u}{2m_+}, \\ &a^2 = -x_+(s,u)x_-(s,u), \qquad |\mathbf{p}_\tau| = \frac{\sqrt{\lambda(u, m_\tau^2, m_+^2)}}{2m_+}, \\ &\epsilon_\nu = \frac{u + s - m_\tau^2 - m_0^2}{2m_+}.\end{aligned} \tag{A28}$$

Recalling the following definitions given by Ginsberg [37],

$$r_\pm = \frac{x_\tau}{m_\tau}\left\{\Omega_+ \pm \sqrt{\Omega_+^2 - \frac{1}{16}m_\tau^2 a^4}\right\}, \tag{A29}$$

$$\Omega_+ = \epsilon_\nu|\mathbf{p}_\tau|^2(s - m_\tau^2) - \frac{1}{4}a^2 E_\tau, \tag{A30}$$

and using the master formula (Eq. (28) in Ref. [37]), we get the final result

$$\begin{aligned}I_{1,1} &= \frac{x_\tau y_\tau}{\sqrt{r_\tau}(1 - x_\tau^2)}\left\{Li_2\left[\frac{-a^2}{4r_+}\right] - Li_2\left[\frac{-4r_-}{a^2}\right]\right. \\ &\quad + \ln[x_\tau]\ln\left[\frac{\lambda^2}{m_\tau m_+}\right] \\ &\quad \left. - \ln[x_\tau]\ln\left[\frac{(y_\tau^2 - 4r_\tau)}{\sqrt{r_\tau}}\frac{x_\tau x_+^2(s,u)}{4r_+}\right]\right\},\end{aligned} \tag{A31}$$

where

$$\begin{aligned}&r_\tau = \frac{m_\tau^2}{m_+^2}, \qquad y_\tau = 1 + r_\tau - \frac{u}{m_+^2}, \\ &x_\tau = \frac{1}{2\sqrt{r_\tau}}\left[y_\tau - \sqrt{y_\tau^2 - 4r_\tau}\right],\end{aligned} \tag{A32}$$

and the very well-known dilogarithm function [41] is defined as follows:

$$Li_2(z) = -\int_0^1 \frac{\ln[1 - zy]}{y}dy. \tag{A33}$$

In computing the one-loop calculation, the 3-point scalar function $C0[m_+^2, u, m_\tau^2, \lambda^2]$ can be evaluated with the LOOPTOOLS package [29]. However, in order to see the exact cancellation of the IR singularity, the function is written in terms of logarithms and dilogarithms with the help of the general form given in Ref. [42],

$$C0[m_+^2, u, m_\tau^2, \lambda^2] = \frac{x_t}{m_+ m_\tau (1 - x_t^2)} \left[-\frac{1}{2} \ln^2[x_t] - \frac{\pi^2}{6} + 2 \ln[x_t] \ln[1 - x_t^2] + \frac{1}{8} \ln^2[r_t] + Li_2\left[1 - \frac{x_t}{\sqrt{r_t}}\right] + Li_2[x_t^2] \right.$$
$$\left. + Li_2[1 - x_t \sqrt{r_t}] - \ln[x_t] \ln\left[\frac{\lambda^2}{m_+ m_\tau}\right]\right]. \tag{A34}$$

The $B0$ function is very well known, and all the specific cases can be obtained easily from its definition [43].

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